INSIGHT

YEAR 12 ***Trial Exam Paper***

2012
SPECIALIST MATHEMATICS
Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions.
- mark allocations
- tips

SECTION 1

Question 1

The ellipse with equation $9x^2 + 16y^2 - 18x + 64y - 71 = 0$ has a

A. centre (1, –2) with major axis of length 4 units.

B. centre (–1, 2) with major axis of length 8 units.

C. centre (1, –2) with minor axis of length 8 units.

D. **centre (1, –2) with major axis of length 8 units**.

E. centre (–1, 2) with major axis of length 16 units.

Answer is D.

Worked solution

$$9x^2 + 16y^2 - 18x + 64y - 71 = 0$$
$$9(x^2 - 2x) + 16(y^2 + 4y) = 71$$
$$9(x^2 - 2x + 1) + 16(y^2 + 4y + 4) = 71 + 9 + 64$$
$$9(x-1)^2 + 16(y+2)^2 = 144$$
$$\frac{(x-1)^2}{16} + \frac{(y+2)^2}{9} = 1$$

$\therefore$ Centre is (1, –2) and **a** = 4, so length of major axis is 2**a** or 8 units.

Tip

- Make sure that 9 and 64 are added to RHS when completing the squares.

Question 2

The curve $y=\left(x-2+\frac{1}{\sqrt{x-2}}\right)\left(x-2-\frac{1}{\sqrt{x-2}}\right)+4$ has a

A. curved asymptote $y = x^2 - 4x - 8$ and vertical asymptote $x = 2$.

B. curved asymptote $y = (x - 2)^2$ and vertical asymptote $x = 2$.

C. **curved asymptote $y = x^2 - 4x + 8$ and vertical asymptote $x = 2$.**

D. curved asymptote $y = x^2 - 4x + 8$ and vertical asymptote $x = -2$.

E. horizontal asymptote $y = 4$ and vertical asymptote $x = 2$.

Answer is C.

Worked solution

First, expand the expression using difference of two squares:

$$y=\left[(x-2)^2-\frac{1}{x-2}\right]+4$$

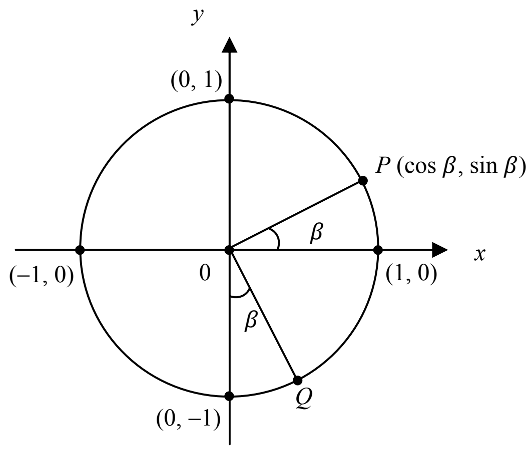
$$y=x^2-4x+8-\frac{1}{x-2}$$

As $x \to +\infty$, $y \to x^2 - 4x + 8$ from above
and as $x \to -\infty$, $y \to x^2 - 4x + 8$ from below, so $y = x^2 - 4x + 8$ is a curved asymptote.

As $x \to 2^+, y \to +\infty$ and
as $x \to 2^-, y \to -\infty$, so $x = 2$ is a vertical asymptote.

Question 3

In the diagram below, P has coordinates $(\cos\beta, \sin\beta)$.
Hence, Q would be the point with coordinates



A. **$(\sin\beta, -\cos\beta)$**

B. $(\cos\beta, \sin\beta)$

C. $(\sin\beta, \cos\beta)$

D. $(\cos\beta, -\sin\beta)$

E. $(-\sin\beta, -\cos\beta)$

Answer is A.

Worked solution

$\angle POQ = 90°$

Using the complementary properties of trigonometric identities, the x coordinate of P becomes the y coordinate of Q (but negative), and the y coordinate of P becomes the x coordinate of Q.

$\therefore$ The coordinates of Q are $(\sin\beta, -\cos\beta)$.

Question 4

If $\sin(x) = -\frac{3}{4}$ and $\pi < x < \frac{3\pi}{2}$ then $\sec(x)$ equals

A. $-\frac{4}{3}$

B. $\boldsymbol{-\frac{4\sqrt{7}}{7}}$

C. $-\frac{\sqrt{7}}{4}$

D. $\frac{4}{\sqrt{7}}$

E. $\frac{3\sqrt{7}}{7}$

Answer is B.

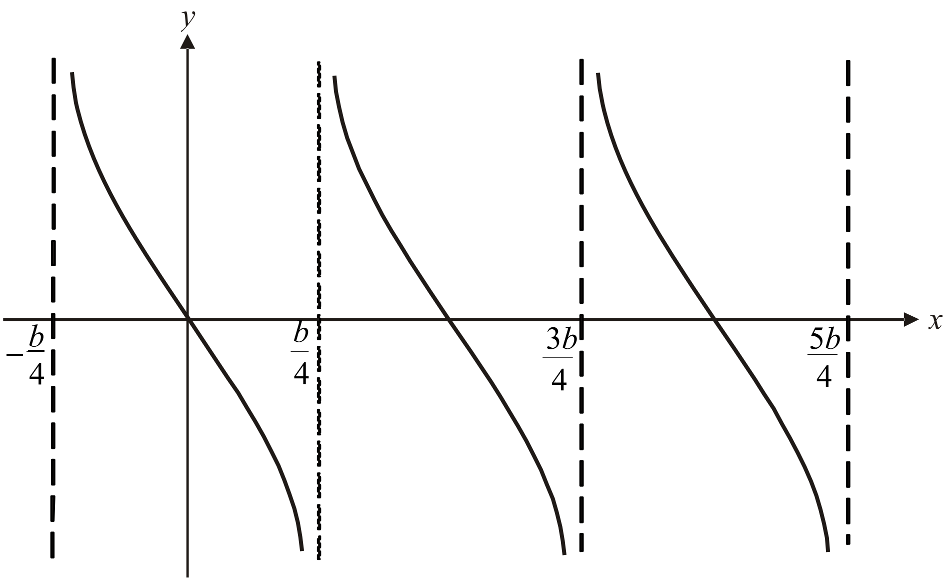
Worked solution

Use $\sin^2 x + \cos^2 x = 1$ to obtain $\cos^2(x) = \frac{7}{16}$.

$\cos(x) = -\frac{\sqrt{7}}{4}$ since x is in third quadrant.

$\therefore\ \sec(x) = -\frac{4}{\sqrt{7}}$ or $-\frac{4\sqrt{7}}{7}$ with rationalised denominator.

Question 5



The function whose graph is shown above, where $b > 0$, could have the rule given by

A. $y = -\tan\left(\frac{2bx}{\pi}\right)$

B. $y = \cot\left(\frac{2\pi x}{b}\right)$

C. $y = -\tan\left(\frac{\pi x}{b}\right)$

D. $\boldsymbol{y = \cot\left[\frac{2\pi}{b}\left(x - \frac{b}{4}\right)\right]}$

E. $y = \cot\left[\frac{2\pi}{b}\left(x - \frac{b}{2}\right)\right]$

Answer is D.

