insight™

Year 12 *Trial Exam Paper*

2014

SPECIALIST MATHEMATICS

Written examination 1

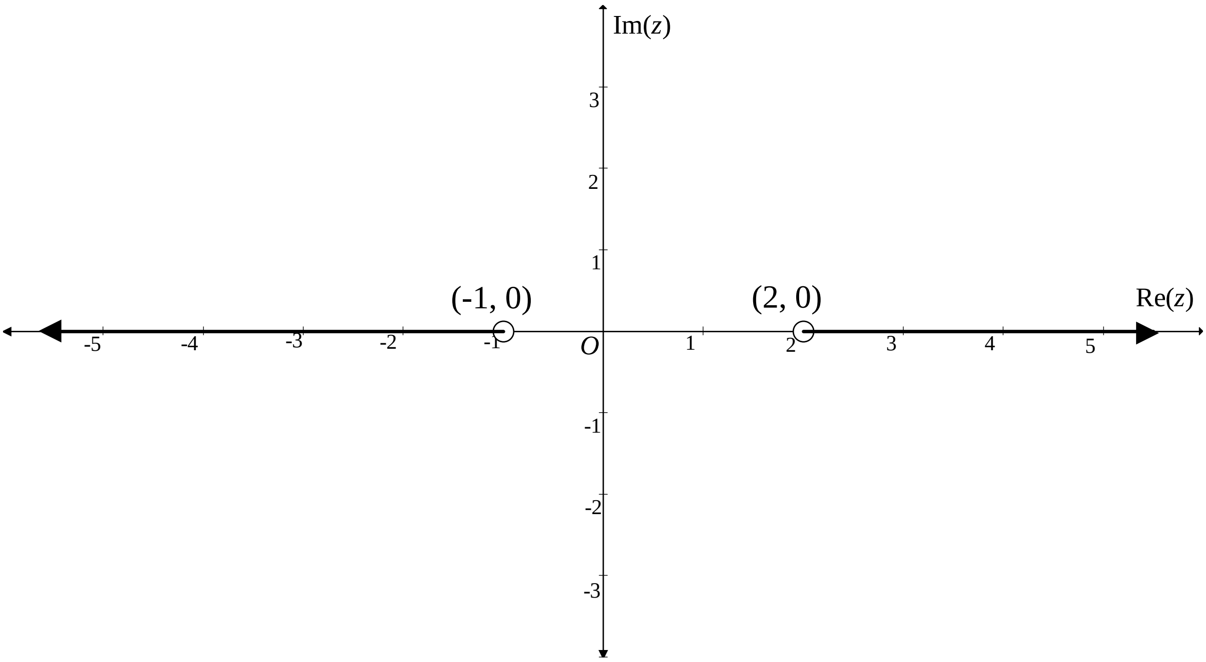
Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

Question 1a.

Worked solution



$\text{Arg}(z-2)=\tan^{-1}\left(\dfrac{y}{x-2}\right)$ is a ray starting from (2, 0) at angle θ to the positive real axis.

$\text{Arg}(z+1)=\tan^{-1}\left(\dfrac{y}{x+1}\right)$ is a ray starting from (–1, 0) at angle θ to the positive real axis.

Since Arg(z – 2) = Arg(z + 1),

$\theta=\tan^{-1}\left(\dfrac{y}{x-2}\right)=\tan^{-1}\left(\dfrac{y}{x+1}\right)$

$\Rightarrow \dfrac{y}{x-2}=\dfrac{y}{x+1}$

$\Rightarrow y=0$

$\therefore\ \theta=\tan^{-1}(0)=0$ or π.

$\Rightarrow$ Arg$(z-2)$ = Arg$(z+1)$ along the real axis.

For $\theta=0$, $x>2\cap x>-1\Rightarrow x>2$.

For $\theta=\pi$, $x<2\cap x<-1\Rightarrow x<-1$.

Mark allocation: 3 marks

- 1 mark for justifying $\theta=0$ or π.
- 1 mark for the correct ray from $(2,\infty)$.
- 1 mark for the correct ray from $(-\infty,-1)$.

Question 1b.

Worked solution

$$\begin{aligned} P(z) &= z^3 - (4+i)z^2 + (5+4i)z - 5i \\ &= (z-i)(z^2 - 4z + 5) \\ &= (z-i)(z^2 - 4z + 4 + 1) \\ &= (z-i)[(z-2)^2 + 1] \end{aligned}$$

$\Rightarrow P(z) = (z-i)(z-2+i)(z-2-i)$

$\therefore$ The other factors are $(z-2+i)$ and $(z-2-i)$.

