Year 2014

VCE

Specialist Mathematics

Trial Examination 2

Solutions

KILBAHA MULTIMEDIA PUBLISHING PO BOX 2227 KEW VIC 3101 AUSTRALIA	TEL: (03) 9018 5376 FAX: (03) 9817 4334 kilbaha@gmail.com http://kilbaha.com.au

- While every care has been taken, no guarantee is given that these answers are free from error. Please contact us if you believe you have found an error.

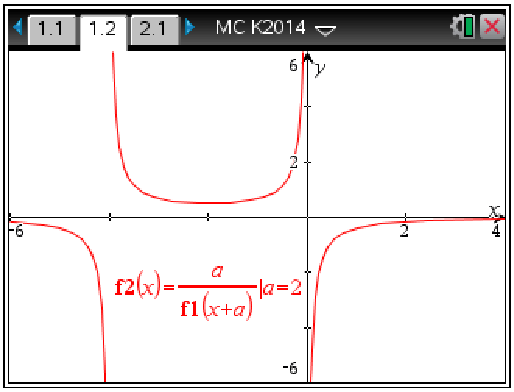
SECTION 1

ANSWERS

1	A	B	C	**D**	E
2	**A**	B	C	D	E
3	A	B	C	**D**	E
4	A	B	**C**	D	E
5	A	B	C	D	**E**
6	A	**B**	C	D	E
7	A	B	C	D	**E**
8	A	B	C	**D**	E
9	A	**B**	C	D	E
10	A	**B**	C	D	E
11	A	B	**C**	D	E
12	A	B	C	D	**E**
13	A	B	C	**D**	E
14	A	B	C	**D**	E
15	A	B	**C**	D	E
16	A	B	**C**	D	E
17	A	**B**	C	D	E
18	**A**	B	C	D	E
19	A	**B**	C	D	E
20	**A**	B	C	D	E
21	A	B	C	D	**E**
22	**A**	B	C	D	E

SECTION 1

Question 1 **Answer D**



$f(x)=a^2-x^2=(a+x)(a-x)$

$y=\dfrac{a}{f(x+a)}=\dfrac{a}{(x+2a)(-x)}=\dfrac{-a}{x(x+2a)}$

vertical asymptotes occur when $x(x+2a)=0$

vertical asymptotes at $x=0$ and $x=-2a$

the turning point is at $x=-a \Rightarrow y=\dfrac{a}{a^2}=\dfrac{1}{a}$

$\left(-a,\dfrac{1}{a}\right)$ and is a minimum turning point.

Question 2 **Answer A**

$\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$. The asymptote is $y=-\dfrac{x}{2}$

and passes through the centre, when

$y=-1 \Rightarrow x=2$, so that the centre of the hyperbola

is at $(2,-1)$, so that $h=2$ and $k=-1$.

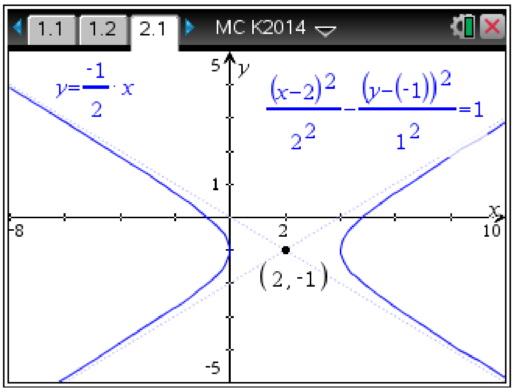
The distance from the centre to the point

where the hyperbola touches the *y*-axis is 2,

so that $a=2$. The asymptotes are

$y-k=\pm\dfrac{b}{a}(x-h)$ and have a gradient

of $\dfrac{b}{a}=\dfrac{1}{2}$, since $a=2$ it follows that $b=1$.



Question 3 **Answer D**

$y=\tan^{-1}(x)$ has a range $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$.

For $y=a\tan^{-1}(bx)+c$, the value of b does not affect the range (only the domain)

For a range equal to $\dfrac{6}{\pi}\times\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)=(-3,3)$ so that $a=\dfrac{6}{\pi}$ and $c=0$

Question 4 **Answer C**

The roots occur in complex conjugate pairs, since the coefficients are all real.

Let $u = \alpha + i$ and $\overline{u} = \alpha - i$ be the roots of $z^2 + bz + c = 0$.

The sum of the roots $u + \overline{u} = 2\alpha = -b$ and the product of the roots are

$u.\overline{u} = (\alpha + i)(\alpha - i) = \alpha^2 + 1 = c$

For the quadratic, let $v = \alpha + 1 - i$ and $\overline{v} = \alpha + 1 + i$.

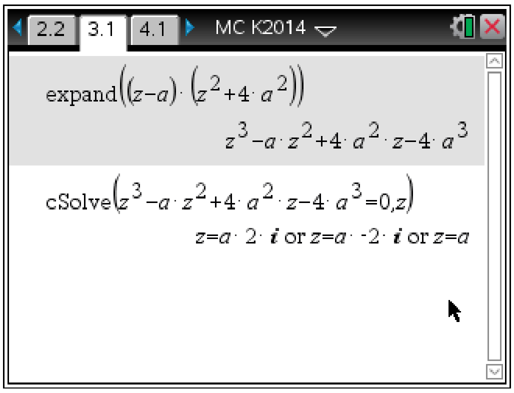
The sum of these roots are $v + \overline{v} = 2\alpha + 2 = 2 - b$ and the product of these roots are

$v.\overline{v} = (\alpha + 1 - i)(\alpha + 1 + i) = (\alpha + 1)^2 + 1 = \alpha^2 + 2\alpha + 2 = c - b + 1$. The quadratic is

$z^2 - (2 - b)z + c - b + 1 = z^2 + (b - 2)z + c - b + 1 = 0$

Question 5 **Answer E**

by the conjugate root theorem
$z = 2ai$ is also a root.
$(z + 2ai)(z - 2ai) = z^2 - 4a^2 i^2 = z^2 + 4a^2$
expanding
$P(z) = (z - a)(z^2 + 4a^2)$
$P(z) = z^3 - az^2 + 4a^2 z - 4a^3$



Question 6 **Answer B**

$a + bi = r\text{cis}(\theta)$ where $r = \sqrt{a^2 + b^2}$

$(a + bi)^2 = r^2\text{cis}(2\theta)$

$a^2 + 2abi + b^2 i^2 = a^2 - b^2 + 2abi = r^2\text{cis}(2\theta)$ take the conjugate

$a^2 - b^2 - 2abi = r^2\text{cis}(-2\theta)$ multiply both sides by $i = 1\text{cis}\left(\frac{\pi}{2}\right)$

$i(a^2 - b^2 - 2abi) = ir^2\text{cis}(-2\theta) = 1\text{cis}\left(\frac{\pi}{2}\right) \times r^2\text{cis}(-2\theta)$

$(a^2 - b^2)i - 2abi^2 = 2ab + (a^2 - b^2)i = r^2\text{cis}\left(\frac{\pi}{2} - 2\theta\right)$

$\text{Arg}\left(2ab + (a^2 - b^2)i\right) = \frac{\pi}{2} - 2\theta$

Question 7 **Answer E**

Alex: $\int_0^{\frac{\pi}{4}} \sin^3(2x)\cos^3(2x)\,dx$,

let $u = \sin(2x)$ $\frac{du}{dx} = 2\cos(2x) \Rightarrow \cos(2x)\,dx = \frac{1}{2}du$

terminals, $x = \frac{\pi}{4}$ $u = \sin\left(\frac{\pi}{2}\right) = 1$ $x = 0$ $u = \sin(0) = 0$

$$\int_0^{\frac{\pi}{4}} \sin^3(2x)\cos^2(2x)\cos(2x)\,dx = \int_0^{\frac{\pi}{4}} \sin^3(2x)\left(1-\sin^2(2x)\right)\cos(2x)\,dx = \frac{1}{2}\int_0^1 u^3\left(1-u^2\right)du$$