Worked solution

Options B, D and E have the correct period of $\frac{\pi}{\frac{2\pi}{b}} = \frac{b}{2}$.

The graph is then $y = \cot\left(\frac{2\pi x}{b}\right)$, which has been translated $\frac{b}{4}$ to the right,

resulting in $y = \cot\left[\frac{2\pi}{b}\left(x - \frac{b}{4}\right)\right]$.

Question 6

For $y = k - a\sin^{-1}(2x - b)$, $a > 0$, $b > 0$, the maximal domain and range are

A. domain $\left[\frac{-b}{2}, \frac{b}{2}\right]$, range $\left[\frac{a-k\pi}{2}, \frac{a+k\pi}{2}\right]$

B. domain $\left[\frac{b-1}{2}, \frac{b+1}{2}\right]$, range $\left[\frac{a-k\pi}{2}, \frac{a+k\pi}{2}\right]$

C. **domain** $\left[\frac{b-1}{2}, \frac{b+1}{2}\right]$**, range** $\left[\frac{2k-a\pi}{2}, \frac{2k+a\pi}{2}\right]$

D. domain $\left[\frac{-b}{2}, \frac{b}{2}\right]$, range $\left[\frac{k-2\pi}{2}, \frac{k+2\pi}{2}\right]$

E. domain $\left[\frac{-b}{2}-1, \frac{b}{2}+1\right]$, range $\left[k-\frac{a\pi}{2}, k+\frac{a\pi}{2}\right]$

Answer is C.

Worked solution

Maximal domain is found by letting

$-1 \le 2x - b \le 1$

$b - 1 \le 2x \le b + 1$

$\frac{b-1}{2} \le x \le \frac{b+1}{2}$

And if $x = \frac{b-1}{2}$, then $y = k - a\sin^{-1}(-1) = k - \left(-\frac{a\pi}{2}\right) = \frac{2k+a\pi}{2}$

$x = \frac{b+1}{2}$, then $y = k - a\sin^{-1}(1) = k - \left(\frac{a\pi}{2}\right) = \frac{2k-a\pi}{2}$

Hence, range is $\left[\frac{2k-a\pi}{2}, \frac{2k+a\pi}{2}\right]$, since $a > 0$.

Question 7

If $z = 1 + i$, then $\text{Arg}(i^3 z)$ is

A. $\frac{7\pi}{4}$

B. $\frac{5\pi}{4}$

C. $-\frac{3\pi}{4}$

D. $\frac{3\pi}{4}$

E. $\boldsymbol{-\frac{\pi}{4}}$

Answer is E.

Worked solution

$i^3 = -i$

then, $i^3 z = -i(1 + i) = -i - i^2 = 1 - i$

Hence, $\text{Arg}(i^3 z) = \text{Arg}(1 - i) = -\frac{\pi}{4}$.

Tip

- *Use the principal argument* $-\pi < \text{Arg}\, z \le \pi$.

Question 8

The complex relation $\text{Re}(z - 1) \times \text{Im}(z - 1) = 1$ can be represented on a Cartesian plane as a

A. **hyperbola with asymptotes $x = 1$, $y = 0$.**

B. hyperbola with asymptotes $y = \pm x$.

C. hyperbola with asymptotes $x = 0, y = 1$.

D. straight line with equation $y = 1 - x$.

E. hyperbola with equation $y = \dfrac{1}{x-1} + 1$.

Answer is A.

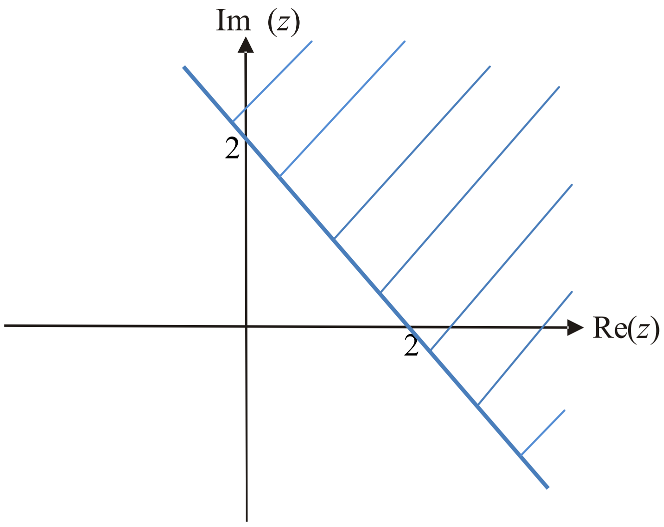
Worked solution

$z = x + yi$

Complex relation becomes:

$$\text{Re}(x + yi - 1) \times \text{Im}(x + yi - 1) = 1$$
$$\text{Re}[(x-1) + yi] \times \text{Im}[(x-1) + yi] = 1$$
$$(x-1) \times y = 1$$
$$y = \frac{1}{x-1}$$, which has asymptotes $x = 1$ and $y = 0$.

Question 9



The shaded region on the Argand diagram above represents

A. $|z+2+2i| \geq |z|$

B. $|z-2-2i| \leq |z-2|$

C. $|z-2| \geq |z-2i|$

D. $|z-2i| \leq |z-2|$

E. $|\boldsymbol{z}| \geq |\boldsymbol{z}-\boldsymbol{2}-\boldsymbol{2i}|$

Answer is E.

Worked solution

The distance from any point on the line to (0, 0), which is represented by $|z|$, is equal to the distance from any point on the line to the point (2, 2), which is represented by $|z-2-2i|$.

Hence, the equation of the line is $|z| = |z-2-2i|$ and the equation of the shaded region is then $|z| \geq |z-2-2i|$.

Alternatively, the relationships given can be converted to Cartesian form:

$$|z| \geq |z-2-2i|$$

$$|x+yi| \geq |x+yi-2-2i|$$

$$|x+yi| \geq |(x-2)+(y-2)i$$

$$\sqrt{x^2+y^2} \geq \sqrt{(x-2)^2+(y-2)^2}$$

$$x^2+y^2 \geq x^2-4x+4+y^2-4y+4$$

$$4x+4y \geq 8$$

$$y \geq -x+2$$

SECTION 1 – continued

Question 10

Given that $z+i$ is a factor of $P(z)=z^3-z^2+z-1$, which one of the following statements is **not** true?

A. $P(z)=0$ has three solutions.

B. $P(-i)=0$.

C. **$P(z)=0$ has two real solutions.**

D. $P(z)$ can be factorised over C.

E. $P(i)=0$.

Answer is C.

Worked solution

Since P(z) has real coefficients, the conjugate root theorem applies. If $(z+i)$ is a factor, then $(z-i)$ is a factor. Both these factors give $\pm i$ as solutions, leaving one real solution as the third solution. Hence, the incorrect statement is C.

