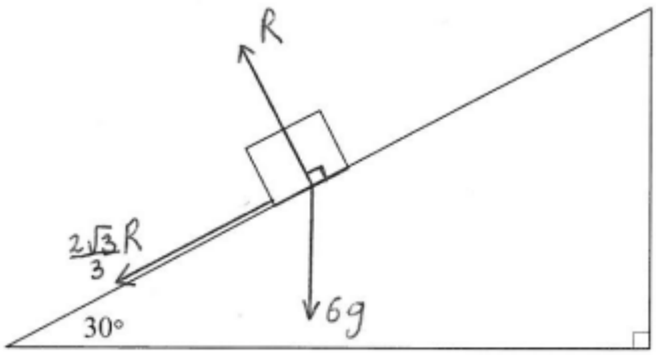
The Mathematical Association of Victoria

SPECIALIST MATHEMATICS 2014

Trial Written Examination 1 - SOLUTIONS

Question 1

a.



All three forces (it is acceptable for the friction force to be labelled $F_{friction}$ *etc.*) **[A1]**

Total 1 mark

Comment:

The forces acting on the body are:

- Normal reaction force R perpendicular to the plane.
- Weight force $= mg = 6g$ down.
- Friction force of size $\mu R = \frac{2\sqrt{3}}{3}R$ (because the body is sliding) and the force is acting down the plane because the body is sliding up the plane and the friction force opposes the motion.

b.

● Resolve forces parallel to the plane (the direction of motion of the body 'up the plane' is taken as the positive direction):

$F_{net} = ma = 6a$.

$F_{net} = -\mu R - mg\sin(30^\circ) = \frac{-2\sqrt{3}}{3} - 3g$.

Therefore: $6a = \frac{-2\sqrt{3}}{3}R - 3g$. .... (1) **[A1]**

● Resolve forces perpendicular to the plane (upwards is taken as the positive direction):

$F_{net} = 0$.

$F_{net} = R - mg\cos(30^\circ) = R - 3\sqrt{3}g$.

Therefore: $0 = R - 3\sqrt{3}g$

$\Rightarrow R = 3\sqrt{3}g$. .... (2)

Substitute equation (2) into equation (1):

$$6a = \frac{-2\sqrt{3}}{3}\left(3\sqrt{3}g\right) - 3g = -9g$$

$\Rightarrow a = -\frac{3g}{2}$. **[A1]**

Substitute $a = -\frac{3g}{2}$, $u = 3$ m/s and $v = 0$ m/s into $2as = v^2 - u^2$ and solve for s:

$s = \frac{3}{g}$ metres. **[A1]**

Total 4 marks

Question 2

a.

Method 1: Convert the complex numbers into polar form.

$u = -2\sqrt{3} - 2i = 4\text{cis}\left(-\frac{5\pi}{6}\right)$. Only the argument is required. **[A1]**

$v = -1 + i\sqrt{3} = 2\text{cis}\left(\frac{2\pi}{3}\right)$. Only the argument is required. **[A1]**

$-\frac{5\pi}{6} - \frac{2\pi}{3} = -\frac{9\pi}{6} = -\frac{3\pi}{2}$.

Therefore:

$\text{Arg}\left(\frac{u}{v}\right) = -\frac{3\pi}{2} + 2\pi = \frac{\pi}{2}$. **[A1]**

Total 3 marks

Method 2:

$$\frac{-2\sqrt{3} - 2i}{i\sqrt{3} - 1} = \frac{(-2\sqrt{3} - 2i)}{(-1 + i\sqrt{3})} \times \frac{(-1 - i\sqrt{3})}{(-1 - i\sqrt{3})}$$

$= \frac{2\sqrt{3} + 6i + 2i - 2\sqrt{3}}{4}$ **[A1]**

$= \frac{8i}{4} = 2i$. **[A1]**

$\text{Arg}(2i) = \frac{\pi}{2}$. **[A1]**

Total 3 marks

b.

Method 1:

Since all the coefficients are real, it follows from the conjugate root theorem that $-\frac{3}{2} - i\sqrt{2}$ is also a root of $p(z)$.

Therefore $z + \frac{3}{2} - i\sqrt{2}$ and $z + \frac{3}{2} + i\sqrt{2}$ are linear factors of $p(z)$.