Brenda: $\int_0^{\frac{\pi}{4}} \sin^3(2x)\cos^3(2x)\,dx$,

let $u = \cos(2x)$ $\frac{du}{dx} = -2\sin(2x) \Rightarrow \sin(2x)\,dx = -\frac{1}{2}du$

terminals, $x = \frac{\pi}{4}$ $u = \cos\left(\frac{\pi}{2}\right) = 0$ $x = 0$ $u = \cos(0) = 1$

$$\int_0^{\frac{\pi}{4}} \sin(2x)\sin^2(2x)\cos^3(2x)\,dx = \int_0^{\frac{\pi}{4}} \left(1-\cos^2(2x)\right)\cos^3(2x)\sin(2x)\,dx$$

$$= -\frac{1}{2}\int_1^0 \left(1-u^2\right)u^3\,du = \frac{1}{2}\int_0^1 \left(1-u^2\right)u^3\,du$$

Claire: $\int_0^{\frac{\pi}{4}} \sin^3(2x)\cos^3(2x)\,dx = \frac{1}{8}\int_0^{\frac{\pi}{4}} \left(2\sin(2x)\cos(2x)\right)^3 dx = \frac{1}{8}\int_0^{\frac{\pi}{4}} \sin^3(4x)\,dx$

$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} \sin(4x)\sin^2(4x)\,dx = \frac{1}{8}\int_0^{\frac{\pi}{4}} \sin(4x)\left(1-\cos^2(4x)\right)dx$$

let $u = \cos(4x)$ $\frac{du}{dx} = -4\sin(4x) \Rightarrow \sin(4x)\,dx = -\frac{1}{4}du$

terminals $x = \frac{\pi}{4}$ $u = \cos(\pi) = -1$ $x = 0$ $u = \cos(0) = 1$

$$\int_0^{\frac{\pi}{4}} \sin^3(2x)\cos^3(2x)\,dx = -\frac{1}{32}\int_1^{-1} \left(1-u^2\right)du = \frac{1}{32}\int_{-1}^{1} \left(1-u^2\right)du$$

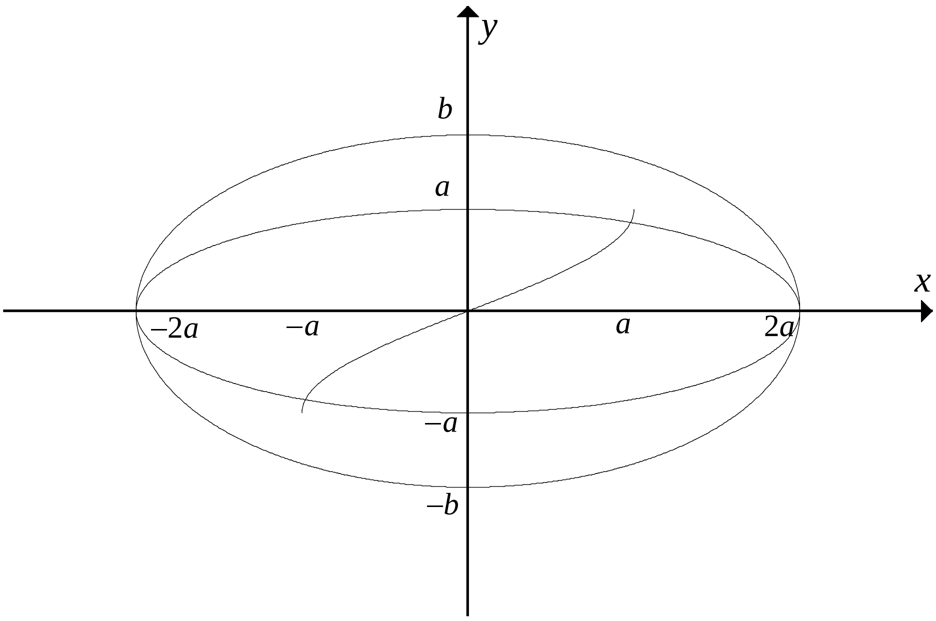
$\int_0^{\frac{\pi}{4}} \left((\sin(2\cdot x))^3 \cdot (\cos(2\cdot x))^3 \right) dx$	$\frac{1}{24}$
$\frac{1}{2}\cdot \int_0^1 \left(u^3 \cdot (1-u^2) \right) du$	$\frac{1}{24}$
$\frac{1}{32}\cdot \int_{-1}^1 (1-u^2)\, du$	$\frac{1}{24}$

All are correct.

Question 8 **Answer D**

The domain of $\frac{2a}{\pi}\sin^{-1}\left(\frac{x}{a}\right)$ is $[-a,a]$ and the range is $[-a,a]$. Its endpoints are $(-a,-a)$ and (a,a). The ellipse $\frac{x^2}{4a^2}+\frac{y^2}{b^2}=1$ has its centre at the origin and has a domain $[-2a,2a]$ and a range $[-b,b]$. When $x=a$ on the ellipse $\frac{a^2}{4a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1-\frac{1}{4}=\frac{3}{4} \Rightarrow y=\pm\frac{\sqrt{3}b}{2}$.

For the ellipse to intersect exactly twice, we require $a>\frac{\sqrt{3}b}{2}$.



Question 9 **Answer B**

$|z+2a| = 2|z-ai|$ let $z = x+yi$

$|(x+2a)+yi| = 2|x+(y-a)i|$

$\sqrt{(x+2a)^2+y^2} = 2\sqrt{x^2+(y-a)^2}$

$(x+2a)^2+y^2 = 4\left[x^2+(y-a)^2\right]$

$x^2+4ax+4a^2+y^2 = 4\left[x^2+y^2-2ay+a^2\right]$

$3x^2-4ax+3y^2-8ay=0$

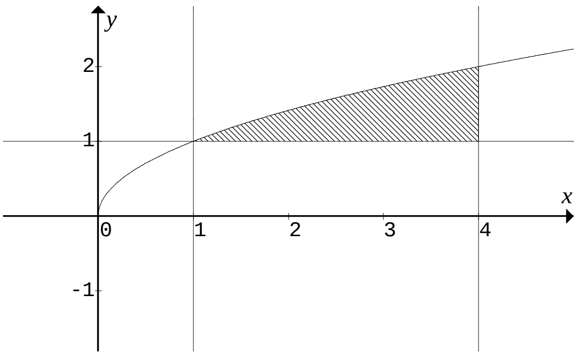
$x^2-\frac{4ax}{3}+y^2-\frac{8ay}{3}=0$ $\Rightarrow$ a circle, however lets find the centre and radius.

$x^2-\frac{4ax}{3}+\frac{4a^2}{9}+y^2-\frac{8ay}{3}+\frac{16a^2}{9}=\frac{20a^2}{9}$

$\left(x-\frac{2a}{3}\right)^2+\left(y-\frac{4a}{3}\right)^2=\frac{20a^2}{9}$

This represents a circle centre at $\left(\frac{2a}{3},\frac{4a}{3}\right)$ and radius $\frac{\sqrt{20}\,a}{3}$.