Question 11

If $\dfrac{da}{dy} = 9 + a^2$ and $y = 0$ when $a = 0$, then y is equal to

A. $\tan^{-1}\left(\dfrac{a}{3}\right)$

B. $\dfrac{1}{3}\tan^{-1}(3a)$

C. $3\tan^{-1} a$

D. $\boldsymbol{\dfrac{1}{3}\tan^{-1}\left(\dfrac{a}{3}\right)}$

E. $\dfrac{1}{3}\tan\left(\dfrac{a}{3}\right)$

Answer is D.

Worked solution

$$\frac{da}{dy} = 9 + a^2$$

$$\frac{dy}{da} = \frac{1}{9 + a^2}$$

$$y = \frac{1}{3}\int \frac{3}{9 + a^2}\, da$$

$$y = \frac{1}{3}\tan^{-1}\left(\frac{a}{3}\right) + c$$

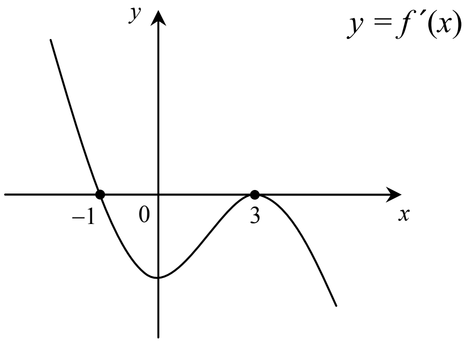
$y = 0$ when $a = 0$, so $c = 0$.

Hence, $y = \dfrac{1}{3}\tan^{-1}\left(\dfrac{a}{3}\right)$.

Tip

- *Need to write* $\dfrac{1}{9 + a^2}$ *as* $\dfrac{1}{3}\left(\dfrac{3}{9 + a^2}\right)$.

Question 12



If $f(x)$ is an antiderivative of $f'(x)$, the graph of $y = f(x)$ has

A. a stationary point of inflexion at $x = 0$ and a local maximum at $x = -1$

B. stationary points of inflexion at $x = 0$ and $x = 3$ and a local maximum at $x = -1$

C. **a stationary point of inflexion at $x = 3$ and a local maximum at $x = -1$**

D. a local minimum at $x = 0$ and a local maximum at $x = 3$

E. a stationary point of inflexion at $x = 3$ and a local minimum at $x = -1$

Answer is C.

Worked solution

When $x < 3, f'(x) < 0$; and when $x > 3, f'(x) < 0$.

So $x = 3$ is a stationary point of inflexion on the graph of $y = f(x)$.

When $x < -1, f'(x) > 0$; and when $x > -1, f'(x) < 0$.

So $x = -1$ is a maximum on the graph of $y = f(x)$.

Question 13

For the curve $xy^3 - 2x^2 = 7$

A. $\dfrac{dy}{dx} = \dfrac{3x - 4y^3}{3xy^2}$

B. $\dfrac{dy}{dx} = \dfrac{7 + 4x}{6x}$

C. $\dfrac{dy}{dx} = \dfrac{4x - y^3}{6xy^2}$

D. $\dfrac{dy}{dx} = \dfrac{y^3}{3xy^2}$

E. $\mathbf{\dfrac{dy}{dx} = \dfrac{4x - y^3}{3xy^2}}$

Answer is E.

Worked solution

$$\frac{d}{dx}(x.y^3) - \frac{d}{dx}(2x^2) = \frac{d}{dx}(7)$$

$$\left[\frac{d}{dx}(x).y^3 + \frac{d}{dx}(y^3).x\right] - 4x = 0$$

$$y^3 + \frac{d}{dy}(y^3)\frac{dy}{dx}.x = 4x$$

$$3xy^2\frac{dy}{dx} = 4x - y^3$$

$$\frac{dy}{dx} = \frac{4x - y^3}{3xy^2}$$

Tip

- *Must recognise a product when differentiating implicitly.*

Question 14

Using an appropriate substitution, $\int \frac{e^{2x}}{1+e^x}\,dx$ can be written as

A. $\int \left(1-\frac{1}{u}\right) du$

B. $\int \left(\frac{u}{1+u^2}\right) du$

C. $\int \left(\frac{u^2}{1+u}\right) du$

D. $\int \left(\frac{2u}{1+u}\right) du$

E. $\int \left(1+\frac{1}{u}\right) du$

Answer is A.

Worked solution

Let $u = 1 + e^x$

$\frac{du}{dx} = e^x$

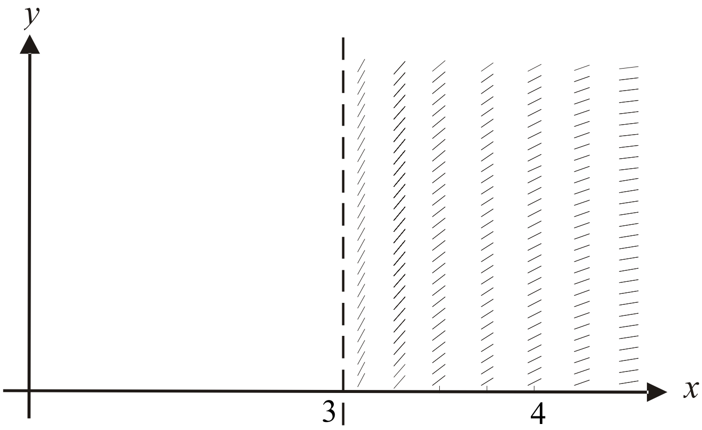
Then $\int \frac{e^{2x}}{1+e^x}\,dx$ can be written as

$= \int \frac{e^x}{1+e^x} . e^x \, dx$

$= \int \left(\frac{u-1}{u}\right) . \frac{du}{dx} \, dx$

$= \int \left(1-\frac{1}{u}\right) du$

Question 15



The direction or slope field for a particular first-order differential equation is shown above.
The differential equation could be

A. $\frac{dy}{dx} = \frac{1}{x+3}$

B. $\boldsymbol{\frac{dy}{dx} = \frac{1}{x-3}}$

C. $\frac{dy}{dx} = \log_e(x-3)$

D. $\frac{dy}{dx} = \log_e(x+3)$

E. $\frac{dy}{dx} = \sqrt{x-3}$

Answer is B.

Worked solution

The solution to the slope field is of the form:

$y = a \log_e(x-3) + k,\ y > 0.$

So, $\frac{dy}{dx}$ could be $\frac{1}{x-3}$ if $a = 1$ and $k \in R$.

Question 16

If $\underset{\sim}{u} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$, the vector resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$ is

A. $\frac{9}{7}\underset{\sim}{i} - \frac{6}{7}\underset{\sim}{j} + \frac{3}{7}\underset{\sim}{k}$

B. $\mathbf{2\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}}$

C. $2\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

D. $2\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$

E. $\frac{9}{7}\underset{\sim}{i} + \frac{6}{7}\underset{\sim}{j} + \frac{3}{7}\underset{\sim}{k}$

Answer is B.

