SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2

*(*TSSM's *2012 trial exam updated for the current study design)*

<u>SOLUTIONS</u>

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: A

Explanation

The gradient, $m = \pm\frac{2}{3}$, therefore $a = 3$ and $b = 2$ (multiples of 3 and 2 are excluded by given choices). **Options B and E are therefore eliminated.**

The x-intercepts for each asymptote respectively are $\left(\frac{11}{2},\, 0\right)$ and $\left(-\frac{7}{2},\, 0\right)$.

The x-coordinate of the centre of the hyperbola is $\frac{1}{2}\left(\frac{11}{2} - \frac{7}{2}\right) = 1$.

The y-intercepts for each asymptote respectively are $\left(0,\, -\frac{11}{3}\right)$ and $\left(0,\, -\frac{7}{3}\right)$.

The y-coordinate of the centre of the hyperbola is $\frac{1}{2}\left(-\frac{11}{3} - \frac{7}{3}\right) = -3$.

The centre of the hyperbola is therefore $(1,\, -3)$. **Options C and D are therefore eliminated,**

It follows that the equation of the hyperbola could be

$$\frac{(y+3)^2}{4} - \frac{(x-1)^2}{9} = 1 \Rightarrow 9(y+3)^2 - 4(x-1)^2 = 36$$

Question 2

Answer: D

Explanation:

For two vertical asymptotes, require two solutions to the equation $2x^2 + kx + 5 = 0$

The discriminant, $\Delta > 0 \Rightarrow k^2 - 40 > 0 \Rightarrow k \in \left(-\infty, -2\sqrt{10}\right) \cup \left(2\sqrt{10}, \infty\right)$

Question 3

Answer: C

Explanation:

$$\begin{aligned} 3x - 6 &\in [-1, 1] \\ 3x &\in [5, 7] \\ x &\in \left[\frac{5}{3}, \frac{7}{3}\right] \end{aligned}$$

$$\begin{aligned} \cos^{-1}(3x-6) &\in [0, \pi] \\ 2\cos^{-1}(3x-6) &\in [0, 2\pi] \\ 2\cos^{-1}(3x-6) - 1 &\in [-1, 2\pi - 1] \end{aligned}$$

Question 4

Answer: B

Explanation:

$$\begin{aligned} z^4 &= -8i \\ |z|^4 \operatorname{cis} 4\theta &= 8 \operatorname{cis}\left(-\frac{\pi}{2}\right) \\ |z|^4 &= 8 \\ |z| &= 2^{\frac{3}{4}} \\ 4\theta &= -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2} \\ \theta &= -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8} \\ &= -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8} \end{aligned}$$

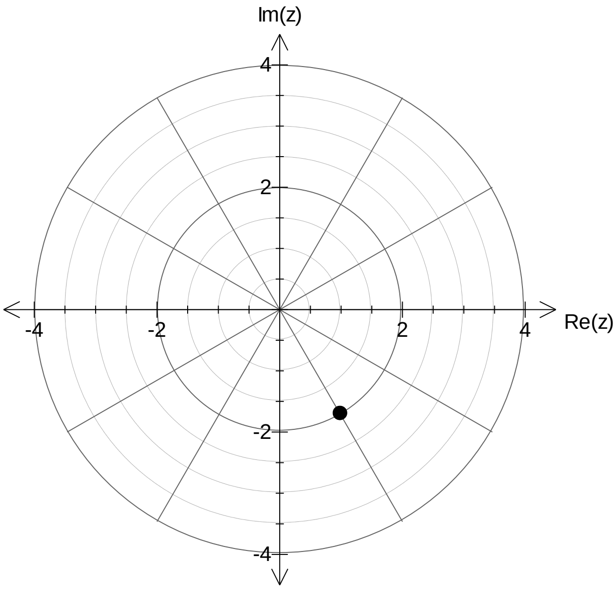
Therefore, the fourth roots are $2^{\frac{3}{4}} \operatorname{cis}\left(-\frac{5\pi}{8}\right)$, $2^{\frac{3}{4}} \operatorname{cis}\left(-\frac{\pi}{8}\right)$, $2^{\frac{3}{4}} \operatorname{cis}\left(\frac{3\pi}{8}\right)$ and $2^{\frac{3}{4}} \operatorname{cis}\left(\frac{7\pi}{8}\right)$

Question 5

Answer: C

Explanation:

$$\bar{z} = -2 - 2\sqrt{3}\,i$$
$$= 4\text{cis}\left(-\frac{2\pi}{3}\right)$$
$$\sqrt{\bar{z}} = \bar{z}^{\frac{1}{2}}$$
$$= 2\text{cis}\left(-\frac{\pi}{3}\right)$$



Question 6

Answer: B

Explanation:

$$\frac{\left(-1-i\sqrt{3}\right)^2}{\left(\sqrt{3}+i\right)^3} = \frac{4\text{cis}\left(-\frac{4\pi}{3}\right)}{8\text{cis}\left(\frac{\pi}{2}\right)}$$
$$= \frac{1}{2}\text{cis}\left(-\frac{4\pi}{3}-\frac{\pi}{2}\right)$$
$$= \frac{1}{2}\text{cis}\left(-\frac{11\pi}{6}\right)$$

