Year 2013

VCE

Specialist Mathematics

Trial Examination 1

Solutions

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Question 1

$\int \frac{3x-5}{\sqrt{25-9x^2}}\,dx$ separate out into two integrals

$= 3\int \frac{x}{\sqrt{25-9x^2}}\,dx - 5\int \frac{1}{\sqrt{25-9x^2}}\,dx$ M1

let $u = 25-9x^2$ let $v = 3x$

$\frac{du}{dx} = -18x$ $\frac{dv}{dx} = 3$ M1

$= -\frac{3}{18}\int u^{-\frac{1}{2}}\,du - \frac{5}{3}\int \frac{1}{\sqrt{25-v^2}}\,dv$

$= -\frac{1}{6}\left(2u^{\frac{1}{2}}\right) + \frac{5}{3}\cos^{-1}\left(\frac{v}{5}\right) + c$ A1

$= -\frac{1}{3}\sqrt{25-9x^2} + \frac{5}{3}\cos^{-1}\left(\frac{3x}{5}\right) + c$ for $|x| < \frac{5}{3}$ $b = \frac{5}{3}$ A1

alternatively $= -\frac{1}{3}\sqrt{25-9x^2} - \frac{5}{3}\sin^{-1}\left(\frac{3x}{5}\right) + c$ for $|x| < \frac{5}{3}$ $b = \frac{5}{3}$

Question 2

$y = Ax^2e^{-3x}$ differentiating using the product rule M1

$\frac{dy}{dx} = A\left(2xe^{-3x} - 3x^2e^{-3x}\right) = Ae^{-3x}\left(2x-3x^2\right)$ differentiating using the product rule again

$\frac{d^2y}{dx^2} = -3Ae^{-3x}\left(2x-3x^2\right) + Ae^{-3x}\left(2-6x\right) = Ae^{-3x}\left[-3\left(2x-3x^2\right)+\left(2-6x\right)\right]$

$\frac{d^2y}{dx^2} = Ae^{-3x}\left(9x^2-12x+2\right)$ A1

substituting into $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 8e^{-3x}$ M1

$Ae^{-3x}\left[\left(9x^2-12x+2\right)+6\left(2x-3x^2\right)+9x^2\right] = 8e^{-3x}$

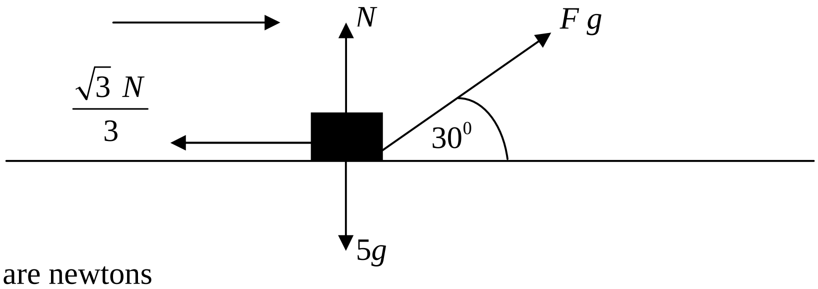
$Ae^{-3x}\left[9x^2-12x+2+12x-18x^2+9x^2\right] = 8e^{-3x}$

$2Ae^{-3x} = 8e^{-3x}$

$2A = 8$

$A = 4$ A1

Question 3



all the forces are newtons

resolving parallel to the plane (1) $Fg\cos\left(30^0\right)-\frac{\sqrt{3}}{3}N=5a$

resolving perpendicular to the plane (2) $Fg\sin\left(30^0\right)+N-5g=0$

$(2)\ \Rightarrow\ N=5g-Fg\sin\left(30^0\right)=5g-\frac{Fg}{2}=\frac{g}{2}(10-F)$ A1

so if $F>10\ \Rightarrow\ N<0$, so that the box leaves the table. A1

$(1)\ \Rightarrow\ Fg\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}\left(\frac{g}{2}(10-F)\right)=5a$