Worked solution

$(\underset{\sim}{u}.\hat{\underset{\sim}{v}}).\hat{\underset{\sim}{v}}$

$= \left[\left(3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) . \frac{1}{\sqrt{3}}\left(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right)\right] . \frac{1}{\sqrt{3}}\left(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right)$

$= \frac{6}{\sqrt{3}} . \frac{1}{\sqrt{3}}\left(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right)$

$= 2\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$

Question 17

A particle moves along a straight line such that its acceleration at time t (seconds) is given by $\ddot{x} = 3t^2 - 4t + 5$ m/s^2.

Initially, the particle is at a fixed point, O, ($x = 0$) with a velocity of –2 m/s^2.

The position of the particle from O at time t is

A. 6

B. $\mathbf{\frac{1}{4}t^4 - \frac{2}{3}t^3 + \frac{5}{2}t^2 - 2t}$

C. –6

D. $3t^4 - \frac{2}{3}t^3 + \frac{5}{2}t^2 - 2t$

E. $\frac{1}{4}t^4 - t^3 + \frac{5}{2}t^2 - 2t$

Answer is B.

Worked solution

$\ddot{x} = 3t^2 - 4t + 5$

$\dot{x} = t^3 - 2t^2 + 5t + c_1$

When $t = 0,\ \dot{x} = -2,$ so $c_1 = -2,$

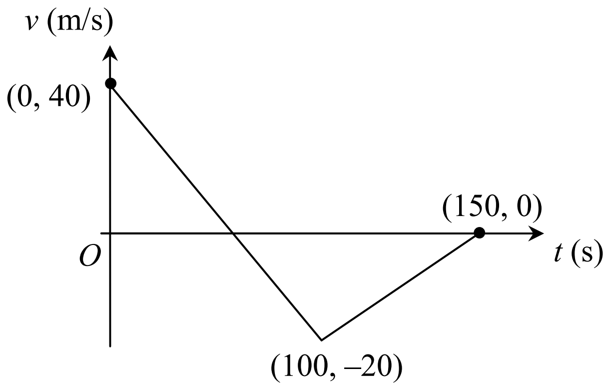
$\therefore\ \dot{x} = t^3 - 2t^2 + 5t - 2$

$x = \frac{1}{4}t^4 - \frac{2}{3}t^3 + \frac{5}{2}t^2 - 2t + c_2$

When $t = 0,\ x = 0,$ so $c_2 = 0.$

$\therefore\ x = \frac{1}{4}t^4 - \frac{2}{3}t^3 + \frac{5}{2}t^2 - 2t$

Question 18



The velocity – time graph of a particle moving in a straight line starting from a fixed position, O, is shown above. If the initial velocity is 40 m/s in an easterly direction, where is the particle located 150 seconds later?

A. 1000 m west of O

B. $\frac{6500}{3}$ m east of O

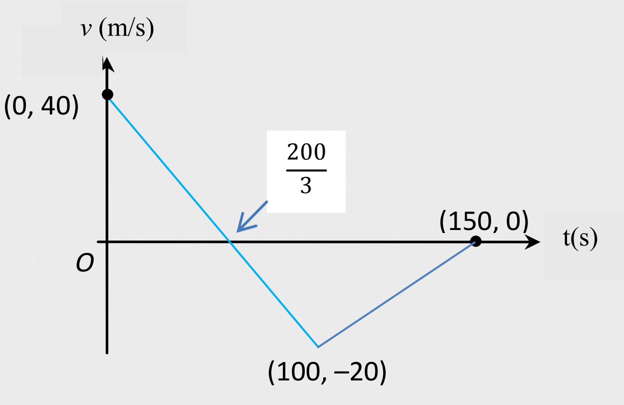
C. 2000 m east of O

D. **500 m east of *O***

E. 1000 m east of O

Answer is D.

Worked solution



Intercept is $\frac{200}{3}$.

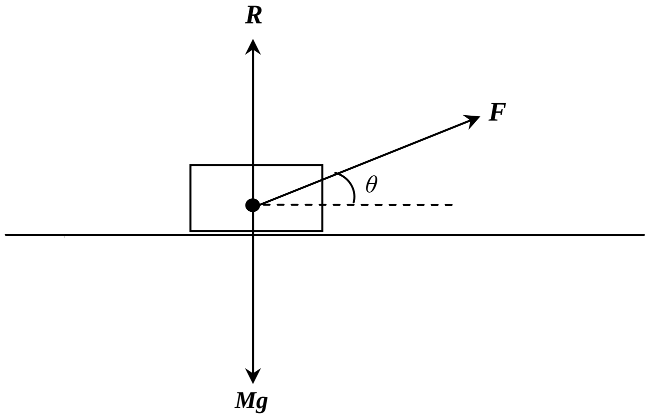
Area under triangle representing easterly displacement $= \frac{1}{2} \times \frac{200}{3} \times 40 = \frac{4000}{3}$.

Area under triangle representing westerly displacement $= \frac{1}{2} \times \frac{250}{3} \times 20 = \frac{2500}{3}$.

Resultant displacement is $\frac{4000}{3} - \frac{2500}{3} = 500$ m east of O.

Question 19

A body of mass M kg is being pulled along a smooth horizontal table by means of a force, F, acting at an angle $\theta°$ to the horizontal.
The diagram below indicates the forces acting on the body.



Which statement regarding the magnitude of the forces is true?

A. $R - Mg = 0$

B. $R - F\sin\theta - Mg = 0$

C. $\boldsymbol{R + F\sin\theta - Mg = 0}$

D. $R + F\cos\theta - Mg = 0$

E. $R - F\cos\theta - Mg = 0$

Answer is C.

Worked solution

Resolving the forces vertically:

$\boldsymbol{R}$ and $\boldsymbol{F}\sin\theta$ act upwards, whereas Mg acts downwards.

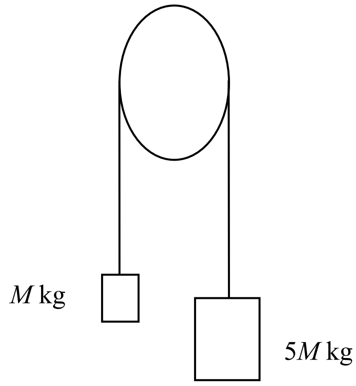
Hence, the magnitudes of these opposing forces must be:

$$R + F\sin\theta = Mg$$

$$\therefore\ R + F\sin\theta - Mg = 0$$

Question 20

The diagram below shows a smooth pulley with masses of M kg and $5M$ kg attached to each end of an inextensible string.



The magnitude of the acceleration of the $5M$ kg mass is

A. $\frac{2}{3}$ m/s^2

B. $\frac{4g}{3}$ m/s^2

C. $\mathbf{\frac{2g}{3}}$ **m/s^2**

D. $\frac{4}{3}$ m/s^2

E. $3g$ m/s^2

Answer is C.

