NAME: ____________________________

VCE SPECIALIST MATHEMATICS

Practice Written Examination 2

Reading time: 15 minutes   Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	5	5	58
			80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 30 pages.
- 5 page formula booklet.
- Answer sheet for multiple-choice questions.

Instructions
- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Take the **acceleration due to gravity** to have magnitude $g\ m/s^2$, where $g = 9.8$.

Question 1

The graph with equation $y = \frac{1}{-3-x+2x^2}$ has asymptotes given by

A. $x = -1, x = \frac{3}{2}$

B. $x = 1, x = -\frac{3}{2}$

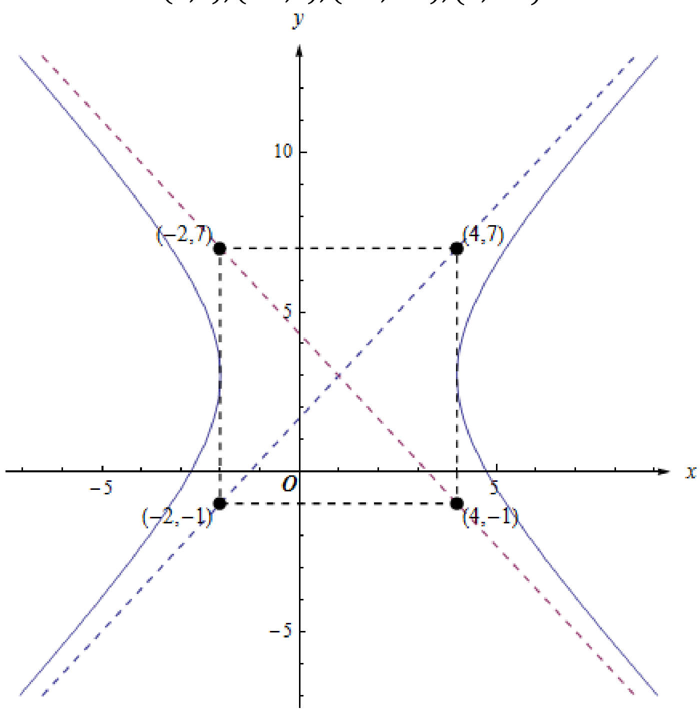
C. $x = -1, x = \frac{3}{2}, y = 0$

D. $x = 1, x = -\frac{3}{2}, y = 0$

E. $x = -1, x = -\frac{3}{2}, y = 0$

Question 2

A rectangle is drawn with corners at $(4,7), (-2,7), (-2,-1), (4,-1)$.



A hyperbola is drawn outside the box so that its vertices touch the box at the midpoints of the vertical sides as shown in the diagram above. The equation of the hyperbola could be

A. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

B. $\dfrac{(x+1)^2}{9} - \dfrac{(y+3)^2}{16} = 1$

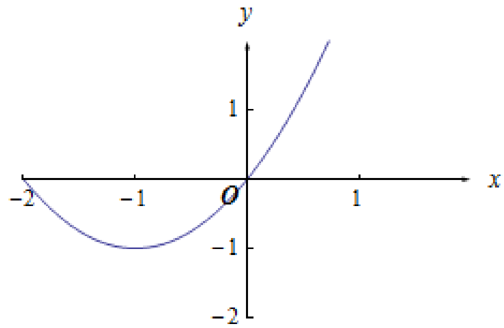
C. $\dfrac{(x-1)^2}{9} - \dfrac{(y-3)^2}{16} = 1$

D. $\dfrac{(x-1)^2}{3} - \dfrac{(y-3)^2}{4} = 1$

E. $\dfrac{(y-3)^2}{16} - \dfrac{(x-1)^2}{9} = 1$

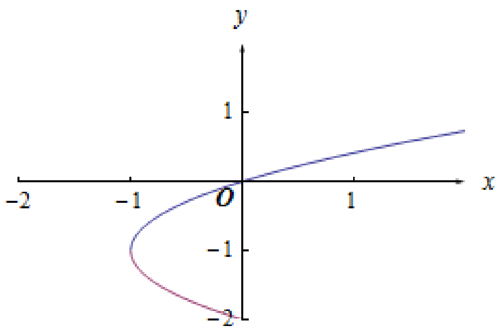
Question 3

The graph of $y = f(x)$ is shown below.

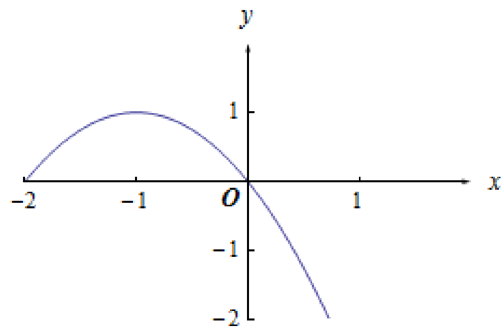


All of the axes below have the same scale as the axes in the diagram above.
The graph of $y = \frac{1}{f(x)}$ is best represented by

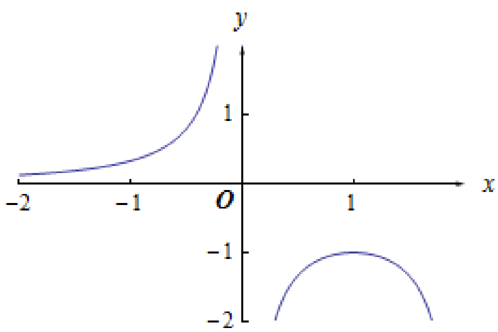
A.



B.



C.



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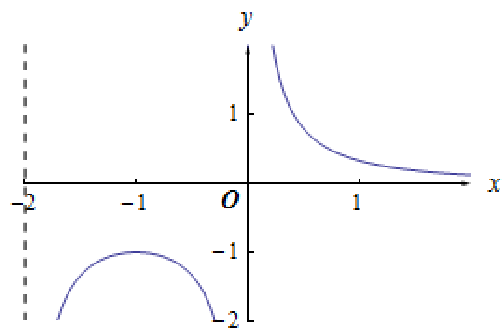
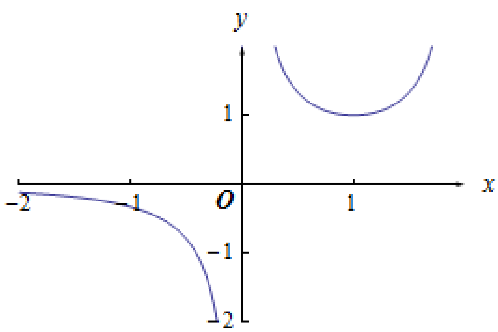


E.



Question 4
The domain and range of the function with rule $f(x) = 2\arcsin(2x+1) - \pi$ are respectively

A. $\mathbb{R}, [-2\pi, 0]$

B. $\mathbb{R}, (-2\pi, 0)$

C. $(-1,0), (-2\pi, 0)$

D. $[-1,0], [-2\pi, 0]$

E. $[-1,0], [-\pi, \pi]$

Question 5
If $z = \frac{1}{\sqrt{2}}\text{cis}(-\frac{\pi}{3})$ then $(\bar{z})^{-11}$ is equal to

A. $\dfrac{1}{32\sqrt{2}}\text{cis}\left(\dfrac{i11\pi}{3}\right)$

B. $32\sqrt{2}\text{cis}\left(\dfrac{i11\pi}{3}\right)$

C. $32\sqrt{2}\text{cis}\left(-\dfrac{i\pi}{3}\right)$

D. $\dfrac{1}{\sqrt{2}}\text{cis}\left(\dfrac{\pi}{3}\right)$

E. $32\sqrt{2}\text{cis}\left(\dfrac{\pi}{3}\right)$

Question 6
For any complex number z, the location on an Argand diagram of the complex number $u = i^2\bar{z}$ can be found by

A. reflecting z in $\text{Re}(z) = 0$

B. reflecting z in $\text{Im}(z) = 0$ and then reflecting in $\text{Re}(z) = 0$

C. rotating z through π in an anticlockwise direction about the origin

D. reflecting z in $\text{Re}(z) = 0$ and then rotating anticlockwise through π about the origin

E. reflecting z in $\text{Im}(z) = 0$.

Question 7

The set of points in the complex plane defined by $(z - 2i)(\bar{z} + 2i) = 2$ corresponds to

A. the point given by $z = 2$

B. the line $\text{Re}(z) = 2$

C. the circle with centre $2i$ and radius 2

D. the circle with centre $2i$ and radius 4

E. the circle with centre $2i$ and radius $\sqrt{2}$.

Question 8

If $z \in \mathbb{C}$ with $\text{Re}(z) \neq 0, \text{Im}(z) \neq 0$, which of the following does **not** represent a real number?