$\frac{g\sqrt{3}}{3}\left(F-\frac{1}{2}(10-F)\right)=5a$ M1

$\frac{g\sqrt{3}}{6}(3F-10)=5a$

$\frac{10}{3}<F<10\quad\Rightarrow\quad a>0$ the box moves with constant acceleration. A1

Question 4

$z=2\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)=2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)=2\sqrt{2}\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=-2+2i$ A1

The conjugate $\overline{z}=-2-2i$ is also a root, by the conjugate root theorem,

the sum of the roots $z+\overline{z}=-4$, the product of the roots $z\,\overline{z}=4-4i^2=8$

since b is real $\left(z^2+4z+8\right)$ is a factor M1

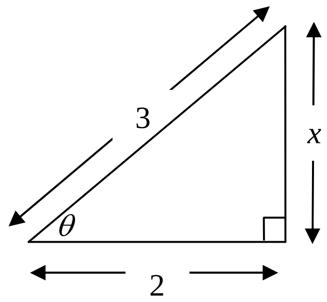
$P(z)=z^3+z^2+bz-24=\left(z^2+4z+8\right)(z-3)$

expanding $z:\ b=8-12=-4$

$b=-4$, and all the roots are $z=-2\pm 2i$, $z=3$ A1

Question 5

$\cos(\theta) = \frac{2}{3}$ draw a triangle



by Pythagoras

$x = \sqrt{3^2 - 2^2} = \sqrt{5}$

$\sin(\theta) = \frac{\sqrt{5}}{3}$

$z = 3\,\text{cis}(\theta)$

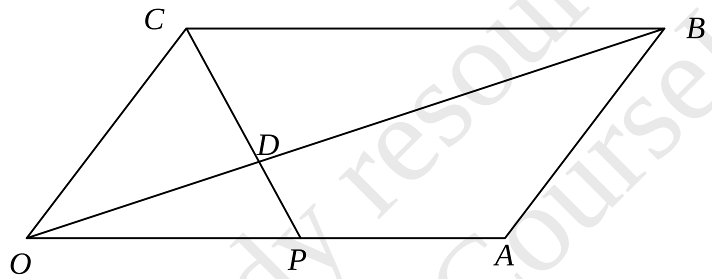
$z^2 = 9\,\text{cis}(2\theta) = 9(\cos(2\theta) + i\sin(2\theta))$ M1

$z^2 = 9(\cos^2(\theta) - \sin^2(\theta)) + 18\sin(\theta)\cos(\theta)i$

$z^2 = 9\left(\frac{4}{9} - \frac{5}{9}\right) + 18 \times \frac{\sqrt{5}}{3} \times \frac{2}{3} i$

$z^2 = -1 + 4\sqrt{5}\,i$ A1

Question 6



$OABC$ is a parallelogram, $\overrightarrow{OA} = \overrightarrow{CB}$ and $\overrightarrow{OC} = \overrightarrow{AB}$

since P is the midpoint of OA, $\overrightarrow{OP} = \overrightarrow{PA} = \frac{1}{2}\overrightarrow{OA}$, given that $\overrightarrow{PD} = \frac{1}{3}\overrightarrow{PC}$ A1

consider $\overrightarrow{OD} = \overrightarrow{OP} + \overrightarrow{PD}$

$= \frac{1}{2}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{PC} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{3}\left(\overrightarrow{PO} + \overrightarrow{OC}\right)$ M1

$= \frac{1}{2}\overrightarrow{OA} + \frac{1}{3}\left(\overrightarrow{OC} - \frac{1}{2}\overrightarrow{OA}\right)$

$= \left(\frac{1}{2} - \frac{1}{6}\right)\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OC}$ since $\overrightarrow{OC} = \overrightarrow{AB}$ M1

$= \frac{1}{3}\left(\overrightarrow{OA} + \overrightarrow{AB}\right)$

So that $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}$ this shows that O, D and B are collinear. A1