Question 10 **Answer B**



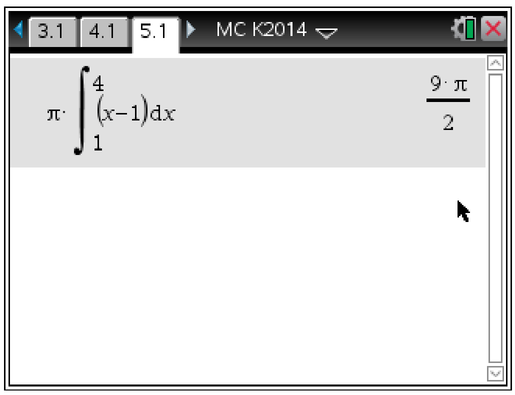
$V_x = \pi\int_a^b \left(y_2^2-y_1^2\right)dx$

$a=1$, $b=4$, $y_2=\sqrt{x}$, $y_1=1$

$V = \pi\int_1^4 (x-1)\,dx = \pi\left[\frac{1}{2}x^2-x\right]_1^4$

$V = \pi\left[(8-1)-\left(\frac{1}{2}-1\right)\right]$

$V = \frac{9\pi}{2}$



Question 11 **Answer C**

The length of u is 2 and the length of v is 3. v is a rotation from u by 270^0, that is i^3.

So $\frac{3ui^3}{2} = v \quad \Rightarrow -3ui = 2v$ so that $2v + 3ui = 0$

Question 12 **Answer E**

$\cos(\theta) = \frac{\underset{\sim}{u}.\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} \quad \Rightarrow \quad \cos\left(120^0\right) = -\frac{1}{2} = \frac{\underset{\sim}{u}.\underset{\sim}{v}}{3\times 4} \Rightarrow \underset{\sim}{u}.\underset{\sim}{v} = -6$ **A.** is true.

The scalar resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$ is equal to $\underset{\sim}{u}.\hat{\underset{\sim}{v}} = \frac{\underset{\sim}{u}.\underset{\sim}{v}}{|\underset{\sim}{v}|} = \frac{-6}{4} = -\frac{3}{2}$ **B.** is true

The scalar resolute of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{u}$ is equal to $\underset{\sim}{v}.\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{v}.\underset{\sim}{u}}{|\underset{\sim}{u}|} = \frac{-6}{3} = -2$ **C.** is true.

$\left|\underset{\sim}{u}+\underset{\sim}{v}\right|^2 = \left(\underset{\sim}{u}+\underset{\sim}{v}\right).\left(\underset{\sim}{u}+\underset{\sim}{v}\right) = \underset{\sim}{u}.\underset{\sim}{u} + 2\underset{\sim}{u}.\underset{\sim}{v} + \underset{\sim}{v}.\underset{\sim}{v} = |\underset{\sim}{u}|^2 + 2\underset{\sim}{u}.\underset{\sim}{v} + |\underset{\sim}{v}|^2 = 3^2 + 2\times{}^{-}6 + 4^2 = 13$

$\left|\underset{\sim}{u}+\underset{\sim}{v}\right| = \sqrt{13}$ **D.** is true

$\left|\underset{\sim}{v}-\underset{\sim}{u}\right| = 1$ **E.** is false.

Question 13 **Answer D**

use implicit differentiation on $x^2 + 2\tan^{-1}\left(\frac{y}{2}\right) + y^2 = 5 + \frac{\pi}{2}$, gives

$2x + \left(\frac{4}{4+y^2} + 2y\right)\frac{dy}{dx} = 0$ when $x = -1$ and $y = 2$

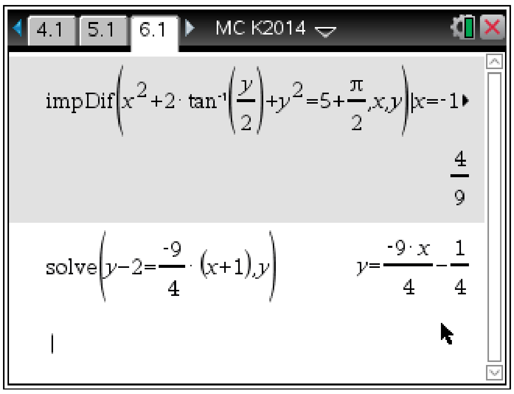
$-2 + \left(\frac{4}{4+4} + 4\right)\frac{dy}{dx} = 0$

$\frac{9}{2}\frac{dy}{dx} = 2 \quad m_T = \frac{dy}{dx} = \frac{4}{9}$

$m_N = -\frac{9}{4}$ at $(-1, 2)$

$N: \ y - 2 = -\frac{9}{4}(x+1)$

$y = -\frac{9x}{4} - \frac{1}{4}$



Question 14 **Answer D**

$\underset{\sim}{a} = x\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{a}.\underset{\sim}{b} = x - 2 - 2 = x - 4$

When $x = 4$ then $\underset{\sim}{a}.\underset{\sim}{b} = 0$, so that the vector $\underset{\sim}{a}$ is perpendicular to the vector $\underset{\sim}{b}$ **A.** is true.

$-2\underset{\sim}{a} = \underset{\sim}{b} \Rightarrow -2x = 1 \Rightarrow x = -\frac{1}{2}$, then the vector $\underset{\sim}{a}$ is parallel to the vector $\underset{\sim}{b}$. **B.** is true.

$|\underset{\sim}{a}| = \sqrt{x^2+1+1} = \sqrt{x^2+2}$ $\quad |\underset{\sim}{b}| = \sqrt{1+4+4} = \sqrt{9} = 3$

If $|\underset{\sim}{a}| = |\underset{\sim}{b}| \Rightarrow \sqrt{x^2+2} = 3 \Rightarrow x^2 + 2 = 9 \Rightarrow x^2 = 7$

If $x = \pm\sqrt{7}$ then the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are equal in length, **C.** is true.

If $x = \sqrt{2}$ then $|\underset{\sim}{a}| = 2$, so that $|\underset{\sim}{a}| + |\underset{\sim}{b}| = 5$, **E.** is true.

$\underset{\sim}{a} + \underset{\sim}{b} = (x+1)\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$, if $x = 0$, $|\underset{\sim}{a} + \underset{\sim}{b}| = \sqrt{1+1+1} = \sqrt{3}$, **D.** is false

Question 15 **Answer C**

$\frac{d^2y}{dx^2} = 5x^4 - 5$

$\frac{dy}{dx} = \int (5x^4 - 5)\,dx = x^5 - 5x + c$

when $x = 1$ $\frac{dy}{dx} = 0$

$0 = 1 - 5 + c \Rightarrow c = 4$

$\frac{dy}{dx} = x^5 - 5x + 4$

$\frac{d^2y}{dx^2} = 0 \Rightarrow x^4 = 1 \quad x^2 = 1 \quad x = \pm 1$

when $x = 1$ $y'' = 0$ and $y' = 0 \Rightarrow x = 1$

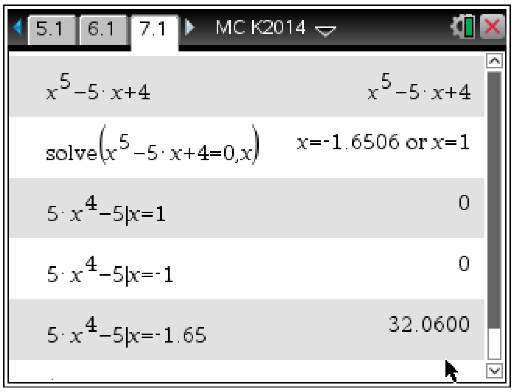
is a stationary point of inflexion.

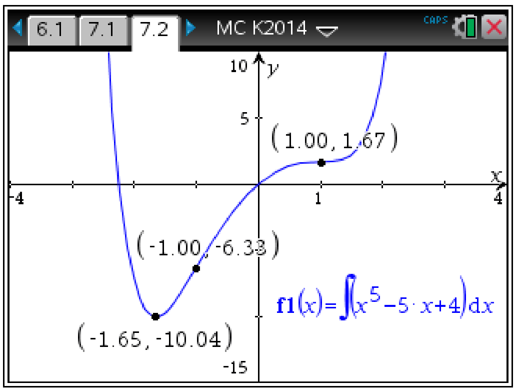
when $x = -1$ $y'' = 0$ and $y' > 0 \Rightarrow x = -1$

is a point of inflexion.

