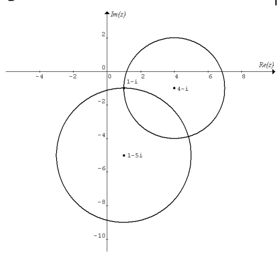
2012 Specialist Maths Trial Exam 2 Solutions

Section 1

1	2	3	4	5	6	7	8	9	10	11
A	C	E	D	E	E	A	E	C	A	B

12	13	14	15	16	17	18	19	20	21	22
D	B	B	B	B	A	C	C	B	D	E

Q1 A possible complex number is $z = 1 - i$, $|z| = \sqrt{2}$ A



Q2 $z = -i\,cis(2)$ is the clockwise rotation of $cis(2)$ about the origin *O* by $\frac{\pi}{2}$. .: $Arg(z) = 2 - \frac{\pi}{2}$ C

Q3 Let $z = x + yi$.

$az^2 + b|z|^2 - c = 0$, $a(x^2 - y^2 + 2xyi) + b(x^2 + y^2) - c = 0$

.: $[(a+b)x^2 + (b-a)y^2 - c] + 2axyi = 0$

.: $(a+b)x^2 + (b-a)y^2 - c = 0$ and $xy = 0$

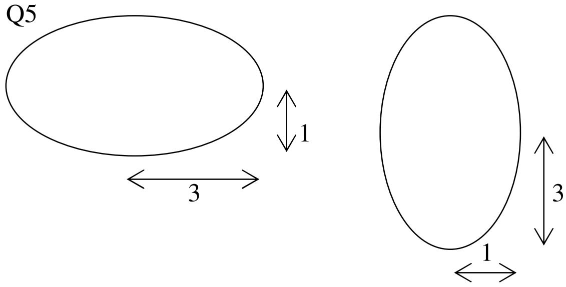
.: either $x = 0$ and $y = \pm\sqrt{\frac{c}{b-a}}$ if $b > a$

or $y = 0$ and $x = \pm\sqrt{\frac{c}{b+a}}$ E

Q4 $0 < Arg[(1+i)z] < \frac{\pi}{2}$, $0 < Arg(1+i) + Arg(z) < \frac{\pi}{2}$,

$0 < \frac{\pi}{4} + Arg(z) < \frac{\pi}{2}$, $-\frac{\pi}{4} < Arg(z) < \frac{\pi}{4}$ D

Q5



$\frac{(x-h)^2}{9} + (y-k)^2 = 1$ or $(x-h)^2 + \frac{(y-k)^2}{9} = 1$ E

Q6 $y = \frac{1}{a + bx + ax^2}$ has a vertical and a horizontal asymptotes when $a + bx + ax^2$ is a perfect square, i.e. $\Delta = b^2 - 4a^2 = 0$

.: $\frac{a}{b} = \pm\frac{1}{2}$ E

Q7 Asymptotes: $y = 2 \pm \left(\frac{1}{2}x + 1\right)$, $y - 2 = \pm\frac{1}{2}(x+2)$

.: the two asymptotes intersect at $(-2,2)$ which is the centre of the hyperbola, also $\frac{b}{a} = \frac{1}{2}$.

A possible equation of the hyperbola is $\frac{(x+2)^2}{2^2} - \frac{(y-2)^2}{1^2} = 1$,

i.e. $(x+2)^2 - 4(y-2)^2 - 4 = 0$ A

Q8 $\sec(a) + \operatorname{cosec}(b) = 0$, $\sec(a) = -\operatorname{cosec}(b)$

.: $\cos(a) = -\sin(b)$, $\cos(a) = -\cos\left(\frac{\pi}{2} - b\right)$

Since $\pi < a < \frac{3\pi}{2}$ and $0 < b < \frac{\pi}{2}$, .: $a = \pi + \left(\frac{\pi}{2} - b\right) = \frac{3\pi}{2} - b$

.: $a + b = \frac{3\pi}{2}$ E

Q9 The domain of the inverse of f is the range of f.