Therefore a quadratic factor of $p(z)$ is

$$\left(z + \frac{3}{2} - i\sqrt{2}\right)\left(z + \frac{3}{2} + i\sqrt{2}\right) = \left(\left[z + \frac{3}{2}\right] - i\sqrt{2}\right)\left(\left[z + \frac{3}{2}\right] + i\sqrt{2}\right)$$

$$= \left(z + \frac{3}{2}\right)^2 + 2$$

$$= z^2 + 3z + \frac{17}{4}.$$ **[A1]**

By equating the coefficients of $\left(z^2 + 3z + \frac{17}{4}\right)(\alpha z - \beta)$ with $p(z)$ (the coefficient of z^3 is 4 and the constant term is -34) it follows that $(4z - 8)$ is also linear factor of $p(z)$. Therefore:

$$p(z) = \left(z^2 + 3z + \frac{17}{4}\right)(4z - 8)$$ **[M1]**

$$= (4z^2 + 12z + 17)(z - 2)$$

$$= 4z^3 + 4z^2 - 7z - 34.$$

Therefore $a = 4$ and $b = -7$. Both values. **[A1]**

Total 3 marks

Method 2:

Let two of the roots be α and β.

Then a quadratic factor is $(z - \alpha)(z - \beta) = z^2 - (\alpha + \beta)z + \alpha\beta$.

$\alpha = -\frac{3}{2} + i\sqrt{2}$ (given) and $\beta = \overline{\alpha} = -\frac{3}{2} - i\sqrt{2}$ (from the conjugate root theorem).

Therefore:

$\alpha + \beta = -3$.

$\alpha\beta = \left(-\frac{3}{2}\right)^2 + \left(\sqrt{2}\right)^2 = \frac{17}{4}$.

So a quadratic factor is $z^2 + 3z + \frac{17}{4}$.

Question 3

Substitute $u = 3 - 2x$: **[M1]**

- $\frac{du}{dx} = -2 \Rightarrow dx = \frac{du}{-2}$.
- $x = \frac{3-u}{2}$.
- $x = \frac{1}{2} \Rightarrow u = 2$ and $x = 1 \Rightarrow u = 1$.
- $\int_{1/2}^{1} \frac{x-1}{\sqrt{3-2x}}\, dx$

$$= \int_{2}^{1} \frac{\frac{3-u}{2} - 1}{\sqrt{u}} \left(\frac{du}{-2}\right)$$ **[M1]**

$$= -\frac{1}{4}\int_{2}^{1} \frac{1-u}{\sqrt{u}}\, du$$

$$= -\frac{1}{4}\int_{2}^{1} u^{-1/2} - u^{1/2}\, du$$ **[M1]**

$$= -\frac{1}{4}\left[2u^{1/2} - \frac{2}{3}u^{3/2}\right]_{2}^{1}$$

$$= \frac{\sqrt{2}-2}{6}.$$ **[A1]**

Round the final total DOWN to the nearest integer

Total 4 marks

Question 4

a.

$F = ma = 3a$.

$v = 1 - x^2$

$\Rightarrow a = v\dfrac{dv}{dx} = (1 - x^2)(-2x)$.

Therefore:

$F = 3(1 - x^2)(-2x) = -6x(1 - x^2)$. **[A1]**

Substitute $x = \dfrac{2}{3}$:

$F = -\dfrac{60}{27} = -\dfrac{20}{9}$

Answer: $F = -\dfrac{60}{27} = -\dfrac{20}{9}$. **[A1]**

Total 2 marks

b.

$v = 1 - x^2$

$\Rightarrow \dfrac{dx}{dt} = 1 - x^2$

$\Rightarrow \dfrac{dt}{dx} = \dfrac{1}{1 - x^2}$ **[M1]**

$\Rightarrow t = \displaystyle\int_0^{2/3} \left|\frac{1}{1 - x^2}\right| dx$ (because the total time is the area of the region bounded by $y = \dfrac{1}{1 - x^2}$ and the x-axis between $x = 0$ and $x = \dfrac{2}{3}$)

$= \displaystyle\int_0^{2/3} \frac{1}{1 - x^2}\, dx$ (since $\dfrac{1}{1 - x^2} > 0$ for $0 < x < \dfrac{2}{3}$)

$= \displaystyle\int_0^{2/3} \frac{A}{1 - x} + \frac{B}{1 + x}\, dx$. **[M1]**

Partial fraction calculation:

$$\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$= \frac{A(1+x) + B(1-x)}{1-x^2}$$

$\Rightarrow 1 = A(1+x) + B(1-x)$ for all values of x.