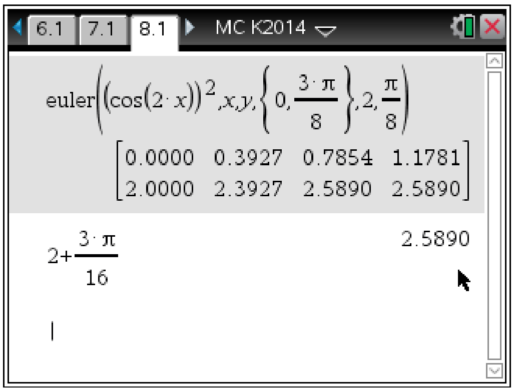
$y' = 0 \Rightarrow x = -1.65 \quad y'' > 0 \Rightarrow x = -1.65$

is a minimum turning point.





Question 16 **Answer C**



$$\frac{dy}{dx} = f(x) = \cos^2(2x)$$

$h = \frac{\pi}{8}$, $x_0 = 0$ and $y_0 = 2$, $x_1 = \frac{\pi}{8}$, $x_2 = \frac{\pi}{4}$

using Euler's Method

$$y_1 = y_0 + h\,f(x_0)$$

$$y_1 = 2 + \frac{\pi}{8}\cos^2(0) = 2 + \frac{\pi}{8}$$

$$y_2 = y_1 + h\,f(x_1)$$

$$y_2 = 2 + \frac{\pi}{8} + \frac{\pi}{8}\cos^2\left(\frac{\pi}{4}\right) = 2 + \frac{\pi}{8} + \frac{\pi}{16} = 2 + \frac{3\pi}{16}$$

$$y_3 = y_2 + h\,f(x_2)$$

$$y_3 = 2 + \frac{3\pi}{16} + \frac{\pi}{8}\cos^2\left(\frac{\pi}{2}\right) = 2 + \frac{3\pi}{16} + 0 = 2 + \frac{3\pi}{16}$$

Question 17 **Answer B**

$$v = \frac{dx}{dt} = e^{\sqrt{t}}$$

$$x = \int_0^t e^{\sqrt{u}}\,du + c$$

when $x = 3$, $t = 1$

$$3 = \int_0^1 e^{\sqrt{u}}\,du + c \;\Rightarrow\; c = 3 - \int_0^1 e^{\sqrt{u}}\,du$$

$$x = x(t) = \int_0^t e^{\sqrt{u}}\,du + 3 - \int_0^1 e^{\sqrt{u}}\,du$$

$$x(2) = \int_0^2 e^{\sqrt{u}}\,du - \int_0^1 e^{\sqrt{u}}\,du + 3$$

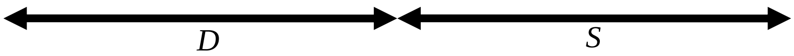
$$x(2) = \int_0^2 e^{\sqrt{u}}\,du + \int_1^0 e^{\sqrt{u}}\,du + 3$$

$$x(2) = \int_1^2 e^{\sqrt{u}}\,du + 3$$

Question 18 **Answer A**

$u = 3V$ $\quad v = V$ $\quad v = 0$

$t = 0$ $\quad t = T$



use $v^2 = u^2 + 2as$

with $v = V$, $u = 3V$ $s = D$

$V^2 = 9V^2 + 2aD$

$a = -\dfrac{4V^2}{D}$

use $s = \left(\dfrac{u+v}{2}\right)t$

with $v = 0$, $u = 3V$ $s = D + S$ and $t = T$

$D + S = \dfrac{3V}{2} \times T \Rightarrow T = \dfrac{2(D+S)}{3V}$

use $v^2 = u^2 + 2as$

with $v = 0$, $u = V$ $s = S$ and $a = -\dfrac{4V^2}{D}$

$0 = V^2 - 2 \times \dfrac{4V^2}{D} \times S \Rightarrow D = 8S$

Question 19 **Answer B**

resolving downwards for the 2 kg mass hanging over the edge of the table

(1) $2g - T = 2a$

resolving for the mass M on the table

(2) $N - Mg = 0$

(3) $T - \mu N = Ma$

$(2) \Rightarrow N = Mg$ into (3) $T - \mu Mg = Ma$

adding to eliminate T

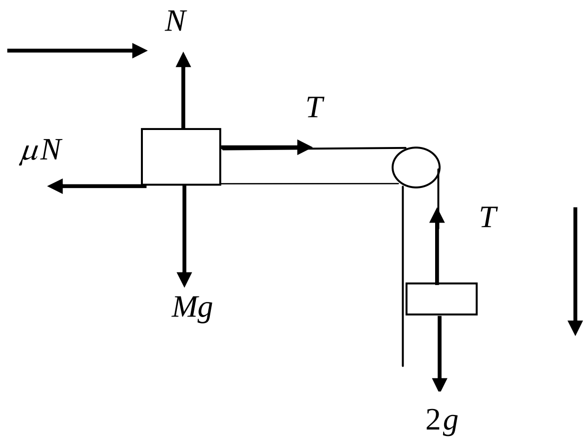
$2g - \mu Mg = (M+2)a$ but $\mu = \dfrac{1}{3}$

$2g - \dfrac{1}{3}Mg = \dfrac{g(6-M)}{3} = (M+2)a$

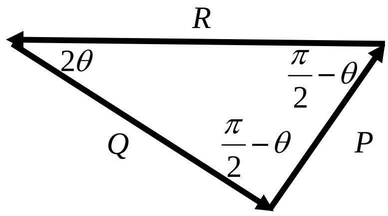
$a = \dfrac{g(6-M)}{3(M+2)}$

$a > 0 \Rightarrow 0 < M < 6$

$a = 0 \Rightarrow M = 6$



Question 20 **Answer A**



$$\frac{Q}{\sin\left(\frac{\pi}{2}-\theta\right)}=\frac{R}{\sin\left(\frac{\pi}{2}-\theta\right)}=\frac{P}{\sin(2\theta)}$$

$$\frac{Q}{\cos(\theta)}=\frac{R}{\cos(\theta)}=\frac{P}{2\sin(\theta)\cos(\theta)}$$

$Q=R$ and $P=2R\sin(\theta)$

Alternatively resolving vertically $P\cos(\theta)-Q\sin(2\theta)=0$

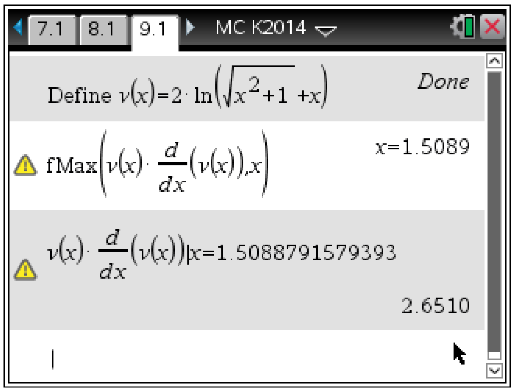
$P\cos(\theta)=2Q\sin(\theta)\cos(\theta) \Rightarrow P=2Q\sin(\theta)$ (1)

resolving horizontally $P\sin(\theta)+Q\cos(2\theta)-R=0$

$P\sin(\theta)+Q\left(1-2\sin^2(\theta)\right)=R$ from (1) $P=2Q\sin(\theta)$

$2Q\sin^2(\theta)+Q-2Q\sin^2(\theta)=R \Rightarrow Q=R$ and $P=2R\sin(\theta)$

Question 21 **Answer E**



$v=2\log_e\left(\sqrt{x^2+1}+x\right)$

$$\frac{dv}{dx}=\frac{2}{\sqrt{x^2+1}}$$

$$a(x)=v\frac{dv}{dx}=\frac{4\log_e\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}}$$