A. $z^2 - 2\text{Re}(z)\text{Im}(z)i$

B. $\frac{z}{\bar{z}}$

C. $\sqrt{z\bar{z}}$

D. $(\bar{z} - z)(z - \bar{z})$

E. $\arg(z)$

Question 9

$\frac{dy}{dx} = \arctan(bx), y(a) = c$, and you may assume that $a \geq 0, b \geq 0, h > 0$. If a single step of Euler's method is used to approximate $y(a + h)$, then

A. $y(a + h) \approx \arctan(bx) + hb\sec^2(bx)$ but is an overestimate of the true value of $y(a + h)$

B. $y(a + h) \approx \arctan(bx) + hb\sec^2(bx)$ but is an underestimate of the true value of $y(a + h)$

C. $y(a + h) \approx c + h\arctan(ab)$ but is an overestimate of the true value of $y(a + h)$

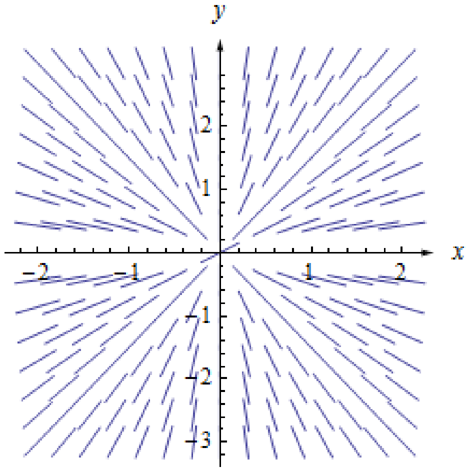
D. $y(a + h) \approx c + h\arctan(ab)$ but is an underestimate of the true value of $y(a + h)$

E. $y(a + h) \approx \arctan(a) + h/(1 + a^2b^2)$ but is an underestimate of the true value of $y(a + h)$.

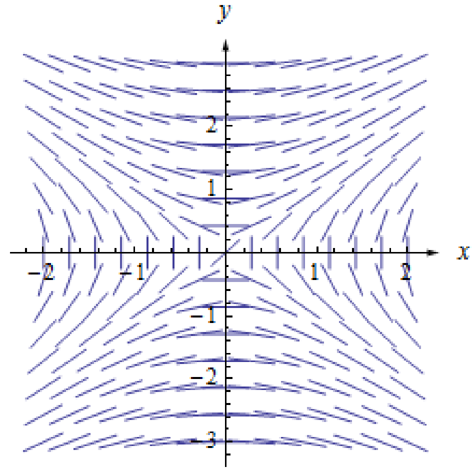
Question 10

The diagram that best represents the direction field of the differential equation $\frac{dy}{dx} = \frac{y}{x}$ is

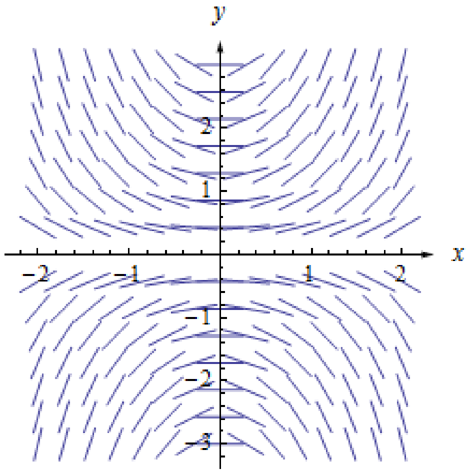
A.



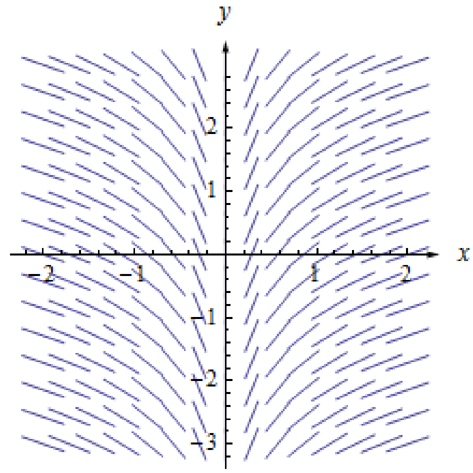
B.



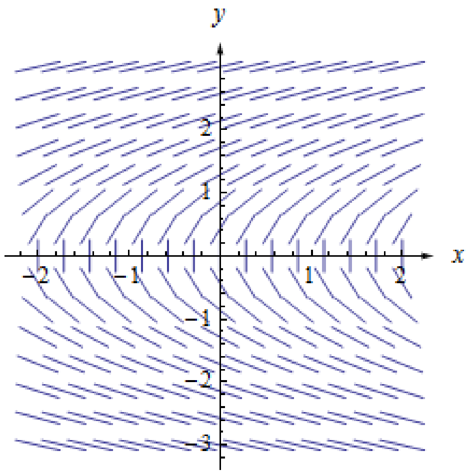
C.



D.



E.



Question 11

If $\frac{d^2y}{dx^2} = -x^2 - x$ and $\frac{dy}{dx} = 0$ at $x = 0$, then the graph of y will have

A. a local maximum at $x = 0$

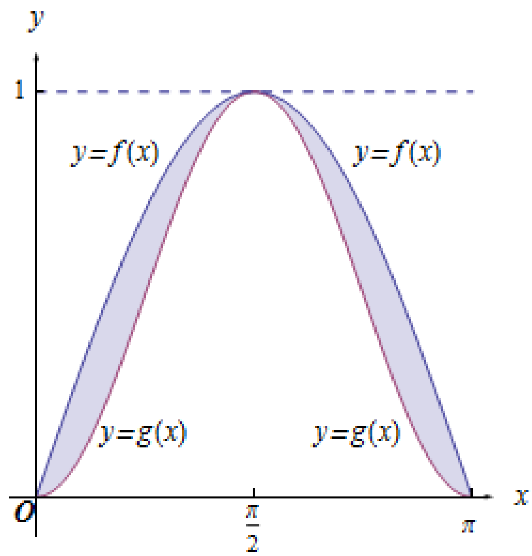
B. a local minimum at $x = -1$ and a local maximum at $x = 0$

C. stationary points of inflection at $x = -1$ and $x = 0$, and a local maximum at $x = -\frac{3}{2}$

D. a stationary point of inflection at $x = 0$, no other points of inflection and a local minimum at $x = -\frac{3}{2}$

E. non-stationary and stationary points of inflection at $x = -1$ and $x = 0$ respectively, and a local maximum at $x = -\frac{3}{2}$.

Question 12



The shaded region above, is symmetric about $x = \frac{\pi}{2}$. The upper curve has equation $y = f(x)$ and the lower curve has equation $y = g(x)$. The volume, V, of the solid of revolution formed by rotating the shaded region about the line $y = 1$ is

A. $V = \pi \int_0^{\pi} \left(f(x) - g(x)\right)^2 dx$

B. $V = \pi \int_0^{\pi} \left(f(x)\right)^2 - \left(g(x)\right)^2 dx$

C. $V = 2\pi \int_0^{\frac{\pi}{2}} \left(g(x) - f(x)\right)\left(2 - f(x) - g(x)\right) dx$

D. $V = 2\pi \int_0^{\frac{\pi}{2}} \left(f(x) - g(x)\right)\left(2 - f(x) - g(x)\right) dx$

E. $V = 2\pi \int_0^{\frac{\pi}{2}} (1 - f(x))^2 - (1 - g(x))^2 dx$

Question 13

$I = \int_0^{\frac{\pi}{4}} \tan^4(x)\sec^4(x)\,dx$. Using the substitution $u = \tan(x)$, I is equal to

A. $I = \int_0^{\frac{\pi}{4}} u^4(1+u^2)dx$

B. $I = \int_0^{1} u^4(1+u^2)du$

C. $I = \int_0^{\frac{\pi}{4}} u^4(1+u^2)^2du$

D. $I = \int_0^{1} u^4(1+u^2)^2du$

E. $I = \int_0^{1} u^2(1+u^2)^2du$

Question 14

$y - 2xy^2 = -1$. When $x = 1$, $\frac{dy}{dx} =$

A. $\frac{1}{6}\ only$

B. $-\frac{2}{3}\ only$

C. $\frac{1}{6}\ or -\frac{2}{3}$

D. $-\frac{1}{6}$

E. $\frac{2}{3}$

Question 15

If vector $\mathbf{a} = 2m\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ is in the same direction as vector $\mathbf{b} = 2\mathbf{i} + \mathrm{m}\mathbf{j} - \mathbf{k}$, but has length 6 units, then

A. $m = 2$

B. $m = -2$

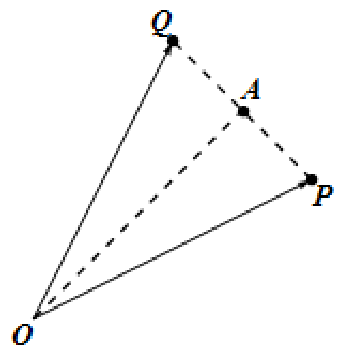
C. $m = \pm 2$

D. $m = -3i$

E. $m = -1$

Question 16

Point P, A and Q in the diagram below, have position vectors $\mathbf{p}, \mathbf{a}$ and $\mathbf{q}$ respectively, relative to the origin O.