Mark allocation: 2 marks

- 1 mark for the correct factor $(z-2+i)$.
- 1 mark for the correct factor $(z-2-i)$.

Question 2

Worked solution

$$y = \frac{\log_e x}{\sqrt{x}}$$

$$x = 1 \Rightarrow y = 0$$

$$V = \pi \int_1^e \left(\frac{\log_e x}{\sqrt{x}} \right)^2 .dx$$

$$= \pi \int_1^e \frac{(\log_e x)^2}{x} .dx$$

Let $u = \log_e x$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$x = e \Rightarrow u = \log_e (e) = 1$$

$$x = 1 \Rightarrow u = \log_e (1) = 0$$

$$V = \pi \int_0^1 u^2 . \frac{du}{dx} .dx$$

$$= \pi \int_0^1 u^2 .du$$

$$= \pi \left[\frac{u^3}{3} \right]_0^1$$

$$= \pi \left(\frac{1}{3} - 0 \right)$$

$$= \frac{\pi}{3} \text{ cubic units}$$

Mark allocation: 4 marks

- 1 mark for the correct integral representing the volume.
- 1 mark for using the appropriate substitution to antidifferentiate.
- 1 mark for correctly antidifferentiating.
- 1 mark for the correct answer.

Question 3

Worked solution

$$ye^{x^2+2x} + xe^y = 2x+1$$

$$\frac{d}{dx}(ye^{x^2+2x} + xe^y) = \frac{d}{dx}(2x+1)$$

$$\Rightarrow (2x+2).e^{x^2+2x}.y + \frac{dy}{dx}.e^{x^2+2x} + e^y + xe^y.\frac{dy}{dx} = 2$$

$$\Rightarrow (2x+2).e^{x^2+2x}.y + e^y + \frac{dy}{dx}(e^{x^2+2x} + xe^y) = 2$$

$$\frac{dy}{dx}(e^{x^2+2x} + xe^y) = 2 - (2x+2).e^{x^2+2x}.y - e^y$$

$$\frac{dy}{dx} = \frac{2-(2x+2).e^{x^2+2x}.y - e^y}{(e^{x^2+2x} + xe^y)}$$

Substituting (0, 1) gives:

$$\frac{dy}{dx} = \frac{2-2e^0.1-e^1}{(e^0+0)}$$

$$= -e$$

$\therefore$ The gradient of the normal is $\frac{1}{e}$.

$\therefore$ $y-1 = \frac{1}{e}(x-0) = e^{-1}x$ is the equation of the normal.

$\Rightarrow y = 1 + e^{-1}x$

Mark allocation: 3 marks

- 1 mark for differentiating correctly.
- 1 mark for correctly evaluating $\frac{dy}{dx}$ at (0, 1).
- 1 mark for the correct equation of the normal.

Tip

- *The relation must be differentiated implicitly to obtain the gradient.*

Question 4a.

Worked solution

$r = \tan(t)i + tj$, where $t \in \left[0, \frac{\pi}{2}\right)$.

$x = \tan t$

$\Rightarrow t = \tan^{-1} x$

$y = t$

$\therefore y = \tan^{-1} x$

$t \in \left[0, \frac{\pi}{2}\right) \Rightarrow 0 \le \tan^{-1} x < \frac{\pi}{2}$

$\Rightarrow \tan 0 \le x < \tan \frac{\pi}{2}$

$\therefore$ Domain is $[0, \infty)$.

Mark allocation: 2 marks

- 1 mark for the correct Cartesian equation.
- 1 mark for the correct domain.

Question 4b.