There are two options for finding the values of A and B:

Option 1: Substitute convenient values of x into $1 = A(1+x) + B(1-x)$.	**Option 2:** Use simultaneous equations.
Substitute $x = 1$: $1 = 2A \Rightarrow A = \frac{1}{2}$.	Expand and group like terms: $1 = (A-B)x + A + B$.
Substitute $x = -1$: $1 = 2B \Rightarrow B = \frac{1}{2}$.	Equate coefficients of powers of x: $0 = A - B$. …. (1)
	Equate constant terms: $1 = A + B$. …. (2)
	Solve equations (1) and (2) simultaneously: $A = \frac{1}{2}$ and $B = \frac{1}{2}$.

$A = \frac{1}{2}$ and $B = \frac{1}{2}$. **[A1]**

Therefore:

$$t = \frac{1}{2}\int_0^{2/3} \frac{1}{1-x} + \frac{1}{1+x}\, dx$$

$$= \frac{1}{2}\left[\log_e\left|\frac{1+x}{1-x}\right|\right]_0^{2/3}$$

$$= \frac{1}{2}\log_e\left(\frac{1+\frac{2}{3}}{1-\frac{2}{3}}\right) = \frac{1}{2}\log_e(5) \text{ seconds.}$$

Unit not essential. **[A1]**

Total 4 marks

Question 5

Use implicit differentiation with respect to *x*:

$$3(y+1)^2 - 2xy - x^2 = 7$$

$$\Rightarrow 6(y+1)\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} - 2x = 0.$$ **[M1]**

Substitute $\frac{dy}{dx} = 0$ and simplify:

$$y = -x.$$ **[M1]**

Solve the pair of equations

$$3(y+1)^2 - 2xy - x^2 = 7 \quad \ldots.(1)$$

$$y = -x \quad \ldots.(2)$$

simultaneously for *y*. Substitute equation (2) into equation (1):

$$3(y+1)^2 - 2(-y)y - (-y)^2 = 7$$

Expand, re-arrange and simplify:

$$\Rightarrow 2y^2 + 3y - 2 = 0$$ **[A1]**

$$\Rightarrow (y+2)(2y-1) = 0$$

$$\Rightarrow y = -2,\ \frac{1}{2}.$$

Therefore the equations of the tangents are

$y = -2$ and $y = \frac{1}{2}$. **[A1]**

Total 4 marks

Alternate method (very inefficient and time consuming):

$$3(y+1)^2 - 2xy - x^2 = 7$$

Expand, re-arrange and simplify:

$$\Rightarrow 3y^2 + (6-2x)y - (4+x^2) = 0$$

Use the quadratic formula to solve for *y*:

$$y = \frac{2x - 6 \pm 2\sqrt{(3-x)^2 + 3(4+x^2)^2}}{6}$$

Now calculate $\frac{dy}{dx}$ and solve $\frac{dy}{dx} = 0$ *etc.*

Question 6

$\underset{\sim}{a} \cdot \underset{\sim}{b} = -4 - t + 2 = -2 - t$. .... (1)

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta) = 3\sqrt{5+t^2}\cos(\theta)$. .... (2) **[M1]**

Equate equations (1) and (2):

$$-2 - t = 3\sqrt{5+t^2}\cos(\theta)$$

$$\Rightarrow (-2-t)^2 = 9(5+t^2)\cos^2(\theta)$$ **[M1]**

$$\Rightarrow (2+t)^2 = 9(5+t^2)\cos^2(\theta). \quad .... (3)$$

$$\sin(\theta) = \frac{4\sqrt{5}}{9}$$

$$\Rightarrow \sin^2(\theta) = \frac{80}{81}$$

$$\Rightarrow \cos^2(\theta) = \frac{1}{81}.$$

Substitute into equation (3):

$$\Rightarrow (2+t)^2 = \frac{1}{9}(5+t^2)$$ **[A1]**

$$\Rightarrow 9(t+2)^2 = 5 + t^2$$

$$\Rightarrow 8t^2 + 36t + 31 = 0.$$

Option 1: Complete the square.