Worked solution

Equations of motion on each mass are:

$5Mg - T = 5Ma$ (1)

$T - Mg = Ma$ (2)

Adding equations (1) and (2) gives:

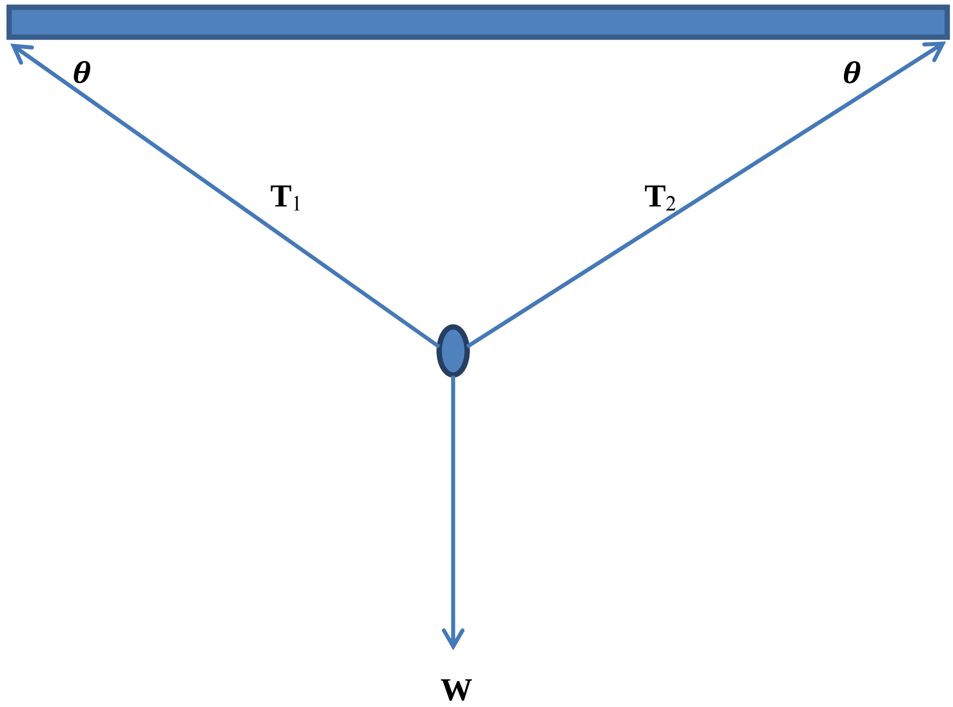
$4Mg = 6Ma$

$\therefore\ a = \frac{2g}{3}$ m/s^2

Tip

- *For connected particles, the equations of motion need to be written for each particle separately, knowing that the acceleration is the same in both equations.*

Question 21



An object in **equilibrium** is suspended from a beam by two ropes, each making an angle of θ with the horizontal. The tension forces acting on the object are $\mathbf{T}_1$ and $\mathbf{T}_2$ and **W** is the weight force. Which one of the following statements is true?

A. $\mathbf{T}_1 = \mathbf{T}_2$

B. $\mathbf{W} = \mathbf{T}_1 \cos \theta° + \mathbf{T}_2 \cos \theta°$

C. $\mathbf{T_1 + T_2 = -W}$

D. $\mathbf{T}_1 \cos \theta° = \mathbf{T}_2 \sin \theta°$

E. $\mathbf{T}_1 + \mathbf{T}_2 = 2\mathbf{W}$

Answer is C.

Worked solution

The forces acting on the object are in equilibrium; therefore, the resultant force is zero.

$\Sigma \mathbf{F} = \mathbf{0}$

$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{W} = \mathbf{0}$

$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{W}$

Question 22

An object of mass 3 kg comes to a complete stop from 20 m/s over a horizontal distance of 60 m. If the acceleration acting on the object is constant, then the **magnitude** of the resultant force acting on the object is

A. **10 N**

B. $\frac{10}{3}$ N

C. 19.6 N

D. –10 N

E. $\frac{10}{3}$ N

Answer is A

Worked solution

$u = 20$, $s = 60$ and $v = 0$.

Using $v^2 = u^2 + 2as$

$0 = 400 + 120a$

$a = -\frac{10}{3}$

$\therefore$ The magnitude of the force is $|ma| = 3\times\frac{10}{3} = 10$ N.

END OF SECTION 1

END OF SECTION 1

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SECTION 2

Question 1

Let $f : [a, b] \rightarrow R$, where $f(x) = (a - x)^2$.

a. If $f(1) = 0$ and $f(b) = 9$, show that $a = 1$ and $b = 4$.

Worked solution

$f(1) = 0$, so $(a - 1)^2 = 0$, $\therefore\ a = 1$.

$f(b) = 9$, so $(1 - b)^2 = 9$

$1 - b = \pm 3$

$b = -2$ or 4

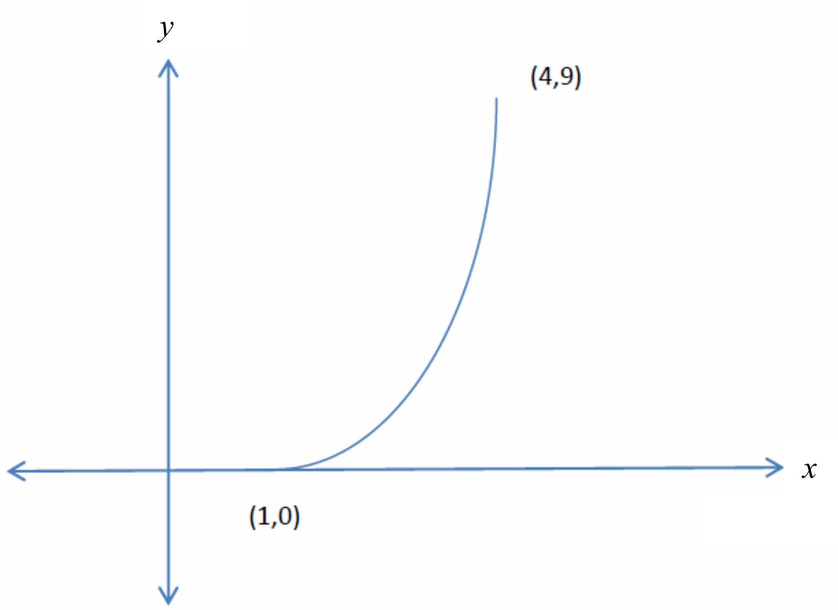
$b = 4$, since $b > a$.

2 marks

Mark allocation

- 1 mark for relevant equations.
- 1 mark for finding a and b correctly.

b. Sketch the graph of the function, clearly labelling the endpoints with their co-ordinates.



Worked solution

The right-hand branch of a minimum point parabola needs to be sketched with the minimum point (1, 0) as one endpoint and (4, 9) as the other endpoint.

1 mark

Mark allocation

- 1 mark for a right branch minimum point parabola with (1, 0) and (4, 9) clearly labelled.

c. Find the area enclosed between the graph of the function and the y-axis. (All units are in centimetres.)