If point Q is obtained by **reflecting** point P in the dashed line OA, then $\mathbf{q}$ is equal to

A. $\mathbf{a} - \dfrac{\mathbf{p} \cdot \mathbf{a}}{\mathbf{p} \cdot \mathbf{p}}\mathbf{p}$

B. $\mathbf{p} - \dfrac{\mathbf{p} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\mathbf{a}$

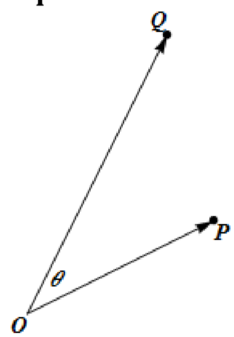
C. $2\dfrac{\mathbf{p} \cdot \mathbf{a}}{\mathbf{p} \cdot \mathbf{p}}\mathbf{p} - \mathbf{a}$

D. $2\dfrac{\mathbf{p} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\mathbf{a} - \mathbf{p}$

E. $\mathbf{a} - \dfrac{1}{2}(\mathbf{a} + \mathbf{p})$

Question 17

Point P and Q in the diagram below, have position vectors **p** and **q** respectively, relative to the origin O. In this diagram θ is the angle between **p** and **q.**



Then $\cos(\theta)$ is equal to

A. $\dfrac{\mathbf{p}\cdot\mathbf{p}+\mathbf{q}\cdot\mathbf{q}-(\mathbf{p}-\mathbf{q})\cdot(\mathbf{p}-\mathbf{q})}{2}$

B. $\dfrac{\mathbf{p}\cdot\mathbf{p}+\mathbf{q}\cdot\mathbf{q}-(\mathbf{q}-\mathbf{p})\cdot(\mathbf{q}-\mathbf{p})}{\sqrt{(\mathbf{p}\cdot\mathbf{p})(\mathbf{q}\cdot\mathbf{q})}}$

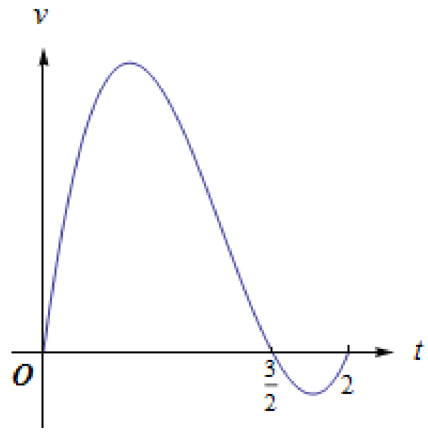
C. $\dfrac{\mathbf{p}\cdot\mathbf{p}+\mathbf{q}\cdot\mathbf{q}-(\mathbf{q}-\mathbf{p})\cdot(\mathbf{q}-\mathbf{p})}{2\,(\mathbf{p}\cdot\mathbf{p})(\mathbf{q}\cdot\mathbf{q})}$

D. $\dfrac{\mathbf{p}\cdot\mathbf{p}+\mathbf{q}\cdot\mathbf{q}-(\mathbf{q}-\mathbf{p})\cdot(\mathbf{q}-\mathbf{p})}{2\sqrt{(\mathbf{p}\cdot\mathbf{p})(\mathbf{q}\cdot\mathbf{q})}}$

E. $\dfrac{\mathbf{p}\cdot\mathbf{p}+\mathbf{q}\cdot\mathbf{q}+(\mathbf{q}-\mathbf{p})\cdot(\mathbf{q}-\mathbf{p})}{2\sqrt{(\mathbf{p}\cdot\mathbf{p})(\mathbf{q}\cdot\mathbf{q})}}$

Question 18

The velocity–time graph for the first 2 seconds of the motion of a particle that is moving in a straight line with respect to the origin is shown below.



The particle's velocity v is measured in m/s. Initially the particle is a m from the origin.
The **displacement** of the particle in the first 2 seconds of its motion is given by

A. $a + \int_0^{\frac{3}{2}} v\, dt - \int_{\frac{3}{2}}^{2} v\, dt$

B. $a - \int_0^{\frac{3}{2}} v\, dt + \int_{\frac{3}{2}}^{2} v\, dt$

C. $a + \int_0^{2} |v|\, dt$

D. $a + \int_0^{2} v\, dt$

E. $\int_0^{2} v\, dt$

Question 19

A body is moving in a straight line with acceleration proportional to its velocity. The acceleration is $2\ \text{ms}^{-2}$, when the velocity is $v = -1\ \text{ms}^{-1}$. Given that $v = -1$ when $t = 2$, the equation for v in terms of t is

A. $v = -e^{4-2t}$

B. $v = e^{4-2t}$

C. $v = -e^{2t-4}$

D. $v = -2e^{t-2}$

E. $v = -e^{t-2}$

Question 20

A particle of mass 2 kg is acted on by a variable force, so that the rate of change of its velocity v with respect to x is **inversely proportional** its velocity (ie. $\frac{dv}{dx} \propto \frac{1}{v}$). When the velocity is $2\ \text{ms}^{-1}$, $\frac{dv}{dx} = 2\ \text{s}^{-1}$. The force acting on the particle, in newtons, is

A. 1 newton

B. 2 newton

C. 4 newton

D. 8 newton

E. 16 newton

Question 21

Particles A, of mass 2.0 kg, and B, of mass 1.0 kg, are attached to the ends of a light inextensible string. Each particle is 0.5 m above the ground level. The string passes over a smooth pulley, and the system is released from rest. Neglecting all resistance to motion, find the **speed** of particle A at the instant it reaches ground level.

A. $\frac{g}{3}$ ms^{-1}

B. $\sqrt{\frac{g}{3}}$ ms^{-1}

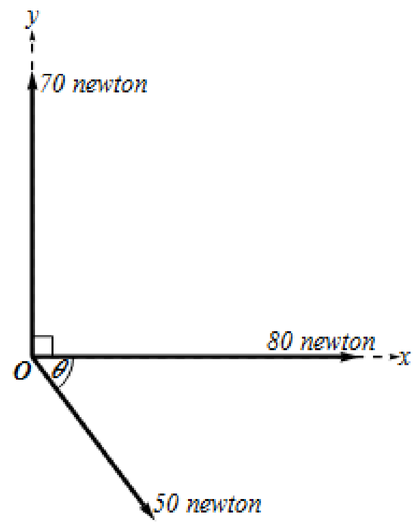
C. $-\frac{2g}{3}$ ms^{-1}

D. $\sqrt{\frac{2g}{3}}$ ms^{-1}

E. $-\sqrt{\frac{2g}{3}}$ ms^{-1}

Question 22

Constant coplanar forces of magnitudes $50\ N$, $80\ N$ and $70\ N$ act at O on a point mass of 10 kg, in the directions shown in the diagram below. The angle θ is such that $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$. In terms of the standard unit vectors, **i** and **j** along the x −axis and y −axis respectively, the **acceleration**, **a**, of the point mass is



A. $\mathbf{a} = 110\mathbf{i} + 30\mathbf{j}$

B. $\mathbf{a} = 100\mathbf{i} + 40\mathbf{j}$

C. $\mathbf{a} = 40\mathbf{i} + 100\mathbf{j}$

D. $\mathbf{a} = 11\mathbf{i} + 3\mathbf{j}$

E. $\mathbf{a} = 10\mathbf{i} + 4\mathbf{j}$

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
Unless otherwise specified an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude $g\ m/s^2$, where $g = 9.8$.