Therefore, $\text{Im}\left(\frac{\left(-1-i\sqrt{3}\right)^2}{\left(\sqrt{3}+i\right)^3}\right) = \frac{1}{2}\sin\left(-\frac{11\pi}{6}\right) = \frac{1}{4}$

Question 7

Answer: E

Explanation:

$$\sec A = \frac{5}{2} \Rightarrow \cos A = \frac{2}{5} \Rightarrow \sin A = \frac{\sqrt{21}}{5}$$

$$\operatorname{cosec} B = -3 \Rightarrow \sin B = -\frac{1}{3} \Rightarrow \cos B = \frac{2\sqrt{2}}{3}$$

$$\begin{aligned}\sin(A+B) &= \sin A \cos B + \cos A \sin B \\ &= \frac{\sqrt{21}}{5} \times \frac{2\sqrt{2}}{3} - \frac{2}{5} \times \frac{1}{3} \\ &= \frac{2}{15}\left(\sqrt{42} - 1\right)\end{aligned}$$

Question 8

Answer: C

Explanation:

$$\underset{\sim}{b} - \underset{\sim}{c} = 3\underset{\sim}{j} - 2\underset{\sim}{k}$$

The scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b} - \underset{\sim}{c}$ is equal to

$$\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}\right) \cdot \frac{1}{\sqrt{13}}\left(3\underset{\sim}{j} - 2\underset{\sim}{k}\right) = -\frac{11}{\sqrt{13}}$$

Therefore, the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b} - \underset{\sim}{c}$ is equal to

$$-\frac{11}{\sqrt{13}} \times \frac{1}{\sqrt{13}}\left(3\underset{\sim}{j} - 2\underset{\sim}{k}\right) = -\frac{11}{13}\left(3\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

Question 9

Answer: D

Explanation:

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \frac{5}{9}\overrightarrow{AB} \\ &= \overrightarrow{OA} + \frac{5}{9}\left(\overrightarrow{OB} - \overrightarrow{OA}\right) \\ &= \frac{4}{9}\overrightarrow{OA} + \frac{5}{9}\overrightarrow{OB} \\ &= \frac{1}{9}\left(4\overrightarrow{OA} + 5\overrightarrow{OB}\right) \\ &= -\frac{1}{9}\left(4\overrightarrow{AO} + 5\overrightarrow{BO}\right)\end{aligned}$$

Question 10

Answer: A

Explanation:

$$3x - 5y + 10 = 0 \Rightarrow y = \frac{3}{5}x + 2$$

A line that is perpendicular to the line $3x - 5y + 10 = 0$ has a gradient of $-\frac{5}{3}$.

Therefore, a vector, $\underset{\sim}{v}$ that is perpendicular to the line $3x - 5y + 10 = 0$ can be expressed as $\underset{\sim}{v} = 3\underset{\sim}{i} - 5\underset{\sim}{j}$. It follows that a vector perpendicular to the line $3x - 5y + 10 = 0$ with magnitude 12 units is

$$\begin{aligned}12 \times \hat{\underset{\sim}{v}} &= 12 \times \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|} \\ &= 12 \times \frac{\pm 1}{\sqrt{34}}\left(3\underset{\sim}{i} - 5\underset{\sim}{j}\right) \\ &= \pm\frac{6\sqrt{34}}{17}\left(3\underset{\sim}{i} - 5\underset{\sim}{j}\right)\end{aligned}$$

Question 11

Answer: C

Explanation

$$x = 1 + \cos 2t$$
$$y = \sin t$$
$$x = 1 + 1 - 2\sin^2 t$$
$$= 2 - 2\sin^2 t$$
$$= 2 - 2y^2$$
$$2y^2 = 2 - x$$
$$y^2 = \frac{2-x}{2}$$
$$y = \pm\sqrt{\frac{2-x}{2}}$$

Since $t \in \left[0, \frac{\pi}{2}\right] \Rightarrow y \in [0, 1]$

Therefore, $y = \sqrt{\frac{2-x}{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{4-2x}}{2}$

Question 12

Answer: E

Explanation:

At the point of intersection,

$2 - x^2 = \sqrt{x} \Rightarrow x = 1$

The volume, V is

$$V = \pi\int_0^1 \left(2 - x^2\right)^2 dx - \pi\int_0^1 \left(\sqrt{x}\right)^2 dx$$
$$= \pi\int_0^1 \left(2 - x^2\right)^2 - x\,dx$$