Worked solution

Make x the subject $x = 1 + \sqrt{y}$ (selecting the correct branch)

Required area $= \int_0^9 (1+\sqrt{y})\,dy$

$A = 27 \text{ cm}^2$ (using CAS or calculus techniques)

Alternatively:

Required area $= (4\times 9) - \int_1^4 (1-x)^2\,dx$

$A = 36 - 9$ (using CAS or calculus techniques)

$A = 27 \text{ cm}^2$

2 marks

Mark allocation

- 1 mark for $x = 1 + \sqrt{y}$ (selecting the correct branch).
- 1 mark for 27 cm^2.

Alternatively:

- 1 method mark for subtracting $\int_1^4 (1-x)^2\,dx$ from 36 cm^2.
- 1 mark for 27 cm^2.

d. The function f is rotated about the y-axis to generate a vessel. If the volume of the vessel is $\frac{m\pi}{n}$ cm^3 find the values of m and n where m and n are both integers.

Worked solution

$\text{Volume} = \pi\int_0^9 x^2\,dy$

$V = \pi\int_0^9 \left(1+\sqrt{y}\right)^2\,dy$

Using CAS or calculus techniques $V = \frac{171\pi}{2}$ cm^3; hence, $m = 171$ and $n = 2$.

2 marks

Mark allocation

- 1 mark for $V = \pi\int_0^9 \left(1+\sqrt{y}\right)^2\,dy$.
- 1 mark for $m = 171$ and $n = 2$.

e. Water now enters the vessel at a constant rate of 2 cm^3/s. How long does it take for the vessel to be filled to 20% of its capacity?
(Write the answer in minutes, to 2 decimal places.)

Worked solution

20% of capacity is $\frac{171\pi}{2} \div 5 = \frac{171\pi}{10}$ cm^3.

Time taken is $\frac{171\pi}{10}$ cm^3 $\div$ 2 cm^3/s $= \frac{171\pi}{20}$ s.

$\frac{171\pi}{20}$ s $\div 60 = 0.45$ min.

2 marks

Mark allocation

- 1 mark for determining 20% of capacity and dividing by 2 cm^3/s.
- 1 mark for correctly determining answer in minutes, to 2 decimal places.

f. Write an expression that will determine the height (h cm) of water in the vessel when it is filled to 20% of its capacity and determine the value of h, to 2 decimal places.

Worked solution

$$\pi\int_0^h \left(1+\sqrt{y}\right)^2 dy = \frac{171\pi}{10}$$

$\therefore$ $h = 3.35$ cm (using CAS or calculus techniques)

2 marks

Mark allocation

- 1 mark for expression $\pi\int_0^h \left(1+\sqrt{y}\right)^2 dy = \frac{171\pi}{10}$.
- 1 mark for calculating $h = 3.35$ cm.

Total: 2 + 1 + 2 + 2 + 2 + 2 = 11 marks

Question 2

The position of a tugboat relative to an island, O, at any time (t hours) is given by

$$\underset{\sim}{r}(t) = \left(t + \frac{1}{t}\right)\underset{\sim}{i} + \left(t - \frac{1}{t}\right)\underset{\sim}{j},\ t > 0,$$

where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors east and north of the island, respectively. An oil rig, R, is located 5 km north of the island.

(All distances are in km and the island can be considered as the origin of a Cartesian plane.)

a. Show that the path of the tugboat can be described by the Cartesian equation

$\frac{x^2}{4} - \frac{y^2}{4} = 1$ and state the domain and range of the path.

Worked solution

$x = t + \frac{1}{t}$ so $x^2 = t^2 + 2 + \frac{1}{t^2}$ (1)

$y = t - \frac{1}{t}$ so $y^2 = t^2 - 2 + \frac{1}{t^2}$ (2)

Subtracting equation (2) from (1) gives:

$x^2 - y^2 = 4$

$\frac{x^2}{4} - \frac{y^2}{4} = 1$ and a graph of x versus t shows that when $t > 0$, $x \geq 2$.

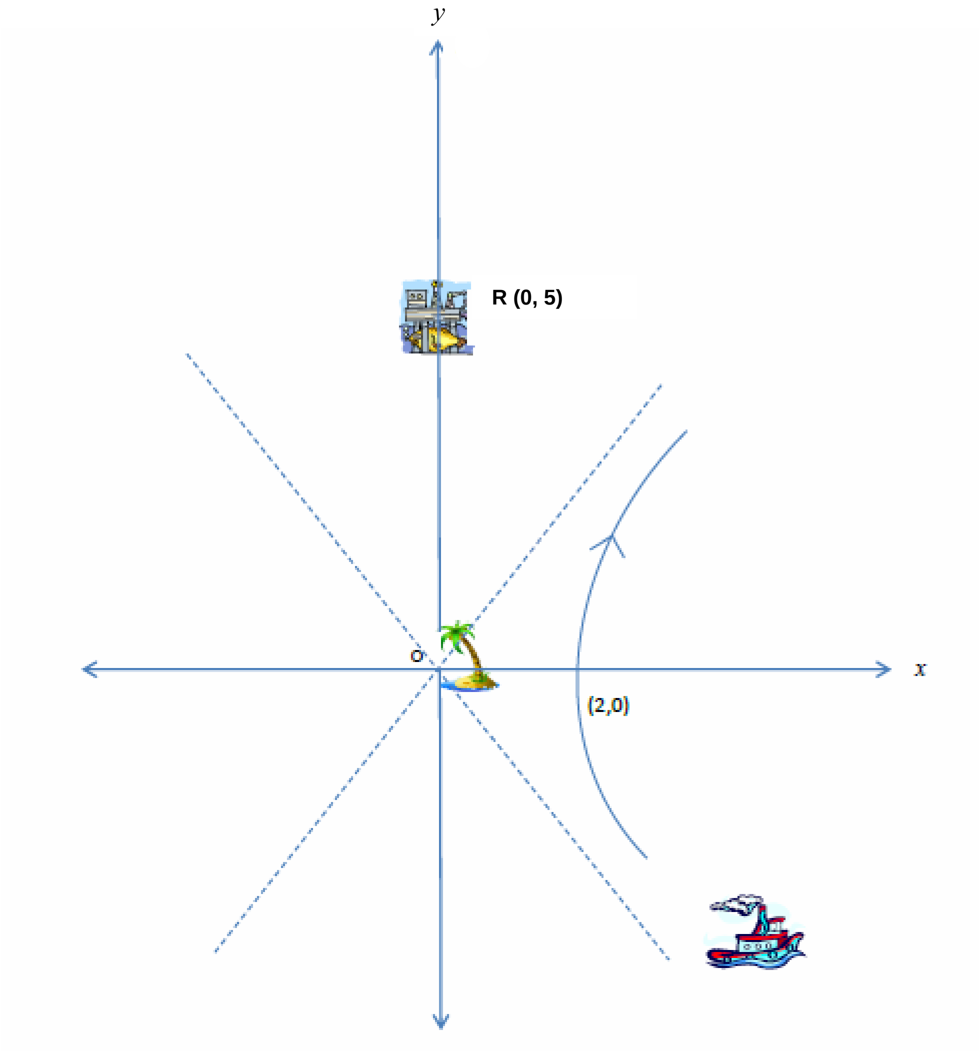
and a graph of y versus t shows that when $t > 0$, $y \in R$.

