SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1

*(***TSSM's** ***2013 trial exam updated for the current study design)***

SOLUTIONS

Question 1

$$\int \frac{4x-1}{x^2+9}\,dx = \int\left(\frac{4x}{x^2+9} - \frac{1}{x^2+9}\right)dx$$

$$= \int \frac{4x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$$

$$= \int \frac{2}{u}\,du - \frac{1}{3}\int \frac{3}{x^2+9}\,dx$$

$u = x^2+9$

$\frac{du}{dx} = 2x$

$2\frac{du}{dx} = 4x$

[M2]

$$= 2\log_e\left(x^2+9\right) - \frac{1}{3}\text{Tan}^{-1}\left(\frac{x}{3}\right) + c$$ [A1]

Question 2

$$2x^2 - x\sin y + y = 10$$

$$4x - \left(\sin y + x\cos y\frac{dy}{dx}\right) + \frac{dy}{dx} = 0$$

$$4x - \sin y - x\cos y\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(1 - x\cos y\right) = \sin y - 4x$$

$$\frac{dy}{dx} = \frac{\sin y - 4x}{1 - x\cos y}$$

[M2]

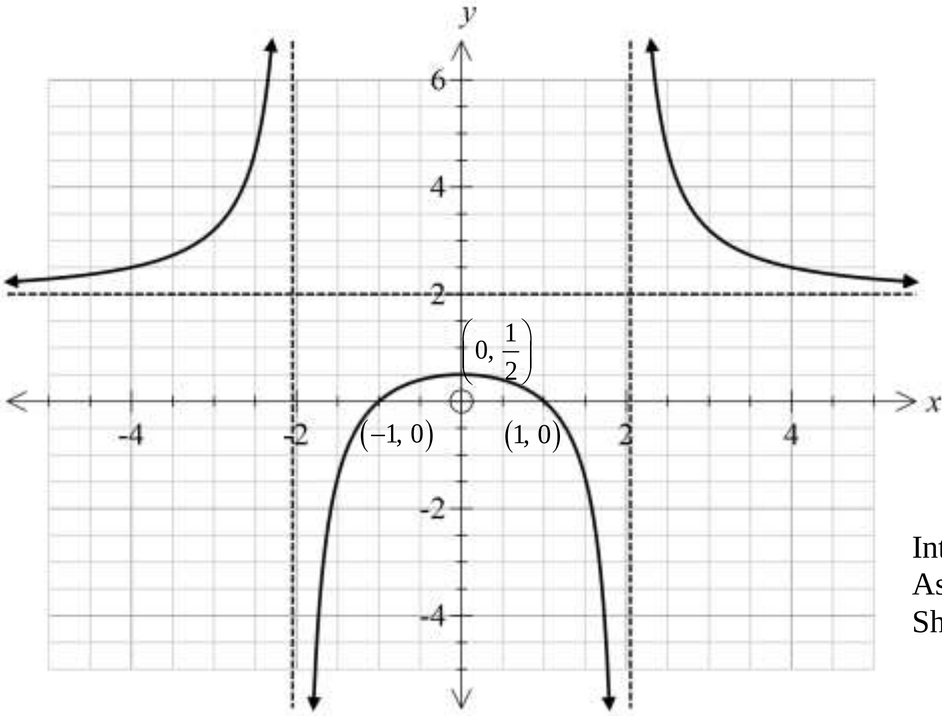
At $\left(1, \frac{\pi}{4}\right)$, $\frac{dy}{dx} = \dfrac{\frac{\sqrt{2}}{2} - 4}{1 - \frac{\sqrt{2}}{2}} = \dfrac{\sqrt{2}-8}{2-\sqrt{2}} = -3\sqrt{2} - 7$ [A1]

Question 3

a.

$$f(x)=\frac{2x^2-2}{x^2-4}=\frac{2\left(x^2-4\right)+6}{x^2-4}=2+\frac{6}{x^2-4}$$ [A1]

b.



Intercepts [A1]
Asymptotes [A1]
Shape [A1]

c.

$$\frac{6}{x^2-4}\equiv\frac{A}{x-2}+\frac{B}{x+2}$$

$$6=A(x+2)+B(x-2)$$

$$A=\frac{3}{2},\ B=-\frac{3}{2}$$ [M2]

$$A=2\times\int_0^1\left(2+\frac{6}{x^2-4}\right)dx$$

$$=2\times\int_0^1\left(2+\frac{3}{2(x-2)}-\frac{3}{2(x+2)}\right)dx$$

$$=\left[4x+3\log_e\left|\frac{x-2}{x+2}\right|\right]_0^1$$

$$=4+3\log_e\left(\frac{1}{3}\right)$$ [A1]

