The Mathematical Association of Victoria
SOLUTIONS: Trial Exam 2013
SPECIALIST MATHEMATICS
Written Examination 2

SECTION 1: Multiple Choice

ANSWERS

1. C	**2. E**	**3. C**	**4. E**	**5. D**	**6. B**
7. D	**8. E**	**9. C**	**10. E**	**11. D**	**12. A**
13. C	**14. C**	**15. B**	**16. A**	**17. D**	**18. C**
19. A	**20. B**	**21. B**	**22. D**		

Question 1 **Answer: C**

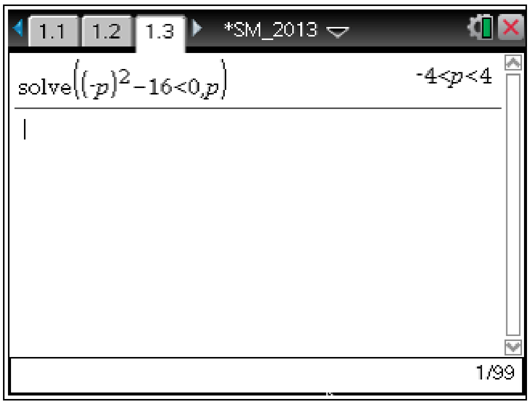
If the domain of g is R, the graph of g does **not** have vertical asymptotes. Therefore the graph of $y = 2x^2 - px + 2$ does **not** have x-axis intercepts, hence its discriminant $\left(\Delta = b^2 - 4ac\right)$ is negative.

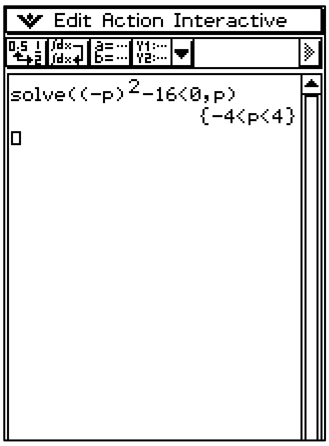
$\Delta = (-p)^2 - 4 \times 2 \times 2 < 0$

$p^2 < 16$

$-4 < p < 4$

$p \in (-4, 4)$





Question 2 **Answer: E**

Let $u = 2t$

$x = \cos(2u) = 1 - 2\sin^2(u)$

$y = 6\sin^2(u)$

Therefore

$y = 3(1-x)$, or

$3x + y - 3 = 0$, which is the equation of a straight line.

Question 3 **Answer: C**

The domain of $y = \sin^{-1}(x)$ is $[-1,1]$

For $\frac{x+1}{3} = -1,\ x = -4$

and $\frac{x+1}{3} = 1,\ x = 2$

Maximal domain of g is $[-4, 2]$

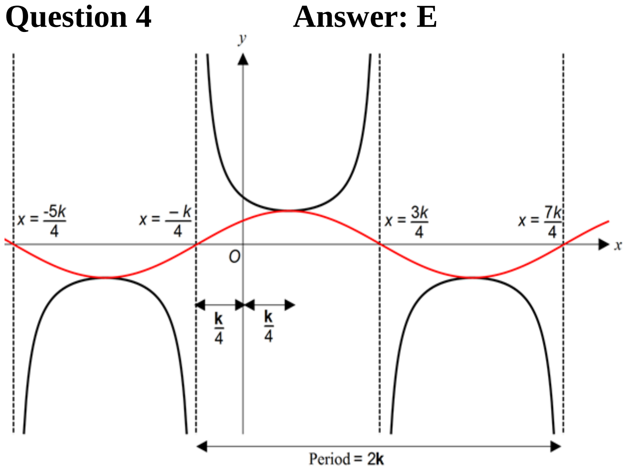
The range of $y = \sin^{-1}(\theta)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The range of $y = 2\sin^{-1}(\theta)$ is $[-\pi, \pi]$ (multiply by a factor of 2)

The range of $y = 2\sin^{-1}(\theta) - \frac{\pi}{4}$ is

$\left[-\frac{5\pi}{4}, \frac{3\pi}{4}\right]$ (subtract $\frac{\pi}{4}$)

Question 4 **Answer: E**



The graph could be the **reciprocal** of a cosine graph of the form $y = \cos(n(x-b))$, with:

- period $= 2k$

$$\frac{2\pi}{n} = 2k$$

$$n = \frac{\pi}{k}$$

- Translated $\frac{k}{4}$ units to the right

$$b = \frac{k}{4}$$

$$f(x) = \frac{1}{\cos\left(\frac{\pi}{k}\left(x - \frac{k}{4}\right)\right)}$$

$$f(x) = \sec\left(\frac{\pi}{k}\left(x - \frac{k}{4}\right)\right)$$

Note: the graph could also be the reciprocal of a sine graph: $f(x) = \dfrac{1}{\sin\left(\frac{\pi}{k}\left(x + \frac{k}{4}\right)\right)}$.

However, $f(x) = \text{cosec}\left(\frac{\pi}{k}\left(x + \frac{k}{4}\right)\right)$ is **not** one of the options.

Question 5 **Answer: D**

If $(-3, 2)$ is the centre, the equation will be

$$\frac{(x+3)^2}{9} + \frac{(y-2)^2}{4} = 1$$

$$4(x+3)^2 + 9(y-2)^2 = 36$$

Expanding the brackets gives

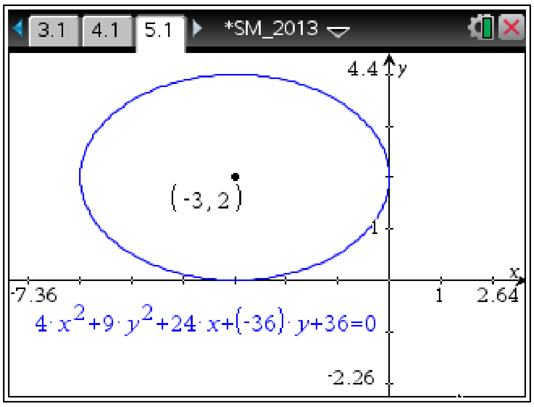
$$4(x^2 + 6x + 9) + 9(y^2 - 4y + 4) = 36$$

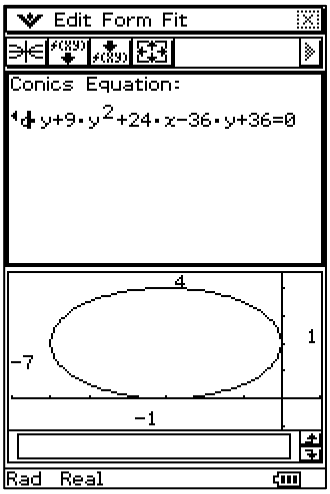
$$4x^2 + 24x + 36 + 9y^2 - 36y + 36 = 36$$

$$4x^2 + 24x + 9y^2 - 36y + 36 = 0$$

Comparing with $4x^2 + mx + 9y^2 - ny + 36 = 0$

$m = 24$ and $n = 36$





Question 6 **Answer: B**

Method 1: 'By hand'

Euler's method: $y_{n+1} = y_n + h\,f(x_n)$, where

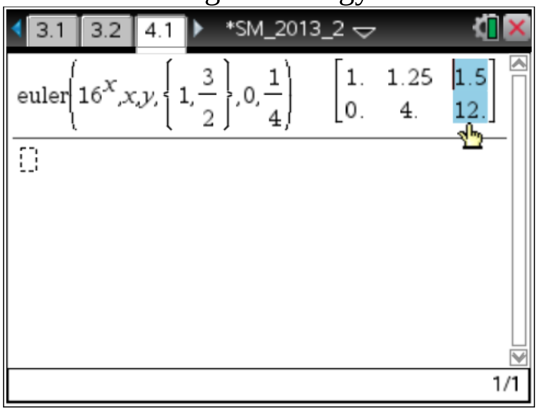
$f(x_n) = 16^x$, $x_0 = 1$, $y_0 = 0$ and $h = \frac{1}{4}$.

