The Mathematical Association of Victoria

SOLUTIONS Trial Exam 2013

SPECIALIST MATHEMATICS

Written Examination 1

Question 1

$80a = T - 80g$

$\Rightarrow T = 80a + 80g = 80(a+g)$. **[M1]**

Require $T = 80(a+g) < 200g$:

$80(a+g) < 200g$

$\Rightarrow a + g < \frac{5g}{2}$

$\Rightarrow a < \frac{5g}{2} - g = \frac{3g}{2}$ m/s^2. **[A1]**

Total 2 marks

Question 2

$(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}) \cdot (a\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k}) = 2a + 6 - 6 = 2a$. .... (1)

$(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}) \cdot (a\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k}) = |2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}||a\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k}|\cos\left(\frac{2\pi}{3}\right)$

$= \sqrt{14}\sqrt{a^2 + 40}\left(-\frac{1}{2}\right)$. .... (2) **[M1]**

Equate equations (1) and (2):

$2a = -\frac{1}{2}\sqrt{14}\sqrt{a^2+40}$ **[M1]**

$\Rightarrow -4a = \sqrt{14}\sqrt{a^2+40}$

$\Rightarrow 16a^2 = 14(a^2 + 40)$

$\Rightarrow 2a^2 = 14 \times 40 \Rightarrow a^2 = 7 \times 40 \Rightarrow a = \pm\sqrt{280} = \pm 2\sqrt{70}$.

Case 1: $a = \sqrt{280} \Rightarrow \cos(\theta) = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ where θ is the angle between $2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $a\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k}$.

Therefore $a = \sqrt{280}$ is rejected.

Case 2: $a = -\sqrt{280} = -2\sqrt{70}$. **[A1]**

Total 3 marks

Justification for rejecting $a = \sqrt{280}$ is not required. Simplification of surd is not required.

Question 3

a.

Use implicit differentiation:

$$\frac{1}{2}\frac{d}{dx}(y-1)^2 - \frac{2}{k}(x+2) = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dy}(y-1)^2 \times \frac{dy}{dx} - \frac{2}{k}(x+2) = 0$$ **[M1]**

$$\Rightarrow (y-1) \times \frac{dy}{dx} - \frac{2}{k}(x+2) = 0$$

$$\Rightarrow (y-1) \times \frac{dy}{dx} = \frac{2}{k}(x+2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(x+2)}{k(y-1)}.$$ **[A1]**

Total 2 marks

b.

The gradient of the line $7y - 3x = -11$ is $\frac{3}{7}$

therefore the gradient of the **normal** to the hyperbola at a point where $x = -1$ is $\frac{3}{7}$

therefore the gradient of the **tangent** to the hyperbola at a point where $x = -1$ is $-\frac{7}{3}$.

Substitute $\frac{dy}{dx} = -\frac{7}{3}$ and $x = -1$ into $\frac{dy}{dx} = \frac{2(x+2)}{k(y-1)}$ (answer from **part a.**):

$$-\frac{7}{3} = \frac{2}{k(y-1)}. \quad \text{..... (1)}$$ **[M1]**

Substitute $x = -1$ into the given normal $7y - 3x = -11$:

$$7y + 3 = -11$$

$$\Rightarrow y = -2.$$ **[A1]**

Therefore $7y - 3x = -11$ is normal to the hyperbola at the point $(-1, \ -2)$.

Substitute $y = -2$ into equation (1):

$$-\frac{7}{3} = \frac{2}{-3k}$$

$$\Rightarrow k = \frac{2}{7}.$$ **[A1]**

Total 3 marks

Question 4

$$\underset{\sim}{a}(t) = \frac{d\underset{\sim}{v}}{dt} = \left(\frac{1}{1+t^2}\right)\underset{\sim}{i} - \frac{1}{(t+1)^2}\underset{\sim}{j} - \left(\frac{1}{t+1}\right)\underset{\sim}{k}$$

$$\Rightarrow \underset{\sim}{v}(t) = \int \left(\frac{1}{1+t^2}\right)\underset{\sim}{i} - \frac{1}{(t+1)^2}\underset{\sim}{j} - \left(\frac{1}{t+1}\right)\underset{\sim}{k}\, dt$$

$$= \tan^{-1}(t)\,\underset{\sim}{i} + \left(\frac{1}{t+1}\right)\underset{\sim}{j} - \log_e(t+1)\,\underset{\sim}{k} + \underset{\sim}{C}$$ (Note that $|t+1| = t+1$ for $t \geq 0$) **[M1]**

where $\underset{\sim}{C}$ is the arbitrary constant vector of integration.

Substitute $\underset{\sim}{v}(0) = \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$:

$$\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} = \tan^{-1}(0)\,\underset{\sim}{i} + \left(\frac{1}{0+1}\right)\underset{\sim}{j} - \log_e(0+1)\,\underset{\sim}{k} + \underset{\sim}{C}$$

$$\Rightarrow \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} = \underset{\sim}{j} + \underset{\sim}{C}$$

$$\Rightarrow \underset{\sim}{C} = \underset{\sim}{i} - 2\,\underset{\sim}{j} + \underset{\sim}{k}.$$

Therefore:

$$\underset{\sim}{v}(t) = \tan^{-1}(t)\,\underset{\sim}{i} + \left(\frac{1}{t+1}\right)\underset{\sim}{j} - \log_e(t+1)\,\underset{\sim}{k} + \underset{\sim}{i} - 2\,\underset{\sim}{j} + \underset{\sim}{k}$$ **[A1]**

$$= \left(1 + \tan^{-1}(t)\right)\underset{\sim}{i} + \left(\frac{1}{t+1} - 2\right)\underset{\sim}{j} + \left(1 - \log_e(t+1)\right)\underset{\sim}{k}.$$

Components do not need to be collected.

Total 2 marks

Question 5

$$\left|\frac{z+1-2i}{z+2-i}\right| = 1$$

$$\Rightarrow \frac{|z+1-2i|}{|z+2-i|} = 1$$ **[M1]**

$$\Rightarrow |z+1-2i| = |z+2-i|.$$

Method 1: Algebraic approach.

Substitute $z = x + iy$:

$$|x+iy+1-2i| = |x+iy+2-i|$$

$$\Rightarrow |(x+1)+i(y-2)| = |(x+2)+i(y-1)|$$

$$\Rightarrow \sqrt{(x+1)^2+(y-2)^2} = \sqrt{(x+2)^2+(y-1)^2}$$

$$\Rightarrow (x+1)^2+(y-2)^2 = (x+2)^2+(y-1)^2$$ **[M1]**

$$\Rightarrow 2x+1-4y+4 = 4x+4-2y+1$$

$$\Rightarrow y = -x.$$ **[A1]**

Total 3 marks

Method 2: Geometric approach.

$$|z+1-2i| = |z+2-i|$$

$$\Rightarrow |z-(-1+2i)| = |z-(-2+i)|.$$

This is recognised as the perpendicular bisector of the line segment joining $z = -1+2i$ and $z = -2+i$.

Coordinates of midpoint: $\left(-\frac{3}{2}, \frac{3}{2}\right)$.

Gradient of line segment: $m_1 = 1$.

Gradient of perpendicular bisector: $\frac{-1}{m_1} = -1$.

Midpoint $\left(-\frac{3}{2}, \frac{3}{2}\right)$ **and gradient** $m = -1$ **[M1]**

Substitute into the standard form $y - y_1 = m(x - x_1)$ of a line:

$$y - \frac{3}{2} = -\left(x + \frac{3}{2}\right)$$

$$\Rightarrow y = -x.$$ **[A1]**

Total 3 marks

Question 6

a.

$$\frac{1}{x^3-2x^2+x}=\frac{1}{x(x-1)^2}\equiv\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}.$$ **[M1]**

$\Rightarrow 1\equiv A(x-1)^2+Bx(x-1)+Cx$

$\Rightarrow A=1,\ \ B=-1,\ \ C=1.$

Answer: $\frac{1}{x}-\frac{1}{x-1}+\frac{1}{(x-1)^2}.$ **[A1]**

Total 2 marks

b.