As $x \to 1^+$, $y \to \sec\pi + a\pi = a\pi - 1$

At $x = \frac{5}{3}$, $y = \sec\frac{4\pi}{3} + a\pi = a\pi - 2$

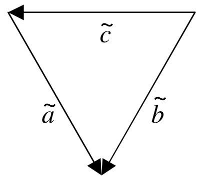
.: the domain of the inverse of f is $[a\pi - 2, a\pi - 1)$. C

Q10 $\tan^{-1}(x - a + b) - \tan^{-1}(x - a) = \pi$

$\tan^{-1}(x - a + b)$ and $\tan^{-1}(x - a)$ have the same range

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, .: $\tan^{-1}(x - a + b) - \tan^{-1}(x - a) \neq \pi$ for $x \in R$ A

Q11 Since $\tilde{a} - \tilde{b} + \tilde{c} = \tilde{0}$, .: unit vectors $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ form an equilateral triangle as shown below:



$\tilde{a}.\tilde{b} - \tilde{b}.\tilde{c} + \tilde{c}.\tilde{a}$

$= \cos 60° - \cos 60° + \cos 120°$

$= -\frac{1}{2}$

B

Q12 Let $\tilde{j} + c\tilde{k} = m(\tilde{k} + a\tilde{i}) + n(\tilde{i} + b\tilde{j})$ where *m* and *n* are non-zero real numbers.

$\tilde{j} + c\tilde{k} = (ma + n)\tilde{i} + nb\tilde{j} + m\tilde{k}$

.: $ma + n = 0$, $nb = 1$ and $m = c$

.: $ca + n = 0$, $n = -ca$

.: $abc = -1$ D

Q13 $\tilde{p} = \pm(\tilde{i} + \tilde{j} + \tilde{k})$ are vectors which make equal angle θ with each of the orthogonal unit vectors $\tilde{i}$, $\tilde{j}$ and $\tilde{k}$.

$\cos\theta = \pm\frac{1}{\sqrt{1+1+1}}$, .: $\theta \approx 55°$ and $125°$ B

Q14 Given $f(0)=1.2$, by CAS $f(0.2)=\int_0^{0.2}\frac{1}{\cos^4 x}dx+1.2\approx 1.4$,

$f(0.4)=\int_0^{0.4}\frac{1}{\cos^4 x}dx+1.2\approx 1.6$, $f(0.6)=\int_0^{0.6}\frac{1}{\cos^4 x}dx+1.2\approx 2$

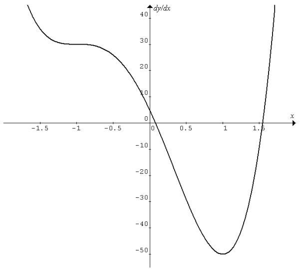
$f(0.8)=\int_0^{0.8}\frac{1}{\cos^4 x}dx+1.2\approx 2.6$, $f(1)=\int_0^{1}\frac{1}{\cos^4 x}dx+1.2\approx 4$

$\textit{Average value}\approx\frac{1.2+1.4+1.6+2+2.6+4}{6}\approx 2$ B

Q15 At a point of inflection the gradient of the curve is a local maximum/minimum.

$y=3x^5+5x^4-10x^3-30x^2+5x-10$

$\frac{dy}{dx}=15x^4+20x^3-30x^2-60x+5$



$\frac{dy}{dx}$ has a local minimum at $x=1$. B

Q16 When $y=2$ and $0\le x\le\frac{3}{2}$, $\tan^2 x=4$, $x\approx 1.10715$

By CAS, $\int_0^{1.10715}\tan^2 x dx\approx 0.8929$

$\textit{Area of the required region}\approx 4\times 1.10715-0.8929\approx 3.5$ B

Q17 The quadratic equation has exactly one solution, .: $\Delta=0$

$k^2-4(1)(1)=0$, .: $k=2$

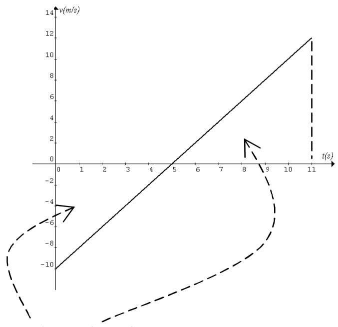
$\left(\frac{dy}{dx}+1\right)^2=0$, .: $\frac{dy}{dx}=-1$, $y=-x+c$

$(1,1)$, .: $c=2$, .: $x+y=2$ A

Q18 $a={}^{+}2$, $u={}^{-}10$, $s={}^{+}16-{}^{+}5={}^{+}11$, v?