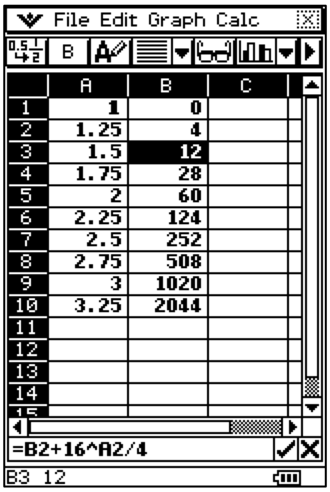
$$x_2 = x_0 + 2h = 1 + 2 \times \frac{1}{4} = \frac{3}{2}$$

$$y_1 = 0 + \frac{1}{4} \times 16^1 = 4 \text{ and } x_1 = x_0 + h = 1 + \frac{1}{4} = \frac{5}{4}$$

$$y_2 = 4 + \frac{1}{4} \times 16^{\frac{5}{4}} = 12 \text{ and } x_2 = \frac{5}{4} + \frac{1}{4} = \frac{3}{2}$$

Method 1: Using technology





Question 7 **Answer: D**

The shape of the graph could be of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a, b are positive real constants.

By implicit differentiation and making $\frac{dy}{dx}$ the subject:

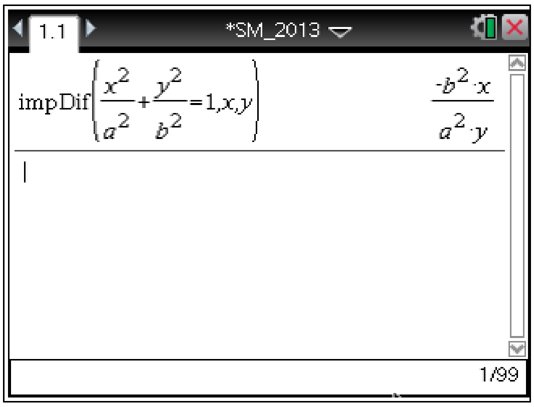
$$\frac{2x}{a^2}+\frac{2y}{b^2}\times\frac{dy}{dx}=0$$

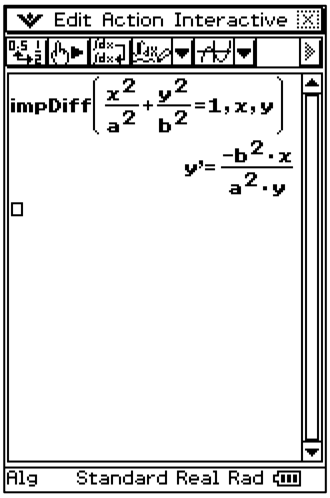
$$\frac{dy}{dx}=-\frac{2x}{a^2}\times\frac{b^2}{2y}$$

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

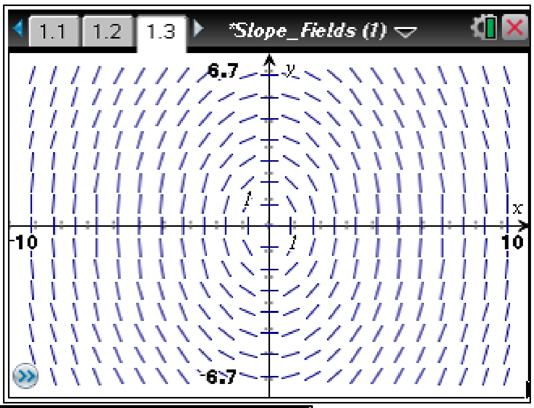
$\frac{dy}{dx}=-\frac{mx}{y}$, where $m=\frac{b^2}{a^2}$.

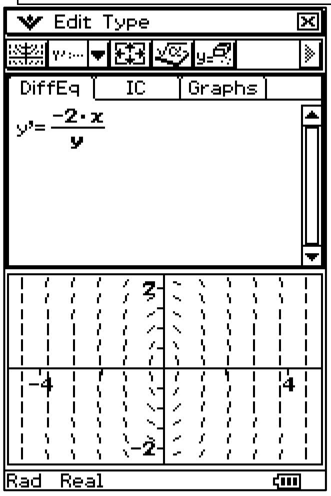
Note that Option C cannot be correct because m must be a positive number.





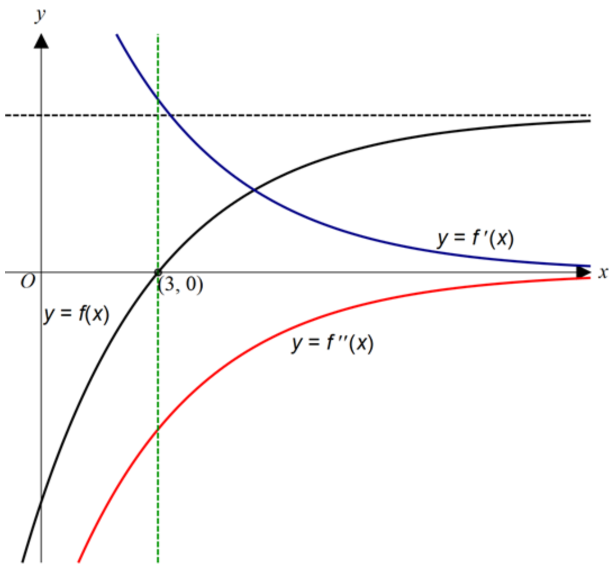
Test this by graphing the slope field for, say, $\frac{dy}{dx}=-\frac{2x}{y}$ (since $m>1$).





Question 8 Answer: E

Consider the sketch graphs of $y = f'(x)$ and $y = f''(x)$.



The gradient of the graph of *f* is always positive, therefore $f'(3) > 0$.

The gradient of the graph of $y = f'(x)$ is always negative, therefore $f''(3) < 0$.

Since $f(3) = 0$, $f'(3) > f(3) > f''(3)$, and in particular, $f''(3) < f'(3)$.

Question 9 Answer: C

$1 - |z+i| = 0$ can be rewritten as $|z+i| = 1$, which is a **circle** of radius 1, centred at $(0, -1)$.
The graphs for all other options are straight lines, with cartesian equations:

A. $y = 0$ B. $y = -\frac{1}{2}$

D. $y = 0$ E. $x + 2y = 0$ or $y = -\frac{x}{2}$

Question 10 Answer: E

$$u^{n-1} = u^n \times u^{-1} = \frac{ai}{1-i}$$

$$\frac{ai}{1-i} \times \frac{1+i}{1+i} = \frac{ai(1+i)}{1-i^2}$$
$$= \frac{a(-1+i)}{2}$$
$$= \frac{-a}{2}(1-i)$$

Question 11 Answer: D

If $(z+i)$ is a factor, then $(z-i)$ is also a factor (conjugate root theorem).
Since $P(z)$ has **integer** coefficients, if $(z+1-\sqrt{2})$ is a linear factor, it would arise from a quadratic factor which is the product of $(z+1-\sqrt{2})$ and $(z+1+\sqrt{2})$ (otherwise, in the expansion of linear factors, irrational coefficients will occur as a consequence of multiplying by $\sqrt{2}$).

The least factors of $P(z)$ are therefore

$(z+1)(z+i)(z-i)(z+1-\sqrt{2})(z+1+\sqrt{2})$.

Hence the degree of $P(z)$ is at least 5.