$\frac{dy}{dx}-\frac{x+1}{\sqrt{1-3x^2}}=0\Rightarrow\frac{dy}{dx}=\frac{x+1}{\sqrt{1-3x^2}}$

$\Rightarrow y=\int\frac{x+1}{\sqrt{1-3x^2}}\,dx$

$=\int\frac{x}{\sqrt{1-3x^2}}\,dx+\int\frac{1}{\sqrt{1-3x^2}}\,dx.$ **[M1]**

- Let $I_1=\int\frac{x}{\sqrt{1-3x^2}}\,dx.$

Substitute $u=1-3x^2\Rightarrow dx=\frac{du}{-6x}$:

$I_1=\int\frac{x}{\sqrt{u}}\left(\frac{du}{-6x}\right)=\frac{-1}{6}\int\frac{1}{\sqrt{u}}\,du$ **[M1]**

$=\frac{-1}{6}\int u^{-1/2}\,du=\frac{-1}{3}u^{1/2}$

$=\frac{-1}{3}\sqrt{1-3x^2}.$

- Let $I_2 = \int \frac{1}{\sqrt{1-3x^2}}\,dx$.

Substitute $u = \sqrt{3}x \Rightarrow dx = \frac{du}{\sqrt{3}}$.

$$I_2 = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{\sqrt{3}}\sin^{-1}(u) = \frac{1}{\sqrt{3}}\sin^{-1}(\sqrt{3}x).$$

Therefore $y = \frac{-1}{3}\sqrt{1-3x^2} + \frac{1}{\sqrt{3}}\sin^{-1}(\sqrt{3}x) + C$. [**A1**]

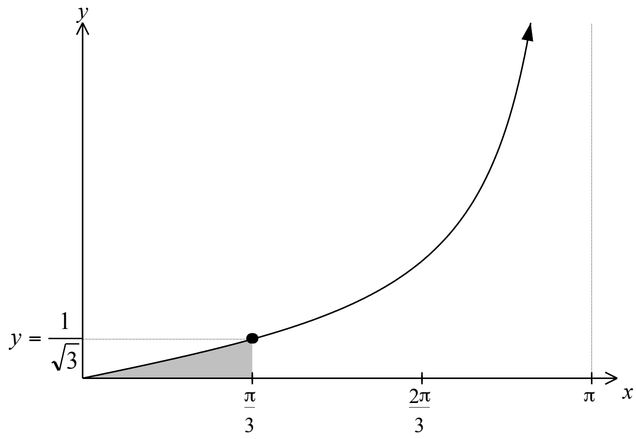
Deduct 1 mark if the arbitrary constant of integration is not included in the final answer.

Total 3 marks

Question 7

$y=\dfrac{1}{\sqrt{3}} \Rightarrow \dfrac{1}{\sqrt{3}}=\tan\left(\dfrac{x}{2}\right) \Rightarrow \dfrac{x}{2}=\dfrac{\pi}{6}$

$\Rightarrow x=\dfrac{\pi}{3}$. **[M1]**



$V=\pi\displaystyle\int_a^b y^2\,dx$

$=\pi\displaystyle\int_0^{\pi/3}\tan^2\left(\frac{x}{2}\right)dx$ **[M1]**

Substitute from the identity $1+\tan^2(A)=\sec^2(A)$:

$=\pi\displaystyle\int_0^{\pi/3}\sec^2\left(\frac{x}{2}\right)-1\,dx$

$=\pi\left[2\tan\left(\dfrac{x}{2}\right)-x\right]_0^{\pi/3}$ **[M1]**

$=\pi\left(2\tan\left(\dfrac{\pi}{6}\right)-\dfrac{\pi}{3}\right)$

$=\dfrac{2\sqrt{3}\pi-\pi^2}{3}$. **[M1]**

Total 4 marks

Question 8

a.

The expression $x = v - v^2$ suggests using the form $a = v\dfrac{dv}{dx}$ for acceleration.

Method 1:

$$x = v - v^2 \Rightarrow \frac{dx}{dv} = 1 - 2v$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{1-2v}$$ **[M1]**

$$\Rightarrow a = v\frac{dv}{dx} = \frac{v}{1-2v}.$$ **[A1]**

Method 2:

Use implicit differentiation.

$$x = v - v^2$$

$$\Rightarrow 1 = \frac{dv}{dx} - 2v\frac{dv}{dx}$$ **[M1]**

$$\Rightarrow 1 = \frac{dv}{dx}(1-2v)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{1-2v}$$

$$\Rightarrow a = v\frac{dv}{dx} = \frac{v}{1-2v}.$$ **[A1]**

Method 3:

$$x = v - v^2$$

$$\Rightarrow \frac{dx}{dt} = \frac{dv}{dt} - 2v\frac{dv}{dt}$$ **[M1]**

$$\Rightarrow v = \frac{dv}{dt}(1-2v)$$

$$\Rightarrow \frac{v}{1-2v} = \frac{dv}{dt}$$

$$\Rightarrow a = \frac{dv}{dt} = \frac{v}{1-2v}.$$ **[A1]**

Total 2 marks

b.

$\frac{dv}{dt} = \frac{v}{1-2v}$

$\Rightarrow \frac{dt}{dv} = \frac{1-2v}{v} = \frac{1}{v} - 2$

$\Rightarrow t = \int \frac{1}{v} - 2\, dv$ **[M1]**

$= \log_e |v| - 2v + C$.

Substitute $v = 1$ and $t = 0$: $C = 2$.

Therefore $t = \log_e |v| - 2v + 2$.

Substitute $v = \frac{1}{2}$:

$t = \log_e\left(\frac{1}{2}\right) + 1 = 1 - \log_e(2)$. **[A1]**

Total 2 marks

Question 9

Substitute the double angle formulae into $1+\cos(2\theta)=\sqrt{3}\sin(2\theta)$:

$1+\cos^2(\theta)-\sin^2(\theta)=2\sqrt{3}\sin(\theta)\cos(\theta)$

$\Rightarrow 1-\sin^2(\theta)+\cos^2(\theta)=2\sqrt{3}\sin(\theta)\cos(\theta)$ **[M1]**

$\Rightarrow 2\cos^2(\theta)=2\sqrt{3}\sin(\theta)\cos(\theta)$

$\Rightarrow \cos^2(\theta)=\sqrt{3}\sin(\theta)\cos(\theta)$

$\Rightarrow \cos^2(\theta)-\sqrt{3}\sin(\theta)\cos(\theta)=0$

$\Rightarrow \cos(\theta)(\cos(\theta)-\sqrt{3}\sin(\theta))=0.$ **[M1]**

Apply the Null Factor Law:

Case 1: $\cos(\theta)=0$

$\Rightarrow \theta=\dfrac{(2n-1)\pi}{2},\quad n\in Z.$

Over $-\pi\le\theta\le\pi$: $\theta=-\dfrac{\pi}{2},\dfrac{\pi}{2}.$ **[A1]**

Case 2: $\cos(\theta)-\sqrt{3}\sin(\theta)=0$

$\Rightarrow \cos(\theta)=\sqrt{3}\sin(\theta)$

$\Rightarrow \tan(\theta)=\dfrac{1}{\sqrt{3}}$

$\Rightarrow \theta=\dfrac{\pi}{6}+n\pi,\quad n\in Z.$

Over $-\pi\le\theta\le\pi$: $\theta=-\dfrac{5\pi}{6},\dfrac{\pi}{6}.$ **[A1]**

Total 4 marks

Question 10

a.

i. $r=\sqrt{1^2+(-\sqrt{3})^2}=\sqrt{4}=2.$

$\tan(\theta)=-\sqrt{3}$ and θ lies in the fourth quadrant: $\theta=-\dfrac{\pi}{3}.$

$1-i\sqrt{3}=2\text{cis}\left(-\dfrac{\pi}{3}\right).$ **[A1]**

ii. **Part i.** suggests working in polar form.

$\sqrt{3}+i = 2\text{cis}\left(\frac{\pi}{6}\right)$. **[M1]**

Therefore:

$\left(\sqrt{3}+i\right)^m = \left(1-i\sqrt{3}\right)^m$

$\Rightarrow \left[2\text{cis}\left(\frac{\pi}{6}\right)\right]^m = \left[2\text{cis}\left(-\frac{\pi}{3}\right)\right]^m$

$\Rightarrow 2^m \text{cis}\left(\frac{m\pi}{6}\right) = 2^m \text{cis}\left(-\frac{m\pi}{3}\right)$. **[M1]**

From here there are two options:

Option 1:

$\text{cis}\left(\frac{m\pi}{6}\right) = \text{cis}\left(-\frac{m\pi}{3}\right)$

$\Rightarrow \frac{m\pi}{6} = -\frac{m\pi}{3} + 2k\pi, \quad k \in Z$

$\Rightarrow m = 4k, \quad k \in Z^+$ (since $m \in Z^+$ is given in the question). **[A1]**

Option 2:

$\frac{\left[2\text{cis}\left(\frac{m\pi}{6}\right)\right]^m}{\left[2\text{cis}\left(-\frac{m\pi}{3}\right)\right]^m} = 1 \qquad \Rightarrow \left(\frac{2\text{cis}\left(\frac{m\pi}{6}\right)}{2\text{cis}\left(-\frac{m\pi}{3}\right)}\right)^m = 1$

$\Rightarrow \left[\text{cis}\left(\frac{\pi}{2}\right)\right]^m = 1 = \text{cis}(2k\pi), \quad k \in Z$

$\Rightarrow \text{cis}\left(\frac{m\pi}{2}\right) = \text{cis}(2k\pi), \quad k \in Z$

$\Rightarrow \frac{m\pi}{2} = 2k\pi$

$\Rightarrow m = 4k, \quad k \in Z^+$ (since $m \in Z^+$ is given in the question). **[A1]**

Total 3 marks

b.

Method 1: Cartesian form approach.

Substitute $z = x + iy \Rightarrow \bar{z} = x - iy$:

$z^2 = i\bar{z}$

$\Rightarrow (x + iy)^2 = i(x - iy)$

$\Rightarrow x^2 + 2xyi - y^2 = ix + y$

$\Rightarrow x^2 - y^2 + 2xyi = y + ix$. **[M1]**

Equate real and imaginary parts:

$x^2 - y^2 = y$. .... (1)

$2xy = x$

$\Rightarrow 2xy - x = 0$

$\Rightarrow x(2y - 1) = 0$. .... (2)

Both equations [M1]

From equation (2) (using the null factor law):

Case 1: $x = 0$.

Substitute $x = 0$ into equation (1): $-y^2 = y$

$\Rightarrow y^2 + y = 0$

$\Rightarrow y(y+1) = 0$

$\Rightarrow y = 0, -1$.

Therefore $z = 0$ or $z = -i$. **[A1]**

Case 2: $2y - 1 = 0 \Rightarrow y = \frac{1}{2}$.

Substitute $y = \frac{1}{2}$ into equation (1): $x^2 - \frac{1}{4} = \frac{1}{2}$

$\Rightarrow x^2 = \frac{3}{4}$

$\Rightarrow x = \pm\frac{\sqrt{3}}{2}$.

Therefore $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ or $z = -\frac{\sqrt{3}}{2} + \frac{i}{2}$. **[A1]**

Solutions: $z = 0$, $z = -i$, $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$, $z = -\frac{\sqrt{3}}{2} + \frac{i}{2}$.

The solutions from the two cases do not need to be consolidated into a final answer.

Total 4 marks

Method 2: Polar form approach.

Substitute $z = r\text{cis}(\theta) \Rightarrow \bar{z} = r\text{cis}(-\theta)$:

$z^2 = i\bar{z}$

$\Rightarrow r^2\text{cis}(2\theta) = ir\text{cis}(-\theta)$

$\Rightarrow r^2\text{cis}(2\theta) = r\text{cis}\left(\frac{\pi}{2}\right)\text{cis}(-\theta)$

$\Rightarrow r^2\text{cis}(2\theta) = r\text{cis}\left(\frac{\pi}{2} - \theta\right)$.

Equate moduli:

$r^2 = r$

$\Rightarrow r^2 - r = 0$

$\Rightarrow r(r-1) = 0$

$\Rightarrow r = 0, 1$. **[M1]**

Equate arguments:

$2\theta = \frac{\pi}{2} - \theta + 2k\pi, \quad k \in Z$

$3\theta = \frac{\pi}{2} + 2k\pi$

$\Rightarrow \theta = \frac{\pi}{6} + \frac{2k\pi}{3}$. **[M1]**

Case 1: $r = 0$.

$z = 0$.

Case 2: $r = 1$.

$z = \text{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right), \quad k \in Z$.

$\underline{k = 0}$: $z = \text{cis}\left(\frac{\pi}{6}\right)$.

$\underline{k = 1}$: $z = \text{cis}\left(\frac{5\pi}{6}\right)$.

Convert into the form $a + ib$: $z = \frac{\sqrt{3}}{2} + \frac{i}{2}, \quad z = -\frac{\sqrt{3}}{2} + \frac{i}{2}$. **[A1]**

$\underline{k = -1}$: $z = \text{cis}\left(-\frac{3\pi}{6}\right) = \text{cis}\left(-\frac{\pi}{2}\right) = -i$.

$z = 0, \quad z = -i$. **[A1]**