Question 1

a. Given that $\sin\left(\frac{\pi}{10}\right) = \frac{1}{4}(-1+\sqrt{5})$ show that $\cos\left(\frac{\pi}{10}\right) = \sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}$

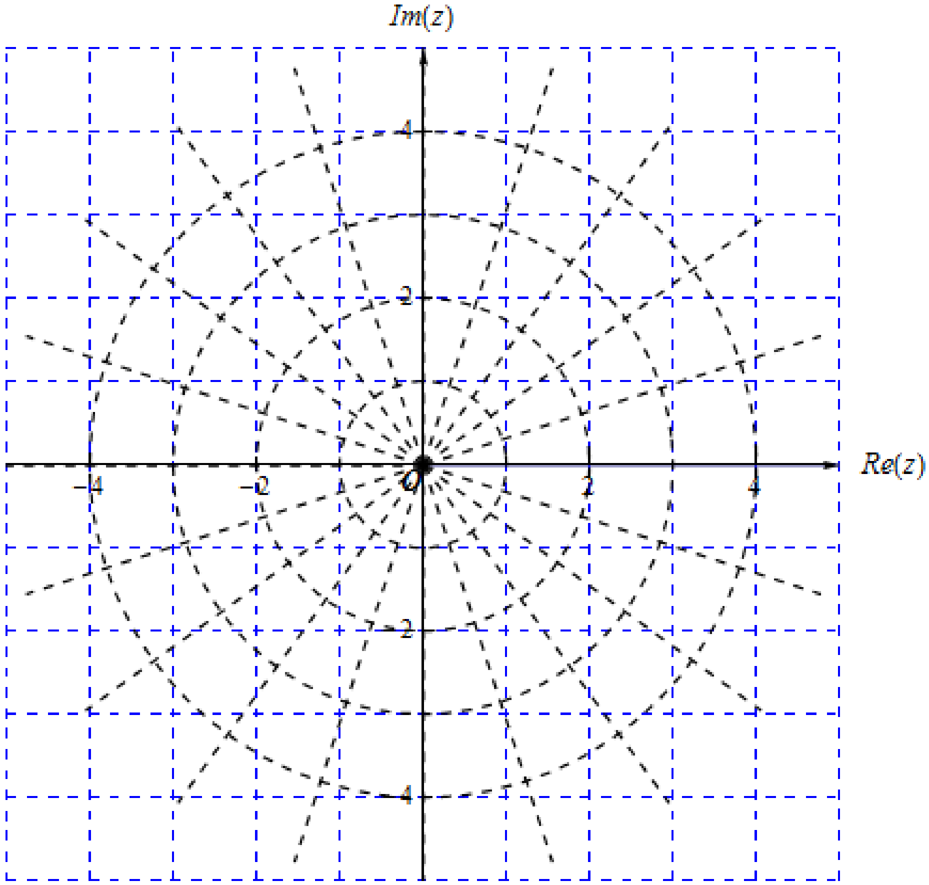
2 marks

b. **i.** Express in polar form $w = \sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}} + i\frac{1}{4}(-1+\sqrt{5})$

ii. Write down w^3 in polar form.

1 + 1 = 2 marks

c. On the Argand diagram below, shade the region defined by
$\{z: Arg(w) \le Arg(z) \le Arg(w^3)\} \cap$ by $\{z: 2 < |z| < 4\}$



2 marks

d. Find the area of the shaded region in **part c.**

1 mark

e. **i.** Find the value(s) of n such that $\text{Im}(w^n) = 0$,where $w = \sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}} + i\,\frac{1}{4}(-1 + \sqrt{5})$

ii. Find w^n for the value(s) of n found in **part i.**

3 + 1 = 4 marks

Question 2

An aircraft of mass 40,000 kg attempts an emergency landing on a runway of length 1 km. The aircraft touches down on the runway at point A (100 m from the start of the runway) and the pilot immediately applies a **uniform deceleration** of a ms^{-2} using the wheel brakes. Two points further along the runway are B, C, where the distances AB and BC are 95 m and 160 m respectively. The speed of the aircraft at touchdown is v_A ms^{-1}. The aircraft takes 1 s to cover the distance AB and 2 s to cover the distance BC.

a. Find the speed v_B of the aircraft at point B in terms of a and v_A.

1 mark

b. Hence, find the value of a and u.

2 marks

c. Find the speed v_C at point C.

1 mark

At point C, the pilot releases the wheel brakes but applies a **reverse thrust** of the jet engines of magnitude $F(\text{t}) = 1600\ v^2(\text{t}) + 1600$ newton, until the aircraft comes to rest at point D. In this equation, v is the velocity t seconds after passing point C.

d. Show that $\frac{dt}{dv} = -\frac{25}{v(t)^2+1}$

1 mark

e. Use a **definite integral** to determine the duration of reverse thrust required to bring the plane to rest.

2 marks

f. Hence, show that $v(\text{t}) = \tan\left(\frac{1}{25}(-t + 25\arctan(70)\right)$

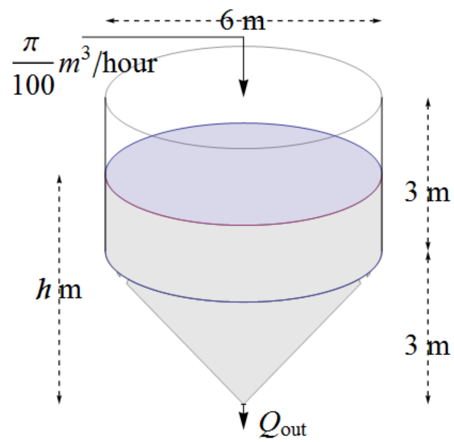
2 marks

g. Evaluate a **definite integral** to find the distance of point D from the start of the runway. (Express answer correct to nearest metre)

2 marks

Question 3

A farmer fabricates a cylindrical water tank with a conical bottom to provide water to a cattle trough. The cylinder is of diameter 6 m and height 3 m, and sits atop an inverted cone with diameter 6 m and height 3 m (see diagram below). Water is pumped into the top of the tank at a **constant rate** $\frac{\pi}{100}$ m^3/hour and flows out of a small hole at the bottom of the tank at any time t hours at a rate (Q_{out} m^3/hour).



The volume, V m^3, of water in the tank when the height, h m, of water above the bottom is given by the hybrid function

$$V(h) = \begin{cases} \frac{\pi h^3}{3} & h \in [0,3] \\ 9\pi(h-2) & h \in [3,6] \end{cases}$$

a. Hence, find the hybrid formula for $\frac{dV}{dt}$ as a function of $\frac{dh}{dt}$

2 marks

Q_{out} is proportional to the square root of the height, h m, of water above the bottom. At time $t = 0$, the depth of water is 4m and the rate at which the water flows out is $\frac{\pi}{50}$ m^3/hour.

b. Explain how the information given above leads to the differential equation $\frac{dh}{dt} = \frac{1-\sqrt{h}}{900}$.

3 marks

c. Use the substitution $h = (1-u)^2$ to find $\int \frac{1}{1-\sqrt{h}}\, dh$.

2 marks

d. **Hence**, write down a **definite integral** for the time needed for the water depth to decrease from 4 m to 3 m.

1 mark

e. Use calculus to evaluate the integral expression found in part ***e***.

2 marks

f. Assuming $h \in [0,3]$, show that $\frac{dh}{dt} = \frac{1-\sqrt{h}}{100\,h^2}$

2 marks

g. Find $\lim_{t\to\infty} h(t)$.

1 mark

Question 4

An object of mass m kg is dropped from rest at a great height in a constant gravitational field of size $g\ \text{m/s}^2$. An air resistance of magnitude kv^2 newtons acts on the object, where v m/s is the speed of the object after it has fallen x m.

a. Draw a free body diagram, showing all forces acting on the object.

1 mark

b. Hence, show that $\frac{dv}{dx} = \frac{mg - kv^2}{mv},\ v(0) = 0\ \text{m/s}$.

2 marks

c. Find the speed v m/s in terms of the distance travelled x m.

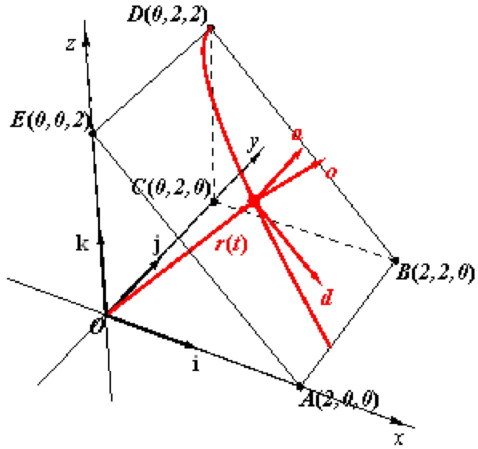
4 marks

d. The terminal velocity is defined $\lim_{x\to\infty} v(x)$. Find the terminal velocity of the object in terms of m, g and k.

1 mark

Question 5

The diagram shows a ramp $ABDE$ on a horizontal base $OABC$ of side length 2 m. Unit vectors **i**, **j** and **k** and parallel to OA, OC and OE respectively.



Players project a wooden hockey puck of mass m kg in the direction along the top of the ramp (ie. along DE). The purpose of a game is to land the puck as close to point A as possible without going off the edge OAE.

a. Find a unit vector **a** in the direction from point E to point D.

1 mark

b. Find a unit vector **d** in the direction from point E to point A.