$a(x)$ has a maximum when $x=1.51$

$F_{\max}=ma=10\times a(1.51)=26.5$

Question 22 **Answer A**

when $x=2$ and $y=1 \Rightarrow m=1$ when $x=2$ and $y=-1 \Rightarrow m=-1$

when $x=0$ and $y=1 \Rightarrow m=-1$ when $x=0$ and $y=-1 \Rightarrow m=1$

is only satisfied by $m=\dfrac{dy}{dx}=\dfrac{x-1}{y}$

alternatively, the solution curves are hyperbolas, of the form,

$\dfrac{(x-1)^2}{a^2}-y^2=1$ using implicit differentiation, $\dfrac{2(x-1)}{a^2}-2y\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx}=\dfrac{x-1}{a^2y}$ let $a^2=1$

END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a. $\cos\left(\frac{5\pi}{12}\right)=\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$

$\cos\left(\frac{5\pi}{12}\right)=\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)-\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$

M1

$\cos\left(\frac{5\pi}{12}\right)=\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}$

$\cos\left(\frac{5\pi}{12}\right)=\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$ A1

b.i. $\overrightarrow{OA}=\underset{\sim}{i}+\underset{\sim}{j}$ and $\overrightarrow{OB}=-\underset{\sim}{i}+\sqrt{3}\,\underset{\sim}{j}$ A1

ii. $\left|\overrightarrow{OA}\right|=\sqrt{2}$ $\left|\overrightarrow{OB}\right|=2$ $\overrightarrow{OA}.\overrightarrow{OB}=-1+\sqrt{3}$

$\cos(\theta)=\frac{\overrightarrow{OA}.\overrightarrow{OB}}{\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|}=\frac{-1+\sqrt{3}}{2\sqrt{2}}$

$\cos(\theta)=\frac{\sqrt{3}-1}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$ M1

$\theta=\cos^{-1}\left(\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)\right)=\frac{5\pi}{12}$

$\theta=75^0$ A1

iii. $\text{Area}=\frac{1}{2}\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\sin(\theta)$

$\text{Area}=\frac{1}{2}\times\sqrt{2}\times2\times\sin\left(\frac{5\pi}{12}\right)=\sqrt{2}\times\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$ M1

$\text{Area}=\frac{1+\sqrt{3}}{2}$ A1

iv. $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AB}$

$=\overrightarrow{OA}+\frac{1}{2}\left(\overrightarrow{OB}-\overrightarrow{OA}\right)=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OB}\right)$

M1

$=\frac{1}{2}\left(\left(\underset{\sim}{i}+\underset{\sim}{j}\right)+\left(-\underset{\sim}{i}+\sqrt{3}\,\underset{\sim}{j}\right)\right)$

$=\frac{1}{2}\left(1+\sqrt{3}\right)\underset{\sim}{j}$ A1

c.i. $a=1+i=\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$ $|a|=\sqrt{2}$ $\alpha=\frac{\pi}{4}$ A1

$b=-1+i\sqrt{3}=2\operatorname{cis}\left(\frac{2\pi}{3}\right)$ $|b|=2$ $\beta=\frac{2\pi}{3}$ A1

ii. If $z=x+iy$ $\operatorname{Re}(z)=x$ $\operatorname{Im}(z)=y$,

the line $\operatorname{Im}(z)=m\operatorname{Re}(z)+k$ is $y=mx+k$

m is the gradient of the line joining $A(1,1)$ and $B\left(-1,\sqrt{3}\right)$ M1

$m(AB)=\frac{1-\sqrt{3}}{2}$ the line is $y-1=\left(\frac{1-\sqrt{3}}{2}\right)(x-1)$

$y=\left(\frac{1-\sqrt{3}}{2}\right)x-\left(\frac{1-\sqrt{3}}{2}\right)+1=\left(\frac{1-\sqrt{3}}{2}\right)x+\frac{1}{2}\left(\sqrt{3}+1\right)$

so that $m=\frac{1}{2}\left(1-\sqrt{3}\right)$ and $k=\frac{1}{2}\left(\sqrt{3}+1\right)$ A1

iii. $R=\frac{1}{4}\left(a\bar{b}-\bar{a}b\right)i$

$R=\frac{1}{4}\left((1+i)\left(-1-i\sqrt{3}\right)-(1-i)\left(-1+i\sqrt{3}\right)\right)i$

$R=\frac{1}{4}\left(-1-i\sqrt{3}-i-i^2\sqrt{3}-\left(-1+i\sqrt{3}+i-i^2\sqrt{3}\right)\right)i$ M1

$R=\frac{1}{4}\left(2i\left(-\sqrt{3}-1\right)\right)i$

$R=\frac{1+\sqrt{3}}{2}$ and is equal to the area of the triangle *AOB* from **b.iii.** A1

iv. The circle $|z-c|=r$, has its centre at *C*,

the midpoint of *AB*, so $c=\frac{1}{2}\left(1+\sqrt{3}\right)i$ A1

$\overrightarrow{AB}=-2\underset{\sim}{i}+\left(\sqrt{3}-1\right)\underset{\sim}{j}$ and $\left|\overrightarrow{AB}\right|=\sqrt{(-2)^2+\left(\sqrt{3}-1\right)^2}=\sqrt{8-2\sqrt{3}}$

and the radius of the circle is $r=\frac{1}{2}|AB|=\frac{1}{2}\sqrt{2\left(4-\sqrt{3}\right)}$ A1

Question 2

a. $N=\dfrac{1000}{1+9e^{-kt}}=1000\left(1+9e^{-kt}\right)^{-1}$

$\dfrac{dN}{dt}=1000\times 9ke^{-kt}\left(1+9e^{-kt}\right)^{-2}=\dfrac{1000\times 9ke^{-kt}}{\left(1+9e^{-kt}\right)^{2}}$

but $1+9e^{-kt}=\dfrac{1000}{N} \quad \Rightarrow \quad 9e^{-kt}=\dfrac{1000}{N}-1=\dfrac{1000-N}{N}$ M1

$\dfrac{dN}{dt}=\dfrac{1000\times 9ke^{-kt}}{\left(1+9e^{-kt}\right)^{2}}$

$\dfrac{dN}{dt}=k\left(\dfrac{1000}{1+9e^{-kt}}\right)^{2}\times\dfrac{9e^{-kt}}{1000}$ A1

$\dfrac{dN}{dt}=\dfrac{kN^2}{1000}\left(\dfrac{1000-N}{N}\right)$

$\dfrac{dN}{dt}=kN\left(1-\dfrac{N}{1000}\right)$

b. initial number $N_0=N(0)=\dfrac{1000}{1+9e^{0}}=100$

ultimate number, since $k>0$ $\lim\limits_{t\to\infty}N(t)=1000$ A1

c. $N(20)=900$

$900=\dfrac{1000}{1+9e^{-20k}} \Rightarrow 900+8100e^{-20k}=1000$

$e^{-20k}=\dfrac{100}{8100}=\dfrac{1}{81} \quad \Rightarrow \quad e^{20k}=81$

$k=\dfrac{1}{20}\log_e(81)=\dfrac{1}{5}\log_e(3)$ A1

d. $\dfrac{dN}{dt}=k\left(N-\dfrac{N^2}{1000}\right)$

$\dfrac{d^2N}{dt^2}=\dfrac{d}{dt}\left(\dfrac{dN}{dt}\right)=\dfrac{d}{dN}\left(k\left(N-\dfrac{N^2}{1000}\right)\right)\dfrac{dN}{dt}$ M1

$\dfrac{d^2N}{dt^2}=k^2N\left(1-\dfrac{N}{500}\right)\left(1-\dfrac{N}{1000}\right)$ A1

$\dfrac{d^2N}{dt^2}=0 \Rightarrow N=500 \Rightarrow t=10$

inflexion point at $(10,500)$ A1

e. $\frac{dN}{dt} = rN \Rightarrow N = N_0 e^{rt}$ $N(0) = 50 \Rightarrow N_0 = 50$