Question 13

Answer: B

Explanation:

$$u = 1 - 2x \Rightarrow x = \frac{1-u}{2}$$

$$\frac{du}{dx} = -2$$

$$-\frac{1}{2}\frac{du}{dx} = 1$$

$$x = -1,\ u = 3$$
$$x = 2,\quad u = -3$$

Therefore,

$$\int_{-1}^{2} \frac{x}{\sqrt{1-2x}}\,dx = \int_{3}^{-3} \frac{1-u}{2} \times \frac{1}{\sqrt{u}} \times -\frac{1}{2}\frac{du}{dx}\,dx$$

$$= \int_{3}^{-3} \frac{u-1}{4\sqrt{u}}\,du$$

$$= \frac{1}{4}\int_{-3}^{3} \frac{1-u}{\sqrt{u}}\,du$$

Question 14

Answer: D

Explanation:

$$\frac{d}{dx}\left(x^2 y - y^2\right) = 0$$

$$2xy + x^2\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(x^2 - 2y\right) = -2xy$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 - 2y}$$

The gradient is undefined when $x^2 - 2y = 0$

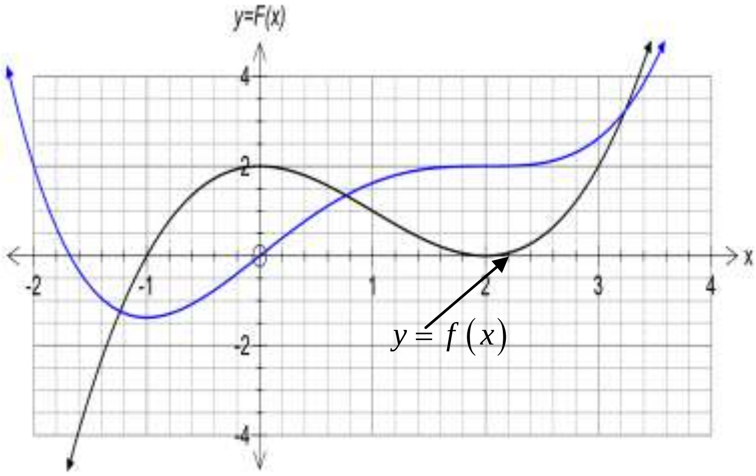
This condition is satisfied by the point $\left(-2\sqrt{2},\ 4\right)$

Question 15

Answer: B

Explanation

The graph at right represents a possible function, $y = f(x)$. It shows that a local minimum occurs at $(-1, 0)$, a non-stationary point of inflection at $(0, 2)$ and a stationary point occurs at $(2, 0)$



Question 16

Answer: A

Explanation

Tabulating, we have

n	x_n	y_n	$\frac{dy}{dx} = 2\cos^{-1}\left(\frac{x_n}{3}\right)$
0	1.00	2.000000	2.461919
1	1.20	2.492384	2.318559
2	1.40	2.956096	2.170556
3	1.60	3.390207	

Question 17

Answer: E

Explanation:

$\text{Inflow} = 0$

$\text{Outflow} = \frac{6x}{300+2t} = \frac{3x}{150+t}$

Therefore, $\frac{dx}{dt} = 0 - \frac{3x}{150+t} = \frac{-3x}{150+t}$ where $x = 1200$ grams when $t = 0$

Question 18

Answer: A

Explanation:

The gradient of the line joining the points $(10,\ 40)$ and $(20,\ -20)$ is -6. Therefore, the equation of the line joining the points $(10,\ 40)$ and $(20,\ -20)$ is

$v-40=-6(t-10)\Rightarrow v=100-6t$

When $v=0\Rightarrow 100-6t=0\Rightarrow t=\dfrac{50}{3}$

Therefore, the total distance travelled by the particle is

$$\frac{1}{2}\times\frac{50}{3}\times 40+\frac{1}{2}\times\left(30-\frac{50}{3}\right)\times 20=\frac{1400}{3}$$

Question 19

Answer: E

Explanation:

If the vertical displacement of the two objects are denoted s_1 and s_2, and if t represents the time in seconds that have elapsed after the projection of the first object, then

$s_1=20t-4.9t^2$ and $s_2=18(t-2)-4.9(t-2)^2$

At the point of collision, $s_1=s_2\Rightarrow 20t-4.9t^2=18(t-2)-4.9(t-2)^2$

Solving, we have $t=3.16$ seconds with the collision point $20\times3.16-4.9\times3.16^2=14.3$ metres above ground level.

Therefore, the objects will collide approximately 3.16 seconds after the projection of the first object.

Question 20

Answer: B

Explanation:

$$a(v) = v^2 + 4$$

$$v\frac{dv}{dx} = v^2 + 4$$

$$\frac{dv}{dx} = \frac{v^2+4}{v}$$

$$\frac{dx}{dv} = \frac{v}{v^2+4}$$

$$x = \int \frac{v}{v^2+4}\,dv$$

$$= \frac{1}{2}\log_e\left|v^2+4\right| + c$$

Since $v(1) = 2$, then

$$1 = \frac{1}{2}\log_e 8 + c \Rightarrow c = 1 - \frac{1}{2}\log_e 8$$

Therefore,

$$x = \frac{1}{2}\log_e \frac{v^2+4}{8}$$

$$\log_e \frac{v^2+4}{8} = 2(x-1)$$

$$v^2 = 4(2e^{2(x-1)} - 1) \Rightarrow v = \pm 2\sqrt{2e^{2(x-1)} - 1}$$

$$v = 2\sqrt{2e^{2(x-1)} - 1}$$

Question 21

Answer: B

Explanation:

The resultant force,

$$\underset{\sim}{F} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3$$

$$= 2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k} - \underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k} + \underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$$

$$= 2\underset{\sim}{i} + 5\underset{\sim}{j} - \underset{\sim}{k}$$

$$\left|\underset{\sim}{F}\right| = \sqrt{2^2 + 5^2 + (-1)^2}$$

$$= \sqrt{30}$$

Therefore, the mass of the particle, $m = \frac{\sqrt{30}}{2.5} = \frac{2\sqrt{30}}{5}$ kg

Question 22

Answer: D

Explanation:

$$25 \times 9.8 \times \sin(46°) = 25 \times a$$
$$a = 7.05$$

SECTION 2

Question 1

a.

$$\begin{aligned}\text{cis}\left(\theta-\frac{3\pi}{2}\right)&=\cos\left(\theta-\frac{3\pi}{2}\right)+i\sin\left(\theta-\frac{3\pi}{2}\right)\\&=\cos\left(\frac{3\pi}{2}-\theta\right)-i\sin\left(\frac{3\pi}{2}-\theta\right)\\&=-\sin\theta-\left(-i\cos\theta\right)\\&=-\sin\theta+i\cos\theta\end{aligned}$$

[M2]

b.

$$\begin{aligned}\text{LHS}&=\frac{1}{\text{cis}\left(\theta-\frac{3\pi}{2}\right)}\\&=\left(\text{cis}\left(\theta-\frac{3\pi}{2}\right)\right)^{-1}\\&=\text{cis}\left(\frac{3\pi}{2}-\theta\right)\\&=-\sin\theta-i\cos\theta\\&=-\left(\sin\theta+i\cos\theta\right)\\&=-\left(\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right)\\&=-\text{cis}\left(\frac{\pi}{2}-\theta\right)\\&=\text{RHS}\end{aligned}$$

[M3]

c.

Using technology,

$$\left(-\sin\theta+i\cos\theta\right)^4=1-8\sin^2\theta\cos^2\theta+4\sin\theta\cos\theta\left(2\cos^2\theta-1\right)i$$ [A1]

d.

$$\begin{aligned}\text{LHS} &= \left(\text{cis}\left(\theta - \frac{3\pi}{2}\right)\right)^4 \\ &= \text{cis}\left(4\left(\theta - \frac{3\pi}{2}\right)\right) \\ &= \text{cis}(4\theta - 6\pi) \\ &= \text{cis}(4\theta) \\ &= \text{RHS}\end{aligned}$$

[M2]

e.

From parts **c.** and **d.**

$$\text{cis}(4\theta) = (-\sin\theta + i\cos\theta)^4$$

$$\cos(4\theta) + i\sin(4\theta) = 1 - 8\sin^2\theta\cos^2\theta + 4\sin\theta\cos\theta\left(2\cos^2\theta - 1\right)i$$

Equating coefficients,

$$\begin{aligned}\cos(4\theta) &= 1 - 8\sin^2\theta\cos^2\theta \\ &= 1 - 8\sin^2\theta\left(1 - \sin^2\theta\right) \\ &= 1 - 8\sin^2\theta + 8\sin^4\theta\end{aligned}$$

[M1] [A1]

f.

$$\cot\theta = -\frac{5}{2} \Rightarrow \sin\theta = \frac{2}{\sqrt{29}}$$

$$\cos(4\theta) = 1 - 8\left(\frac{2}{\sqrt{29}}\right)^2 + 8\left(\frac{2}{\sqrt{29}}\right)^4 = \frac{41}{841}$$

[M1] [A1]

Question 2

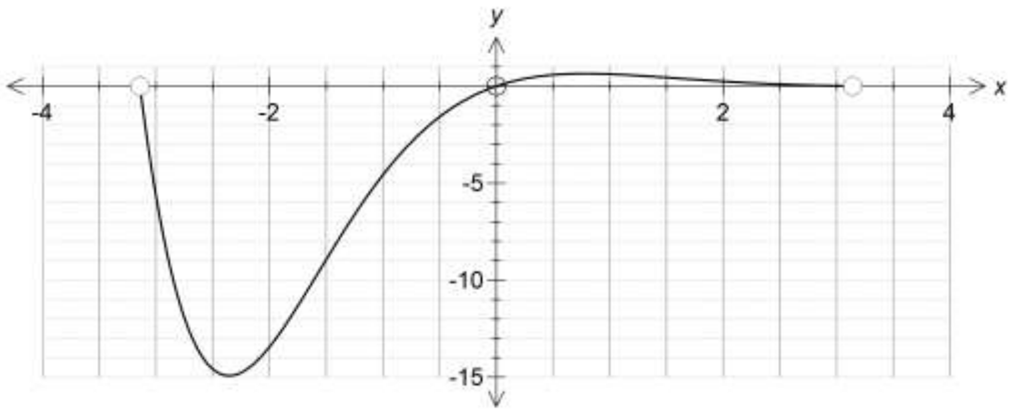
a.