Worked solution

$f(x) = \tan^{-1} x$

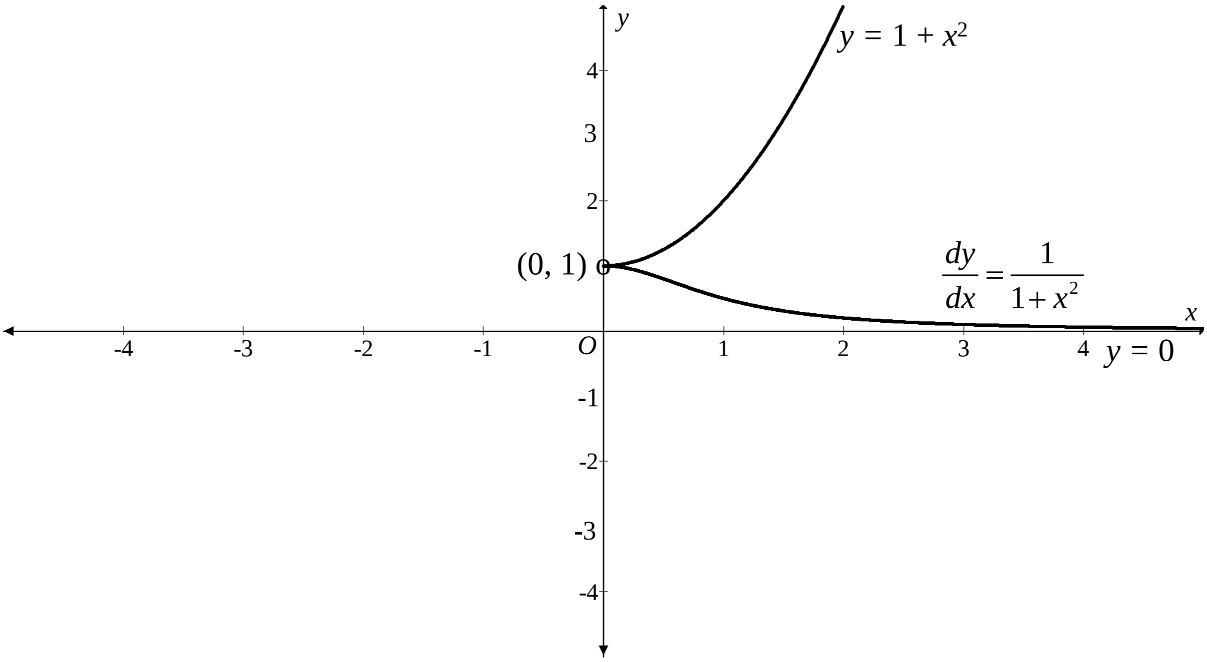
$f'(x) = \frac{1}{1+x^2}$

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 4c.

Worked solution



Mark allocation: 2 marks

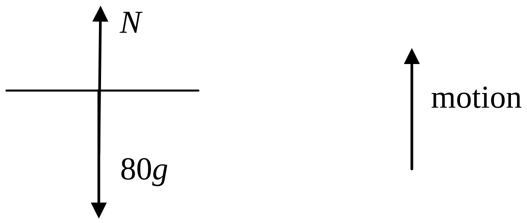
- 1 mark for the correct maximum turning point and asymptotes.
- 1 mark for a correctly shaped curve.

Tip

- *Sketch the graph of* $y = 1 + x^2$*, then obtain the graph of its reciprocal.*

Question 5a.i.

Worked solution



$$R = N - 80g = 80 \times 0.2$$
$$\Rightarrow N = 80(g + 0.2) = 80 \times 10 = 800$$
$$\therefore \text{ Scale reading is } \frac{800}{g} \text{ kg.}$$

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 5a.ii.

Worked solution

$$N = 60g$$
$$\therefore R = 60g - 80g = 80 \times a$$
$$80a = -20g$$
$$\Rightarrow a = \frac{-20g}{80} = \frac{-g}{4}$$
$$\therefore \text{ The acceleration is } \frac{-g}{4} \text{ m/s}^2.$$

Mark allocation: 1 mark

- 1 mark for the correct answer.

Tip

- *The normal reaction is equal to the reading on the scales.*

Question 5b.

Worked solution

$$R = 500g - 0.6x^2 = 500a$$

$$\Rightarrow a = g - 0.0012x^2$$

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - 0.0012x^2$$

$$\Rightarrow \frac{v^2}{2} = gx - 0.0004x^3 + c$$

$$v^2 = 2gx - 0.0008x^3 + 2c$$

But when $x = 0$, $v = 0$.

$\Rightarrow 2c = 0$

$\therefore v^2 = 2gx - 0.0008x^3$

When $x = 50$:

$$v^2 = 100g - 0.0008 \times 125 \times 1000$$

$$v^2 = 100g - 100 = 100(g - 1)$$

$\therefore v = 10\sqrt{g-1}$

So $a = 10$ and $b = 1$.

Mark allocation: 3 marks

- 1 mark for the correct acceleration.
- 1 mark for correctly expressing v^2 in terms of x.
- 1 mark for evaluating a and b correctly.