$v^2=u^2+2as$, $v={}^{+}12$

$v=2t-10$



$\textit{Total distance}=25+36=61\text{ m}$ C

Q19 Given $v=2-e^x$, when $a=v\frac{dv}{dx}=(2-e^x)(-e^x)=0$,

$e^x=2$, $x=\log_e 2$ and $v=0$. .: the particle is moving in the same direction from $x=0$ to $x=\log_e 2$.

$\textit{Distance}=\log_e 2\approx 0.7\text{ m}$ C

Q20 $a={}^{+}9.8\sin 30°={}^{+}4.9$, $s={}^{+}1.3$, $u=0$, v?

$v^2=u^2+2as$, $v\approx{}^{+}3.57$

$\textit{Momentum}=mv\approx 1.3\times{}^{+}3.57\approx{}^{+}4.6\text{ kg m s}^{-1}$ B

Q21 The particle exerts a downward force on the inclined plane equal in magnitude to the weight of the particle, .: the upward reaction force of the inclined plane on the particle is also equal in magnitude to the weight of the particle according to Newton's third law, $mg=2.04\times 9.8\approx 20\text{ N}$ upward D

Q22 *Acceleration of the crate* = *acceleration of the chain*

Consider the forces on the chain:

1000 N ← 10 kg → 1010 N

$a=\frac{R}{m}=\frac{1010-1000}{10}=1\text{ m s}^{-2}$ E

Section 2

Q1a $x^2+4y^2=4\xrightarrow{(1)}x^2+4\left(y-\frac{1}{2}\right)^2=4$

$\xrightarrow{(2)}x^2+4\left(\frac{1}{2}y-\frac{1}{2}\right)^2=4$, and in simplified form

$x^2+(y-1)^2=4$.

Q1b Let $y=0$, $x^2=3$, $x=\pm\sqrt{3}$

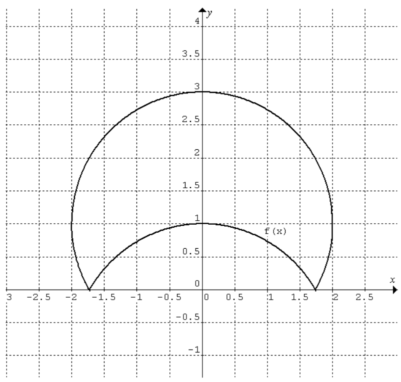
The x-intercepts are $(-\sqrt{3},0)$ and $(\sqrt{3},0)$.

Q1ci $x^2+(y-1)^2=4$, $(y-1)^2=4-x^2$, $y-1=\pm\sqrt{4-x^2}$

.: $g(x)=1-\sqrt{4-x^2}$,

.: $f(x)=|g(x)|=\sqrt{4-x^2}-1$ for $-\sqrt{3}<x<\sqrt{x}$

Q1cii



Q1d Volume V_1 of solid by formed by rotating $f(x)$:

$y=\sqrt{4-x^2}-1$, $\sqrt{4-x^2}=y+1$, $x^2=4-(y+1)^2$

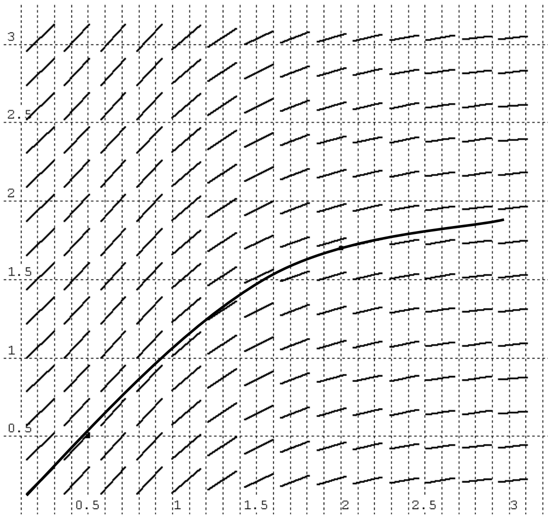
$$V_1=\int_0^1 \pi x^2\,dy=\pi\int_0^1\left(4-(y+1)^2\right)dy=\frac{5\pi}{3}$$