$$\begin{aligned} f(x) &= \frac{1}{e^{x}\sec x} \\ &= e^{-x}\cos x \\ f'(x) &= -e^{-x}\cos x - e^{-x}\sin x \\ &= -e^{-x}(\cos x + \sin x) \\ f''(x) &= -e^{-x}(-\sin x + \cos x) + e^{-x}(\cos x + \sin x) \\ &= 2e^{-x}\sin x \end{aligned}$$

At a point of inflection, $f''(x) = 0$

Therefore, $2e^{-x}\sin x = 0 \Rightarrow x = -\pi,\ 0,\ \pi$ [M2]

And sketching $y = f''(x)$, we have



Since $y = f''(x)$ changes sign for $x \in (-\infty, -\pi) \cup (-\pi, \pi) \cup (\pi, \infty)$ then $x = -\pi,\ 0,\ \pi$ gives rise to three non-stationary points of inflection.

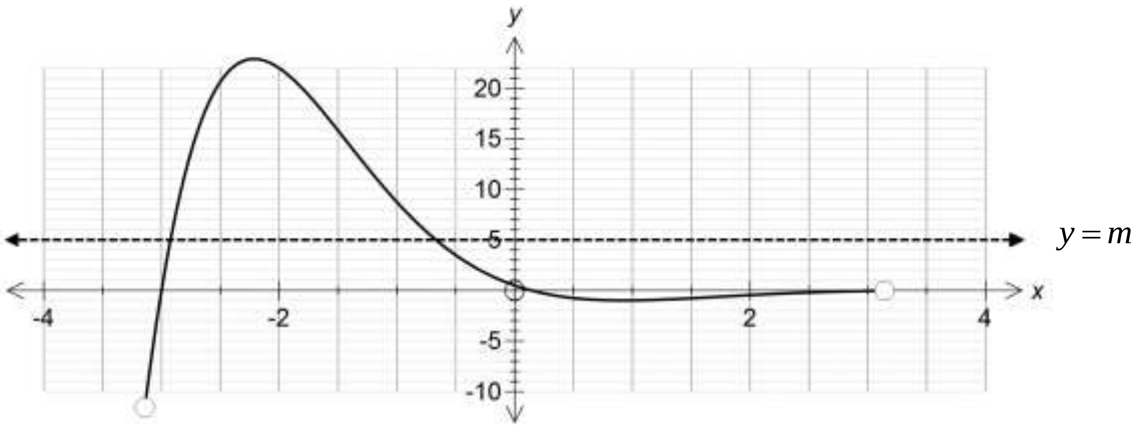
The points of inflection are $(-\pi, -e^{\pi})$, $(0, 1)$ and $\left(\pi, -\frac{1}{e^{\pi}}\right)$ [A1]

b.

i.

$$\left.\begin{aligned} 2f''(x)+\frac{1}{2}f'(x)+m&=0 \\ 2\left(2e^{-x}\sin x\right)+\frac{1}{2}\left(-e^{-x}\left(\cos x+\sin x\right)\right)+m&=0 \\ \frac{1}{2}e^{-x}\cos x-\frac{7}{2}e^{-x}\sin x&=m \end{aligned}\right\} \text{[M1]}$$

Sketching $y=\frac{1}{2}e^{-x}\cos x-\frac{7}{2}e^{-x}\sin x$, we have



The local minimum is $(0.9273, -0.9891)$. Therefore, there will be three solutions to the differential equation for $m \in (-0.989, 0)$ [M1]

ii

Solving $\frac{7}{2}e^{-x}\sin x-\frac{1}{2}e^{-x}\cos x-\frac{1}{4}=0 \Rightarrow x=-3.00,\ 0.23,\ 2.40$ [A1]

c.

The transformed equation is $g(x)=\dfrac{-2}{e^{x-\frac{\pi}{8}}\sec\left(x-\frac{\pi}{8}\right)}$ [A1]

d.

From the calculator,

the points of intersection are $(-1.28, 1.04)$ and $(1.87, -0.04)$ [A2]

e.

From the calculator,

The area, $A = \int_{-1.28}^{1.87} \frac{1}{e^x \sec x} - \frac{2}{e^{x-\frac{\pi}{8}} \sec\left(x - \frac{\pi}{8}\right)} dx = 7.3$ square units [A1]

Question 3

a.

Let the point where the tangent touches the curve be $\left(a,\ 2a^2+1\right)$. It follows that the gradient of the line joining the points $\left(a,\ 2a^2+1\right)$ and $\left(1,\ -3\right)$ is

$$\frac{2a^2+1-(-3)}{a-1} = \frac{2a^2+4}{a-1}$$

Also, $\frac{dy}{dx} = 4x$. At $\left(a,\ 2a^2+1\right)$, $\frac{dy}{dx} = 4a$

Therefore,

$$\frac{2a^2+4}{a-1} = 4a$$

$$a^2 - 2a - 2 = 0$$

$$a = \frac{2+\sqrt{12}}{2}$$

$$= \sqrt{3}+1$$

And $y = 2\left(\sqrt{3}+1\right)^2 + 1 = 4\sqrt{3}+9$

[M2]

Therefore, the point where the tangent touches the curve is $\left(\sqrt{3}+1,\ 4\sqrt{3}+9\right)$ [A1]

b.

At $\left(1,\ -3\right)$,

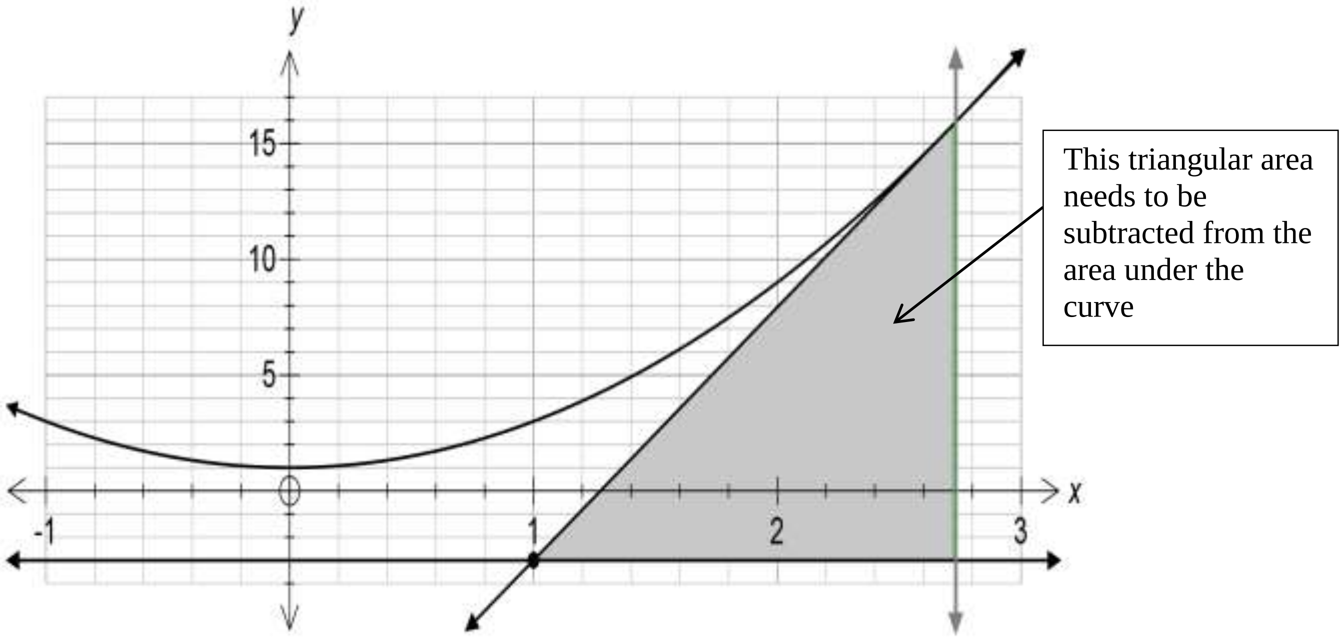
$$y - (-3) = 4\left(\sqrt{3}+1\right)(x-1)$$

$$y = 4\left(\sqrt{3}+1\right)x - 4\sqrt{3} - 7$$

[A1]

c.

Sketching, we have



The required area, A is calculated as

$$A = \int_0^{\sqrt{3}+1} 2x^2 + 1 - (-3)\,dx - \frac{1}{2}\times\left(4\sqrt{3}+9-(-3)\right)\times\left(\sqrt{3}+1-1\right)$$ [M2]

$$= \int_0^{\sqrt{3}+1} 2x^2 + 4\,dx - \frac{1}{2}\times\left(4\sqrt{3}+12\right)\times\sqrt{3}$$

$$= 8\sqrt{3} + \frac{32}{3} - \left(6\sqrt{3}+6\right)$$ [M1]

$$= \frac{2\left(3\sqrt{3}+7\right)}{3} \text{ square units}$$

d.

Since $y=4\left(\sqrt{3}+1\right)x-4\sqrt{3}-7 \Rightarrow x=\dfrac{y+4\sqrt{3}+7}{4\left(\sqrt{3}+1\right)}$

And since $y=2x^2+1 \Rightarrow x^2=\dfrac{y-1}{2}$ [M1]

Therefore, the volume, V is calculated as

$$V=\pi\int_{-3}^{4\sqrt{3}+9}\left(\frac{y+4\sqrt{3}+7}{4\left(\sqrt{3}+1\right)}\right)^2 dy-\pi\int_{1}^{4\sqrt{3}+9}\frac{y-1}{2}dy \quad \text{[M2]}$$

$$=4\pi\left(\sqrt{3}+2\right) \text{ cubic units}$$

Question 4

a.

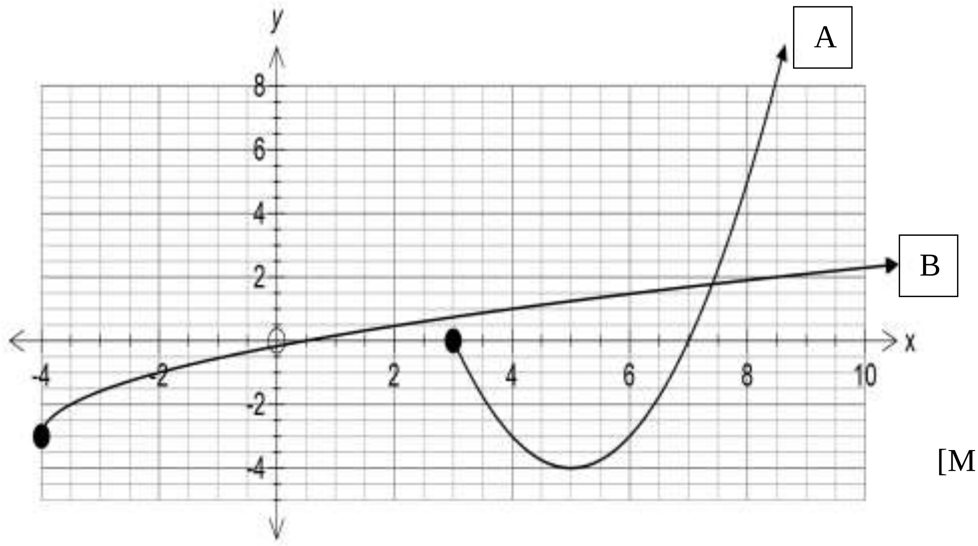
For car B, the parametric equations are $x=\dfrac{t^2-8}{2}$ and $y=t-3$

Therefore,

$$x=\frac{(y+3)^2-8}{2}$$
$$(y+3)^2=2x+8$$
$$y+3=\pm\sqrt{2x+8}$$
$$y=-3+\sqrt{2x+8} \text{ since } y\in[-3,\infty)$$
[M1]

where $2x+8\geq 0 \Rightarrow x\in[-4,\infty)$ [M1]

b.



[M1]

c.

Equating the x-coordinates for both cars, $\dfrac{t^2-8}{2}=t+3 \Rightarrow t=4.873$.

Therefore, $y_A=4.873^2-4\times4.873=4.254$ and $y_B=4.873-3=1.873$

Since the y-coordinates for both cars do not coincide at the same point in time, the cars do not collide

[M2]

d.

Note that since the cars do not collide, we cannot use vector methods to find the required angle. Instead, we need to use coordinate geometry.

At the point of intersection,

$x^2 - 10x + 21 = -3 + \sqrt{2x+8} \Rightarrow x = 7.403$

For car A, $\frac{dy}{dx} = 2x - 10$ and for car B, $\frac{dy}{dx} = \frac{1}{\sqrt{2x+8}}$ [M1]

At $x = 7.403$,

For car A, the gradient, $m_A = 4.81$ and for car B, $m_B = 0.209$

Therefore, at the point of intersection, the acute angle between the two paths, [M1]

θ is given by $\theta = \tan^{-1}(4.81) - \tan^{-1}(0.21) = 66°$ [A1]

(Alternatively, you could find the times for both position vectors corresponding to x = 7.403, and then find the angle between their respective velocity vectors at those particular times.)

e.

The distance between the cars, d at any time, $t \geq 0$ is given by

$$\sqrt{\left(\frac{t^2 - 2t - 14}{2}\right)^2 + (t^2 - 5t + 3)^2}$$ [M1]

f.

$$\sqrt{\left(\frac{t^2 - 2t - 14}{2}\right)^2 + (t^2 - 5t + 3)^2} = 5 \Rightarrow t = 3.47 \text{ and } 5.31 \text{ seconds}$$ [A1]

g.

$\underset{\sim}{v}_A = \underset{\sim}{i} + (2t - 4)\underset{\sim}{j}$

$\underset{\sim}{v}_B = t\underset{\sim}{i} + \underset{\sim}{j}$ [A1]

h.

Require $\underset{\sim}{v}_A \cdot \underset{\sim}{v}_B = 0 \Rightarrow \left(\underset{\sim}{i} + (2t-4)\underset{\sim}{j}\right)\cdot\left(t\underset{\sim}{i} + \underset{\sim}{j}\right) = 0$ [M1]

i.e. $t + 2t - 4 = 0 \Rightarrow 3t - 4 = 0 \Rightarrow t = \frac{4}{3}$ seconds [A1]

Question 5

a.

$v^2 = 2g \times 90 \Rightarrow v = \sqrt{2g \times 90} = 42 \text{ ms}^{-1}$ [A1]

b.

The equation of motion is

$$5v^2 - 85g = -85a$$ [M1]

Therefore,

$$a = \frac{85g - 5v^2}{85}$$ [M1]

$$= \frac{17g - v^2}{17}$$

c.

$$v\frac{dv}{dx}=\frac{17g-v^2}{17}$$
$$\frac{dv}{dx}=\frac{17g-v^2}{17v}$$
$$\frac{dx}{dv}=\frac{17v}{17g-v^2}$$
$$x=\int\frac{17v}{17g-v^2}\,dv$$

Let $u=17g-v^2$

$$\frac{du}{dv}=-2v\Rightarrow-\frac{1}{2}\frac{du}{dv}=v$$

Therefore, [M2]

$$x=\int\frac{17v}{17g-v^2}\,dv$$
$$=-\frac{17}{2}\int\frac{1}{u}\,du$$
$$=-\frac{17}{2}\log_e|u|+c$$
$$=-\frac{17}{2}\log_e\left|17g-v^2\right|+c$$

[M1]

When $x=0,\ v=42$

Therefore,

$$0=-\frac{17}{2}\log_e|-1597.4|+c$$
$$c=\frac{17}{2}\log_e|-1597.4|$$

It follows that

$$x=\frac{17}{2}\log_e\left|\frac{1597.4}{v^2-17g}\right|$$
$$e^{-\frac{2x}{17}}=\frac{v^2-17g}{1597.4}$$
$$v^2=1597.4e^{-\frac{2x}{17}}+17g$$

[M1]

$$v=\sqrt{1597.4e^{-\frac{2x}{17}}+166.6}$$

d.

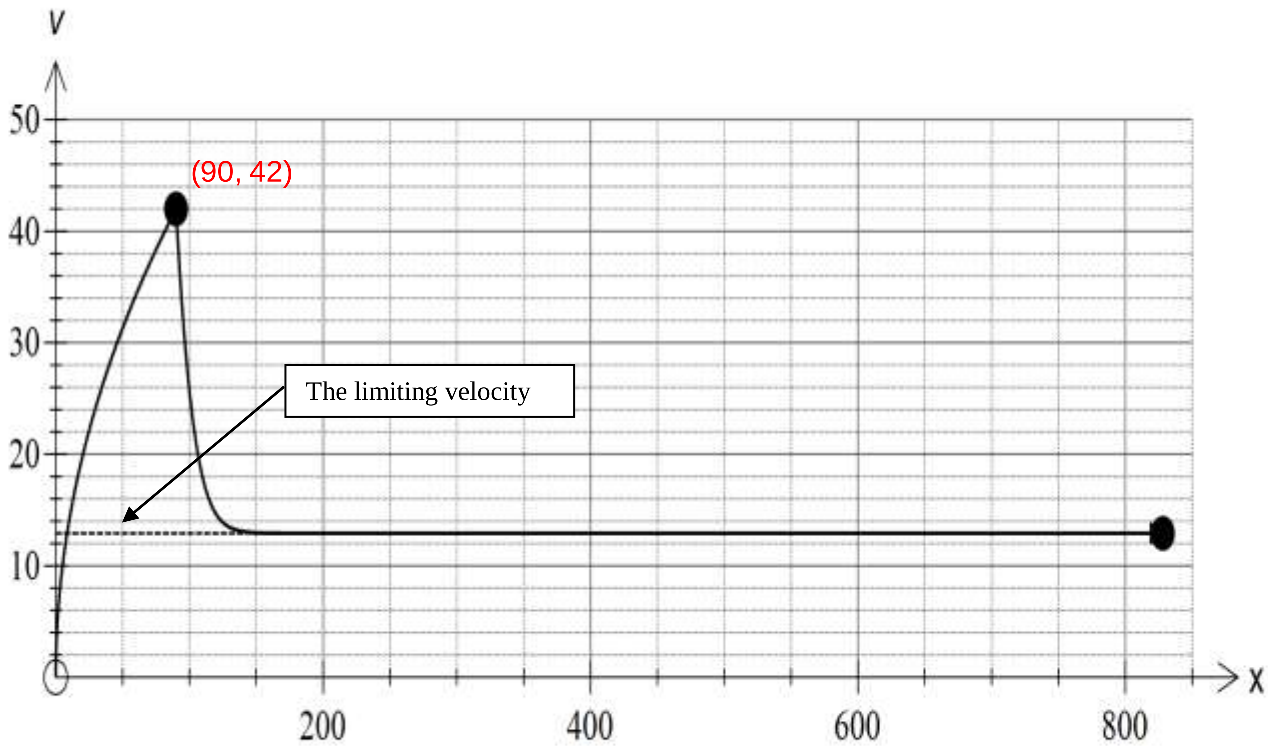
As $x \to \infty,\ e^{-\frac{2x}{17}} \to 0 \Rightarrow v \to \sqrt{166.6} = 12.91\text{ms}^{-1}$ [A1]

e.

Solving $\sqrt{1597.4e^{-\frac{2x}{17}} + 166.6} = 25.8 \Rightarrow x = 9.89$ [M1]

The distance that Alex is from ground level at this point is $738 - 9.89 \approx 728$ m [A1]

f.



[M3]