Question 12 **Answer: A**

Method 1 – 'by hand'

$$2\frac{dx}{dt}-x^2-4=0$$

$$\frac{dt}{dx}=\frac{2}{4+x^2}$$

$$t=\int\left(\frac{2}{4+x^2}\right)dx$$

$$t=\tan^{-1}\left(\frac{x}{2}\right)+c$$

$x=2$ at $t=0$

$$0=\tan^{-1}\left(\frac{2}{2}\right)+c$$

$$c=-\frac{\pi}{4}$$

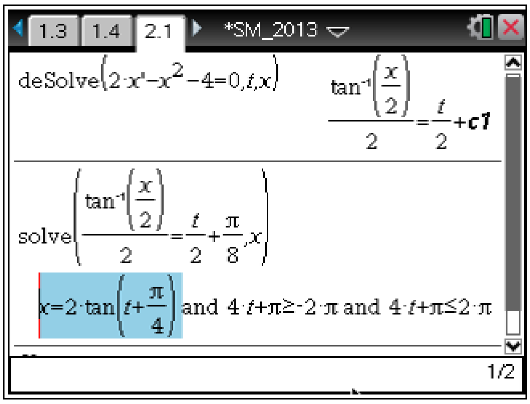
$$t=\tan^{-1}\left(\frac{x}{2}\right)-\frac{\pi}{4}$$

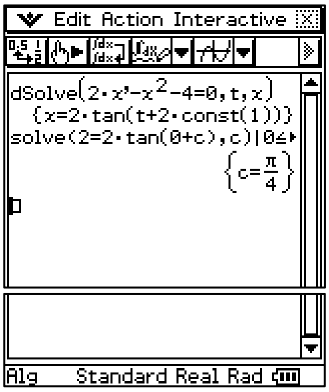
$$x=2\tan\left(t+\frac{\pi}{4}\right)$$

Method 2 – using technology,

$2\frac{dx}{dt}-x^2-4=0$, $x=2$ at $t=0$

$$x=2\tan\left(t+\frac{\pi}{4}\right)$$



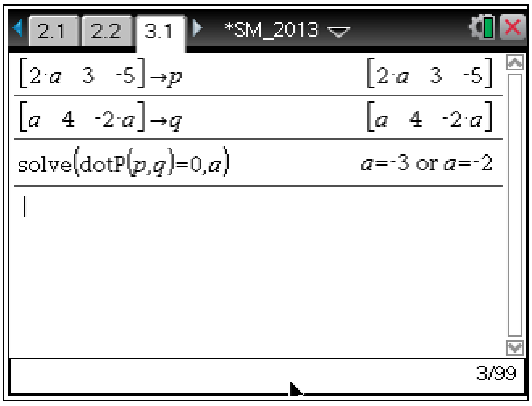


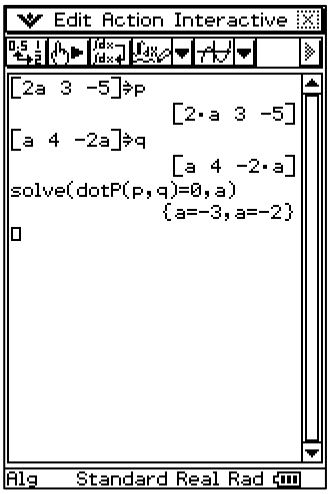
Question 13 **Answer: C**

$$\underset{\sim}{p}\cdot\underset{\sim}{q}=2a^2+12+10a=0$$

$$a^2+5a+6=0$$

$$a=-2 \text{ or } a=-3$$





Question 14 **Answer: C**

$$\overrightarrow{BD}=\overrightarrow{BA}+\overrightarrow{AD}$$

$$=-\underset{\sim}{b}+\underset{\sim}{d}$$

The diagonals of a parallelogram bisect each other, therefore,

$$\overrightarrow{BM}=\frac{1}{2}\overrightarrow{BD}$$

$$=\frac{1}{2}\left(\underset{\sim}{d}-\underset{\sim}{b}\right)$$

Question 15 **Answer: B**

$$\underset{\sim}{r}(t)=\int\underset{\sim}{v}(t)\,dt=\left(12t+c_1\right)\underset{\sim}{i}+\left(18t-3t^2+c_2\right)\underset{\sim}{j}+c_3\underset{\sim}{k}$$

However, $c_1=c_2=c_3=0$ since

$\underset{\sim}{r}(0)=0\underset{\sim}{i}+0\underset{\sim}{j}+0\underset{\sim}{k}$ (the ball is hit at the origin).

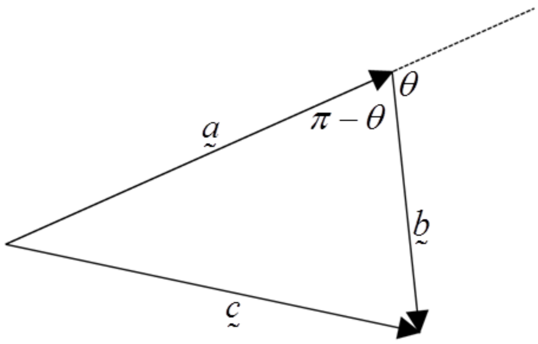
At its maximum height, the vertical component of the velocity is zero.

$(18-6t)\underset{\sim}{j}=0\underset{\sim}{j}$, therefore $t=3$.

The position vector at the maximum height is

$\underset{\sim}{r}(3)=(12\times3)\underset{\sim}{i}+\left(18\times3-3\times\left(3^2\right)\right)\underset{\sim}{j}$

$\underset{\sim}{r}(3)=36\underset{\sim}{i}+27\underset{\sim}{j}$



Question 16 **Answer: A**

$xy+y^2-x^2-11=0$

$\frac{dy}{dx}=\frac{2x-y}{x+2y}$

$\frac{2x-y}{x+2y}=\frac{1}{8}$

$y=\frac{3x}{2}$

Substitute $y=\frac{3x}{2}$ into $xy+y^2-x^2-11=0$

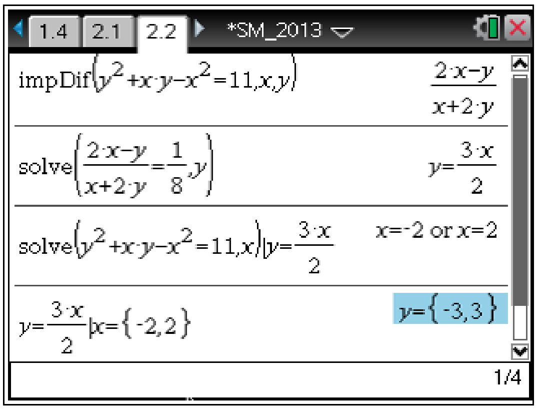
$x^2-4=0$

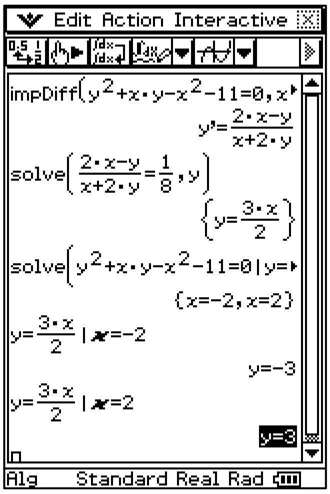
$x=-2$ or $x=2$

Substitute in $y=\frac{3x}{2}$

When $x=-2$ and $y=-3$ or $x=2$ and $y=3$

Coordinates are $(2,3)$ and $(-2,-3)$





Alternatively, substitute the coordinates given in each option into $\frac{2x-y}{x+2y}$. Option **A** coordinates give answers of $\frac{1}{8}$.

Question 17 **Answer: D**

Let $u=\cos^{-1}(x)$.