Question 4

$$z = \left(\frac{-2-2i}{\sqrt{3}-i}\right)^3$$

$$= \left(\frac{2\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)}{2\text{cis}\left(-\frac{\pi}{6}\right)}\right)^3$$

$$= \left(\sqrt{2}\text{cis}\left(-\frac{7\pi}{12}\right)\right)^3$$

$$= 2\sqrt{2}\text{cis}\left(-\frac{7\pi}{4}\right)$$

$$= 2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$$

[M2]

$$= 2+2i$$ [A1]

Question 5

$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$, where $\frac{dV}{dt} = 2$ and $h = 2r \Rightarrow r = \frac{h}{2}$ [M1]

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$= \frac{\pi h^3}{12}$$

$$\frac{dV}{dh} = \frac{\pi h^2}{4} \Rightarrow \frac{dh}{dV} = \frac{4}{\pi h^2} \Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2}\times 2 = \frac{8}{\pi h^2}$$

After 1 minute, $V = 2\,\text{m}^3 \Rightarrow \frac{\pi h^3}{12} = 2 \Rightarrow h = \sqrt[3]{\frac{24}{\pi}}$

[M1]

Therefore,

$\frac{dh}{dt} = \frac{8}{\pi\left(\frac{24}{\pi}\right)^{\frac{2}{3}}} = \sqrt[3]{\frac{8^3}{24^2\pi}} = \sqrt[3]{\frac{2^9}{2^6\times 9\pi}} = \sqrt[3]{\frac{8}{9\pi}}$ [M1]

Question 6

a.

$$\frac{d}{dx}\left(x\,\mathrm{Tan}^{-1}x\right) = \mathrm{Tan}^{-1}x + \frac{x}{1+x^2} \qquad \text{[M1]}$$

b.

$$\int\left(\mathrm{Tan}^{-1}x + \frac{x}{1+x^2}\right)dx = x\,\mathrm{Tan}^{-1}x + c$$

$$\int \mathrm{Tan}^{-1}x\,dx + \int\frac{x}{1+x^2}\,dx = x\,\mathrm{Tan}^{-1}x + c$$

$$\int \mathrm{Tan}^{-1}x\,dx = x\,\mathrm{Tan}^{-1}x - \int\frac{x}{1+x^2}\,dx + c$$

$$= x\,\mathrm{Tan}^{-1}x - \frac{1}{2}\int\frac{1}{u}\,du + c$$

$$= x\,\mathrm{Tan}^{-1}x - \frac{1}{2}\log_e\left(1+x^2\right) + c$$

$$u = 1+x^2$$

$$\frac{du}{dx} = 2x$$

$$\frac{1}{2}\frac{du}{dx} = x$$

Therefore,

$$\int_0^1 \mathrm{Tan}^{-1}x\,dx = \left[x\,\mathrm{Tan}^{-1}x - \frac{1}{2}\log_e\left(1+x^2\right)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}\log_e 2$$

$$= \frac{\pi}{4} - \log_e\sqrt{2} \qquad \text{[A1]}$$

[M2]

Question 7

Resolving forces,

$2g \times sin(30°) = 2 \times a$

$a = 4.9\ m/s^2$ [M1]

$v^2 = 0 + 2 \times 4.9 \times 2 \rightarrow v^2 = 4 \times 4.9$ [M1]

$v = \frac{14}{\sqrt{10}} = \frac{14\sqrt{10}}{10} = \frac{7\sqrt{10}}{5}\ m/s$ [A1]

Question 8

a.

$$\begin{aligned} \underset{\sim}{a}.\underset{\sim}{b} &= |\underset{\sim}{a}||\underset{\sim}{b}|\cos 60 \\ &= |\underset{\sim}{a}|\times 3|\underset{\sim}{a}|\times\frac{1}{2} \\ &= \frac{3}{2}|\underset{\sim}{a}|^2 \end{aligned}$$

[M1]

b. $\overrightarrow{BA} = \underset{\sim}{a} - \underset{\sim}{b}$ [A1]

$$\begin{aligned} \overrightarrow{OD} &= \overrightarrow{OB} + \overrightarrow{BC} + \overrightarrow{CD} \\ &= \underset{\sim}{b} + \underset{\sim}{a} + m\left(\underset{\sim}{a} - \underset{\sim}{b}\right) \\ &= \underset{\sim}{a}\left(1+m\right) + \underset{\sim}{b}\left(1-m\right) \end{aligned}$$

[A1]

c.