Volume of the large sphere $V_2=\frac{4}{3}\pi\left(2^3\right)=\frac{32\pi}{3}$

.: Volume of revolution of the required region

$$=\frac{32\pi}{3}-2\times\frac{5\pi}{3}=\frac{22\pi}{3}$$

Q2a



When $x=2$, $y\approx 1.7$

Q2b

$x=0.5$	$y=0.5$	$\frac{dy}{dx}\approx 1.1765$
$x=1.0$	$y=0.5+0.5\times 1.1765\approx 1.0882$	$\frac{dy}{dx}=1$
$x=1.5$	$y=1.0882+0.5\times 1\approx 1.5882$	$\frac{dy}{dx}\approx 0.5361$
$x=2.0$	$y\approx 1.5882+0.5\times 0.5361\approx 1.9$	

Q2c $y=\int_{0.5}^{2}\frac{1+x^2}{1+x^4}dx+0.5\approx 1.7$ by CAS

Q2di $\frac{\frac{1+x^2}{x^2}}{\frac{1+x^4}{x^2}}=\frac{\frac{1}{x^2}+1}{\frac{1}{x^2}+x^2}$

Q2dii $x^2+\frac{1}{x^2}=\left(x+\frac{b}{x}\right)^2+c=x^2+2b+\frac{b^2}{x^2}+c$,

.: $b^2=1$, $2b+c=0$ and $c>0$

.: $b=-1$ and $c=2$

.: $x^2+\frac{1}{x^2}=\left(x-\frac{1}{x}\right)^2+2$

Q2diii $y=\int\frac{1+x^2}{1+x^4}dx=\int\frac{\frac{1}{x^2}+1}{\left(x-\frac{1}{x}\right)^2+2}dx$

Let $u=x-\frac{1}{x}$

$\frac{du}{dx}=1+\frac{1}{x^2}$

$=\int\frac{\frac{du}{dx}}{u^2+2}dx=\int\frac{1}{2+u^2}du$

$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+c$

$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right)\right)+c$

Q2div When $x=0.5$, $y=0.5$

$0.5=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\left(0.5-\frac{1}{0.5}\right)\right)+c$, $c=1.0762$

.: $y=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right)\right)+1.0762$

When $x=2$, $y\approx 1.7$

Q3a $\overrightarrow{P_1P_2}=\tilde{b}-\tilde{a}$, $\left|\overrightarrow{P_1P_2}\right|^2=\overrightarrow{P_1P_2}\bullet\overrightarrow{P_1P_2}=\left(\tilde{b}-\tilde{a}\right)\bullet\left(\tilde{b}-\tilde{a}\right)$

$=\tilde{a}\bullet\tilde{a}+\tilde{b}\bullet\tilde{b}-2\tilde{a}\bullet\tilde{b}=\left|\tilde{a}\right|^2+\left|\tilde{b}\right|^2-2\left|\tilde{a}\right|\left|\tilde{b}\right|\cos\alpha$

Q3b Since the length of arc P_1P_2 equals the length of arc P_2P_3,

.: $\angle P_2OP_3=\angle P_1OP_2=\alpha$

$\left|\overrightarrow{P_2P_3}\right|^2=\left|\tilde{b}\right|^2+\left|\tilde{c}\right|^2-2\left|\tilde{b}\right|\left|\tilde{c}\right|\cos\alpha$

Q3c Since the length of arc P_1P_2 equals the length of arc P_2P_3,

.: $\left|\overrightarrow{P_1P_2}\right|^2=\left|\overrightarrow{P_2P_3}\right|^2$

.: $\left|\tilde{a}\right|^2+\left|\tilde{b}\right|^2-2\left|\tilde{a}\right|\left|\tilde{b}\right|\cos\alpha=\left|\tilde{b}\right|^2+\left|\tilde{c}\right|^2-2\left|\tilde{b}\right|\left|\tilde{c}\right|\cos\alpha$