$N(20) = 450 \Rightarrow 450 = 50e^{20r}$

$r = \frac{1}{20}\log_e(9) = \frac{1}{10}\log_e(3)$ M1

$r \approx 0.1099$

$N = N(t) = 50e^{0.1099t}$ A1

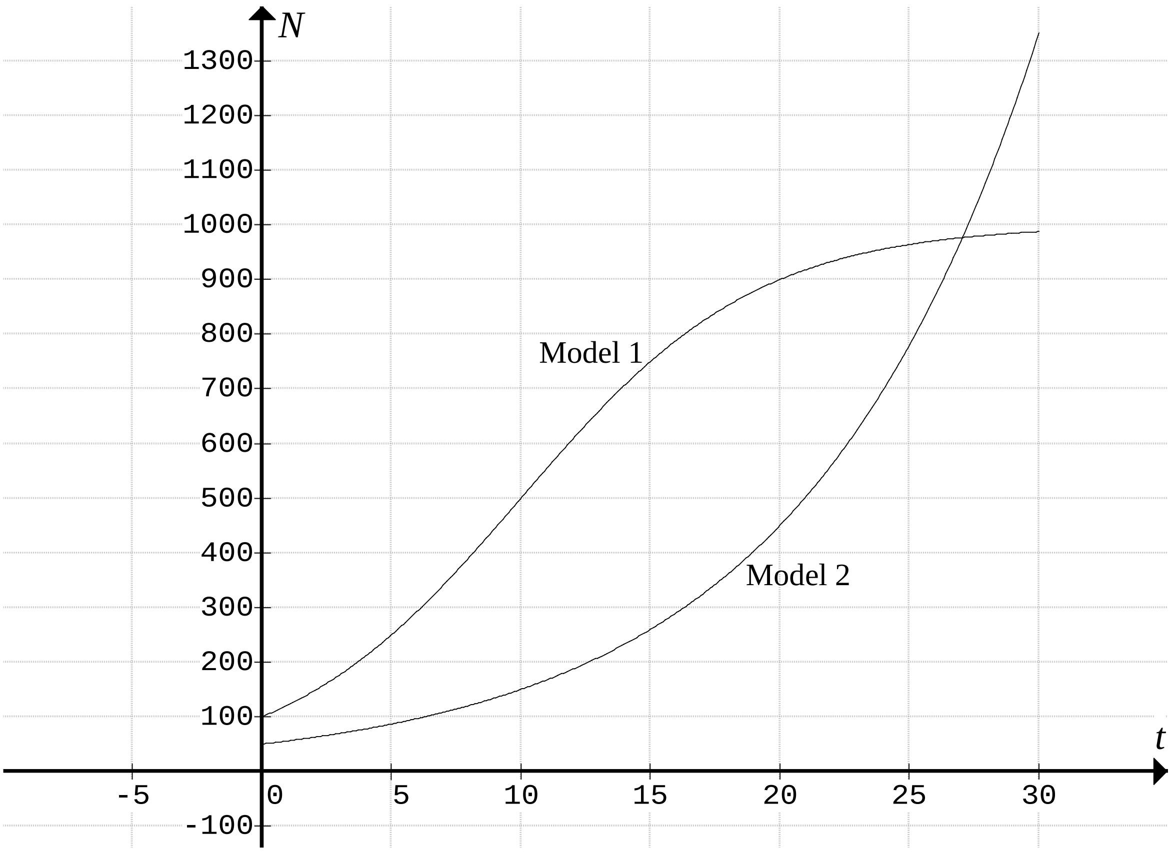
f. solving using CAS $50e^{0.1099t} = \frac{1000}{1+9e^{-0.2197t}}$ using exact values gives

$N = 977$ and $t = 27.1$ A1

g. both graphs correct shape, passing through exact points G2

Model 1 $(0,100)$, $(10,500)$, $(20,900)$, $(27,977)$, $(30,988)$ asymptotes to $N = 1000$

Model 2 $(0,50)$, $(20,450)$, $(27,977)$, $(30,1350)$



Question 3

a. $x = 9\cos(t) \quad y = \dfrac{9\sin^2(t)}{2+\sin(t)}$

$LHS = y^2\left(81 - x^2\right)$ substituting

$$= \frac{81\sin^4(t)}{\left(2+\sin(t)\right)^2}\left(81 - 81\cos^2(t)\right)$$

$$= \frac{81\sin^4(t)}{\left(2+\sin(t)\right)^2}\left(81\left(1-\cos^2(t)\right)\right)$$ M1

$$= \frac{81^2\sin^6(t)}{\left(2+\sin(t)\right)^2}$$

$RHS = \left(x^2 + 18y - 81\right)^2$ substituting

$$= \left(81\cos^2(t) + \frac{18\times 9\sin^2(t)}{2+\sin(t)} - 81\right)^2$$

$$= \left(81\left(\cos^2(t)-1\right) + \frac{162\sin^2(t)}{2+\sin(t)}\right)^2$$ M1

$$= \left(-81\sin^2(t) + \frac{162\sin^2(t)}{2+\sin(t)}\right)^2$$

$$= \left(\frac{-81\sin^2(t)\left(2+\sin(t)\right)+162\sin^2(t)}{2+\sin(t)}\right)^2$$

$$= \left(\frac{-162\sin^2(t)-81\sin^3(t)+162\sin^2(t)}{2+\sin(t)}\right)^2$$ A1

$$= \frac{81^2\sin^6(t)}{\left(2+\sin(t)\right)^2} = LHS$$

b. the gradient is zero $\frac{dy}{dx}=0 \quad \frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{\dot{y}}{\dot{x}}=0$

$x(t)=9\cos(t) \quad \frac{dx}{dt}=\dot{x}=-9\sin(t)$ for $t\in[0,2\pi]$.

$y(t)=\frac{9\sin^2(t)}{2+\sin(t)} \quad \frac{dy}{dt}=\dot{y}=\frac{9\sin(t)(\sin(t)+4)\cos(t)}{(2+\sin(t))^2}$ M1

solving, $\frac{dy}{dx}=\frac{-(\sin(t)+4)\cos(t)}{(2+\sin(t))^2}=0$ since $\sin(t)\neq -4$

$\Rightarrow \cos(t)=0$ with $t\in[0,2\pi]\Rightarrow t=\frac{\pi}{2},\frac{3\pi}{2}$ A1

$x\left(\frac{\pi}{2}\right)=0$, $x\left(\frac{3\pi}{2}\right)=0$ and $y\left(\frac{\pi}{2}\right)=3$, $y\left(\frac{3\pi}{2}\right)=9$

turning points are $t=\frac{\pi}{2}\Rightarrow (0,3) \quad t=\frac{3\pi}{2}\Rightarrow (0,9)$ A1