$$\begin{aligned} \overrightarrow{OD}.\overrightarrow{BA} &= 0 \\ \left(\underset{\sim}{a}(1+m) + \underset{\sim}{b}(1-m)\right).\left(\underset{\sim}{a} - \underset{\sim}{b}\right) &= 0 \\ (1+m)\underset{\sim}{a}.\underset{\sim}{a} - (1+m)\underset{\sim}{a}.\underset{\sim}{b} + (1-m)\underset{\sim}{a}.\underset{\sim}{b} - (1-m)\underset{\sim}{b}.\underset{\sim}{b} &= 0 \\ (1+m)|\underset{\sim}{a}|^2 - (1+m)\times\frac{3}{2}|\underset{\sim}{a}|^2 + (1-m)\times\frac{3}{2}|\underset{\sim}{a}|^2 - (1-m)|\underset{\sim}{b}|^2 &= 0 \\ (1+m)|\underset{\sim}{a}|^2 - (1+m)\times\frac{3}{2}|\underset{\sim}{a}|^2 + (1-m)\times\frac{3}{2}|\underset{\sim}{a}|^2 - (1-m)\times 9|\underset{\sim}{a}|^2 &= 0 \\ (1+m) - \frac{3}{2}(1+m) + \frac{3}{2}(1-m) - 9(1-m) &= 0 \end{aligned}$$

[M2]

$$m = \frac{8}{7}$$

Question 9

$$y = \text{Cos}^{-1}(2x)$$
$$x = \frac{1}{2}\cos y$$
$$x^2 = \frac{1}{4}\cos^2 y$$
[M1]

$$V = \frac{\pi}{4}\int_0^{\pi/4} \cos^2 y\, dy$$
$$= \frac{\pi}{8}\int_0^{\pi/4} \left(\cos(2y)+1\right) dy$$
$$= \frac{\pi}{8}\left[\frac{1}{2}\sin(2y)+y\right]_0^{\pi/4}$$
$$= \frac{\pi}{8}\left(\frac{1}{2}+\frac{\pi}{4}\right)$$
[M1]

$$= \frac{\pi(\pi+2)}{32}$$ [A1]

Question 10

a.

$$x = 2\sec t + 1 \Rightarrow \sec t = \frac{x-1}{2}$$
$$y = 3\tan t - 2 \Rightarrow \tan t = \frac{y+2}{3}$$

Since $\sec^2 t - \tan^2 t = 1$, then

[M1]

$$\left(\frac{x-1}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 1$$
$$\frac{(x-1)^2}{4} - \frac{(y+2)^2}{9} = 1$$
[A1]

Since $0 \le t < \frac{\pi}{2}$ then $1 \le \sec t < \infty \Rightarrow 3 \le x < \infty$

Since $0 \le t < \frac{\pi}{2}$ then $0 \le \tan t < \infty \Rightarrow -2 \le y < \infty$

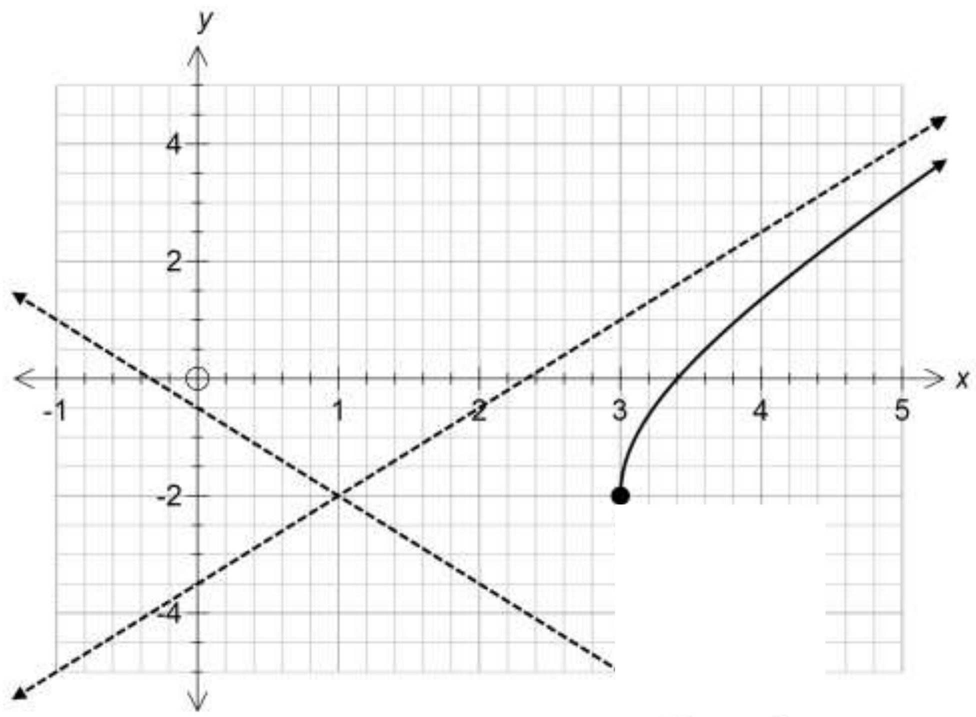
[A1]

b.

$$y+2=\pm\frac{3}{2}(x-1)$$

$$y=\frac{3}{2}x-\frac{7}{2} \text{ and } y=-\frac{3}{2}x-\frac{1}{2}$$ [A1]

c.



Shape and domain and range restrictions [A1]
Asymptotes and direction [A1]