.: $\left|\tilde{a}\right|^2-\left|\tilde{c}\right|^2=2\left|\tilde{b}\right|\left(\left|\tilde{a}\right|-\left|\tilde{c}\right|\right)\cos\alpha$

.: $\cos\alpha=\frac{\left|\tilde{a}\right|^2-\left|\tilde{c}\right|^2}{2\left|\tilde{b}\right|\left(\left|\tilde{a}\right|-\left|\tilde{c}\right|\right)}=\frac{\left|\tilde{a}\right|+\left|\tilde{c}\right|}{2\left|\tilde{b}\right|}$

Q3d Since OP_1 is a diameter, .: $\angle OP_2P_1$ and $\angle OP_3P_1$ are right angles.

.: $\tilde{b}$ is the vector resolute of $\tilde{a}$ in the direction of $\overrightarrow{OP_2}$ and $\tilde{c}$ is the vector resolute of $\tilde{a}$ in the direction of $\overrightarrow{OP_3}$.

Q3e Given the length of $\overline{OP_1}=\left|\tilde{a}\right|=1$, .: $\left|\tilde{b}\right|=1\cos\alpha=\cos\alpha$ and $\left|\tilde{c}\right|=1\cos 2\alpha=\cos 2\alpha$.

Q3f $\cos\alpha=\frac{\left|\tilde{a}\right|+\left|\tilde{c}\right|}{2\left|\tilde{b}\right|}$, $\cos\alpha=\frac{1+\cos 2\alpha}{2\cos\alpha}$, .: $1+\cos 2\alpha=2\cos^2\alpha$

Q3g $\cos\alpha=\frac{\left|\tilde{b}\right|+\left|\tilde{d}\right|}{2\left|\tilde{c}\right|}$ and $\left|\tilde{d}\right|=\cos 3\alpha$ by similar methods,

.: $\cos\alpha=\frac{\cos\alpha+\cos 3\alpha}{2\cos 2\alpha}$,

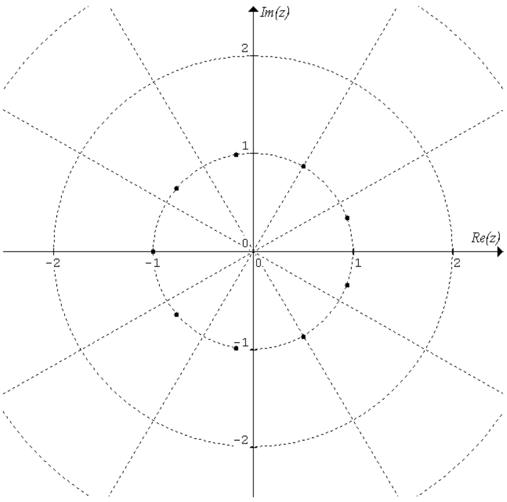
.: $\cos\alpha+\cos 3\alpha=2\cos\alpha\cos 2\alpha$

Q4a $\dfrac{z^9+1}{z^6-z^3+1}=\dfrac{z^9+0z^6+0z^3+1}{z^6-z^3+1}$

$$\begin{array}{r} z^3+1 \\ z^6-z^3+1 \enclose{longdiv}{z^9+0z^6+0z^3+1} \\ -\left(z^9-z^6+z^3\right) \\ \hline z^6-z^3+1 \\ -\left(z^6-z^3+1\right) \\ \hline 0 \end{array}$$

.: $\dfrac{z^9+1}{z^6-z^3+1}=z^3+1$

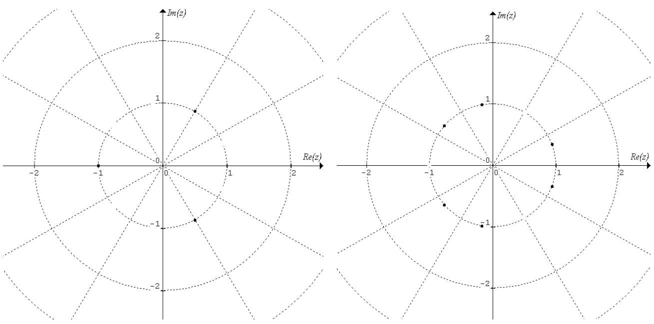
Q4b $z^9+1=0$ has a real root $z=-1$, together with the other 8 roots they space out equally on the unit circle.