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Question 6

Worked solution

$\overrightarrow{AB} = 2b - 2a$

$\Rightarrow \overrightarrow{AE} = \underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{CE} = \overrightarrow{CA} + \overrightarrow{AE} = 2\underset{\sim}{a} + (\underset{\sim}{b} - \underset{\sim}{a}) = \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{CF} = \underset{\sim}{b}$

$\Rightarrow \overrightarrow{AF} = -\overrightarrow{CA} + \overrightarrow{CF} = -2\underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{CM} = k\overrightarrow{CE} = k(\underset{\sim}{a} + \underset{\sim}{b})$

$\overrightarrow{FM} = -\overrightarrow{CF} + \overrightarrow{CM} = -\underset{\sim}{b} + k(\underset{\sim}{a} + \underset{\sim}{b})$

$\Rightarrow \overrightarrow{FM} = k\underset{\sim}{a} + (k-1)\underset{\sim}{b}$

$\overrightarrow{FA} = -\overrightarrow{CF} + \overrightarrow{CA} = 2\underset{\sim}{a} - \underset{\sim}{b}$

But $\overrightarrow{FM}$ is parallel to $\overrightarrow{FA}$.

$$\therefore \frac{k}{2} = \frac{k-1}{-1}$$

$$-k = 2k - 2$$

$$3k = 2$$

$$k = \frac{2}{3}$$

$$\therefore \overrightarrow{CM} = \frac{2}{3}\overrightarrow{CE}$$

$$\Rightarrow \overrightarrow{ME} = \frac{1}{3}\overrightarrow{CE}$$

$\therefore$ Point M divides $\overrightarrow{CE}$ in the ratio $2:1$.

Mark allocation: 4 marks

- 1 mark for the correctly finding $\overrightarrow{CM}$.
- 1 mark for correctly finding $\overrightarrow{FM}$.
- 1 mark for obtaining the correct value of k.
- 1 mark for showing $\overrightarrow{CM} = 2\overrightarrow{CE}$.

Tip

- *If two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel, then $\underset{\sim}{a} = k\underset{\sim}{b}$.*

Question 7

Worked solution

$$\frac{dJ}{dt} \propto J$$

$$\frac{dJ}{dt} = kJ$$

$$\frac{dt}{dJ} = \frac{1}{kJ} = \frac{1}{k} \times \frac{1}{J}$$

$$t = \frac{1}{k} \int \frac{1}{J} dJ$$

$$t = \frac{1}{k} \log_e J + c$$

Or

$$t = A \log_e J + c$$

$$t = 0, J = J_o$$

$$A \log_e J_o + c = 0$$

$$c = -A \log_e J_o$$

$$t = A \log_e J - A \log_e J_o$$

$$t = A \log_e \frac{J}{J_o}$$

When t = 0.1, J is $\frac{1}{10}$th of its original strength, $J = 10^{-1} \times J_o$.

$$0.1 = A \log_e \frac{10^{-1} J_o}{J_o} = -A \log_e 10$$

$$A = \frac{-0.1}{\log_e 10}$$

$$\Rightarrow t = \frac{-0.1}{\log_e 10} \times \log_e \frac{J}{J_o}$$

When J is one-millionth of its original strength, $J = 10^{-6} \times J_o$.

$$t = \frac{-0.1}{\log_e 10} \times \log_e \frac{10^{-6} J_o}{J_o}$$

$$t = \frac{-0.1}{\log_e 10} \times \log_e 10^{-6}$$

$$t = \frac{-0.1}{\log_e 10} \times -6 \log_e 10$$

$$\therefore\ t = 0.6 \text{ s}$$

Mark allocation: 4 marks

- 1 mark for setting up a differential equation for $\frac{dJ}{dt}$ correctly.
- 1 mark for correctly antidifferentiating to express t as a function of J.
- 1 mark for expressing t in terms of J and J_o.
- 1 mark for the correct answer.

Tip

- *Let J_o represent the original value of the electric current, or simply let $J_o = 1$ or any arbitrary number.*

Question 8

Worked solution

$$\sec(4\theta) = 8$$

$$\Rightarrow \cos(4\theta) = \frac{1}{8}$$

$$2\cos^2(2\theta) - 1 = \frac{1}{8}$$

$$2\cos^2(2\theta) = \frac{9}{8}$$

$$\cos^2(2\theta) = \frac{9}{16}$$

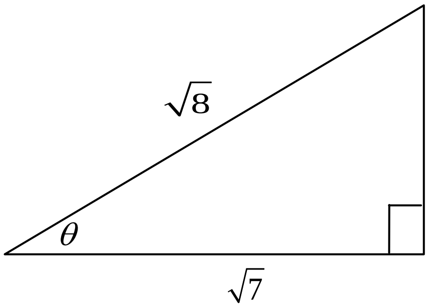
$$\therefore \cos(2\theta) = \frac{3}{4} \text{ since } 0 \le 2\theta \le \frac{\pi}{4}.$$

$$2\cos^2(\theta) - 1 = \frac{3}{4}$$

$$2\cos^2(\theta) = \frac{7}{4}$$

$$\cos^2(\theta) = \frac{7}{8}$$

$$\Rightarrow \cos(\theta) = \frac{\sqrt{7}}{\sqrt{8}} \text{ since } 0 \le \theta \le \frac{\pi}{8}.$$