2 marks

Unit vector $\mathbf{o}$ is the outward normal of the ramp. The components u, v, w of vector $\mathbf{o} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ are chosen so that $\mathbf{a} \cdot \mathbf{o} = 0, \mathbf{d} \cdot \mathbf{o} = 0$ and $\mathbf{o} \cdot \mathbf{o} = 1$.

c. Show that $\mathbf{o} = \frac{1}{\sqrt{2}}\mathbf{i} + 0\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$.

3 marks

Ignoring the effects of friction, the velocity vector, $\mathbf{v}$, of the puck is governed by the equation of motion $m\frac{d\mathbf{v}}{dt} = \mathbf{W} + \mathbf{F}_N$ where $\mathbf{W}$ is weight force on the puck and $\mathbf{F}_N$ is the normal reaction force.

d. Given that $\mathbf{o} \cdot \mathbf{v} = 0$, show that $\mathbf{o} \cdot (\mathbf{W} + \mathbf{F}_N) = 0$.

1 mark

e. **Hence**, show that the magnitude of the normal reaction force, $N = |\mathbf{F}_N|$, is $N = \frac{mg}{\sqrt{2}}$

2 marks

f. **Hence**, show that $\frac{d\mathbf{v}}{dt} = \frac{g}{2}\mathbf{i} - \frac{g}{2}\mathbf{k}$

1 mark

The puck is launched from point D, in the direction from D to E with speed V, and reaches the bottom of the ramp at time T.

g. **Solve**, $\frac{d^2}{dt^2}\mathbf{r}(t) = \frac{g}{2}(\mathbf{i} - \mathbf{k})$ with appropriate initial conditions to find the vector parametric equation $\mathbf{r}(t)$, for the motion of the puck on the domain $t \in [0, T]$.

3 marks

h. **Hence**, choose V if the puck passes through point A.

2 marks

SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.

 Ser2SME2

Specialist Mathematics Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$
cosine rule:	$c^2=a^2+b^2-2\,ab\sin C$

Coordinate geometry

ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	hyperbola: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Circular (trigonometric) functions

$cos^2(x)+sin^2(x)=1$	$\sin(2x)=2\sin(x)\cos(x)$
$1+\tan^2(x)=\sec^2(x)$	$\cot^2(x)+1=\operatorname{cosec}^2(x)$
$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$	$\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$
$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$	$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$
$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$	$\tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	$\tan(2x) = \dfrac{2\tan(x)}{1-\tan^2(x)}$

function	sin^{-1}	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1,1]$	$[-1,1]$	$\mathbb{R}$
range	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0,\pi]$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + iy = r(\cos\theta + i\sin\theta) = r \text{ cis } \theta$	
$\lvert z\rvert = \sqrt{x^2+y^2} = r$	$-\pi < Arg(z) \le \pi$
$z_1 z_2 = r_1 r_2 \text{cis } (\theta_1 + \theta_2)$	$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} cis\ (\theta_1 - \theta_2)$
$z^n = r^n \text{cis } (n\theta)$ (de Moivre's theorem)	

Calculus

$\dfrac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$	$\int x^n \mathrm{d}x = \dfrac{1}{n+1}x^{n+1} + c, n \ne -1$
$\dfrac{\mathrm{d}}{\mathrm{d}x}(e^{ax}) = ae^{ax}$	$\int e^{ax} dx = \dfrac{1}{a}e^{ax} + c$
$\dfrac{\mathrm{d}}{\mathrm{d}x}(\log_e(x)) = \dfrac{1}{x}$	$\int \dfrac{1}{x}\,\mathrm{d}x = \log_e\lvert x\rvert + c$
$\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin(ax)) = a\cos(ax)$	$\int \sin(ax)\,\mathrm{d}x = \dfrac{1}{a}\cos(ax) + c$
$\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos(ax)) = -a\sin(ax)$	$\int \cos(ax)\,\mathrm{d}x = -\dfrac{1}{a}\sin(ax) + c$
$\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan(ax)) = a\sec^2(ax)$	$\int \sec^2(ax)\,\mathrm{d}x = \dfrac{1}{a}\tan(ax) + c$

$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}(ax) + c, a > 0$
$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \cos^{-1}(ax) + c, a > 0$
$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1}(x)) = \frac{1}{1+x^2}$	$\int \frac{a}{a^2+x^2}\,dx = sin^{-1}(ax) + c$

product rule:	$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}}{\mathrm{d}x}(v) + v\frac{\mathrm{d}}{\mathrm{d}x}(u)$
quotient rule:	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$
chain rule:	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Euler's method:	$\frac{dy}{dx} = f(x), x_0 = a \text{ and } y_0 = b$ $\Rightarrow x_{n+1} = x_n + h,\ \ y_{n+1} = y_n + hf(x_n)$
acceleration:	$a = \frac{d^2x}{dx^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
constant (uniform) acceleration:	$v = u + at, s = ut + \frac{1}{2}at^2, v^2 = u^2 + 2as, s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$	
$\lvert\mathbf{r}\rvert = \sqrt{x^2+y^2+z^2} = r$	$\mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2 \cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$
$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$	

Mechanics

momentum:	$\mathbf{p} = m\mathbf{v}$
equation of motion:	$\mathbf{R} = m\mathbf{a}$
friction:	$F \leq \mu N$

Multiple Choice Answer Sheet

Student Name:

Shade the space of the letter that corresponds to your choice of the correct answer.

Question	A	B	C	D	E
Question 1	[ ]	[ ]	[ ]	[ ]	[ ]
Question 2	[ ]	[ ]	[ ]	[ ]	[ ]
Question 3	[ ]	[ ]	[ ]	[ ]	[ ]
Question 4	[ ]	[ ]	[ ]	[ ]	[ ]
Question 5	[ ]	[ ]	[ ]	[ ]	[ ]
Question 6	[ ]	[ ]	[ ]	[ ]	[ ]
Question 7	[ ]	[ ]	[ ]	[ ]	[ ]
Question 8	[ ]	[ ]	[ ]	[ ]	[ ]
Question 9	[ ]	[ ]	[ ]	[ ]	[ ]
Question 10	[ ]	[ ]	[ ]	[ ]	[ ]
Question 11	[ ]	[ ]	[ ]	[ ]	[ ]
Question 12	[ ]	[ ]	[ ]	[ ]	[ ]
Question 13	[ ]	[ ]	[ ]	[ ]	[ ]
Question 14	[ ]	[ ]	[ ]	[ ]	[ ]
Question 15	[ ]	[ ]	[ ]	[ ]	[ ]
Question 16	[ ]	[ ]	[ ]	[ ]	[ ]
Question 17	[ ]	[ ]	[ ]	[ ]	[ ]
Question 18	[ ]	[ ]	[ ]	[ ]	[ ]
Question 19	[ ]	[ ]	[ ]	[ ]	[ ]
Question 20	[ ]	[ ]	[ ]	[ ]	[ ]
Question 21	[ ]	[ ]	[ ]	[ ]	[ ]
Question 22	[ ]	[ ]	[ ]	[ ]	[ ]

Solution Pathway

Below are sample answers. Please consider the merit of alternative responses.

SECTION 1

1. C	**2. C**	**3. D**	**4. D**	**5. E**
6. E	**7. E**	**8. B**	**9. C**	**10. A**
11. E	**12. D**	**13. B**	**14. C**	**15. A**
16. D	**17. D**	**18. E**	**19. A**	**20. D**
21. B	**22. D**			

Question 1 Answer C

$Degree\ of\ numerator = 0\ and\ degree\ of\ denominator = 2 \Rightarrow y = 0\ is\ an\ asymptote$

$Denominator = (x+1)(2x-3) \Rightarrow x = -1\ and\ x = \frac{3}{2}\ asymptotes$

Question 2 Answer C

$midpoint\ of\ box = (h,k) = (1,3)$

$(box\ half\ width, box\ half\ height) = (3,4) = (a,b)$

$vertices = (-1,3) and\ (4,3) \Rightarrow \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Question 3 Answer D

$f(x) = 0 \Rightarrow x = -2, x = 0 \ \Rightarrow y = \frac{1}{f(x)}\ has\ asymptotes\ x = -2, x = 0$

$(x, f(x)) = (-1,-1) \Rightarrow \left(x, \frac{1}{f(x)}\right) = (-1,-1)$

Question 4 Answer D

$Transformation\ y = \arcsin(x) \to y = 2\arcsin(2x+1) - \pi\ \ is\ (x,y) \to (\frac{x-1}{2}, 2y-\pi)$

$domain = [-1,1], range = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to domain = [-1,0], range = [-2\pi, 0]$

Question 5 Answer E

$z = \frac{1}{\sqrt{2}}\text{cis}\left(-\frac{\pi}{3}\right) \Rightarrow \bar{z} = \frac{1}{\sqrt{2}}\text{cis}\left(\frac{\pi}{3}\right) \Rightarrow \bar{z}^{-1} = \sqrt{2}\text{cis}\left(-\frac{\pi}{3}\right) \Rightarrow (\bar{z}^{-1})^{11} = \left(\sqrt{2}\right)^{11}\text{cis}\left(-11\frac{\pi}{3}\right)$