$$8t^2 + 36t + 31 = 8\left(t^2 + \frac{9}{2}t + \frac{31}{8}\right) \quad = 8\left(\left[t + \frac{9}{4}\right]^2 - \frac{81}{16} + \frac{31}{8}\right) \quad = 8\left[t + \frac{9}{4}\right]^2 - \frac{81}{2} + 31 \quad = 8\left[t + \frac{9}{4}\right]^2 - \frac{19}{2}:$$

$$8\left[t + \frac{9}{4}\right]^2 - \frac{19}{2} = 0$$

$$\Rightarrow \left[t + \frac{9}{4}\right]^2 = \frac{19}{16}$$

$\Rightarrow t + \dfrac{9}{4} = \pm\dfrac{\sqrt{19}}{4}$

$\Rightarrow t = \dfrac{-9 \pm \sqrt{19}}{4}$. **[A1]**

Option 2: Use the quadratic formula.

$\Delta = 36^2 - 32 \times 31$

$= 4^2(9^2 - 2 \times 31) = 16 \times 19$.

Therefore:

$t = \dfrac{-36 \pm \sqrt{16 \times 19}}{16} = \dfrac{-36 \pm 4\sqrt{19}}{16} = \dfrac{-9 \pm \sqrt{19}}{4}$. **[A1]**

Total 4 marks

Question 7

$\underset{\sim}{v} = \dfrac{d\underset{\sim}{r}}{dt} = \left(2t\cos(t) - t^2\sin(t)\right)\underset{\sim}{i} - \left(\sin(t) + t\cos(t)\right)\underset{\sim}{j}$. **[A1]**

Substitute $t = \pi$:

$\underset{\sim}{v} = \left(2\pi\cos(\pi) - \pi^2\sin(\pi)\right)\underset{\sim}{i} - \left(\sin(\pi) + \pi\cos(\pi)\right)\underset{\sim}{j}$

$= -2\pi\,\underset{\sim}{i} + \pi\,\underset{\sim}{j}$. **[A1]**

Speed $= \left|\underset{\sim}{v}\right| = \sqrt{(-2\pi)^2 + \pi^2}$

$= \sqrt{5}\pi$. **[A1]**

Total 3 marks

Question 8

$m_{normal} = \frac{-1}{m_{\text{tangent}}}$ and $m_{\text{tangent}} = \frac{dy}{dx}$.

Use the chain rule to get $\frac{dy}{dx}$.

Let $u = \frac{3}{x}$.

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ where:

- $\frac{du}{dx} = -\frac{3}{x^2}$
- $y = \cos^{-1}(u)$.
- $\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$.

Therefore:

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \times \left(-\frac{3}{x^2}\right)$$

$$= \frac{-1}{\sqrt{1-\left(\frac{3}{x}\right)^2}} \times \left(-\frac{3}{x^2}\right).$$ **[M1]**

Substitute $x = -2\sqrt{3}$:

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{3}{-2\sqrt{3}}\right)^2}} \times \left(-\frac{3}{12}\right)$$

$$= \frac{-1}{\sqrt{1-\frac{3}{4}}} \times \left(-\frac{3}{12}\right)$$

$$= \frac{1}{2}$$ **[A1]**

$\Rightarrow m_{normal} = -2$.

Substitute $x=-2\sqrt{3}$ into $y=\cos^{-1}\left(\frac{3}{x}\right)$:

$$y=\cos^{-1}\left(\frac{3}{-2\sqrt{3}}\right)=\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}.$$

Equation of normal:

$y-\frac{5\pi}{6}=-2(x+2\sqrt{3}) \quad \Rightarrow y=-2x-4\sqrt{3}+\frac{5\pi}{6}.$ Any correct form. **[A1]**

Total 3 marks

Comment:

The following is NOT required but is included as a teaching point (see line *).

$$\frac{dy}{dx}=\frac{-1}{\sqrt{1-\left(\frac{3}{x}\right)^2}}\times\left(-\frac{3}{x^2}\right)$$

$$=\frac{3}{x^2\sqrt{1-\frac{9}{x^2}}} \qquad =\frac{3}{x^2\sqrt{\frac{x^2-9}{x^2}}}$$

$$=\frac{3\sqrt{x^2}}{x^2\sqrt{x^2-9}} \qquad =\frac{3|x|}{x^2\sqrt{x^2-9}} \ \textbf{ not } \ \frac{3x}{x^2\sqrt{x^2-9}} \qquad *$$

$$=\frac{3}{|x|\sqrt{x^2-9}}.$$

Substitute $x=-2\sqrt{3}$:

$$\frac{dy}{dx}=\frac{3}{2\sqrt{3}\sqrt{12-9}}=\frac{1}{2}.$$

Question 9

a.