Q4c $\dfrac{2\pi}{9}$

Q4di $z^9+1=\left(z^3+1\right)\left(z^6-z^3+1\right)=\left(z^3+1\right)P_2(z)=0$

The roots of $z^3+1=0$ (left) and the roots of $P_2(z)=0$ (right) are shown below:



The roots of $P_2(z)=0$ are:

$z=cis\left(\pm\dfrac{\pi}{9}\right)$, $z=cis\left(\pm\dfrac{5\pi}{9}\right)$ and $z=cis\left(\pm\dfrac{7\pi}{9}\right)$.

Q4dii The roots of $P_2(iz)=0$ are:

$iz=cis\left(\pm\dfrac{\pi}{9}\right)$, $z=\dfrac{cis\left(\pm\dfrac{\pi}{9}\right)}{i}=\dfrac{cis\left(\pm\dfrac{\pi}{9}\right)}{cis\left(\dfrac{\pi}{2}\right)}=cis\left(-\dfrac{7\pi}{18}\right)$ or $cis\left(-\dfrac{11\pi}{18}\right)$

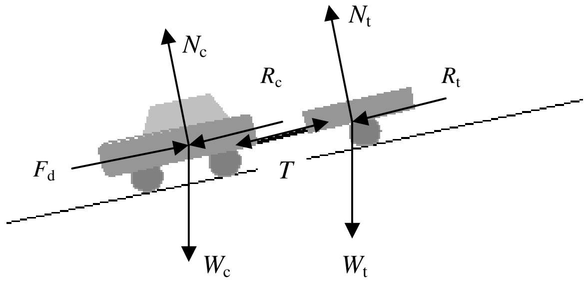
$iz=cis\left(\pm\dfrac{5\pi}{9}\right)$, $z=cis\left(\dfrac{\pi}{18}\right)$ or $cis\left(\dfrac{17\pi}{18}\right)$

$iz=cis\left(\pm\dfrac{7\pi}{9}\right)$, $z=cis\left(\dfrac{5\pi}{18}\right)$ or $cis\left(\dfrac{13\pi}{18}\right)$

Q4e $cis\left(-\dfrac{7\pi}{18}\right)cis\left(-\dfrac{11\pi}{18}\right)cis\left(\dfrac{\pi}{18}\right)cis\left(\dfrac{17\pi}{18}\right)cis\left(\dfrac{5\pi}{18}\right)cis\left(\dfrac{13\pi}{18}\right)$

$=cis\left(-\dfrac{7\pi}{18}-\dfrac{11\pi}{18}+\dfrac{\pi}{18}+\dfrac{17\pi}{18}+\dfrac{5\pi}{18}+\dfrac{13\pi}{18}\right)=cis(\pi)=-1$

Q5a



Q5b *Mass of the car* $=\dfrac{15000}{9.8}$ kg; *mass of the trailer* $=\dfrac{5000}{9.8}$ kg

At $t=0$, $a=0.8\cos^2 0=0.8$

Apply Newton's second law to the car and the trailer: $R=ma$,

$F_d-300-200-15000\sin 20°-5000\sin 20°=\dfrac{15000+5000}{9.8}\times 0.8$

.: $F_d\approx 9.0\times 10^3$ N

Q5c Apply Newton's second law to the trailer:

$T-200-5000\sin 20°=\dfrac{5000}{9.8}\times 0.8$, $T\approx 2.3\times 10^3$ N

Q5d The car and the trailer accelerate for the first 2 s:

$\Delta v=\displaystyle\int_0^2 0.8\cos^2\left(\frac{\pi t}{4}\right)dt=0.8$; there is no change in the velocity in the next second.

The car and the trailer start from rest,

.: the exact speed of the trailer after the first 3 s is 0.8 m s^{-1}.

Q5e *Distance* $=0.8(8-3)=4$ m.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors