THE HEFFERNAN GROUP
P.O. Box 1180
Surrey Hills North VIC 3127
Phone 03 9836 5021
Fax 03 9836 5025
info@theheffernangroup.com.au
www.theheffernangroup.com.au

SPECIALIST MATHS TRIAL EXAMINATION 2 SOLUTIONS 2014

SECTION 1 – Multiple-choice answers

1.	**E**	**7.**	**D**	**13.**	**D**	**19.**	**D**
2.	**B**	**8.**	**E**	**14.**	**C**	**20.**	**E**
3.	**C**	**9.**	**C**	**15.**	**D**	**21.**	**B**
4.	**A**	**10.**	**E**	**16.**	**A**	**22.**	**A**
5.	**A**	**11.**	**A**	**17.**	**C**		
6.	**D**	**12.**	**E**	**18.**	**B**		

SECTION 1 - Multiple-choice solutions

Question 1

First, we require an equation of the form $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ (rather than one of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$). So eliminate options A and C.

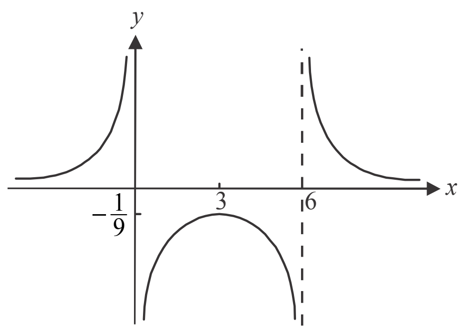
Second, if the graph with equation $\frac{(y-k)^2}{a^2}-\frac{x^2}{b^2}=1$ is translated left or right, then the maximum and minimum points on the branches of the hyperbola are no longer on the *y*-axis.
So eliminate options B and D.
The answer is E.

Question 2

Since the graph has vertical asymptotes of $x=6$ and $x=0$ (ie the *y*-axis), we have

$$y=\frac{1}{ax^2+bx}$$
$$=\frac{1}{x(ax+b)}$$

that is, $c=0$ and $6a+b=0$ so $b=-6a$.
Also, by symmetry, the maximum turning point occurs at $\left(3,-\frac{1}{9}\right)$.



So, $-\frac{1}{9}=\frac{1}{3(3a-6a)}$

$$-\frac{1}{9}=\frac{1}{-9a}$$
$$a=1$$

So $b=-6$
The answer is B.

Question 3

Since the period of *f* is π, we can eliminate options D and E which both have periods of 2π.

For option A, $f(x) = \text{cosec}\left(2x - \frac{\pi}{4}\right)$ so $f\left(\frac{\pi}{4}\right) = \sqrt{2}$.

Since this function is defined at $x = \frac{\pi}{4}$ we can eliminate option A.

For option B, $f(x) = \sec\left(2x - \frac{\pi}{2}\right)$ so $f\left(\frac{\pi}{4}\right) = 1$

Since this function is defined at $x = \frac{\pi}{4}$ we can eliminate option B.

The answer is C.

Question 4

The range of the function $y = \arctan\left(\frac{x}{a}\right)$ is $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The range of the function $y = 2\arctan\left(\frac{x}{a}\right)$ is $y \in (-\pi, \pi)$.

The range of the function $y = 2\arctan\left(\frac{x}{a}\right) - b$ is $y \in (-\pi - b, \pi - b)$.

The answer is A.

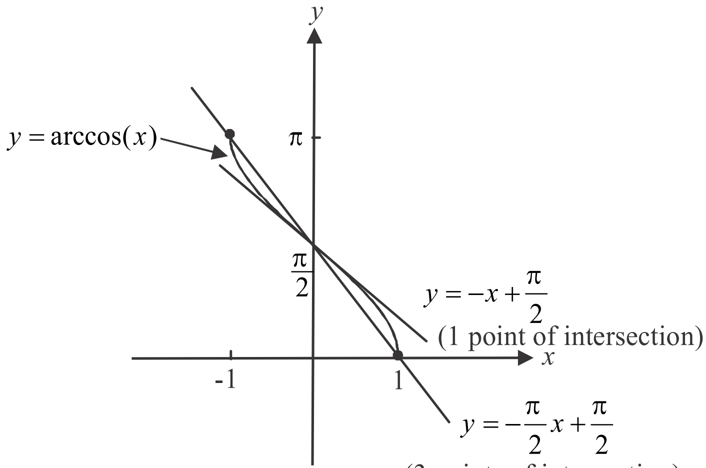
Question 5

Draw a graph.

A straight line through the points $(-1, \pi)$ and $(1,0)$ has gradient $= -\frac{\pi}{2}$

and it's equation is $y = -\frac{\pi}{2}(x-1)$

$$y = -\frac{\pi}{2}x + \frac{\pi}{2}.$$



This straight line intersects the graph of $y = \arccos(x)$ exactly three times with $\left(0, \frac{\pi}{2}\right)$ being the third point of intersection.

If this straight line is rotated anticlockwise about the point $\left(0, \frac{\pi}{2}\right)$, it will intersect exactly three times with the graph of $y = \arccos(x)$ **until** its gradient is equal to the gradient of $y = \arccos(x)$ at the point where $x = 0$. When this is the case, they will only intersect once.

i.e. for $y = \arccos(x)$, $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$

at $x = 0$, $\frac{dy}{dx} = -1$

So, for $-\frac{\pi}{2} \le a < -1$ there will be three points of intersection.

The answer is A.

Question 6

$$z = r\text{cis}(\theta)$$

$$\frac{1}{(\bar{z})^2} = \frac{1}{(r\text{cis}(-\theta))^2}$$

$$= \frac{\text{cis}(0)}{r^2\text{cis}(-2\theta)} \qquad \text{ie.} \text{cis}(0) = \cos(0) + i\sin(0) = 1$$

$$= \frac{1}{r^2}\text{cis}(2\theta)$$

$$= r^{-2}\text{cis}(2\theta)$$

The answer is D.

Question 7

$\text{Arg}(z) = \frac{\pi}{4}$ does not pass through the origin, it has an excluded endpoint at the origin.

For option B,

$$z\bar{z} = 1, \text{let } z = x + iy$$

$$(x+iy)(x-iy) = 1$$

$x^2 + y^2 = 1$ which is a circle that doesn't pass through the origin.

For option C

$$z + \bar{z} = 1$$

$$x + iy + x - iy = 1$$

$$2x = 1$$

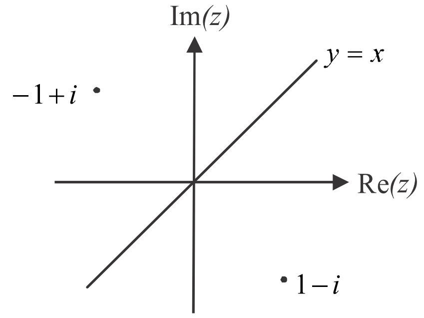
which is a straight line that doesn't pass through the origin.

For option D

$|z - 1 + i| = |z + 1 - i|$ which can be expressed as

$$|z - (1 - i)| = |z - (-1 + i)|$$

This defines the perpendicular bisector of the complex numbers $1 - i$ and $-1 + i$ which is the line $y = x$.



Alternatively, $|z - 1 + i| = |z + 1 - i|$

$$|x + iy - 1 + i| = |x + iy + 1 - i|$$

$$\sqrt{(x-1)^2 + (y+1)^2} = \sqrt{(x+1)^2 + (y-1)^2}$$

$$x^2 - 2x + 1 + y^2 + 2y + 1 = x^2 + 2x + 1 + y^2 - 2y + 1$$

$$-2x + 2y + 2 = 2x - 2y + 2$$

$$4y = 4x$$

$$y = x$$

The answer is D.

Question 8

$$z^3 = \sqrt{3}i \qquad \text{let } z = r\text{cis}(\theta)$$

$$(r\text{cis}(\theta))^3 = \sqrt{3}\text{cis}\left(\frac{\pi}{2}\right)$$

$$r^3\text{cis}(3\theta) = \sqrt{3}\text{cis}\left(\frac{\pi}{2}\right) \qquad \text{(De Moivre)}$$

$$3\theta = \frac{\pi}{2} + 2k\pi, \qquad k \in Z$$

$$\theta = \frac{\pi}{6} + \frac{2k\pi}{3}$$

So, $\theta = \frac{\pi}{6}$ when $k = 0$

$$\theta = \frac{\pi}{6} + \frac{2\pi}{3} \text{ when } k = 1$$

$$= \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{6} - \frac{2\pi}{3} \text{ when } k = -1$$

$$= \frac{-\pi}{2}$$

The solutions will start to repeat with other values of *k*.

The three principal arguments $\left(\text{i.e. where } -\pi < \theta \le \pi\right)$ are $\frac{-\pi}{2}, \frac{\pi}{6}$ and $\frac{5\pi}{6}$.

The answer is E.

Question 9

The coefficients of each of the terms in the equation are real therefore we know that the roots appear in conjugate pairs (conjugate root theorem) with the odd root out having to be real. Only option C shows three pairs of conjugate roots and one root on the Real axis of the Argand diagram.
The answer is C.

Question 10

Method 1 – using CAS

$$\int \sin^2(x)\,dx = \frac{x}{2} - \frac{\sin(x)\cos(x)}{2} + c$$
$$= \frac{x}{2} - \frac{\sin(2x)}{4} + c$$
$$= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + c$$

Method 2 – by hand

$$\int \sin^2(x)dx = \int\left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)dx$$

since $\cos(2x) = 1 - 2\sin^2(x)$ (formula sheet)

$$= \frac{1}{2}\int(1 - \cos(2x))dx$$
$$= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + c$$

The answer is E.

Question 11

$$\int_0^2 (x-1)\sqrt{2-x}\,dx$$
$$= \int_2^0 (1-u)\sqrt{u} \times -1\frac{du}{dx}dx$$
$$= \int_0^2 \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$

let $u = 2 - x$

$\frac{du}{dx} = -1$

and $x = 2 - u$

so $x - 1 = 1 - u$

Also, if $x = 2,\ u = 0$

and if $x = 0,\ u = 2$

The answer is A.

Question 12

For option A, when $x = 0,\ \frac{dy}{dx}$ is undefined whereas on the graph, the gradient is zero.

For option B, when $y = 0,\ \frac{dy}{dx}$ is undefined whereas on the graph, the gradient is defined along the x-axis.

For option C, when $x = 0,\ \frac{dy}{dx}$ is undefined whereas on the graph, the gradient is zero.

For option D, when x is positive, $\frac{dy}{dx}$ is always negative whereas on the graph when x is positive, the gradient is always positive.

Option E is correct since for $x = 0,\ \frac{dy}{dx} = 0,$ for $x < 0,\ \frac{dy}{dx} < 0$ and for $x > 0, \frac{dy}{dx} > 0$.

The answer is E.

Question 13

$f''(x) = 1 - x^2$

$f'(x) = x - \dfrac{x^3}{3} + c$

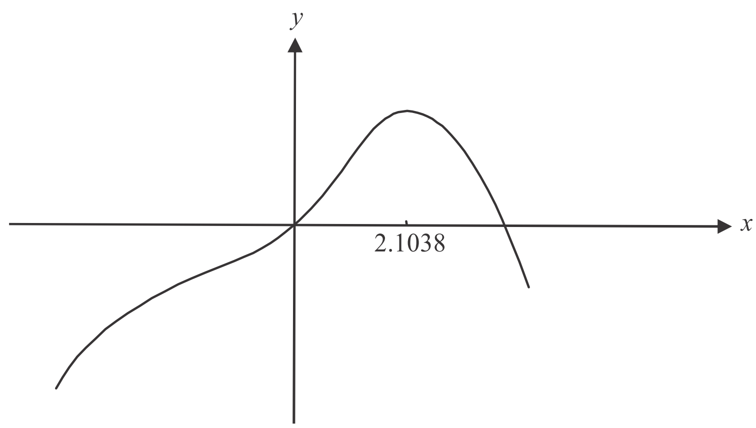
$f'(0) = 1$

$1 = c$

$f'(x) = x - \dfrac{x^3}{3} + 1$

$f(x) = \dfrac{x^2}{2} - \dfrac{x^4}{12} + x + d$

The graph of $y = f(x)$ will only pass through the origin if $d = 0$ so eliminate option C.
Sketch $y = f(x)$.



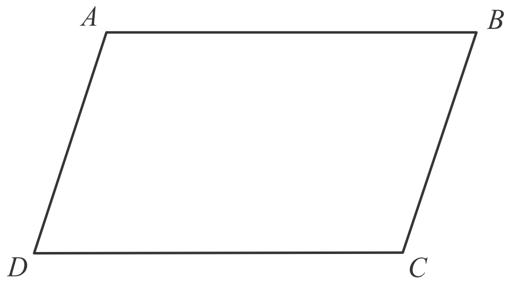
The case shown is for $d = 0$.
Stationary points occur when $f'(x) = 0$.
Solve $f'(x) = 0$ for x using CAS $x = 2.1038$.
This is the only solution so there is only one stationary point and it is a local maximum.
This eliminates option A, B and E.
The answer is D.

Question 14

$$\begin{aligned}\text{scalar resolute} &= \underset{\sim}{u} \bullet \underset{\sim}{\hat{v}}\\ &= (\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}) \bullet \frac{1}{\sqrt{16}}\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}\right)\\ &= \frac{1}{4}\left(2 + 6 - \sqrt{3}\right)\\ &= \frac{8 - \sqrt{3}}{4}\end{aligned}$$

The answer is C.

Question 15



If *ABCD* is a parallelogram then

$$\overrightarrow{AB} = \overrightarrow{DC}$$

$$\overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{DO} + \overrightarrow{OC}$$

$$-2\underset{\sim}{k} + 4\underset{\sim}{j} + m\underset{\sim}{k} = -3n\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{i} + 5\underset{\sim}{j} + n\underset{\sim}{k}$$

$$4\underset{\sim}{j} + (m-2)\underset{\sim}{k} = (3-3n)\underset{\sim}{i} + 4\underset{\sim}{j} + n\underset{\sim}{k}$$

So $3 - 3n = 0$

$n = 1$

and $m - 2 = n$

$m = 3$

Alternatively you could use

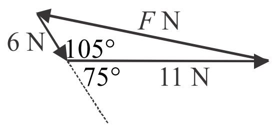
$\overrightarrow{AD} = \overrightarrow{BC}$

and use the same procedure.

The answer is D.

Question 16

The triangle of forces is shown below.



Using the cosine rule,

$$F = \sqrt{6^2 + 11^2 - 2 \times 6 \times 11\cos(105^\circ)}$$

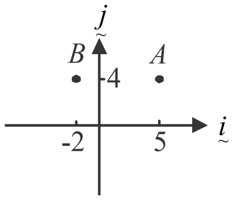
$$= \sqrt{6^2 + 11^2 - 132\cos(105^\circ)}$$

The answer is A.

Question 17

<u>Method 1</u>
If particle A is to be due east of particle B then the $\underset{\sim}{j}$ components of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ will be equal.

That is,
$$t+2=t^2$$
$$t^2-t-2=0$$
$$(t-2)(t+1)=0$$
$$t=2$$

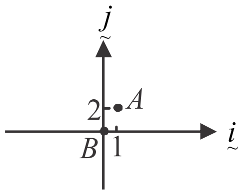


Confirming this, at $t=2$,
$\underset{\sim}{a}=5\underset{\sim}{i}+4\underset{\sim}{j}$ and $\underset{\sim}{b}=-2\underset{\sim}{i}+4\underset{\sim}{j}$.

At $t=2$, the positions of particle A and B are shown and A is due east of B.
The answer is C.

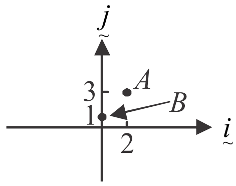
<u>Method 2</u> – trial and error
At $t=0$, $\underset{\sim}{a}=\underset{\sim}{i}+2\underset{\sim}{j}$ and $\underset{\sim}{b}=0\underset{\sim}{i}+0\underset{\sim}{j}$

A is clearly not due east of B.



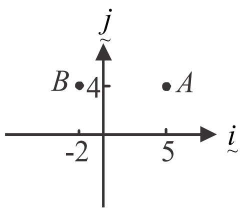
At $t=1$, $\underset{\sim}{a}=2\underset{\sim}{i}+3\underset{\sim}{j}$ and $\underset{\sim}{b}=0\underset{\sim}{i}+\underset{\sim}{j}$

A is again clearly not due east of B.



At $t=2$, $\underset{\sim}{a}=5\underset{\sim}{i}+4\underset{\sim}{j}$ and $\underset{\sim}{b}=-2\underset{\sim}{i}+4\underset{\sim}{j}$

So at $t=2$, A is due east of B.
The answer is C.



Question 18
<u>Method 1</u>

The normal passes through $P(x,y)$ and $(a,0)$ and has a gradient of $\dfrac{y-0}{x-a}=\dfrac{y}{x-a}$.

So the gradient of the tangent at $P(x,y)$; that is, $\dfrac{dy}{dx}=-\left(\dfrac{x-a}{y}\right)$.

$$\frac{dy}{dx}+\frac{x-a}{y}=0$$
The answer is B.

<u>Method 2</u>

The gradient of the normal is $-\dfrac{dx}{dy}$ and it passes through the point $(a,0)$. The equation of the normal is therefore:

$$y-0=-\frac{dx}{dy}(x-a)$$
$$\frac{y}{x-a}=-\frac{dx}{dy}$$

$$\frac{x-a}{y}=-\frac{dy}{dx}$$
$$\frac{dy}{dx}+\frac{x-a}{y}=0$$
The answer is B.

Question 19

$$a=\frac{(v^3-1)^2}{v}$$

$$v\frac{dv}{dx}=\frac{(v^3-1)^2}{v} \qquad \left(a=v\frac{dv}{dx} \text{ from formula sheet}\right)$$

$$\frac{dv}{dx}=\frac{(v^3-1)^2}{v^2}$$

$$\frac{dx}{dv}=\frac{v^2}{(v^3-1)^2}$$

<u>Method 1</u> – using CAS

$$x=\frac{-1}{3(v^3-1)}+c$$

<u>Method 2</u> – by hand

$$x=\int\frac{1}{3}\frac{du}{dv}u^{-2}\,dv \qquad u=v^3-1$$

$$=\frac{1}{3}\int u^{-2}\,du \qquad \frac{du}{dv}=3v^2$$

$$=\frac{1}{3}u^{-1}\times -1+c$$

$$=\frac{-1}{3(v^3-1)}+c$$

When $v=0,\ x=\frac{1}{3}$

$$\frac{1}{3}=\frac{-1}{-3}+c \quad \text{so} \quad c=0$$

$$x=\frac{-1}{3(v^3-1)}$$

$$3x(v^3-1)=-1$$

$$v^3-1=\frac{-1}{3x}$$

$$v^3=\frac{-1}{3x}+1$$

$$v=\sqrt[3]{\frac{3x-1}{3x}}$$

The answer is D.

Question 20

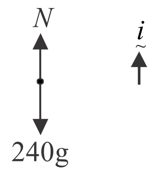
Let the reaction of the base of the crate on the load be *N*.

$$\underset{\sim}{R} = m\,\underset{\sim}{a}$$

$$(N - 240g)\,\underset{\sim}{i} = 240 \times 5\,\underset{\sim}{i}$$

So $N - 240g = 1200$

$$N = 1200 + 240g$$



The answer is E.

Question 21

$u = 6,\qquad v^2 = u^2 + 2as$

$s = 50\qquad 144 = 36 + 100a$

$v = 12\qquad a = \dfrac{27}{25}$

$a = ?$

$$|F| = m|a|$$

$$|F| = 5 \times \frac{27}{25}$$

$$= \frac{27}{5}$$

= 5.4 newtons

The answer is B.

Question 22

During the first five seconds the particle has a positive velocity so it travels $\frac{1}{2} \times 2 \times 3 + 2 \times 3 + \frac{1}{2} \times 3 = 10\frac{1}{2}$ units to the right.

During the last five seconds its velocity is negative and it travels $\frac{1}{2} \times 1 \times 3 + 3 \times 3 + \frac{1}{2} \times 1 \times 3 = 12$ units to the left.

It ends up $1\frac{1}{2}$ units left of its starting point.

We are told that the particle is initially 2 units from the fixed point. This means it could have a possible displacement from this fixed point of 2 or – 2.

A possible displacement from the fixed point could therefore be -3.5.

The answer is A.

SECTION 2

Question 1 (11 marks)

a. $xy - x^3 = 2$

Method 1 – using implicit differentiation

$$xy - x^3 = 2$$

$$1 \times y + x\frac{dy}{dx} - 3x^2 = 0$$

$$x\frac{dy}{dx} = 3x^2 - y$$

$$= 3x^2 - \left(\frac{2 + x^3}{x}\right) \quad \text{since} \quad xy = 2 + x^3$$

$$= 3x^2 - \frac{2}{x} - x^2 \qquad\qquad y = \frac{2 + x^3}{x}$$

$$= 2x^2 - \frac{2}{x}$$

So $\dfrac{dy}{dx} = 2x - \dfrac{2}{x^2}$ as required.

(1 mark)

Method 2

$$xy - x^3 = 2$$

$$xy = 2 + x^3$$

$$y = \frac{2}{x} + x^2$$

$$= 2x^{-1} + x^2$$

$$\frac{dy}{dx} = -2x^{-2} + 2x$$

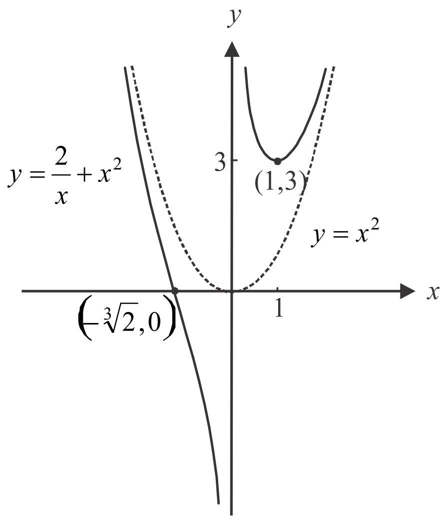
$$= \frac{-2}{x^2} + 2x$$

So $\dfrac{dy}{dx} = 2x - \dfrac{2}{x^2}$ as required.

(1 mark)

b. $xy - x^3 = 2$

$$y = \frac{2}{x} + x^2$$



The y-axis is an asymptote.

Also, as $x \to \pm\infty$, $y \to x^2$ so $y = x^2$ is a curved asymptote.

As $x \to -\infty$, $y \to x^2$ from below.

As $x \to +\infty$, $y \to x^2$ from above.

<u>x-intercepts</u> occur when $y = 0$

Solve $0 = \frac{2}{x} + x^2$ for x

$$x = -\sqrt[3]{2}$$
$$\approx -1.25$$

<u>Stationary points</u> occur when $\frac{dy}{dx} = 0$

Solve $0 = 2x - \frac{2}{x^2}$ for x (from part a.)

$$x = 1$$

so $y = 3$

(1 mark) – correct shape including acknowledgement of both asymptotes
(1 mark) – correct x-intercept
(1 mark) – correct stationary point

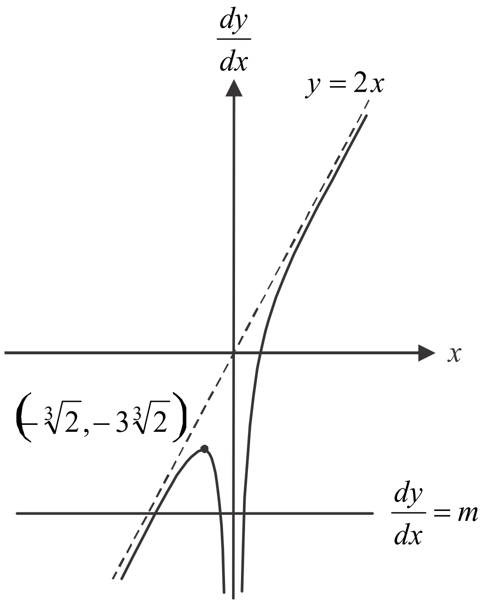
c. $\frac{dy}{dx} = 2x - \frac{2}{x^2}$ from part a.

$$\frac{d^2y}{dx^2} = 2 + \frac{4}{x^3}$$

(1 mark)

d. If there are 3 points on the graph of f where the gradient is m, then there must be 3 solutions to the equation $\frac{dy}{dx} = m$.

Sketch the graph of $y = \frac{dy}{dx}$ on your CAS.



An example of where the graph of $\frac{dy}{dx} = 2x - \frac{2}{x^2}$ intersects with the graph of $\frac{dy}{dx} = m$ three times is shown above.

The stationary point on the graph of $y = \frac{dy}{dx}$ occurs when $\frac{d^2y}{dx^2} = 0$.

Solve $2 + \frac{4}{x^3} = 0$ (using part c.) for x

$$x = -2^{\frac{1}{3}}$$ **(1 mark)**

When $x = -2^{\frac{1}{3}}, \frac{dy}{dx} = -3 \times 2^{\frac{1}{3}}$

$$= -3\sqrt[3]{2}$$ **(1 mark)**

So, there are three points on the graph of f where the gradient is m if $m < -3\sqrt[3]{2}$.

(1 mark)

e. **i.**
$$\text{volume} = \pi \int_1^2 y^2 \, dx$$
$$= \pi \int_1^2 \frac{(2 + x^3)^2}{x^2} dx \text{ since } y = \frac{2 + x^3}{x}$$

(1 mark)- correct integrand
(1 mark) – correct terminals

ii. Using CAS, volume equals $\frac{71\pi}{5}$ cubic units

(1 mark)

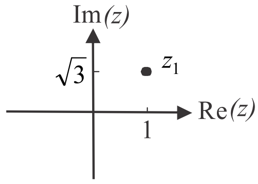
Question 2 (12 marks)

a.

$$z_1 = 1 + \sqrt{3}i$$

$$\text{Arg}(z_1) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$= \frac{\pi}{3}$$



(1 mark)

b. **i.**

$$\bar{z}_1 = 2\text{cis}\left(\frac{-\pi}{3}\right)$$

$$(\bar{z}_1)^6 = \left(2\text{cis}\left(\frac{-\pi}{3}\right)\right)^6$$

$$= 64\text{cis}(-2\pi)$$

$$= 64(\cos(0) + i\sin(0))$$

$$= 64(1 + 0i)$$

$$= 64$$

So $\bar{z}_1$ is a sixth root of 64.

(1 mark)

ii. The six roots of the equation $z^6 = 64$ lie on a circle with centre at $0 + 0i$ and a radius of $64^{\frac{1}{6}} = 2$. They are spaced at intervals of $\frac{2\pi}{6} = \frac{\pi}{3}$ apart.

Since $z_1 = 2\text{cis}\left(\frac{\pi}{3}\right)$ and $\bar{z}_1 = 2\text{cis}\left(\frac{-\pi}{3}\right)$ are both roots, the other four roots will be located at $2\text{cis}\left(\frac{2\pi}{3}\right)$, $2\text{cis}(0)$, $2\text{cis}\left(\frac{-2\pi}{3}\right)$ and $2\text{cis}(\pi)$.

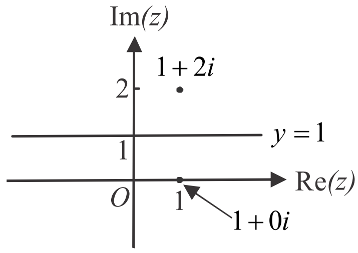
(1 mark)

c. <u>Method 1</u>

$$|z-1-2i| = |z-1|$$
$$|x+iy-1-2i| = |x+iy-1|$$
$$|(x-1)+i(y-2)| = |(x-1)+iy| \qquad \textbf{(1 mark)}$$
$$\sqrt{(x-1)^2+(y-2)^2} = \sqrt{(x-1)^2+y^2}$$
$$(x-1)^2+y^2-4y+4 = (x-1)^2+y^2$$
$$-4y+4 = 0$$
$$y = 1 \qquad \textbf{(1 mark)}$$

<u>Method 2</u>

$|z-1-2i| = |z-1|$ can be expressed as
$|z-(1+2i)| = |z-(1+0i)|$ **(1 mark)**
This defines the perpendicular bisector of the complex numbers $1+2i$ and $1+0i$ which is the line $y=1$. **(1 mark)**



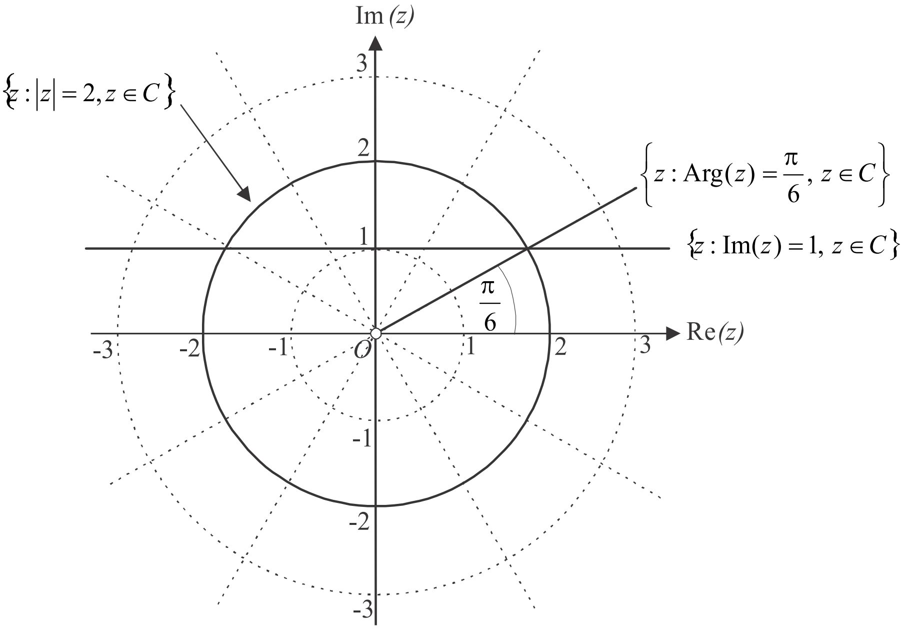
d. Now $\text{Arg}(z) = \text{Arg}(z_1) = \frac{\pi}{3}$ (from part a.)

So,
$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{3}$$
$$\tan\left(\frac{\pi}{3}\right) = \frac{y}{x}$$
$$\sqrt{3} = \frac{y}{x}$$
$$y = \sqrt{3}x \qquad \textbf{(1 mark)}$$

Since the Cartesian equation of L is $y=1$, we have $1=\sqrt{3}x$ so $x = \frac{1}{\sqrt{3}}$.

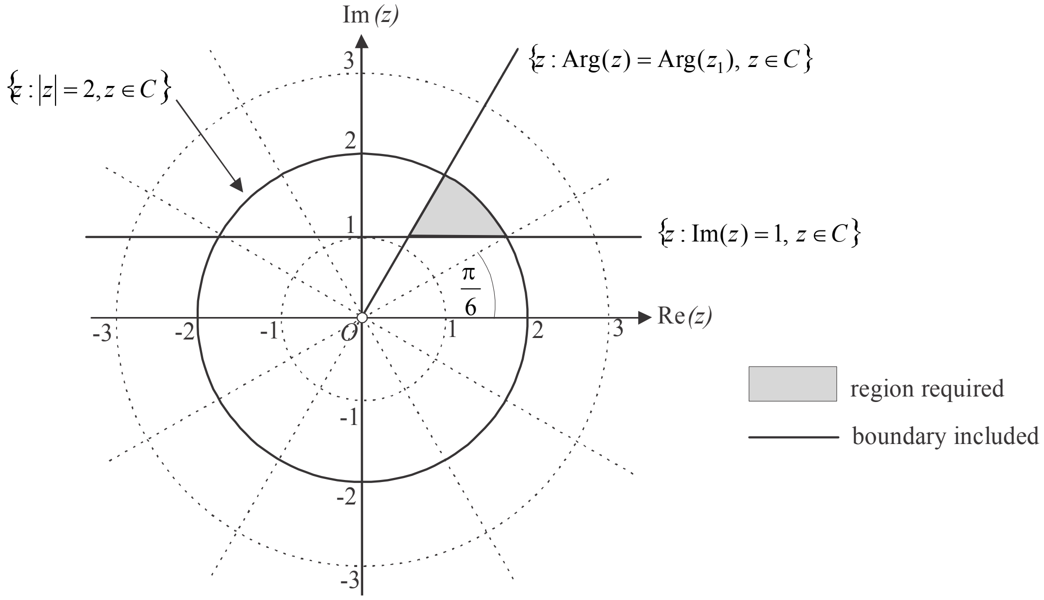
The point of intersection is $\left(\frac{1}{\sqrt{3}}, 1\right)$. **(1 mark)**

e.



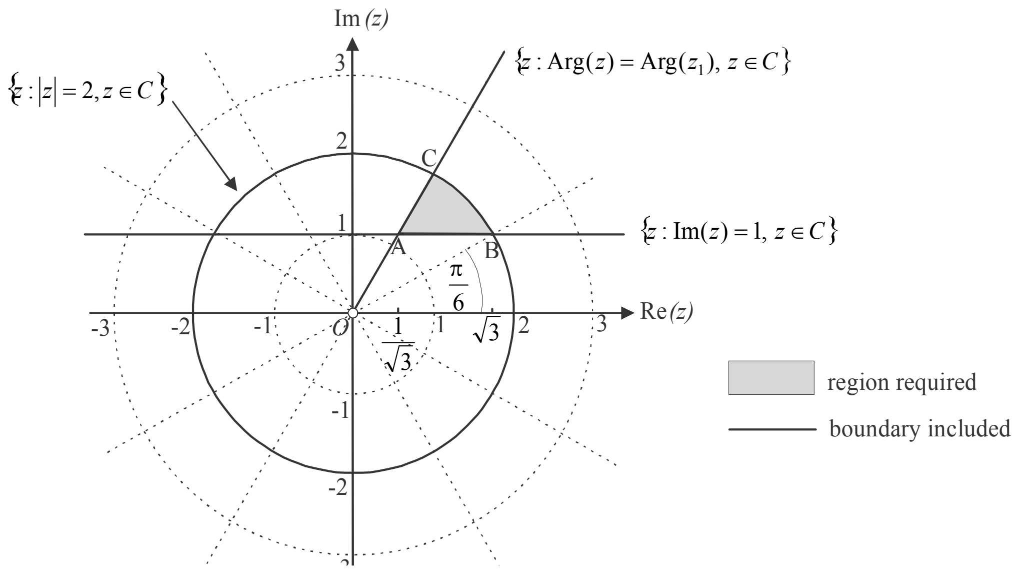
(1 mark) – one correct graph **(1 mark)** – a second correct graph

f.



(1 mark) – correct shading

g.



Note that point A is the point $\left(\frac{1}{\sqrt{3}},1\right)$ from part **d.**, and point B lies on the line $y=1$ and the circle $x^2+y^2=4$ so $x=\sqrt{3}$ and B is the point $\left(\sqrt{3},1\right)$.

Method 1

$$\text{shaded area} = \text{area of sector} OBC - \text{area of } \Delta OAB$$

$$= \frac{1}{12}\times\pi\times 2^2 - \frac{1}{2}\times OA\times OB\times\sin(\angle AOB) \qquad \textbf{(1 mark)}$$

$$= \frac{\pi}{3} - \frac{1}{2}\times\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+1^2}\times 2\times\sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3} - \frac{1}{2}\times\frac{2}{\sqrt{3}}\times 2\times\frac{1}{2}$$

$$= \frac{\pi}{3} - \frac{1}{\sqrt{3}} \text{ square units} \qquad \textbf{(1 mark)}$$

Method 2 – using calculus

Note that point C lies on the line $y=\sqrt{3}x$ which was found in part **d.**

$$\text{shaded area} = \int_{\frac{1}{\sqrt{3}}}^{1}(\sqrt{3}x-1)\,dx+\int_{1}^{\sqrt{3}}(\sqrt{4-x^2}-1)\,dx$$ **(1mark)**

$$=\frac{\pi}{3}-\frac{\sqrt{3}}{3}\text{ square units} \quad \text{(using CAS)}$$ **(1 mark)**

Question 3 (12 marks)

a.

$\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}$

$\overrightarrow{AO}=-\overrightarrow{OA}$

$=-(-2\underset{\sim}{i}+2\underset{\sim}{j})$

$=2(\underset{\sim}{i}-\underset{\sim}{j})$

$\overrightarrow{OB}=5i-5j$

$=5(\underset{\sim}{i}-\underset{\sim}{j})$

Since $\overrightarrow{AO}=\frac{2}{5}\overrightarrow{OB}$, $\overrightarrow{AO}$ and $\overrightarrow{OB}$ are parallel. **(1 mark)**

Since $\overrightarrow{AO}$ and $\overrightarrow{OB}$ are parallel and share the point O, then the vector $\overrightarrow{AB}$ passes through the origin.

(1 mark)

b.

$\overrightarrow{AD}=\overrightarrow{AO}+\overrightarrow{OD}$

$=2\underset{\sim}{i}-2\underset{\sim}{j}-4\underset{\sim}{i}-2\underset{\sim}{j}$

$=-2\underset{\sim}{i}-4\underset{\sim}{j}$

$\overrightarrow{CD}=\overrightarrow{CO}+\overrightarrow{OD}$

$=-4\underset{\sim}{i}+6\underset{\sim}{j}-4\underset{\sim}{i}-2\underset{\sim}{j}$

$=-8\underset{\sim}{i}+4\underset{\sim}{j}$ **(1 mark)**

$\overrightarrow{AD}\bullet\overrightarrow{CD}=(-2\underset{\sim}{i}-4\underset{\sim}{j})\bullet(-8\underset{\sim}{i}+4\underset{\sim}{j})$

$=-2\times{}^{-}8-4\times 4$

$=16-16$

$=0$

So $\overrightarrow{AD}$ is perpendicular to $\overrightarrow{CD}$. **(1 mark)**

c. $\overrightarrow{AD} = -2\underset{\sim}{i} - 4\underset{\sim}{j}$ from part **b.**

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC}\\ &= 2\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{i} - 6\underset{\sim}{j}\\ &= 6\underset{\sim}{i} - 8\underset{\sim}{j}\end{aligned}$$

$$\begin{aligned}|\overrightarrow{AC}| &= \sqrt{36+64}\\ &= 10\end{aligned}$$

$$\begin{aligned}\hat{\overrightarrow{AC}} &= \frac{1}{10}(6\underset{\sim}{i} - 8\underset{\sim}{j})\\ &= \frac{1}{5}(3\underset{\sim}{i} - 4\underset{\sim}{j})\end{aligned}$$

(1 mark)

The vector component of $\overrightarrow{AD}$ parallel to $\overrightarrow{AC}$ is given by

$$\begin{aligned}&\left(\overrightarrow{AD} \bullet \hat{\overrightarrow{AC}}\right)\hat{\overrightarrow{AC}}\\ &= \{(-2\underset{\sim}{i} - 4\underset{\sim}{j}) \bullet \frac{1}{5}(3\underset{\sim}{i} - 4\underset{\sim}{j})\}\frac{1}{5}(3\underset{\sim}{i} - 4\underset{\sim}{j})\\ &= \frac{-6+16}{5} \times \frac{1}{5}(3\underset{\sim}{i} - 4\underset{\sim}{j})\\ &= \frac{2}{5}(3\underset{\sim}{i} - 4\underset{\sim}{j})\end{aligned}$$

(1 mark)

The vector component of $\overrightarrow{AD}$ perpendicular to $\overrightarrow{AC}$ is given by

$$\begin{aligned}&\overrightarrow{AD} - (\overrightarrow{AD} \bullet \hat{\overrightarrow{AC}})\hat{\overrightarrow{AC}}\\ &= -2\underset{\sim}{i} - 4\underset{\sim}{j} - \frac{2}{5}(3\underset{\sim}{i} - 4\underset{\sim}{j})\\ &= -\frac{16}{5}\underset{\sim}{i} - \frac{12}{5}\underset{\sim}{j}\\ &= -\frac{4}{5}(4\underset{\sim}{i} + 3\underset{\sim}{j})\end{aligned}$$

(1 mark)

If you have time, check your answers, i.e.,

$$\begin{aligned}&\frac{2}{5}(3\underset{\sim}{i} - 4\underset{\sim}{j}) - \frac{4}{5}(4\underset{\sim}{i} + 3\underset{\sim}{j})\\ &= -2\underset{\sim}{i} - 4\underset{\sim}{j}\\ &= \overrightarrow{AD}\end{aligned}$$

d. $\overrightarrow{AC} = 6\underset{\sim}{i} - 8\underset{\sim}{j}$ and $\overrightarrow{AD} = -2\underset{\sim}{i} - 4\underset{\sim}{j}$ from part **c.**

$|\overrightarrow{AC}| = 10$ from part **c.** and $|\overrightarrow{AD}| = \sqrt{4+16} = 2\sqrt{5}$

$\overrightarrow{AC} \bullet \overrightarrow{AD} = |\overrightarrow{AC}||\overrightarrow{AD}|\cos(\angle CAD)$ **(1 mark)** – use of scalar product

$-12 + 32 = 20\sqrt{5}\cos(\angle CAD)$

$\cos(\angle CAD) = \dfrac{1}{\sqrt{5}}$ **(1 mark)** correct answer

e. $\overrightarrow{PC} = \overrightarrow{PO} + \overrightarrow{OC}$

$= -\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{i} - 6\underset{\sim}{j}$

$= 3\underset{\sim}{i} - 4\underset{\sim}{j}$

$\overrightarrow{PD} = \overrightarrow{PO} + \overrightarrow{OD}$

$= -\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{i} - 2\underset{\sim}{j}$

$= -5\underset{\sim}{i}$

$|\overrightarrow{PC}| = \sqrt{9+16} = 5$ $\qquad |\overrightarrow{PD}| = 5$

$\overrightarrow{PC} \bullet \overrightarrow{PD} = |\overrightarrow{PC}||\overrightarrow{PD}|\cos(\angle CPD)$

$-15 + 0 = 5 \times 5\cos(\angle CPD)$

$\cos(\angle CPD) = -\dfrac{3}{5}$

(1 mark)

Also, $\cos(\angle CAD) = \dfrac{1}{\sqrt{5}}$ from part **d.**

$\cos(2x) = 2\cos^2(x) - 1$ (formula sheet – double angle identity)

So $\cos(2 \times \angle CAD) = 2\cos^2(\angle CAD) - 1$

$= 2 \times \left(\dfrac{1}{\sqrt{5}}\right)^2 - 1$

$= -\dfrac{3}{5}$ **(1 mark)**

Since $\cos(\angle CPD) = \cos(2 \times \angle CAD)$,

then $\angle CPD = 2\angle CAD$

(1 mark)

Question 4 (10 marks)

a. $f(x) = \arccos\left(\frac{1}{x}\right)$

For f to be defined, $-1 \le \frac{1}{x} \le 1$.

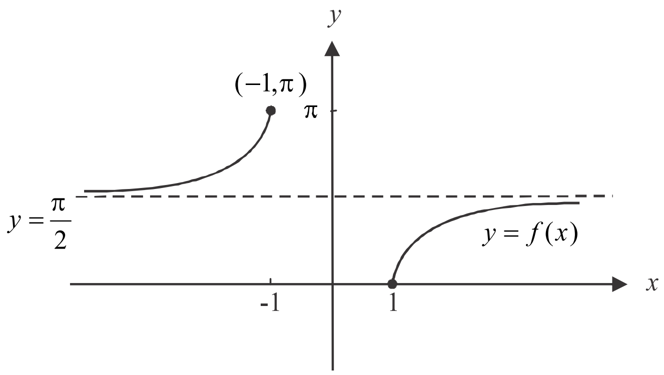
Solving this for x, gives

$x \le -1$ or $x \ge 1$

or $d_f = (-\infty, -1] \cup [1, \infty)$

(1 mark)

b. Use your CAS to sketch the function.



(1 mark) – correct endpoints
(1 mark) – correct shape and asymptote

c. $y = \arccos\left(\frac{1}{x}\right)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \times -x^{-2} \qquad \text{(chain rule)}$$

$$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$$

$$= \frac{1}{x^2\sqrt{\frac{1}{x^2}\left(x^2-1\right)}}$$

$$= \frac{1}{x^2 \times \frac{1}{|x|} \times \sqrt{x^2-1}}$$

$$= \frac{1}{|x|\sqrt{x^2-1}} \quad \text{as required}$$

(1 mark)

d. $y = \arccos\left(\frac{1}{x}\right), \quad x \le -1 \text{ or } x \ge 1$

so $\frac{1}{x} = \cos(y)$

and $x = \frac{1}{\cos(y)}$ **(1 mark)**

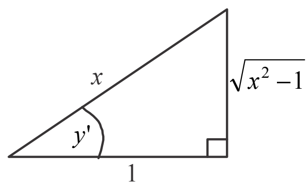
Also, $\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$ from part **c.**.

The differential equation given is

$$\sin(y)\frac{dy}{dx} = \cos^2(y)$$

$$\begin{aligned}\text{LHS} &= \sin(y)\frac{dy}{dx}\\ &= \sin\left(\arccos\left(\frac{1}{x}\right)\right)\frac{1}{|x|\sqrt{x^2-1}}\\ &= \frac{\sqrt{x^2-1}}{|x|} \times \frac{1}{|x|\sqrt{x^2-1}}\\ &= \frac{1}{|x|^2}\\ &= \frac{1}{x^2}\\ &= \cos^2(y)\\ &= \text{RHS}\end{aligned}$$

Let y' be a first quadrant angle such that $\cos(y') = \cos(y)$.



$\sin(y') = \frac{\sqrt{x^2-1}}{x}$

Since y is in the first or **second** quadrant, ie $y \in [0,\pi] \setminus \left\{\frac{\pi}{2}\right\}$ from the graph in part **b.**, $\sin(y)$ must be positive, hence the absolute value of x is needed in the denominator, ie

$\sin(y) = \frac{\sqrt{x^2-1}}{|x|}$

(1 mark)

e.

$$\sin(y)\frac{dy}{dx}=\cos^2(y)$$

$$\frac{dy}{dx}=\frac{\cos^2(y)}{\sin(y)}$$

$$\frac{dx}{dy}=\frac{\sin(y)}{\cos^2(y)}$$

$$x=\int\frac{\sin(y)}{\cos^2(y)}dy$$

(1 mark)

$$=\int-\frac{du}{dy}\cdot u^{-2}dy \qquad \text{where } u=\cos(y)$$

$$=-\int u^{-2}du \qquad \frac{du}{dy}=-\sin(y)$$

$$x=u^{-1}+c, \quad c \text{ is a constant}$$

$$=\frac{1}{\cos(y)}+c$$

$$x-c=\frac{1}{\cos(y)}$$

$$\cos(y)=\frac{1}{x-c}$$

$$y=\arccos\left(\frac{1}{x-c}\right)$$

(1 mark)

f.

$$\sin(y)\frac{dy}{dx}=\cos^2(y)$$

$$\frac{dy}{dx}=\frac{\cos^2(y)}{\sin(y)}$$

$$x_0=2, \qquad y_0=\frac{\pi}{3}$$

$$x_1=2+\frac{1}{2}, \qquad y_1=\frac{\pi}{3}+\frac{1}{2}\times\frac{\cos^2\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)} \qquad \text{(Euler's method – from formula sheet)}$$

$$=2\frac{1}{2} \qquad =\frac{\pi}{3}+\frac{1}{2}\times\frac{1}{4}\div\frac{\sqrt{3}}{2}$$

$$=\frac{\pi}{3}+\frac{1}{4\sqrt{3}}$$

(1 mark)

$$x_2=\frac{5}{2}+\frac{1}{2}, \qquad y_2=\frac{\pi}{3}+\frac{1}{4\sqrt{3}}+\frac{1}{2}\times\frac{\cos^2\left(\frac{\pi}{3}+\frac{1}{4\sqrt{3}}\right)}{\sin\left(\frac{\pi}{3}+\frac{1}{4\sqrt{3}}\right)}$$

$$=3 \qquad =1.26531...$$

The estimate for y is 1.3 (correct to one decimal place).

(1 mark)

Question 5 (13 marks)

a. $\underset{\sim}{r}(t) = \left(35 + 20\sin\left(\frac{\pi t}{12}\right)\right)\underset{\sim}{i} + 5t\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = \frac{5\pi}{3}\cos\left(\frac{\pi t}{12}\right)\underset{\sim}{i} + 5\,\underset{\sim}{j}$ **(1 mark)**

$\dot{\underset{\sim}{r}}(12) = \frac{5\pi}{3}\cos(\pi)\,\underset{\sim}{i} + 5\,\underset{\sim}{j}$

$= \frac{-5\pi}{3}\underset{\sim}{i} + 5\,\underset{\sim}{j}$

$\left|\dot{\underset{\sim}{r}}(12)\right| = \sqrt{\frac{25\pi^2}{9} + 25}$

$= 7.2398$

The snowboarder's speed is 7.24 m/s (correct to 2 decimal places).

(1 mark)

b. From part **a.** $\dot{\underset{\sim}{r}}(t) = \frac{5\pi}{3}\cos\left(\frac{\pi t}{12}\right)\underset{\sim}{i} + 5\,\underset{\sim}{j}$

$\ddot{\underset{\sim}{r}}(t) = \frac{-5\pi^2}{36}\sin\left(\frac{\pi t}{12}\right)\underset{\sim}{i}$ **(1 mark)**

$\left|\ddot{\underset{\sim}{r}}(t)\right| = \sqrt{\frac{25\pi^4}{36^2}\sin^2\left(\frac{\pi t}{12}\right)}$

This is a maximum when $\sin^2\left(\frac{\pi t}{12}\right) = 1$. **(1 mark)**

$\sin\left(\frac{\pi t}{12}\right) = -1$ or $\sin\left(\frac{\pi t}{12}\right) = +1$

$\frac{\pi t}{12} = \frac{3\pi}{2} + 2k\pi,\ k \in Z$ $\qquad$ $\frac{\pi t}{12} = \frac{\pi}{2} + 2k\pi,\ k \in Z$

$\frac{t}{12} = \frac{3}{2} + 2k$ $\qquad$ $\frac{t}{12} = \frac{1}{2} + 2k$

$t = 18 + 24k$ or $t = 6 + 24k$

$t = 6, 18, 30, 42, 54, 66, \ldots$

or $t = 6 + 12k,\ k \in Z^+ \cup \{0\}$

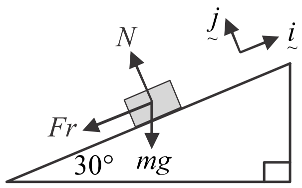
(1 mark)

c. Whilst moving up the launching pad the deceleration is constant with

$u = 7.2$ Since $v^2 = u^2 + 2as$ (for constant acceleration)

$v = 0$ $0 = 51.84 + 10a$

$s = 5$ $a = -5.184\text{m/s}^2$



(1 mark)

$$\underset{\sim}{R} = m\,\underset{\sim}{a}$$

$$(-Fr - mg\sin(30^\circ))\underset{\sim}{i} + (N - mg\cos(30^\circ))\underset{\sim}{j} = ma\underset{\sim}{i}$$

So $-Fr - \frac{mg}{2} = ma$ and $N - mg\cos(30^\circ) = 0$

$$-\mu N - \frac{mg}{2} = -5.184m \qquad N = \frac{\sqrt{3}mg}{2}$$

$$-\mu \times \frac{\sqrt{3}mg}{2} = \frac{mg}{2} - 5.184m$$

(1 mark)

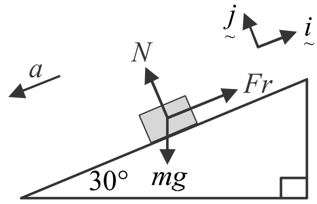
$$\sqrt{3}\mu = \frac{5.184 \times 2}{g} - 1$$

$$\sqrt{3}\mu = \frac{10.368 - g}{g}$$

$$\mu = \frac{10.368 - g}{\sqrt{3}g}$$

(1 mark)

d. After he momentarily comes to rest, the force diagram will be as shown:



$(Fr - mg\sin(30^\circ))\underset{\sim}{i} + (N - mg\cos(30^\circ))\underset{\sim}{j} = -ma\underset{\sim}{i}$

$Fr - \dfrac{mg}{2} = -ma$ and $N = \dfrac{\sqrt{3}mg}{2}$ **(1 mark)**

$\mu N - \dfrac{mg}{2} = -ma$

$ma = \dfrac{mg}{2} - \dfrac{10.368 - g}{\sqrt{3}g} \times \dfrac{\sqrt{3}mg}{2}$

$a = \dfrac{g}{2} - \dfrac{10.368 - g}{2}$

$a = 4.616 \text{ m/s}^2$ down the slope

(1 mark)

e. At the top of the launching pad, the horizontal component of the snowboarder's velocity is $10\cos(30^\circ) = 5\sqrt{3}$ m/s.

Distance travelled horizontally $= 13.78$m

$\text{time taken in air} = \dfrac{13.78\text{m}}{5\sqrt{3}\text{m/s}}$ **(1 mark)**

$= 1.59117...\text{s}$

At the top of the launching pad, the vertical component of the snowboarder's velocity is $10\sin(30^\circ) = 5$ with the vertically upwards direction taken as positive, the details of the snowboarder's vertical motion are:

$u = 5$

$t = 1.59117...$

$s = ?$

$a = -9.8$

Since $s = ut + \dfrac{1}{2}at^2$

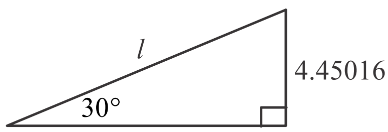
$s = -4.45015...$

(1 mark)

Let the length of the launching pad be l.

$l = \dfrac{4.45016}{\sin(30^\circ)}$

$= 8.9$ m (correct to one decimal place)



(1 mark)