3 marks

Mark allocation

- 1 mark for letting $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$.
- 1 mark for squaring x and y and subtracting to eliminate t, obtaining $x^2 - y^2 = 4$.
- 1 mark for determining domain is $x \geq 2$ and range is $y \in R$.

b. Sketch a graph of the path travelled by the tugboat, clearly indicating the direction of motion and label any intercepts.



Worked solution

The sketch should be the right-hand branch of the hyperbola $\frac{x^2}{4} - \frac{y^2}{4} = 1$, with a vertex at (2, 0), with domain $x \geq 2$ and range $y \in R$, and with an arrow on the hyperbola going upwards to indicate the direction of motion.

2 marks

Mark allocation

- 1 mark for right branch of hyperbola with vertex (2, 0).
- 1 mark for arrow in upwards direction.

c. When will the tugboat be due east of the island?

Worked solution

Solve either $t+\frac{1}{t}=2$ or $t-\frac{1}{t}=0.$

Hence, $t=1$ h.

2 marks

Mark allocation

- 1 mark for setting up either equation.
- 1 mark for solution $t=1$ h.

d. Find the velocity vector of the tugboat at any time t.

Worked solution

$\underset{\sim}{r}(t)=\left(1-\frac{1}{t^2}\right)\underset{\sim}{i}+\left(1+\frac{1}{t^2}\right)\underset{\sim}{j}$

1 mark

Mark allocation

- 1 mark for correctly deriving $\underset{\sim}{r}(t)$.

e. If P is any point on the path of the tugboat, find the vector $\underset{\sim}{RP}$ in terms of t.

Worked solution

$\underset{\sim}{RP}=\underset{\sim}{RO}+\underset{\sim}{OP}$

$\underset{\sim}{RP}=-5\underset{\sim}{j}+\left(t+\frac{1}{t}\right)\underset{\sim}{i}+\left(t-\frac{1}{t}\right)\underset{\sim}{j}$

$\underset{\sim}{RP}=\left(t+\frac{1}{t}\right)\underset{\sim}{i}+\left(t-\frac{1}{t}-5\right)\underset{\sim}{j}$

2 marks

Mark allocation

- 1 mark for using a vector method to obtain $\underset{\sim}{RP}$.
- 1 mark for the correct answer.

f. Find the time, to the nearest minute, when the tugboat is **closest** to the oil rig.

Worked solution

$\underset{\sim}{RP} \cdot \dot{\underset{\sim}{r}}(t) = 0$

$$\left(t+\frac{1}{t}\right)\left(1-\frac{1}{t^2}\right)+\left(t-\frac{1}{\underset{\sim}{t}}-5\right)\left(1+\frac{1}{t^2}\right)=0$$

Using CAS, $t = 2.85078$ since $t > 0$.

Hence, $t = 2$ h and 51 min (to the nearest minute).

3 marks

Mark allocation

- 1 mark for using dot product to show that the position vector $\underset{\sim}{RP}$ and the velocity vector $\dot{\underset{\sim}{r}}(t)$ are perpendicular when the tug is closest to the oil rig.
- 1 mark for equation $\left(t+\frac{1}{t}\right)\left(1-\frac{1}{t^2}\right)+\left(t-\frac{1}{\underset{\sim}{t}}-5\right)\left(1+\frac{1}{t^2}\right)=0$.
- 1 mark for $t = 2$ h and 51 min.

g. Find the times, to the nearest minute, when the island, tugboat and oil rig form the vertices of an isosceles triangle with equal sides of 5 km.

Worked solution

$\left|\underset{\sim}{RO}\right| = 5$

$\left|\underset{\sim}{OP}\right| = \sqrt{\left(t+\frac{1}{t}\right)^2 + \left(t-\frac{1}{t}\right)^2}$

$\left|\underset{\sim}{RP}\right| = \sqrt{\left(t+\frac{1}{t}\right)^2 + \left(t-\frac{1}{t}-5\right)^2}$

<u>There are two situations to consider:</u>

1) Need to show that that $\left|\underset{\sim}{RO}\right| = \left|\underset{\sim}{OP}\right| = 5$ and $\left|\underset{\sim}{RP}\right| \neq 5$.

Now $\left|\underset{\sim}{RO}\right| = 5$ and $\left|\underset{\sim}{OP}\right| = 5$ when

$$\sqrt{\left(t+\frac{1}{t}\right)^2 + \left(t-\frac{1}{t}\right)^2} = 5$$

$$\left(t+\frac{1}{t}\right)^2 + \left(t-\frac{1}{t}\right)^2 = 25$$

Using CAS, $t = 0.283758$ h and 3.52413 h.

Hence, $t = 17$ min; and 3 h and 31 min (to the nearest minute).

When $t = 0.283758$, $\left|\underset{\sim}{RP}\right| = 9.1$; and when $t = 3.52413$, $\left|\underset{\sim}{RP}\right| = 4.2$.

$\therefore$ When $t = 17$ min, and 3 h and 31 min, $\left|\underset{\sim}{RO}\right| = \left|\underset{\sim}{OP}\right| = 5$ and $\left|\underset{\sim}{RP}\right| \neq 5$.

Worked solution (continued)

2) Need to show that that $\left|\underset{\sim}{RO}\right| = \left|\underset{\sim}{RP}\right| = 5$ and $\left|\underset{\sim}{OP}\right| \neq 5$.

Now $\left|\underset{\sim}{RO}\right| = 5$ and $\left|\underset{\sim}{RP}\right| = 5$ when

$$\sqrt{\left(t+\frac{1}{t}\right)^2 + \left(t-\frac{1}{t}-5\right)^2} = 5$$

$$\left(t+\frac{1}{t}\right)^2 + \left(t-\frac{1}{t}-5\right)^2 = 25$$

Using CAS, $t = 1.24297$ h; and 4.77115 h.

Hence, $t = 1$ h and 15 min; and 4 h and 46 min (to the nearest minute).

When $t = 1.24297$, $\left|\underset{\sim}{OP}\right| = 2.1$; and when $t = 4.77115$, $\left|\underset{\sim}{OP}\right| = 6.8$.

$\therefore$ When $t = 1$ h and 15 min, and 4 h and 46 min, $\left|\underset{\sim}{RO}\right| = \left|\underset{\sim}{RP}\right| = 5$ and $\left|\underset{\sim}{OP}\right| \neq 5$.