$\frac{du}{dx}=\frac{-1}{\sqrt{1-x^2}}$

When $x=0$, $u=\cos^{-1}(0)=\frac{\pi}{2}$

When $x=\frac{1}{2}$, $u=\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$.

$$\int_{x=0}^{x=\frac{1}{2}}\left(\frac{\sqrt{\cos^{-1}(x)}}{\sqrt{1-x^2}}\right)dx=\int_{u=\frac{\pi}{2}}^{u=\frac{\pi}{3}}\left(\sqrt{u}\times-\frac{du}{dx}\right)dx$$

$$=-\int_{\frac{\pi}{2}}^{\frac{\pi}{3}}\left(\sqrt{u}\right)du$$

Question 18 **Answer: C**

Method 1

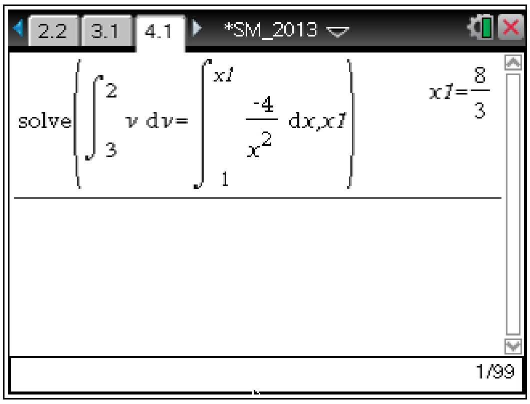
$a=v\frac{d(v)}{dx}=-\frac{4}{x^2}$

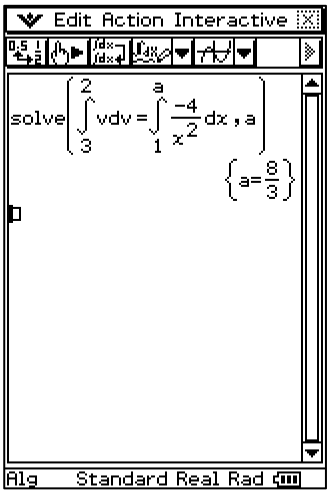
$v=3$ at $x=1$

$\int_3^2 v\,dv=-4\int_1^{x_1}\frac{dx}{x^2}$

Solve for x_1

$x_1=\frac{8}{3}$, therefore $x=\frac{8}{3}$ m





Method 2

$$a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-\frac{4}{x^2}$$

$$v^2=-8\int\frac{dx}{x^2}$$

$$v^2=\frac{8}{x}+c$$

$v=3$ at $x=1$, $c=1$

$$v^2=\frac{8}{x}+1$$

When $v=2$

$$x=\frac{8}{3}\text{m}$$

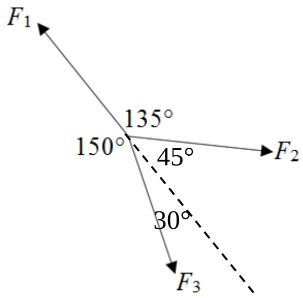
Question 19 **Answer: A**

The displacement gives the distance from the starting point.
The displacement can be found by calculating the signed area bounded by the graph and the horizontal axis.

$$\int_0^3\left(4-t^2\right)dt-\frac{5\times(3+5)}{2}+\frac{4\times10}{2}=3-20+20$$

Distance = 3 m

Question 20 **Answer: B**



Method 1: Lami's theorm

$$\frac{F_2}{\sin\left(150^\circ\right)}=\frac{100}{\sin\left(75^\circ\right)}$$

$F_2\approx52$ newtons

$$\frac{F_3}{\sin\left(135^\circ\right)}=\frac{100}{\sin\left(75^\circ\right)}$$

$F_3\approx73$ newtons

Method 2: resolving vectors
If the particle is in equilibrium,
Resolving forces perpendicular to F_1:

$$F_2\sin\left(45^\circ\right)=F_3\sin\left(30^\circ\right)$$

$$F_2\times\frac{\sqrt{2}}{2}=F_3\times\frac{1}{2}$$

$$F_3=\sqrt{2}F_2 \quad \ldots \text{ equation(1)}$$

Resolving forces parallel to F_1:

$$100=F_2\cos\left(45^\circ\right)+F_3\cos\left(30^\circ\right)$$

$$100=F_2\times\frac{\sqrt{2}}{2}+F_3\times\frac{\sqrt{3}}{2}$$

Substitute equation(1)

$$100=F_2\times\frac{\sqrt{2}}{2}+F_2\times\frac{\sqrt{6}}{2}$$

$$F_2=\frac{200}{\sqrt{2}+\sqrt{6}}\approx52\text{ newtons}$$

$$F_3=\sqrt{2}F_2=\sqrt{2}\times\frac{200}{\sqrt{2}+\sqrt{6}}\approx73\text{ newtons}$$

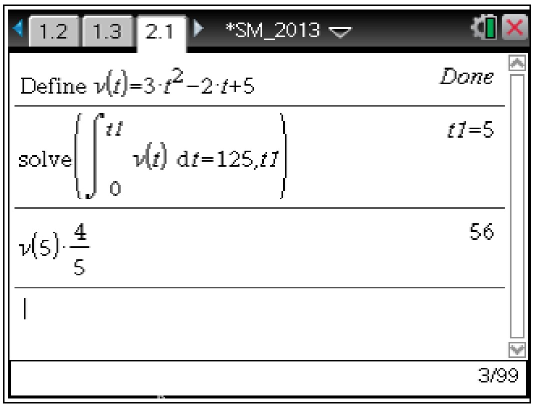
Question 21 **Answer: B**

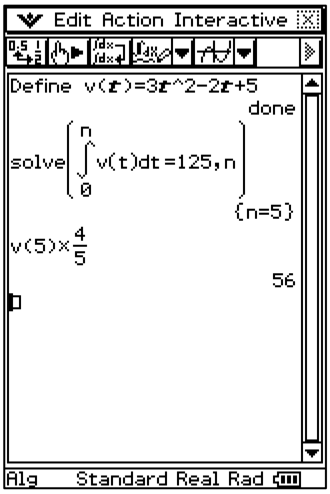
Solve for t_1, $\int_0^{t_1}v(t)\,dt=125$

$t_1=5\text{s}$

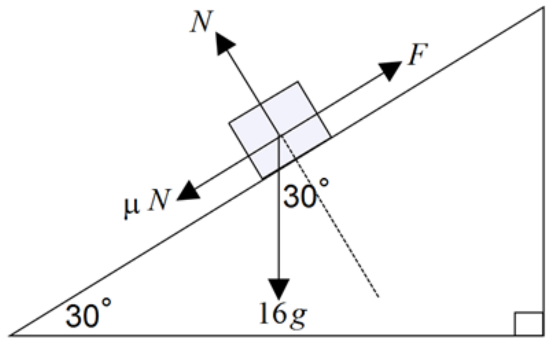
$p=m\times v$

$$p=\frac{4}{5}\times v(5)=56\text{ kg ms}^{-1}$$





Question 22 **Answer: D**



Resolving forces perpendicular to the plane

$N - 16g\cos\left(30^\circ\right) = 0$

$N = 8\sqrt{3}\,g$

Let F be the minimum force required to pull the block up the plane at constant speed (zero acceleration).

Resolving forces parallel to the plane

$F - \mu N - 16g\sin\left(30^\circ\right) = 0$

$F - \frac{1}{4}\times 8\sqrt{3}\,g - 8\,g = 0$

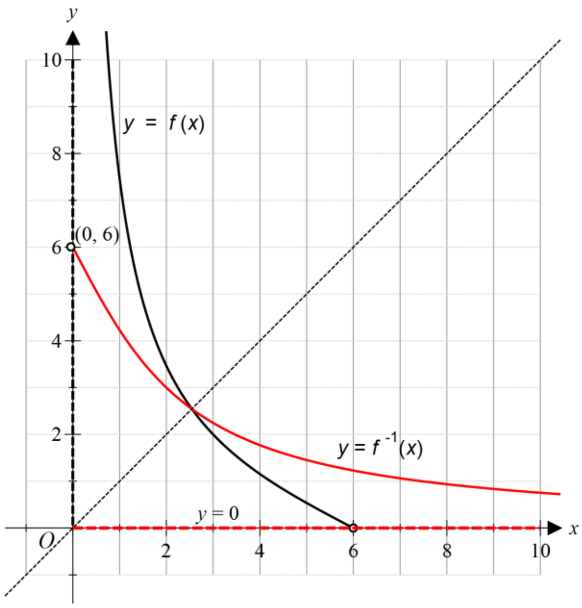
$F = \left(8 + 2\sqrt{3}\right)g$

END OF SECTION 1 SOLUTION

SECTION 2: Extended Response SOLUTIONS

Question 1

1a.



Correct shape 1A, endpoint and asymptote correctly labelled 1A

1b. As $x \to 0$, $\tan^{-1}\left(\frac{2}{x}\right) \to \frac{\pi}{2}$, and $k\tan^{-1}\left(\frac{2}{x}\right) \to 6$. Therefore, $k = 6\times\frac{2}{\pi} = \frac{12}{\pi}$ 1A

1c.

$\frac{d}{dx}\left(\log_e\left(x^2+4\right)\right)=\frac{2x}{x^2+4}$ 1A

Using the product rule,

$\frac{d}{dx}\left(x\tan^{-1}\left(\frac{2}{x}\right)\right)=x\times\frac{-2}{x^2+4}+\tan^{-1}\left(\frac{2}{x}\right)$ 1M

$\frac{d}{dx}\left(\log_e\left(x^2+4\right)\right)=\frac{2x}{x^2+4}$

Therefore,

$\frac{d}{dx}\left(\log_e\left(x^2+4\right)+x\tan^{-1}\left(\frac{2}{x}\right)\right)=\frac{2x}{x^2+4}-\frac{2x}{x^2+4}+\tan^{-1}\left(\frac{2}{x}\right)$ 1A

$=\tan^{-1}\left(\frac{2}{x}\right)$, as required

1d.

$\text{Area}=\lim_{a\to 0}\frac{12}{\pi}\int_a^2\left(\tan^{-1}\left(\frac{2}{x}\right)\right)dx$

$=\lim_{a\to 0}\frac{12}{\pi}\left[\log_e\left(x^2+4\right)+\tan^{-1}\left(\frac{2}{x}\right)\right]_a^2$ 1M

$=\frac{12}{\pi}\left[\left(\log_e(8)+2\tan^{-1}(1)\right)-\left(\log_e(4)+0\times\frac{\pi}{2}\right)\right]$ 1A

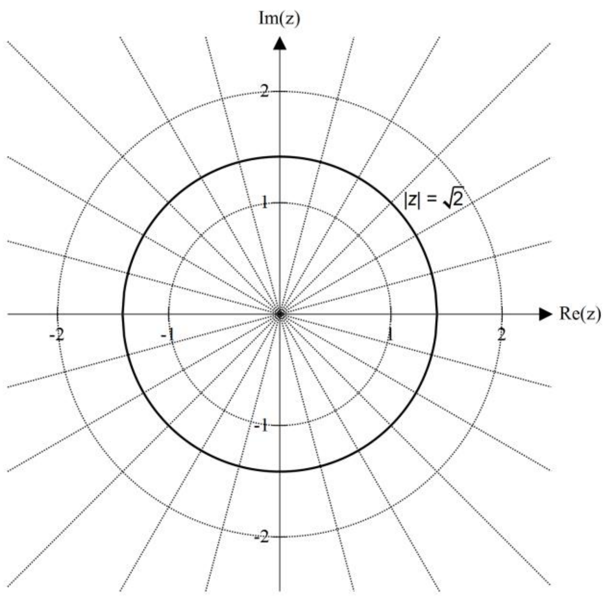
$=\frac{12}{\pi}\left[3\log_e(2)+2\times\frac{\pi}{4}-2\log_e(2)\right]$

$=\frac{12}{\pi}\times\left(\log_e(2)+\frac{\pi}{2}\right)$ 1A

$=\frac{12\log_e(2)+6\pi}{\pi}$, as required

Question 2

2a.



Correct centre and shape 1A, correct radius 1A

2b.

$$|z+i| = \left|z + \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right|$$

$$\sqrt{x^2 + (y+1)^2} = \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(y - \frac{\sqrt{3}}{2}\right)^2} \qquad \text{1M}$$

$$x^2 + y^2 + 2y + 1 = x^2 + x + \frac{1}{4} + y^2 - \sqrt{3}y + \frac{3}{4}$$

$$y\left(2+\sqrt{3}\right) = x$$

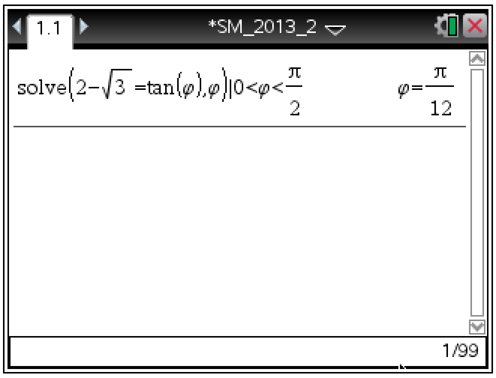
$$y = \frac{2-\sqrt{3}}{2-\sqrt{3}} \times \frac{x}{2+\sqrt{3}} \qquad \text{1A}$$

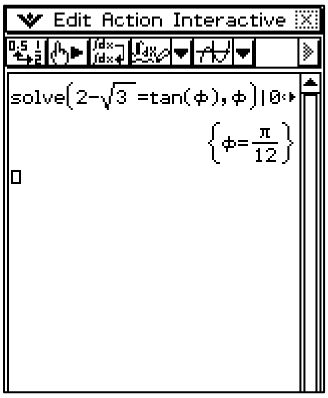
$$y = \left(2-\sqrt{3}\right)x, \quad \text{as required}$$

2c.

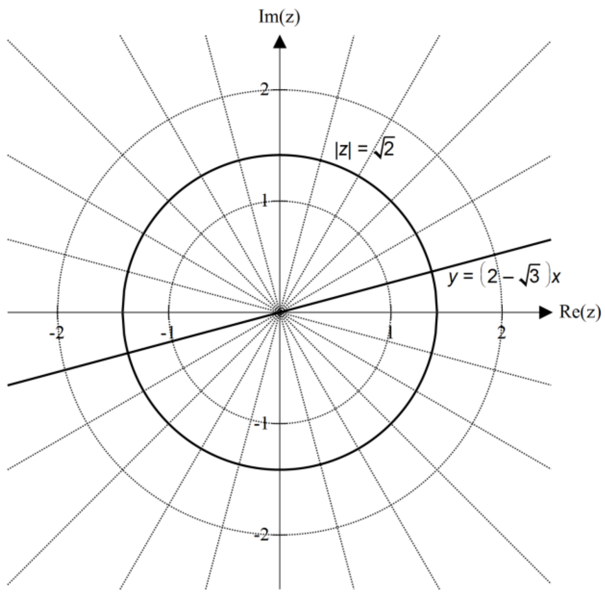
$$m = \tan(\phi), \quad 0 < \phi < \frac{\pi}{2}$$

$$\phi = \tan^{-1}\left(2-\sqrt{3}\right) = \frac{\pi}{12} \qquad \text{1A}$$





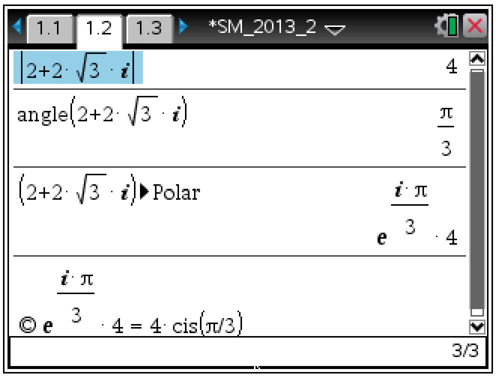
2d.

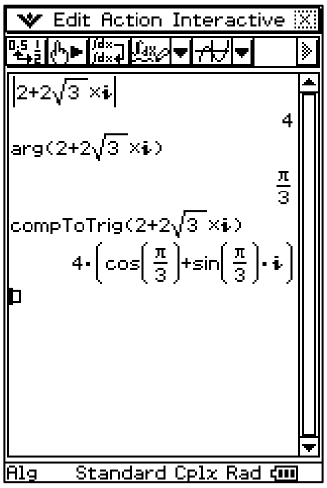


1A

2e.i.

$2+2\sqrt{3}\,i = 4\text{cis}\left(\dfrac{\pi}{3}\right)$ 1A





2e.ii.