Question 7

a. $\cos(3A) = \cos(2A + A)$

$= \cos(2A)\cos(A) - \sin(2A)\sin(A)$

$= \left(2\cos^2(A) - 1\right)\cos(A) - 2\sin^2(A)\cos(A)$ M1

$= \left(2\cos^2(A) - 1\right)\cos(A) - 2\left(1 - \cos^2(A)\right)\cos(A)$

$= 2\cos^3(A) - \cos(A) - 2\cos(A) + 2\cos^3(A)$

$\cos(3A) = 4\cos^3(A) - 3\cos(A)$

b.i. $\underset{\sim}{r}(t) = 2\cos(t)\underset{\sim}{i} + 4\cos(3t)\underset{\sim}{j}$ for $t \ge 0$

$x = 2\cos(t) \qquad y = 4\cos(3t)$

$\cos(t) = \frac{x}{2} \quad , \quad \cos(3t) = \frac{y}{4}$

$\cos(3t) = 4\cos^3(t) - 3\cos(t)$

$\frac{y}{4} = 4\left(\frac{x}{2}\right)^3 - 3\left(\frac{x}{2}\right)$

$y = 2x^3 - 6x$ A1

but since $t \ge 0$, $x = 2\cos(t) \quad \Rightarrow x \in [-2, 2]$

$y = 4\cos(3t) \quad \Rightarrow y \in [-4, 4]$ A1

ii. the particle moves on part of a cubic,

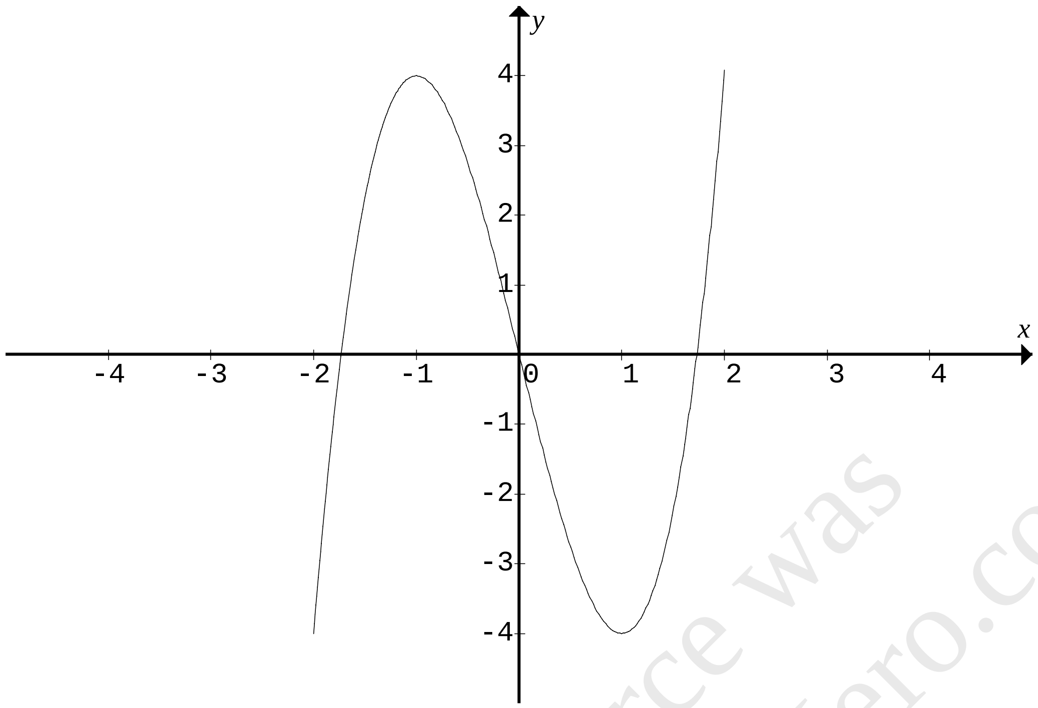
$y = 2x^3 - 6x = 2x\left(x^2 - 3\right) = 2x\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right)$

crosses the x-axis when $y = 0$ at $\left(\sqrt{3}, 0\right)$, $(0, 0)$, $\left(-\sqrt{3}, 0\right)$