$\sqrt{8-7} = 1$ (Pythagoras' theorem)

$$\therefore \tan(\theta) = \frac{1}{\sqrt{7}}$$

Mark allocation: 3 marks

- 1 mark for correctly evaluating $\cos(2\theta)$.
- 1 mark for correctly evaluating $\cos(\theta)$.
- 1 mark for the correct answer.

Tip

- *Use the identity* $\cos(2a) = 2\cos^2(a) - 1$.

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Question 9a.

Worked solution

$$f(x) = \frac{x}{1+x^2}$$

$$\Rightarrow f'(x) = \frac{1\times\left(1+x^2\right) - x\times 2x}{\left(1+x^2\right)^2}$$

$$f'(x) = \frac{1+x^2-2x^2}{\left(1+x^2\right)^2}$$

$$\Rightarrow f'(x) = \frac{\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

Mark allocation: 1 mark

- 1 mark for differentiating correctly.

Question 9b.

Worked solution

x-intercepts: $\quad a(1-x^2) = 0$

$$\Rightarrow a(1-x)(1+x) = 0$$

$$\therefore\ x = -1 \text{ or } x = 1$$

$$\text{Area} = \int_{-1}^{1} \frac{a(1-x^2)}{(1+x^2)^2}.dx$$

$$= \left[\frac{ax}{1+x^2}\right]_{-1}^{1}$$

$$= \frac{a}{2} - \frac{-a}{2}$$

$$= a$$

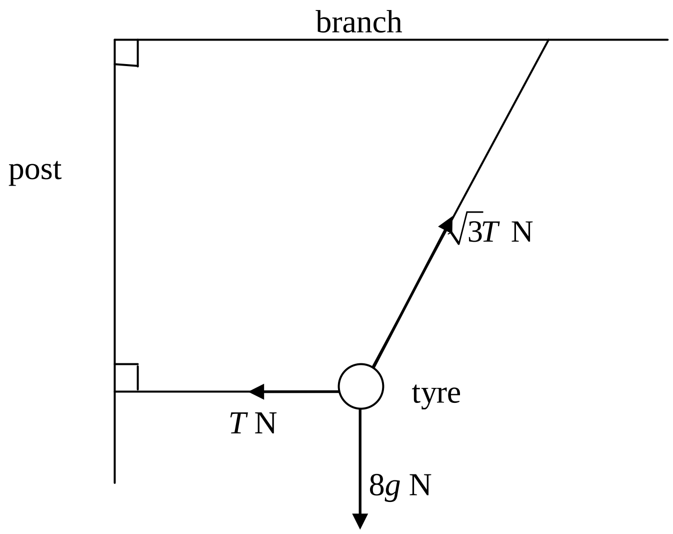
$\therefore$ The area is a square units.

Mark allocation: 3 marks

- 1 mark for correctly finding the *x*-intercepts.
- 1 mark for antidifferentiating correctly.
- 1 mark for the correct answer.

Question 10a.

Worked solution

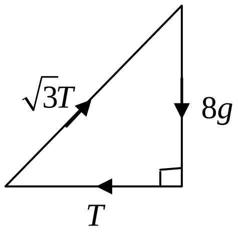


Mark allocation: 1 mark

- 1 mark for labelling the three forces correctly.

Question 10b.

Worked solution



$T^2 + (8g)^2 = (\sqrt{3}T)^2$

$T^2 + 64g^2 = 3T^2$

$\Rightarrow 2T^2 = 64g^2$

$T^2 = 32g^2$

$\therefore\ T = 4\sqrt{2}g$

The tension in the rope attached to the post is $4\sqrt{2}g$ newtons.

Mark allocation: 2 marks

- 1 mark for correctly setting up a triangle of forces.
- 1 mark for the correct answer.

Tip

- *Three forces acting on a particle that is in equilibrium can be expressed as a triangle of forces.*

END OF SOLUTIONS BOOK