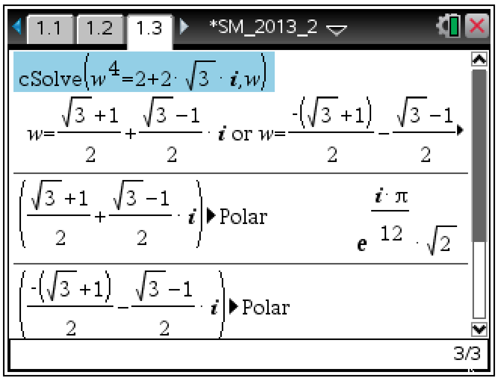
$w^4 = 4\text{cis}\left(\dfrac{\pi}{3}\right)$

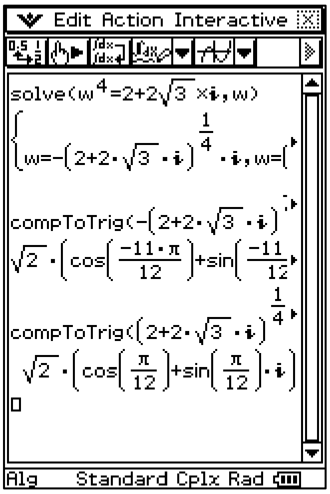
$w = \sqrt[4]{4}\text{cis}\left(\dfrac{\dfrac{\pi}{3}+2k\pi}{4}\right),\ k = 0,1,-1,-2$

$w_1 = \sqrt{2}\text{cis}\left(\dfrac{\pi}{12}\right),\ w_2 = \sqrt{2}\text{cis}\left(\dfrac{7\pi}{12}\right)$ 1A

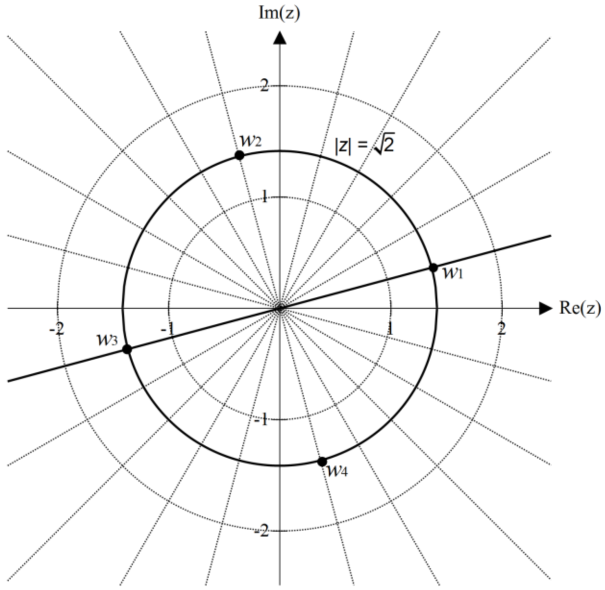
$w_3 = \sqrt{2}\text{cis}\left(\dfrac{-5\pi}{12}\right),\ w_4 = \sqrt{2}\text{cis}\left(\dfrac{-11\pi}{12}\right)$ 1A

At least 1 correct solution: 1mark. All four correct solutions: 2 marks





2e.iii.



1A

Question 3

3a.i.

$\left|\overrightarrow{OP}\right| = \left|\overrightarrow{OQ}\right| = \sqrt{16+5+4} = 5$ 1M

Therefore, $h = 5$, as required

3a.ii.

$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = -\underset{\sim}{p} - \underset{\sim}{q}$

$= 4\underset{\sim}{i} + \sqrt{5}\underset{\sim}{j} - 2\underset{\sim}{k} - 5\underset{\sim}{i}$

$= -\underset{\sim}{i} + \sqrt{5}\underset{\sim}{j} - 2\underset{\sim}{k}$ 1A

3a.iii.

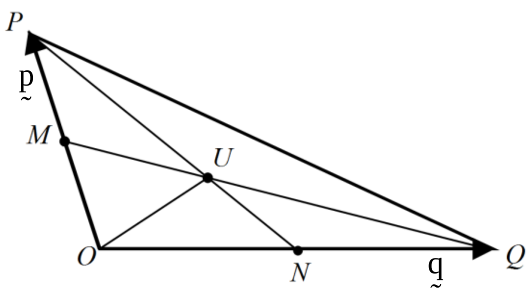
If $\angle QPR$ is a right angle, then $\overrightarrow{QP} \cdot \overrightarrow{PR} = 0$. 1M

$\overrightarrow{QP} = \underset{\sim}{p} - \underset{\sim}{q} = -9\underset{\sim}{i} - \sqrt{5}\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{QP} \cdot \overrightarrow{PR} = (-9 \times -1) + \left(-\sqrt{5} \times \sqrt{5}\right) + (2 \times -2)$ 1A

$= 0$, as required

3b.



3b.i.

$\overrightarrow{PN} = \frac{1}{2}\underset{\sim}{q} - \underset{\sim}{p}$ 1A

3b.ii.

$\overrightarrow{QM} = \frac{1}{2}\underset{\sim}{p} - \underset{\sim}{q}$ 1A

3b.iii.

$\overrightarrow{OU} = \overrightarrow{OP} + \overrightarrow{PU} = \underset{\sim}{p} + \frac{a}{2}\underset{\sim}{q} - a\underset{\sim}{p}$

$= (1-a)\underset{\sim}{p} + \frac{a}{2}\underset{\sim}{q}$ … equation 1

Also,

$\overrightarrow{OU} = \overrightarrow{OQ} + \overrightarrow{QU} = \underset{\sim}{q} + \frac{b}{2}\underset{\sim}{p} - b\underset{\sim}{q}$

$= \frac{b}{2}\underset{\sim}{p} + (1-b)\underset{\sim}{q}$ … equation 2 1M

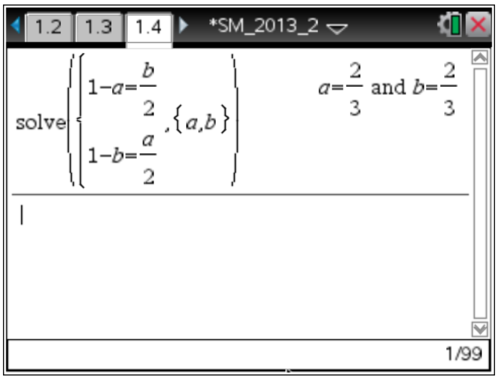
Equating coefficients for equations 1 and 2

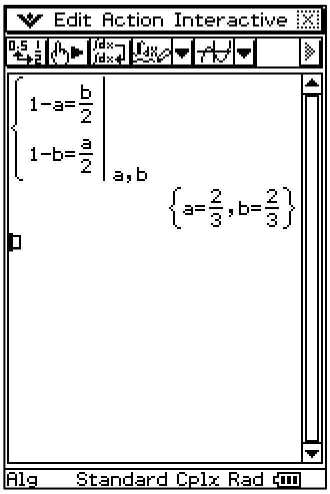
$1 - a = \frac{b}{2}$ … equation 3, and $1 - b = \frac{a}{2}$ … equation 4

Solving equations 3 and 4 simultaneously, 1M

$a = b = \frac{2}{3}$ 1A

Note that this result shows that the centroid *U* of a triangle divides the medians, *PN* and *QM*, into parts in the ratio 2:1.





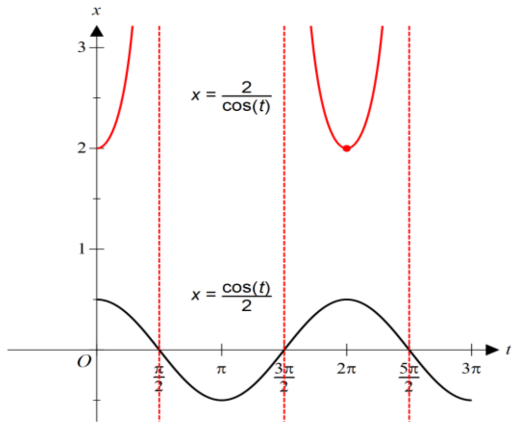
Question 4

4a.

For $\frac{3\pi}{2} < t < \frac{5\pi}{2}$, $0 \le \frac{1}{2}\cos(t) \le \frac{1}{2}$ and hence $2 < \frac{2}{\cos(t)} < \infty$

Since $x = \frac{2}{\cos(t)}$, $2 < x < \infty$, or $x \in [2, \infty)$, as required. 1M

A sketch graph of $x = \frac{2}{\cos(t)}$ *could* be used to show this.



4b.

$x^2 = \left(\frac{2}{\cos(t)}\right)^2 = 4\sec^2(t)$ and $(y-3)^2 = \left(\sqrt{3}\tan(t)\right)^2 = 3\tan^2(t)$ 1M

Substituting into the identity $\tan^2(t)+1=\sec^2(t)$,

$\frac{(y-3)^2}{3}+1=\frac{x^2}{4}$ 1M

$\frac{x^2}{4}-\frac{(y-3)^2}{3}=1$, as required.

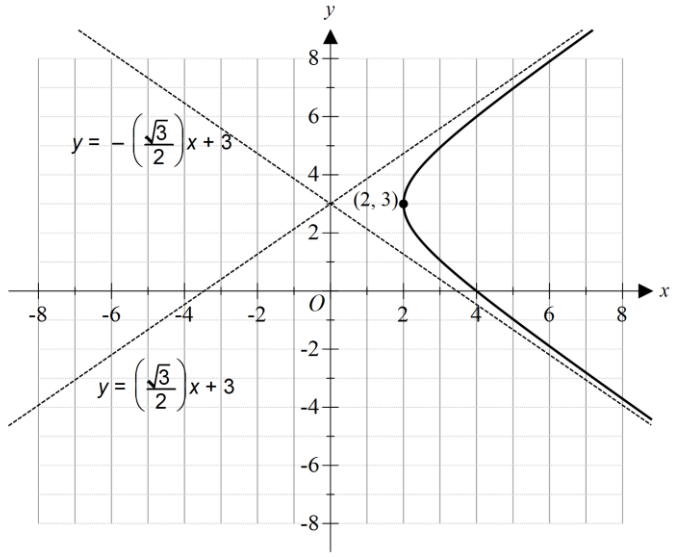
The coordinates of the vertices, $(h+a,k)$, $(h-a,k)$, could be $(2,3)$, $(-2,3)$.

Since $x\in[2,\infty)$, the vertex is $(2,3)$. Hence right-hand branch of the hyperbola. 1M

4c.

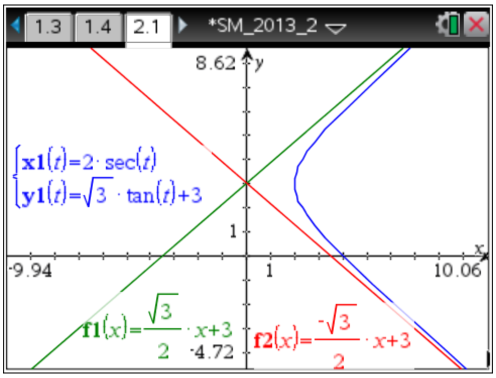
Coordinates of vertex: (2, 3), by considering the translation from $\frac{x^2}{4}-\frac{y^2}{3}=1$ to $\frac{x^2}{4}-\frac{(y-3)^2}{3}=1$

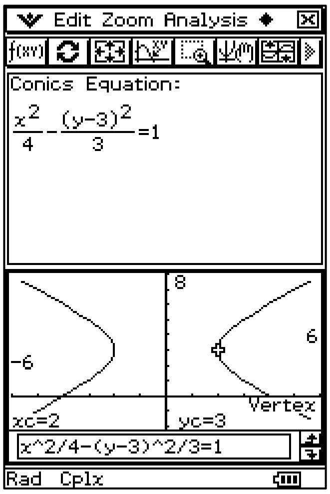
Equation of asymptotes are of the form $y-k=\pm\frac{b}{a}(x-h)$, therefore $y=\pm\frac{\sqrt{3}}{2}x+3$.



Correct shape, with vertex in the correct position: 1A. Asymptotes in correct position and labelled with their correct equations: 1A

A CAS can be used to graph the curve from either the parametric or cartesian equations.



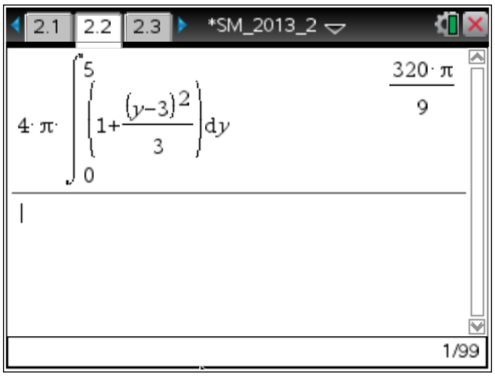


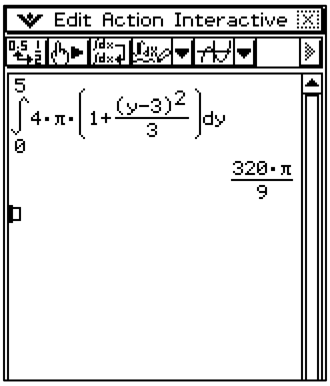
4d.

$$V = \pi \int_0^5 \left(x^2\right) dy$$

$$= 4\pi \int_0^5 \left(1 + \frac{(y-3)^2}{3}\right) dy \qquad \text{1M}$$

$$= \frac{320\pi}{9} \text{ m}^3. \qquad \text{1A}$$





4e.

$\frac{dV}{dt} = 2$ m^3/hour, and $\frac{dV}{dy} = \frac{4\pi}{3}\left(y^2 - 6y + 12\right)$

$$\frac{dy}{dt} = \frac{dy}{dV} \times \frac{dV}{dt}$$

$$= \frac{3}{4\pi\left(y^2 - 6y + 12\right)} \times 2 \qquad \text{1A}$$

Let $y = a$ m when $V = 24\pi$ m^3.

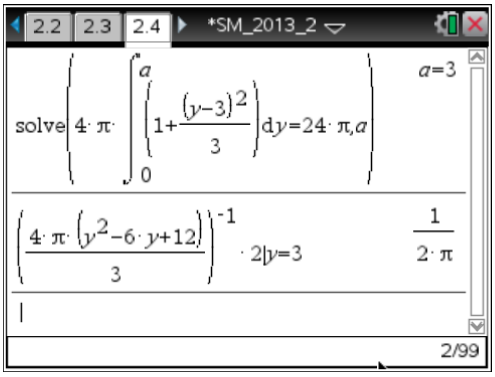
Solve for a,

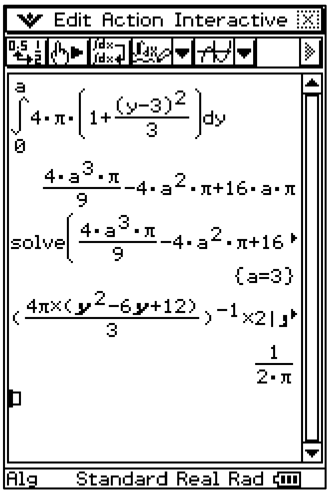
$$24\pi = 4\pi \int_0^a \left(1 + \frac{(y-3)^2}{3}\right) dy \qquad \text{1M}$$

$a = 3$

Substituting $y = 3$,

$$\frac{dy}{dt} = \frac{1}{2\pi} \text{ m/hour} \qquad \text{1A}$$





4f.i.

$$\frac{dN}{dt} = \text{rate of resin inflow} - \text{rate of resin outflow}$$

$$\frac{dN}{dt} = \left(0.02\,\text{tonne/m}^3 \times 2\,\text{m}^3\text{/hour}\right) - \left(\frac{N}{100}\,\text{tonne/m}^3 \times 2\,\text{m}^3\text{/hour}\right)$$ 1M

$$\frac{dN}{dt} = 0.04 - 0.02N \text{ or } \frac{dN}{dt} = \frac{2-N}{50} \quad \text{tonne/hour}$$ 1A

4f.ii.

Solve the differential equation $\frac{dN}{dt} = \frac{2-N}{50}$

$$t = 50\int \frac{dN}{2-N}$$

$$= -50\log_e\left(\frac{|2-N|}{c}\right)$$, where c is an integration constant 1M

When $t = 0$, $N = 10$, therefore

$c = |2-10| = 8$

Therefore,

$$t = 50\log_e\left(\frac{8}{|2-N|}\right)$$

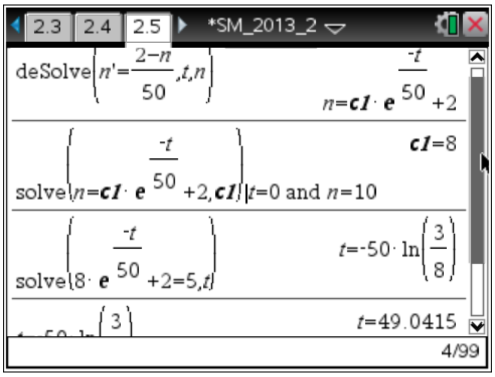
Pumping stops when the concentration of 100 m^3 of solution is 0.05 tonne/m^3.
Therefore, when pumping stops, $N = 5$.

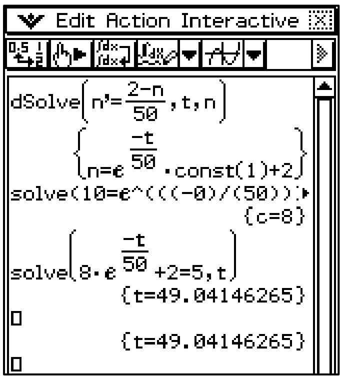
$$t = 50\log_e\left(\frac{8}{|2-5|}\right)$$ 1M

$$t = 50\log_e\left(\frac{8}{3}\right)$$

$t = 49$ hours, correct to the nearest hour. 1A

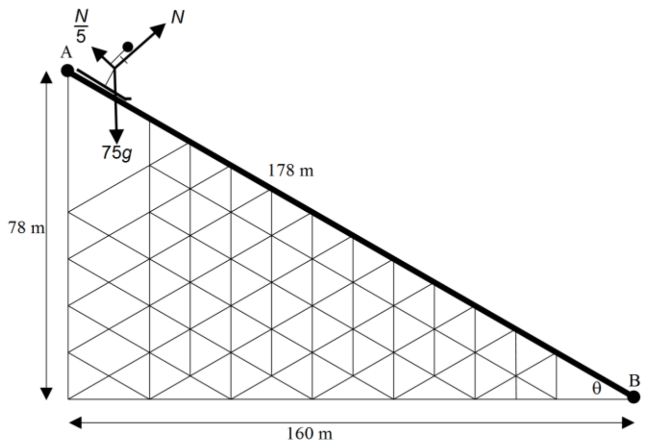
CAS differential equation solver could also be used.





Question 5

5a.



1A

5b.

Let θ be the angle that the ramp makes with the horizontal.

$75a = 75g\sin(\theta) - \frac{1}{5}\times 75g\cos(\theta)$ 1M

$75a = 75g \times \left(\frac{78}{178} - \frac{1}{5}\times\frac{160}{178}\right)$

$a = \frac{23g}{89}$

$a = 2.53\,\text{ms}^{-2}$, correct to two decimal places. 1A

5c.

$v^2 = u^2 + 2as$, where $a = \frac{23g}{89}$, $s = 178$ and $s = 0$. 1M

$v = \sqrt{2\times\frac{23}{89}\times 9.8\times 178} = 30\,\text{ms}^{-1}$, correct to the nearest integer. 1A

5d.

$v = u + at$

$30.02 = 0 + 2.53t$

$t = \frac{30.02}{2.53} = 11.9$ seconds, correct to one decimal place. 1A

Alternatively,

$s=\frac{1}{2}(u+v)t$, therefore $t=\frac{2s}{u+v}$

$t=\frac{2\times 178}{30.0267}=11.9\,\text{seconds}$, correct to one decimal place. 1A

5e.i.

Let u_x and u_y be the horizontal and vertical components, respectively, of the magnitude of Xue's velocity as she leaves *O*.

$u_x=v_0\cos(30^\circ)=\frac{\sqrt{3}v_0}{2}$

$u_y=v_0\sin(30^\circ)=\frac{v_0}{2}$ 1M

Let $\alpha\underset{\sim}{i}+\beta\underset{\sim}{j}$ be the position vector of Xue at time *t* seconds, as she jumps from *O* to *P*.

Using the formula $s=ut+\frac{1}{2}at^2$

$\alpha=u_xt=\frac{\sqrt{3}v_0t}{2}$ and 1A

$\beta=u_yt-\frac{1}{2}\times gt^2=\frac{v_0t}{2}-4.9t^2$ 1A

5e.ii.

At point *P*, $\alpha=80$ and $\beta=-60$, therefore

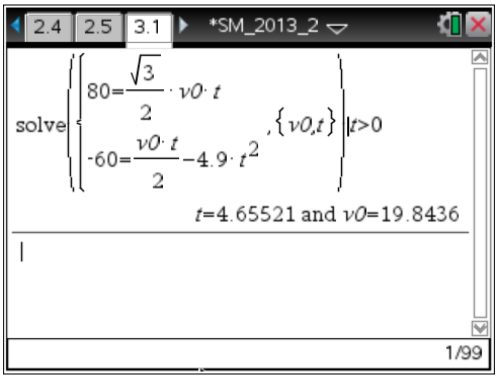
$80=\frac{\sqrt{3}v_0t}{2}$ … equation 1

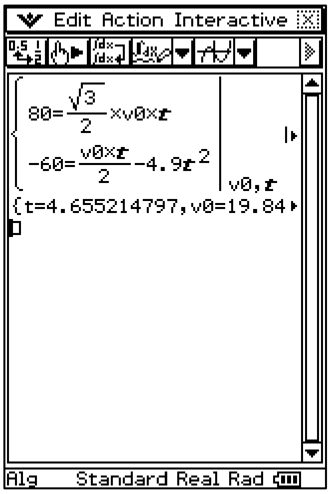
$-60=\frac{v_0t}{2}-4.9t^2$ … equation 2

Solving equations 1 and 2 simultaneously,

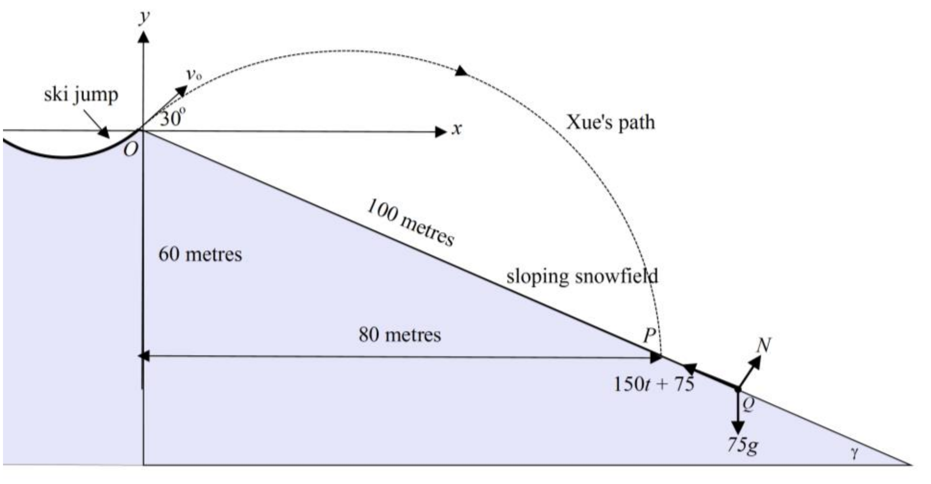
$v_0=19.84\,\text{ms}^{-1}$ 1A

$t=4.66\,\text{s}$ 1A





5f.



From point Q until Xue comes to rest,

$75a = 75g\sin(\gamma) - (150t + 75)$ 1M

$a = \dfrac{3\times 9.8}{5} - 2t - 1 = 4.88 - 2t$

Therefore,

$v = \int (4.88 - 2t)\,dt$ 1M

$v = 4.88t - t^2 + c$

At $t = 0$, $v = 22$. Therefore $c = 22$

When Xue comes to rest,

$t^2 - 4.88t - 22 = 0$

$t = 7.7$ seconds, correct to one decimal place. 1A

END OF SECTION 2 SOLUTIONS