$\frac{dy}{dx} = 6x^2 - 6 = 6\left(x^2 - 1\right) = 6(x + 1)(x - 1)$

when $\frac{dy}{dx} = 0$ there are turning points at $(-1, 4)$ and $(1, -4)$ A1

correct graph, on restricted domain, with key features. G1



Question 8

$v = \frac{dx}{dt} = \frac{5}{100 - 9t^2}$

distance travelled in two seconds $D = \int_0^2 \frac{5}{100-9t^2}\,dt$ A1

by partial fractions

$\frac{5}{100-9t^2} = \frac{A}{10-3t} + \frac{B}{10+3t} = \frac{A(10+3t)+B(10-3t)}{(10-3t)(10+3t)} = \frac{10(A+B)+3t(A-B)}{100-9t^2}$ M1

$(1)\ A+B = \frac{1}{2} \quad (2)\ A-B=0 \quad$ solving $A = B = \frac{1}{4}$

$D = \frac{1}{4}\int_0^2 \left(\frac{1}{10-3t} + \frac{1}{10+3t}\right) dt$

$D = \frac{1}{4}\left[-\frac{1}{3}\log_e(10-3t) + \frac{1}{3}\log_e(10+3t)\right]_0^2$ M1

$D = \frac{1}{12}\left[\log_e\left(\frac{10+3t}{10-3t}\right)\right]_0^2 = \frac{1}{12}\left(\log_e\left(\frac{16}{4}\right) - \log_e(1)\right)$

$D = \frac{1}{12}\log_e(4)$ metres A1

Question 9

a. $x^2 + 4x - 4y^2 = 0$

$x^2 + 4x + 4 - 4y^2 = 4$ completing the square

$(x+2)^2 - 4y^2 = 4$

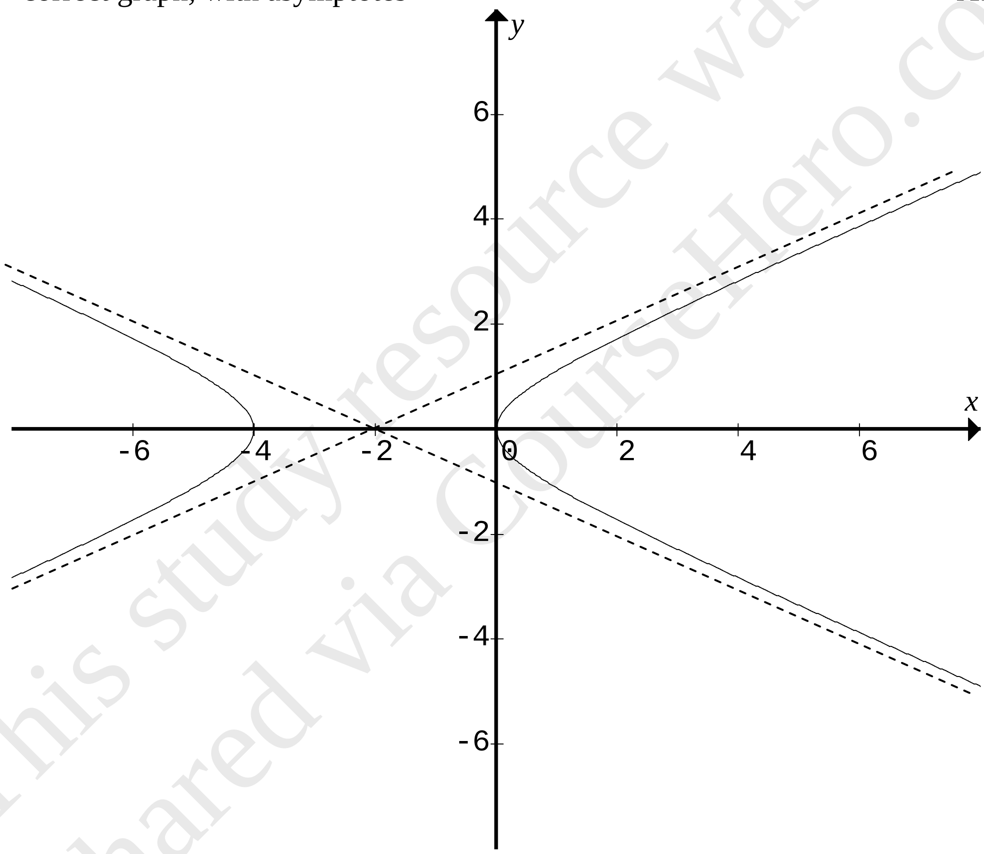
$\frac{(x+2)^2}{4} - y^2 = 1$, hyperbola centre $(-2, 0)$ A1