$(\bar{z})^{-11} = (2)^{11/2}\text{cis}\left(2\pi - 11\frac{\pi}{3}\right) = 32\sqrt{2}\text{cis}\left(\frac{\pi}{3}\right)$

Question 6 Answer E

$z=(a+ib)\Rightarrow \bar{z}=(a-ib)\Rightarrow u=i^2\bar{z}=-1(a-ib)=-a+ib\Rightarrow reflect\ in\ imaginary\ axis$

Question 7 Answer E

$(z-2i)(\bar{z}+2i)=2\Rightarrow (z-2i)\overline{(z-2i)}=2\Rightarrow |z-2i|^2=2\Rightarrow |z-2i|=\sqrt{2}$

Question 8 Answer B

$z=a+ib\Rightarrow \frac{z}{\bar{z}}=\frac{z}{\bar{z}}\times\frac{z}{z}=\frac{z^2}{|z|^2}=\frac{a^2-b^2}{a^2+b^2}+i\frac{2ab}{a^2+b^2}$

Question 9 Answer C

$If\ \frac{dy}{dx}=f(x),\quad y(a)=c\ then\ a\ single\ step\ of\ Euler\ method, y(a+h)\approx y(a)+hf(a)$

$with\ local\ truncation\ error\ E=y(a+h)-y(a)-hf(a)=\frac{h^2}{2}f'(\eta), \eta\in[a,a+h]$

$y(a+h)\approx y(a)+hf(a)\Rightarrow y(a+h)\approx c+h\arctan(ab)$

$f'(\eta)=\frac{b}{1+\eta^2b^2}\Rightarrow E=\frac{h^2}{2}f'(\eta)=\frac{h^2}{2}\frac{b}{1+\eta^2b^2}>0, \eta\in[a,a+h]\Rightarrow overestimate$

Question 10 Answer A

$\frac{dy}{dx}=0\ (ie. horizontal\ tangent) when\ y=0$

$\frac{dy}{dx}=\infty\ (ie. vertical\ tangent) when\ x=0$

$\frac{dy}{dx}=1\ (ie. tangent\ at\ 45°)\ when\ y=x$

$\frac{dy}{dx}=-1\ (ie. tangent\ at-45°)\ when\ y=-x$

Question 11 Answer E

$\frac{d^2y}{dx^2} = -x^2 - x$ and $\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{1}{3}x^3 - \frac{1}{2}x^2 = -x^2\left(\frac{x}{3} + \frac{1}{2}\right)$

$\Rightarrow stationary\ points\ at\ x = 0\ and\ x = -\frac{3}{2}\ only$

$\frac{d^2y}{dx^2} = -x(x+1)\ changes\ sign\ at\ x = -1\ and\ x = 0$

$\Rightarrow p.o.i. at\ x = -1\ and\ x = 0\ only$

$\therefore x = -1\ is\ a\ non - stationary\ p.o.i. and\ x = 0\ is\ a\ stationary\ p.o.i.$

$At\ x = -\frac{3}{2}, \frac{dy}{dx} = 0\ and\ \frac{d^2y}{dx^2} < 0$

$\therefore x = -\frac{3}{2}\ is\ a\ local\ maximum$

Question 12 Answer D

When rotating about $y = 1$, the outer surface is formed by the lower curve i.e. $y = g(x)$ and the inner surface is formed by the upper curve i.e. $y = f(x)$

$$\begin{aligned}\Rightarrow V &= \pi\int_0^{\pi}\left(1 - g(x)\right)^2dx - \pi\int_0^{\pi}\left(1 - f(x)\right)^2dx \\ &= \pi\int_0^{\pi}\left(1 - g(x)\right)^2 - \left(1 - f(x)\right)^2dx \\ &= \pi\int_0^{\pi}\left(f(x) - g(x)\right)\left(2 - f(x) - g(x)\right)dx \\ &= 2\pi\int_0^{\frac{\pi}{2}}\left(f(x) - g(x)\right)\left(2 - f(x) - g(x)\right)dx\end{aligned}$$

Question 13 Answer B

$$\begin{aligned}I &= \int_0^{\frac{\pi}{4}}\tan^4(x)\sec^4(x)\,dx \\ &= \int_0^{\frac{\pi}{4}}\tan^4(x)\,(1 + \tan^2(x))\sec^2(x)\,dx \\ &= \int_0^{\frac{\pi}{4}}\tan^4(x)\,(1 + \tan^2(x))\frac{d\tan(x)}{dx}\,dx\end{aligned}$$

Using $u = \tan(x)$, $x = 0 \Rightarrow u = 0$ and $x = \frac{\pi}{4} \Rightarrow u = 1$

$\therefore I = \int_0^1 u^4(1 + u^2)du$

Question 14 Answer C

$y - 2xy^2 = -1, x = 1 \Rightarrow 2y^2 - y - 1 = 0 \Rightarrow y = -\frac{1}{2}\ or\ y = 1$

Using implicit differentiation

$$\frac{dy}{dx} = \frac{2y^2}{1-4xy} \Rightarrow \left.\frac{2y^2}{1-4xy}\right|_{(1,-\frac{1}{2})} = \frac{1}{6}\ and\ \left.\frac{2y^2}{1-4xy}\right|_{(1,1)} = -\frac{2}{3}$$

Question 15 Answer A

$$\begin{aligned} \mathbf{a} &= 2m\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} = 6\hat{\mathbf{b}} \\ &= 6\frac{2\mathbf{i} + \mathrm{m}\mathbf{j} - \mathbf{k}}{\sqrt{(2\mathbf{i} + \mathrm{m}\mathbf{j} - \mathbf{k}) \cdot (2\mathbf{i} + \mathrm{m}\mathbf{j} - \mathbf{k})}} \\ \therefore\ 2m\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} &= \frac{12\mathbf{i} + 6\mathrm{m}\mathbf{j} - 6\mathbf{k}}{\sqrt{m^2+5}} \end{aligned}$$

$$\begin{aligned} equating\ coefficients\quad 2m &= \frac{12}{\sqrt{m^2+5}} \Rightarrow m \in \{-3i, 2\} \quad ① \\ 4 &= \frac{6m}{\sqrt{m^2+5}} \Rightarrow m \in \{2\} \quad ② \\ -2 &= \frac{-6}{\sqrt{m^2+5}} \Rightarrow m \in \{-2,2\} \quad ③ \end{aligned}$$

$but\ only\ m = 2\ satisfies\ equation$ ①, ② and ③.

Question 16 Answer D

$$\begin{aligned} \overrightarrow{OA} &= \frac{\mathbf{p}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} \\ \overrightarrow{AP} &= \overrightarrow{OP} - \overrightarrow{OA} \\ &= \mathbf{p} - \frac{\mathbf{p}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} \\ \overrightarrow{OQ} &= \overrightarrow{OA} + \overrightarrow{AQ} \\ &= \overrightarrow{OA} - \overrightarrow{AP} \\ &= \frac{\mathbf{p}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} - \left(\mathbf{p} - \frac{\mathbf{p}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\right) \\ &= 2\frac{\mathbf{p}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} - \mathbf{p} \end{aligned}$$

Question 17 Answer D

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$$

$$\overrightarrow{PQ} \cdot \overrightarrow{PQ} = (\mathbf{q} - \mathbf{p}) \cdot (\mathbf{q} - \mathbf{p})$$

$$= \mathbf{p} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{q} - 2\mathbf{p} \cdot \mathbf{q}$$

$$\mathbf{p} \cdot \mathbf{q} = \frac{\mathbf{p} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{q} - (\mathbf{q} - \mathbf{p}) \cdot (\mathbf{q} - \mathbf{p})}{2}$$

$$|\mathbf{p}||\mathbf{q}| \cos(\theta) = \frac{\mathbf{p} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{q} - (\mathbf{q} - \mathbf{p}) \cdot (\mathbf{q} - \mathbf{p})}{2}$$

$$\cos(\theta) = \frac{\mathbf{p} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{q} - (\mathbf{q} - \mathbf{p}) \cdot (\mathbf{q} - \mathbf{p})}{2|\mathbf{p}||\mathbf{q}|}$$

$$= \frac{\mathbf{p} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{q} - (\mathbf{q} - \mathbf{p}) \cdot (\mathbf{q} - \mathbf{p})}{2\sqrt{(\mathbf{p} \cdot \mathbf{p})(\mathbf{q} \cdot \mathbf{q})}}$$

Question 18 Answer E

$$displacement = x(2) - x(0)$$

$$= \int_0^2 v\, dt$$

Question 19 Answer A

$$a \propto v \Rightarrow \frac{dv}{dt} = kv$$

$$(v, a) = (-1, 2) \Rightarrow k = -2$$

$$\Rightarrow \frac{dv}{dt} = -2v$$

$$-2\frac{dt}{dv} = \frac{1}{v}$$

$$-2t = \ln|v| + c$$

$$(t, v) = (2, -1) \Rightarrow c = -4$$

$$|v| = e^{4-2t}$$

$$v < 0 \Rightarrow |v| = -v$$

$$v = -e^{4-2t}$$

Question 20 Answer D

$$\frac{dv}{dx} \propto \frac{1}{v} \Rightarrow \frac{dv}{dx} = \frac{k}{v}$$

$$\left(v, \frac{dv}{dx}\right) = (2,2) \Rightarrow k = 4$$

$$\Rightarrow v\frac{dv}{dx} = 4$$

$$F = ma = 2v\frac{dv}{dx} = 8$$

Question 21 Answer B

$$System\ has\ mass\ 3\ kg, and\ a\ net\ force\ of\ g\ down \Rightarrow a = \frac{1}{3}g\ \mathrm{ms}^{-2}\ down$$

$$v^2 = u^2 + 2as \Rightarrow v = \sqrt{u^2 + 2as} = \sqrt{0^2 + 2 \times \frac{g}{3} \times \frac{1}{2}} = \sqrt{\frac{g}{3}}$$

Question 22 Answer D

$$\Sigma\mathbf{F} = (80 + 50\cos(\theta))\mathbf{i} + (70 - 50\sin(\theta))\mathbf{j}$$

$$\Rightarrow 10\mathbf{a} = 110\mathbf{i} + 30\mathbf{j}$$

$$\Rightarrow \mathbf{a} = 11\mathbf{i} + 3\mathbf{j}$$

SECTION 2

Question 1

a.

$$\cos\left(\frac{\pi}{10}\right) = +\sqrt{1-\sin^2\left(\frac{\pi}{10}\right)}, \quad since\ \cos(\theta) > 0\ \ when\ \theta \in \left(0, \frac{\pi}{2}\right) \quad M1$$

$$= \sqrt{1-\left(\frac{1}{4}(-1+\sqrt{5})\right)^2} \quad M1$$

$$= \sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}$$

2 marks

b. **i.** Express in polar form $w = \sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}} + i\frac{1}{4}(-1+\sqrt{5})$

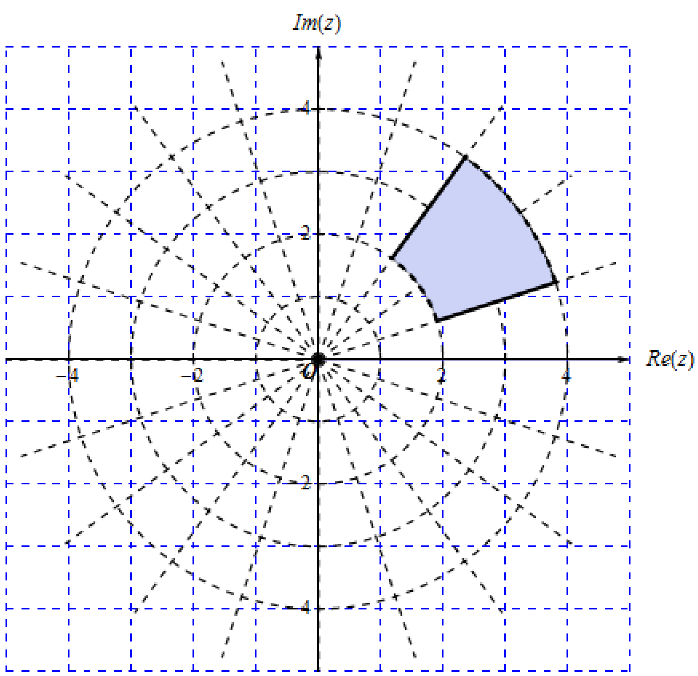
$w = \cos\left(\frac{\pi}{10}\right) + i\ sin\left(\frac{\pi}{10}\right) = cis\left(\frac{\pi}{10}\right) \quad (A1)$

ii. Write down w^3 in polar form.

$w^3 = \left(cis\left(\frac{\pi}{10}\right)\right)^3 = cis\left(\frac{3\pi}{10}\right) \quad (A1)$

1 + 1 = 2 marks

c.



$Correct\ Region \quad (A1)$
$Correct\ Boundaries \quad (A1)$

2 marks

d. $Area = \frac{6\pi}{5}\ square\ units$ $(A1)$

1 mark

e. i.

$$\text{Im}(w^n) = 0 \quad \Rightarrow \quad \text{Im}\left(\text{cis}\left(n\frac{\pi}{10}\right)\right) = 0 \qquad (M1)$$

$$\Rightarrow \quad \sin\left(n\frac{\pi}{10}\right) = 0 \qquad (M1)$$

$$n\frac{\pi}{10} = m\pi, \qquad m \in \mathbb{Z}$$

$$\therefore \quad n = 10m, \qquad m \in \mathbb{Z} \qquad (A1)$$

ii. $w^{10\,m} = \text{cis}(m\pi) = \cos(m\pi) = (-1)^m$ $(A1)$

3 + 1 = 4 marks

Question 2

a. $v_B = v_A - a$ $(A1)$

1 mark

b. $using\ s = ut + 1/2at^2$

$$95 = v_A - \frac{1}{2}a \quad ①$$

$$160 = 2v_A - 4a \quad ②$$

Solving ① and ② simultaneously for of a and v_A gives

$\therefore v_A = 100\ \text{ms}^{-1}\ and\ a = 10\ \text{ms}^{-2}$. $(A2)$

2 marks

c.

$v_C = v_A - 3a = 70\ \text{ms}^{-1}$.

1 mark

d.

$$newton's\ second\ law \quad \Rightarrow \quad -40{,}000\frac{dv}{dt} = 1600\ v^2(\text{t}) + 1600$$

$$rearranging\ and\ reciprocating \quad \Rightarrow \quad \frac{dt}{dv} = -\frac{25}{v(t)^2 + 1} \qquad (M1)$$

1 mark

e.

$$t - 0 = -25\int_{70}^{0} \frac{1}{v^2+1}\,dv \qquad (M1)$$

$$\therefore t = -25[\arctan(v)]_{70}^{0} = 25\arctan(70) \qquad (A1)$$

2 marks

f.

$$t - 0 = -25\int_{70}^{v} \frac{1}{u^2+1}\,du \qquad (M1)$$

$$\Rightarrow \quad t = -25[\arctan(u)]_{70}^{v} = -25(\arctan(v) - \arctan(70)) \qquad (M1)$$

$$\textit{solving for } v \quad v(t) = \tan\left(\frac{1}{25}(-t + 25\arctan(70))\right)$$

2 marks

g.

Reverse thrust is applied at Point C, which is 355 metres from the start of the runway

$$\Rightarrow \quad x = 355 + \int_{0}^{25\arctan(70)} \tan\left(\frac{1}{25}(-t + 25\arctan(70))\right) dt \qquad (M1)$$

$$\therefore \quad x \approx 461 \text{ m} \qquad (A1)$$

2 marks

Question 3

a.

$$\frac{dV(h)}{dt} = \frac{dV(h)}{dh} \times \frac{dh}{dt} = \begin{cases} \pi h^2 \dfrac{dh}{dt} & h \in [0,3] \\ 9\pi \dfrac{dh}{dt} & h \in [3,6] \end{cases}$$

(Derivative $\frac{dV(h)}{dh}$ is defined at $x = 3$)

2 marks

b.

$$\begin{aligned}
rate\ of\ outflow &\propto \sqrt{h} \\
&= k\sqrt{h} \\
&= \frac{\pi}{50} when\ h = 4 \\
\therefore k &= \frac{\pi}{100} && (A1) \\
\frac{dV}{dt} &= rate\ of\ inflow - rate\ of\ outflow && (M1) \\
9\pi\frac{dh}{dt} &= \frac{\pi}{100} - \frac{\pi}{100}\sqrt{h} && (A1) \\
\frac{dh}{dt} &= \frac{1-\sqrt{h}}{900}
\end{aligned}$$

3 marks

c.

$$\begin{aligned}
h = (1-u)^2 \quad &\Rightarrow \quad dh = \frac{dh}{du}du = 2(u-1)du \\
&\Rightarrow \quad \frac{1}{1-\sqrt{h}} = \frac{1}{u} \\
&\Rightarrow \quad \int \frac{1}{1-\sqrt{h}}\,dh = 2\int \left(1 - \frac{1}{u}\right) du \qquad (A1) \\
&\Rightarrow \quad \int \frac{1}{1-\sqrt{h}}\,dh = 2(u - \log_e|u|)|_{u=1-\sqrt{h}} + c \\
&\therefore \quad \int \frac{1}{1-\sqrt{h}}\,dh = -2\left(\sqrt{h} + \log_e\left|1-\sqrt{h}\right|\right) + c \qquad (A1)
\end{aligned}$$

2 marks

d.

$$\frac{dt}{dh} = \frac{900}{1-\sqrt{h}}, t(4) = 0 \quad \Rightarrow \quad t(3) = 0 + 900\int_4^3 \frac{1}{1-\sqrt{h}}\,dh \qquad (A1)$$

1 mark

e.

$$t(3) = t(4) + 900\int_4^3 \frac{1}{1-\sqrt{h}}\,dh$$
$$= 0 - 1800\left[\sqrt{h} + \log_e\left|1-\sqrt{h}\right|\right]_4^3 \qquad (A1)$$
$$= 1800\left(2-\sqrt{3}-\log_e\left(\sqrt{3}-1\right)\right)\text{hours} \qquad (A1)$$

2 marks

f.

$$\frac{dV}{dt} = rate\ of\ inflow - rate\ of\ outflow \qquad (M1)$$
$$\pi h^2\frac{dh}{dt} = \frac{\pi}{100} - \frac{\pi}{100}\sqrt{h} \qquad (A1)$$
$$\therefore \frac{dh}{dt} = \frac{1-\sqrt{h}}{100\,h^2}$$

2 marks

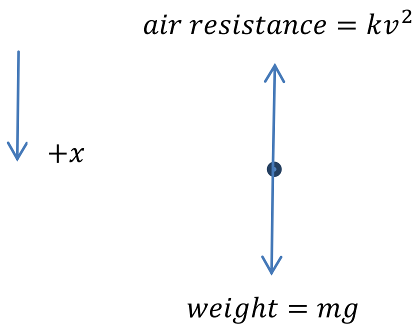
g.

$$\lim_{t\to\infty} h(t) = 1 \qquad (A1)$$

1 mark

Question 4

a.



1 mark

b.

$$Newton's Second\ Law \Rightarrow mv\frac{dv}{dx} = mg - kv^2 \quad (M1)$$

$$rearranging \Rightarrow \frac{dv}{dx} = \frac{mg - kv^2}{mv}$$

$$particle\ falls\ from\ rest \Rightarrow v(0) = 0 \quad (R1)$$

2 marks

c.

$$reciprocate\ both\ sides \quad \frac{dx}{dv} = \frac{mv}{mg - kv^2}, x(0) = 0$$

$$x = \int \frac{mv}{mg - kv^2}\,dv \quad (M1)$$

$$= -\frac{m}{2k}\int \frac{-2kv}{mg - kv^2}\,dv$$

$$= -\frac{m}{2k}\log_e(mg - kv^2) + c, since\ mg - kv^2 > 0 \quad (A1)$$

$$x(0) = 0 \Rightarrow \quad c = \frac{m}{2k}\log_e(mg)$$

$$x(v) = -\frac{m}{2k}\log_e\left(\frac{mg - kv^2}{mg}\right) \quad (A1)$$

$$Solving\ for\ v \quad v(x) = \sqrt{\frac{mg}{k}}\sqrt{1 - e^{-\frac{2k}{m}x}} \quad (A1)$$

4 marks

d.

$$\lim_{x\to\infty} v(x) = \lim_{x\to\infty}\sqrt{\frac{mg}{k}}\sqrt{1 - e^{-\frac{2k}{m}x}} = \sqrt{\frac{mg}{k}} \quad (A1)$$

1 mark

Question 5

a.

$$\mathbf{a} = unit\ vector\ along\ y-axis = \mathbf{j} \qquad \text{(A1)}$$

1 mark

b.

$$\mathbf{d} = \frac{\overrightarrow{OE}-\overrightarrow{OA}}{\left|\overrightarrow{OE}-\overrightarrow{OA}\right|} \qquad (M1)$$

$$= \frac{2\mathbf{i}-2\mathbf{k}}{|2\mathbf{i}-2\mathbf{k}|}$$

$$= \frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{k} \qquad (A1)$$

2 marks

c.

$$\mathbf{a}\cdot\mathbf{o}=0 \Rightarrow \mathbf{j}\cdot(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})=0$$

$$\therefore\ v=0 \qquad (A1)$$

$$\mathbf{d}\cdot\mathbf{o}=0 \Rightarrow \left(\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{k}\right)\cdot(u\mathbf{i}+0\mathbf{j}+w\mathbf{k})=0$$

$$\Rightarrow \frac{u}{\sqrt{2}}-\frac{w}{\sqrt{2}}=0$$

$$\therefore\ w=u \qquad (A1)$$

$$\mathbf{o}\cdot\mathbf{o}=1 \Rightarrow (u\mathbf{i}+0\mathbf{j}+u\mathbf{k})\cdot(u\mathbf{i}+0\mathbf{j}+u\mathbf{k})=1$$

$$\Rightarrow 2u^2=1$$

$$\Rightarrow u=w=\frac{1}{\sqrt{2}} \qquad +\sqrt{\ }\ because\ outward\ normal$$

$$\therefore\ \mathbf{o}=\frac{1}{\sqrt{2}}\mathbf{i}+0\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k} \qquad (A1)$$

3 marks

d.

$$\mathbf{o}\cdot(\mathbf{W}+\mathbf{F}_N)=m\mathbf{o}\cdot\frac{d\mathbf{v}}{dt}=m\frac{d}{dt}(\mathbf{o}\cdot\mathbf{v})=0 \qquad \text{(M1)}$$

1 mark

e.

$$\begin{aligned}\mathbf{o}\cdot(\mathbf{W}+\mathbf{F}_N)=0 \quad \Rightarrow \quad \mathbf{o}\cdot\mathbf{F}_N &= -\mathbf{o}\cdot\mathbf{W} \\ \Rightarrow \quad \mathbf{o}\cdot(N\mathbf{o}) &= -\mathbf{o}\cdot(-mg\mathbf{k}) \\ \Rightarrow \quad N &= mg\mathbf{o}\cdot\mathbf{k} && (A1) \\ \Rightarrow \quad &= mg(\frac{1}{\sqrt{2}}\mathbf{i}+0\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k})\cdot\mathbf{k} && (M1) \\ \therefore \quad N &= \frac{mg}{\sqrt{2}}\end{aligned}$$

2 marks

f.

$$\begin{aligned}m\frac{d\mathbf{v}}{dt}=\mathbf{W}+\mathbf{F}_N \quad \Rightarrow \quad m\frac{d\mathbf{v}}{dt} &= -mg\mathbf{k}+\frac{mg}{\sqrt{2}}\mathbf{o} \\ &= -mg\mathbf{k}+\frac{mg}{\sqrt{2}}(\frac{1}{\sqrt{2}}\mathbf{i}+0\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}) && (M1) \\ &= \frac{g}{2}\mathbf{i}-\frac{g}{2}\mathbf{k}\end{aligned}$$

1 mark

g.

$$\begin{aligned}\frac{d\mathbf{r}}{dt}(t) &= \frac{d\mathbf{r}}{dt}(0)+\int_0^t \frac{g}{2}(\mathbf{i}-\mathbf{k})\mathrm{d}u \\ \Rightarrow \quad \frac{d\mathbf{r}}{dt}(t) &= -\mathrm{V}\mathbf{j}+\frac{g}{2}(\mathbf{i}-\mathbf{k})t && (A1) \\ \mathbf{r}(t) &= \mathbf{r}(0)+\int_0^t -\mathrm{V}\mathbf{j}+\frac{g}{2}(\mathbf{i}-\mathbf{k})t\mathrm{d}u && (M1) \\ &= 2\mathbf{j}+2\mathbf{k}-\mathrm{V}\mathbf{j}t+\frac{g}{4}(\mathbf{i}-\mathbf{k})t^2 \\ \therefore \quad \mathbf{r}(t) &= \frac{g}{4}t^2\mathbf{i}+(2-\mathrm{V}t)\mathbf{j}+\left(2-\frac{g}{4}t^2\right)\mathbf{k} && (A1)\end{aligned}$$

3 marks

h.

At time T the puck is at the bottom of the ramp, so the **k** component of $\mathbf{r}(t)$ is zero.

$$\left(2-\frac{g}{4}T^2\right)=0 \quad \Rightarrow \quad T=2\sqrt{\frac{2}{g}}\ \mathrm{s} \qquad (A1)$$

To pass through point A, the **j** component of $\mathbf{r}(T)$ is zero.

$$(2-\mathrm{V}T)=0 \quad \Rightarrow \quad V=\frac{2}{T}=\sqrt{\frac{g}{2}}\ \mathrm{ms}^{-1} \quad (A1)$$

2 marks