c. $y^2(81-x^2)=(x^2+18y-81)^2$

$y\sqrt{81-x^2}=\pm(x^2+18y-81)$

consider the case when taking the positive sign

$y\sqrt{81-x^2}=x^2+18y-81$

$81-x^2=18y-y\sqrt{81-x^2}=y\left(18-\sqrt{81-x^2}\right)$

$y=f(x)=\frac{81-x^2}{18-\sqrt{81-x^2}}$ and $f(0)=9$

consider the case when taking the negative sign M1

$y\sqrt{81-x^2}=-x^2-18y+81$

$81-x^2=y\sqrt{81-x^2}+18y=y\left(\sqrt{81-x^2}+18\right)$

$y=g(x)=\frac{81-x^2}{18+\sqrt{81-x^2}}$ and $g(0)=3$

both have maximal domain of $[-9,9]$

$g:[-9,9]\to R$, $g(x)=\frac{81-x^2}{18+\sqrt{81-x^2}}$ A1

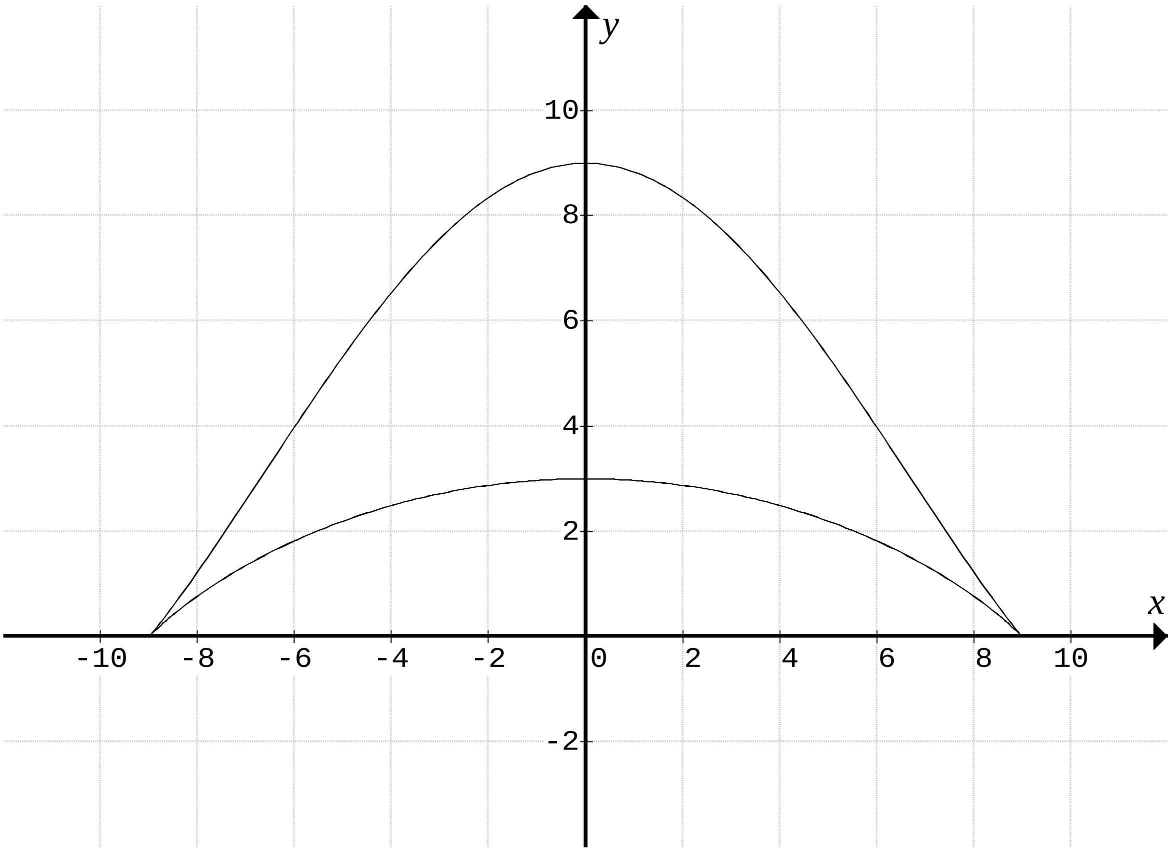
d. graphs correct shape, plotting $f(x)$ and $g(x)$ or parametric $x1(t), y1(t)$ over correct domain $[-9,9]$ passing through exact points

$t=0,2\pi \Rightarrow A(9,0)$, $t=\frac{\pi}{2}\Rightarrow B(0,3)$, $t=\pi \Rightarrow C(-9,0)$, $t=\frac{3\pi}{2}\Rightarrow D(0,9)$ A1

$B(0,3)$ and $D(0,9)$ are a maximum turning points.

The function $f(x)$ passes through *ADC*

The function $g(x)$ passes through *ABC* A1



e. $A=\int_{-9}^{9}\left(f(x)-g(x)\right)dx=2\int_{0}^{9}\left(f(x)-g(x)\right)dx$ by symmetry

$$A=2\int_0^9\left(\frac{81-x^2}{18-\sqrt{81-x^2}}-\frac{81-x^2}{18+\sqrt{81-x^2}}\right)dx$$

$$A=2\int_0^9\left(\frac{\left(81-x^2\right)\left[\left(18+\sqrt{81-x^2}\right)-\left(18-\sqrt{81-x^2}\right)\right]}{\left(18-\sqrt{81-x^2}\right)\left(18+\sqrt{81-x^2}\right)}\right)dx$$ M1

$$A=2\int_0^9\left(\frac{\left(81-x^2\right)2\sqrt{81-x^2}}{\left(18^2-\left(81-x^2\right)\right)}\right)dx$$

$$A=\int_0^9\frac{4\left(81-x^2\right)^{\frac{3}{2}}}{x^2+243}dx \quad \Rightarrow \quad n=\frac{3}{2} \quad b=243$$ A1

f.i. $y=\dfrac{81-x^2}{18-\sqrt{81-x^2}}$ let $u=x^2$

$$y=\frac{81-u}{18-\sqrt{81-u}} \quad \text{solving CAS gives}$$

$$u=x^2=\frac{1}{2}\left(\sqrt{y^3\left(y+72\right)}-y^2-36y+162\right)$$

$$V=\pi\int_0^9 x_2^2\,dy-\pi\int_0^3 x_1^2\,dy$$

$$V=\pi\int_3^9 x^2\,dy$$

$$V=\frac{\pi}{2}\int_3^9\left(\sqrt{y^3\left(y+72\right)}-y^2-36y+162\right)dy$$ A1

f.ii. $V=391.2\text{ units}^3$ A1

Define $x1(t)=9\cdot\cos(t)$	*Done*
Define $y1(t)=\dfrac{9\cdot(\sin(t))^2}{2+\sin(t)}$	*Done*
$\dfrac{d}{dt}(x1(t))$	$-9\cdot\sin(t)$
$\dfrac{d}{dt}(y1(t))$	$\dfrac{9\cdot\sin(t)\cdot(\sin(t)+4)\cdot\cos(t)}{(\sin(t)+2)^2}$
solve$\left(\dfrac{\frac{d}{dt}(y1(t))}{\frac{d}{dt}(x1(t))}=0,t\right)\|0\le t\le 2\cdot\pi$	$t=\dfrac{\pi}{2}$ or $t=\dfrac{3\cdot\pi}{2}$
$x1\left(\dfrac{\pi}{2}\right)$	0
$x1\left(\dfrac{3\cdot\pi}{2}\right)$	0
$y1\left(\dfrac{\pi}{2}\right)$	3
$y1\left(\dfrac{3\cdot\pi}{2}\right)$	9

$y^2\cdot(81-x^2)=(x^2+18\cdot y-81)^2$	$-(x^2-81)\cdot y^2=(x^2+9\cdot(2\cdot y-9))^2$
impDif$\left(y^2\cdot(81-x^2)=(x^2+18\cdot y-81)^2,x,y\right)$	$\dfrac{-x\cdot(2\cdot x^2+y^2+36\cdot y-162)}{x^2\cdot(y+18)+243\cdot(y-6)}$
Define $f1(x)=\dfrac{81-x^2}{18-\sqrt{81-x^2}}$	*Done*
Define $f2(x)=\dfrac{81-x^2}{18+\sqrt{81-x^2}}$	*Done*
$f1(0)$	9
$f2(0)$	3
$y=\dfrac{81-x^2}{18+\sqrt{81-x^2}}\|x^2=u$	$y=\dfrac{-(u-81)}{\sqrt{81-u}+18}$
solve$\left(y=\dfrac{-(u-81)}{\sqrt{81-u}+18},u\right)$	$u=\dfrac{\sqrt{y^3\cdot(y+72)}-y^2-36\cdot y+162}{2}$ or $u=\dfrac{-\left(\sqrt{y^3\cdot(y+72)}+y^2+36\cdot y-162\right)}{2}$
$\pi\cdot\displaystyle\int_3^9 \dfrac{\sqrt{y^3\cdot(y+72)}-y^2-36\cdot y+162}{2}\,dy$	391.2027