crosses the *x*-axis, when

$y = 0$, $(x+2)^2 = 4$, $x + 2 = \pm 2$, $x = 0, -4$ $(0,0)$, $(-4,0)$,

asymptotes $y = \pm\frac{1}{2}(x+2)$ $y = \frac{x}{2} + 1$ and $y = -\frac{1}{2}x - 1$ A1

correct graph, with asymptotes A1



b. $x^2 + 4x - 4y^2 = 0$ using implicit differentiation

$2x + 4 - 8y\frac{dy}{dx} = 0$

$8y\frac{dy}{dx} = 2x + 4 = 2(x+2)$

$\frac{dy}{dx} = \frac{x+2}{4y}$ A1

Question 10

i. $y=\tan(2x)\sec(2x)=\dfrac{\sin(2x)}{\cos^2(2x)}$

asymptotes when denominator is zero, when $\cos(2x)=0 \;\Rightarrow\; 2x=(2n+1)\dfrac{\pi}{2}$

$x=(2n+1)\dfrac{\pi}{4}$ where $n\in Z$ A1

ii. area $A=\int_0^{\frac{\pi}{6}}\tan(2x)\sec(2x)\,dx=\int_0^{\frac{\pi}{6}}\dfrac{\sin(2x)}{\cos^2(2x)}\,dx$

let $u=\cos(2x)$ $\dfrac{du}{dx}=-2\sin(2x)$ M1

terminals when $x=\dfrac{\pi}{6}$ $u=\cos\left(\dfrac{\pi}{3}\right)=\dfrac{1}{2}$ and when $x=0$ $u=\cos(0)=1$

$A=-\dfrac{1}{2}\int_1^{\frac{1}{2}}\dfrac{1}{u^2}\,du=-\dfrac{1}{2}\int_1^{\frac{1}{2}}u^{-2}\,du$ M1

$A=\dfrac{1}{2}\left[\dfrac{1}{u}\right]_1^{\frac{1}{2}}=\dfrac{1}{2}(2-1)$

$A=\dfrac{1}{2}$ A1

iii. volume $V=\pi\int_a^b y^2\,dx$

$V=\pi\int_0^{\frac{\pi}{6}}\tan^2(2x)\sec^2(2x)\,dx$

let $u=\tan(2x)$ $\dfrac{du}{dx}=2\sec^2(2x)$ M1

terminals when $x=\dfrac{\pi}{6}$ $u=\tan\left(\dfrac{\pi}{3}\right)=\sqrt{3}$ and when $x=0$ $u=\tan(0)=0$

$V=\dfrac{\pi}{2}\int_0^{\sqrt{3}}u^2\,du$

$=\dfrac{\pi}{2}\left[\dfrac{1}{3}u^3\right]_0^{\sqrt{3}}=\dfrac{\pi}{6}\left(\left(\sqrt{3}\right)^3-0\right)$

$V=\dfrac{\pi\sqrt{3}}{2}$ A1

END OF SUGGESTED SOLUTIONS