$$f(x) = 1 - 2\text{cosec}\left(\frac{\pi x}{3}\right) = 1 - \frac{2}{\sin\left(\frac{\pi x}{3}\right)}$$

has vertical asymptotes when

$$\sin\left(\frac{\pi x}{3}\right) = 0$$

$$\Rightarrow \frac{\pi x}{3} = n\pi, \quad n \in Z$$

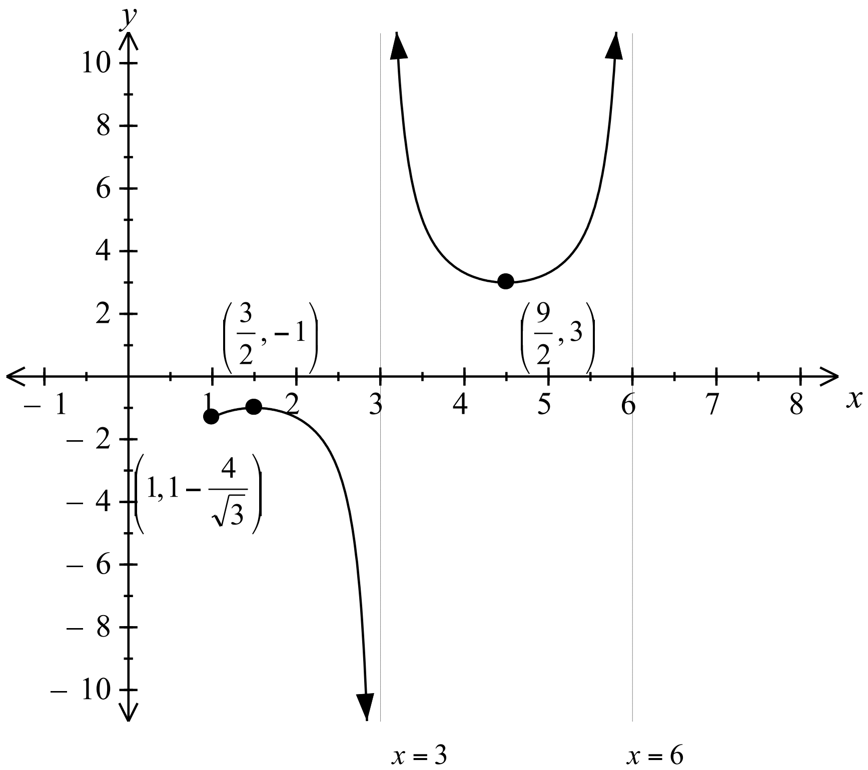
$$\Rightarrow x = 3n.$$

The first two asymptotes for which $x > 1$ are $x = 3$ and $x = 6$.

Therefore $a = 3$ and $b = 6$. Both values. **[A1]**

Total 1 mark

b.



- Vertical asymptotes:

$x = 3$ and $x = 6$ (consequential on answer to **part a.**: $x = a$ and $x = b$). **[A1]**

- Minimum turning point at $\left(\frac{9}{2}, 3\right)$ and maximum turning point at $\left(\frac{3}{2}, -1\right)$. **[A1]**

Calculation:

Method 1:

Maximum turning point when $\sin\left(\frac{\pi x}{3}\right)=1$: $x=\frac{3}{2}$ and $y=-1$.

Minimum turning point when $\sin\left(\frac{\pi x}{3}\right)=-1$: $x=\frac{9}{2}$ and $y=3$.

Method 2:

The turning points lie halfway between the vertical asymptotes.

- Endpoint: $x=1$ and $y=1-\frac{4}{\sqrt{3}}$. Correct 'ball park' location. **[A1]**

Comment: $1-\frac{4}{\sqrt{3}}\approx 1-\frac{4}{2}=-1$ so the y-coordinate of the endpoint should be shown in the 'ball park' of $y=-1$.

- Shape. **[A1]**

Total 4 marks