Question 4

a. $\dot{x} = \dfrac{dx}{dt} = 12 - 0.1x$

$\dfrac{dt}{dx} = \dfrac{1}{12 - 0.1x} = \dfrac{10}{120 - x}$

$\dfrac{1}{10}\int 1\,dt = \int \dfrac{1}{120 - x}\,dx$ M1

$\dfrac{t}{10} = -\log_e(120 - x) + C$

but when $t = 0 \quad x = 0$

$0 = -\log_e(120) + C \quad \Rightarrow \quad C = \log_e(120)$

$\dfrac{t}{10} = -\log_e(120 - x) + \log_e(120)$ M1

$\dfrac{t}{10} = -\log_e\left(\dfrac{120 - x}{120}\right)$

$\dfrac{120 - x}{120} = e^{-\frac{t}{10}}$

$120 - x = 120e^{-\frac{t}{10}}$ A1

$x = x(t) = 120\left(1 - e^{-\frac{t}{10}}\right)$

b. $\dot{y} = \dfrac{dy}{dt} = 12 - gt = 12 - 9.8t$

$y = \int (12 - 9.8t)\,dt$

$y = 12t - 4.9t^2 + C$ A1

but when $t = 0 \quad y = 1 \quad \Rightarrow C = 1$

$y = 12t - 4.9t^2 + 1$

c. hits the ground when $y = 0 \Rightarrow 12t - 4.9t^2 + 1 = 0$

solving $t = 2.52966$

correct to 3 decimal places $T = 2.530$ s A1

d. the range $R = x(T) = 120\left(1 - e^{-\frac{2.52966}{10}}\right) = 26.821\text{m}$ A1

e. at the maximum height $\dot{y}=12-9.8t=0 \Rightarrow t=\dfrac{12}{9.8}=1.22449$

so $t=1.224$ s A1

$H=y(1.22449)=1+12\times1.22449-4.9(1.22449)^2=8.347\text{m}$

and $x(1.22449)=120\left(1-e^{-\frac{1.22449}{10}}\right)=13.830\text{m}$ A1

f. when it hits the ground $T=2.5297$

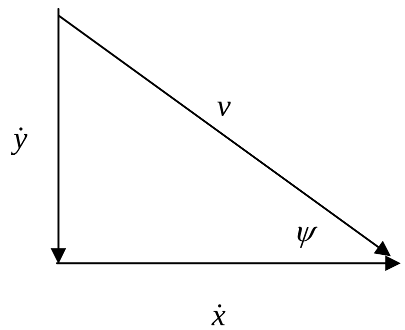
$\dot{x}(t)=12-0.1x=12-0.1\left(120\left(1-e^{-\frac{t}{10}}\right)\right)=12e^{-\frac{t}{10}}$

$\dot{x}(2.5297)=12e^{-\frac{2.5297}{10}}=9.318$

$\dot{y}(t)=12-9.8t$

$\dot{y}(2.52966)=12-9.8\times2.52966=-12.791$

The speed $|\dot{r}(t)|=\sqrt{\dot{x}^2+\dot{y}^2}=\sqrt{(9.318)^2+(-12.791)^2}=15.825\text{m/s}$ A1



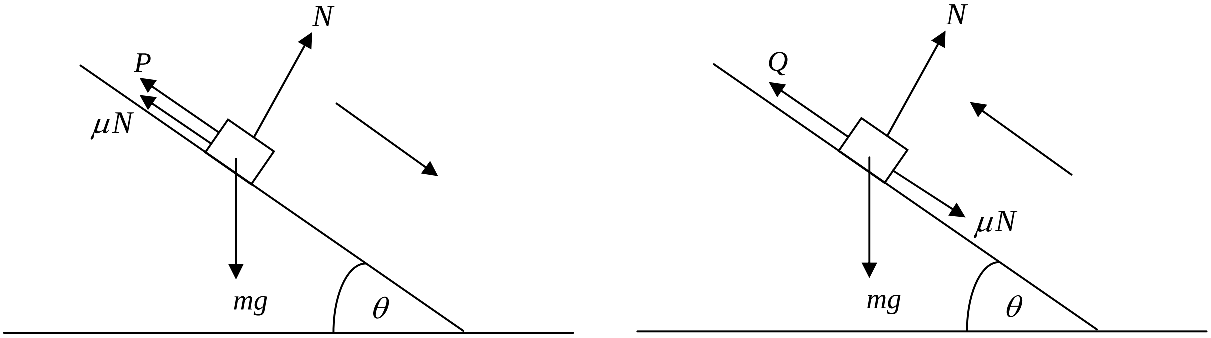
The angle at which it hits the ground $\tan(\psi)=\left|\dfrac{\dot{y}}{\dot{x}}\right|$

$\psi=\tan^{-1}\left(\dfrac{12.791}{9.318}\right)=53.927^0$

$\psi=53^0 56'$ A1

Define $x1(t)=120\cdot\left(1-e^{\frac{-t}{10}}\right)$	*Done*
Define $y1(t)=1+12\cdot t-4.9\cdot t^{2}$	*Done*
$\text{solve}(y1(t)=0,t)\vert t\geq 0$	$t=2.5297$
$tg:=2.5296552616588$	2.5297
$x1(tg)$	26.8206
$\frac{d}{dt}(y1(t))$	$12.0000-9.8000\cdot t$
$\text{solve}\left(\frac{d}{dt}(y1(t))=0,t\right)$	$t=1.2245$
$tt:=1.2244897959184$	1.2245
$y1(tt)$	8.3469
$x1(tt)$	13.8299
$\frac{d}{dt}(x1(t))\vert t=tg$	9.3179
$\frac{d}{dt}(y1(t))\vert t=tg$	-12.7906
$\sqrt{(-12.79062)^{2}+(9.31793)^{2}}$	15.8248
$\tan^{-1}\left(\left\vert\frac{-12.79062}{9.31793}\right\vert\right)$	53.9268
(53.926769293273)▸DMS	53°55'36.3695"

g.



diagrams, with correct forces
in both situations resolving perpendicular to the plane

$N - mg\cos(\theta) = 0 \quad \Rightarrow \quad N = mg\cos(\theta)$ A1

when the football is on the point of moving down the grassy slope

$(1)\ P + \mu N - mg\sin(\theta) = 0$

when the football is on the point of moving up the grassy slope

$(2)\ Q - \mu N - mg\sin(\theta) = 0$ M1

adding $(1)+(2)$ gives

$P + Q = 2mg\sin(\theta)$

$\sin(\theta) = \dfrac{P+Q}{2mg}$ A1

subtracting $(2)-(1)$ gives

$Q - P = 2\mu N = 2\mu mg\cos(\theta)$

$\cos(\theta) = \dfrac{Q-P}{2\mu mg}$

eliminating θ using $\sin^2(\theta) + \cos^2(\theta) = 1$ M1

$$\frac{(P+Q)^2}{4m^2g^2} + \frac{(Q-P)^2}{4\mu^2m^2g^2} = 1$$

$$\frac{(Q-P)^2}{4\mu^2m^2g^2} = 1 - \frac{(P+Q)^2}{4m^2g^2} = \frac{4m^2g^2 - (P+Q)^2}{4m^2g^2} \quad \text{since } Q > P$$

$$\mu = \frac{Q-P}{\sqrt{4m^2g^2 - (P+Q)^2}}$$ A1

END OF SECTION 2 SUGGESTED ANSWERS