4 marks

Mark allocation

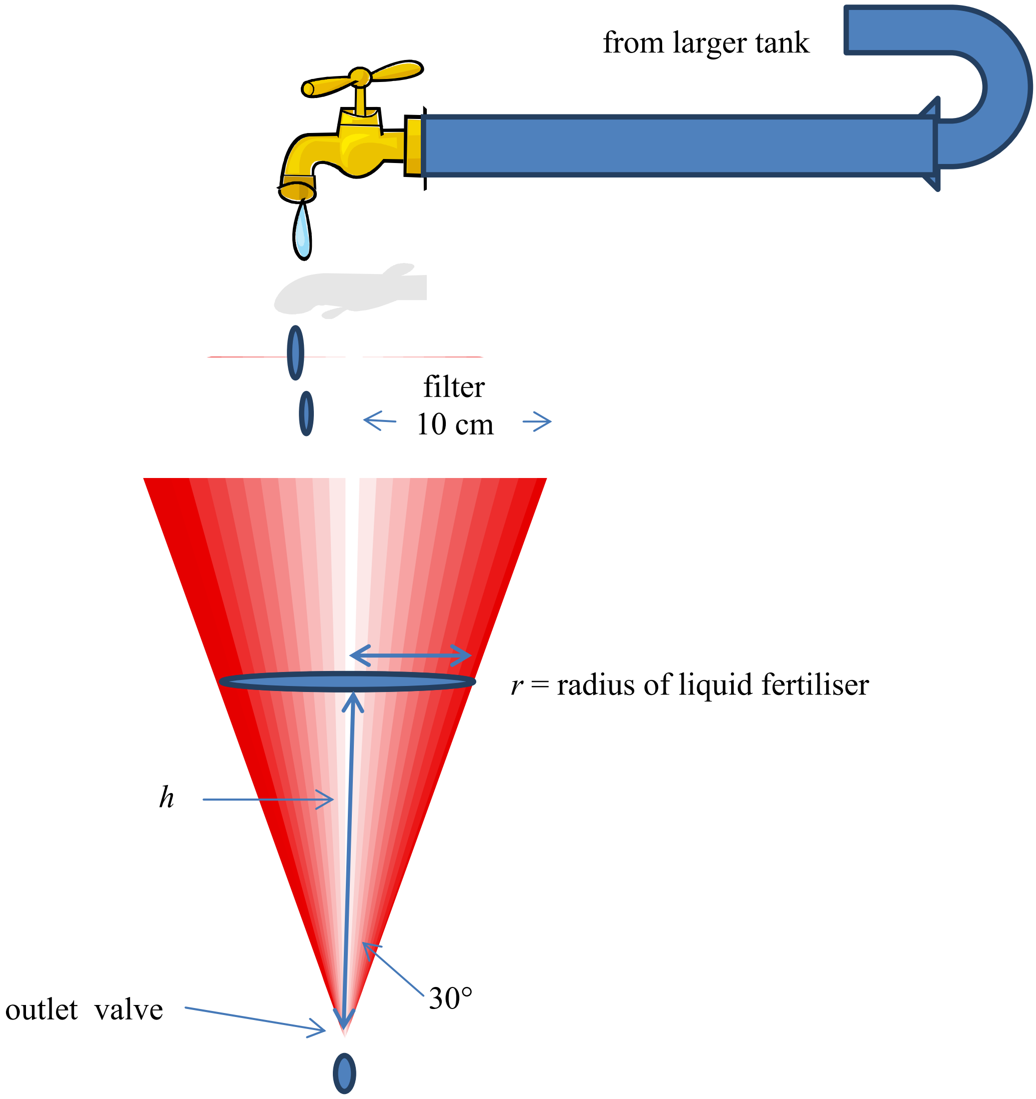
- 1 mark for determining magnitude of $\left|\underset{\sim}{OP}\right| = \sqrt{\left(t+\frac{1}{t}\right)^2 + \left(t-\frac{1}{t}\right)^2}$ and

  $\left|\underset{\sim}{RP}\right| = \sqrt{\left(t+\frac{1}{t}\right)^2 + \left(t-\frac{1}{t}-5\right)^2}$.
- 1 mark for when $t = 17$ min; and 3 h and 31 min, $\left|\underset{\sim}{RO}\right| = \left|\underset{\sim}{OP}\right| = 5$.
- 1 mark for when $t = 1$ h and 15 min; and 4 h and 46 min, $\left|\underset{\sim}{RO}\right| = \left|\underset{\sim}{RP}\right| = 5$.
- 1 mark for showing that the third side $\neq 5$ in both cases. (Give 2 marks for one correct solution.)

Total: 3 + 2 + 2 + 1 + 2 + 3 + 4 = 17 marks

Question 3

A fertiliser drip system contains a filter in the form of an inverted cone with a semi-vertical angle of 30°, as shown in the diagram below. The radius of the filter is 10 cm.



a. Liquid fertiliser from a larger tank pours from a tap into the filter at a constant rate of 2 cm^3/s and drips from an outlet valve in the bottom at $\frac{\sqrt{h}}{5}$ cm^3/s, where h cm is the depth of the water in the filter at any time t seconds.

i. Show that $\frac{dV}{dt} = \frac{10-\sqrt{h}}{5}$.

Worked solution

$\frac{dV}{dt}$ = inflow rate – outflow rate

$$\frac{dV}{dt} = 2 - \frac{\sqrt{h}}{5}$$

$$\frac{dV}{dt} = \frac{10 - \sqrt{h}}{5}$$

1 mark

Mark allocation

- 1 mark for using inflow rate – outflow rate to obtain $\frac{dV}{dt}$.

ii. Find the volume of the liquid fertiliser (V) in terms of its depth (h).

Hence, find $\frac{dh}{dt}$.

Worked solution

Let r = radius of liquid fertiliser.

$V = \frac{1}{3}\pi r^2 h$

Now, $\tan 30° = \frac{r}{h}$.

$\frac{\sqrt{3}}{3} = \frac{r}{h}$

$r = \frac{h\sqrt{3}}{3}$

$V = \frac{1}{3}\pi\left(\frac{h\sqrt{3}}{3}\right)^2 h$

$V = \frac{\pi h^3}{9}$

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$

$\frac{dh}{dt} = \left(\frac{10-\sqrt{h}}{5}\right) \times \frac{3}{\pi h^2}$ (since $\frac{dV}{dh} = \frac{\pi h^2}{3}$)

$\frac{dh}{dt} = \frac{3(10-\sqrt{h})}{5\pi h^2}$

4 marks

Mark allocation

- 1 mark for finding relationship between r and h, i.e. $r = \frac{h\sqrt{3}}{3}$.
- 1 mark for finding $V = \frac{\pi h^3}{9}$.
- 1 mark for using chain rule to find $\frac{dh}{dt}$.
- 1 mark for $\frac{dh}{dt} = \frac{3(10-\sqrt{h})}{5\pi h^2}$.

1 + 4 = 5 marks

b. The outlet valve is now closed and the filter is completely filled from the larger tank. Find the **exact** height of liquid fertiliser in the filter when it is completely full.

Worked solution

$$\tan 30° = \frac{10}{h}$$

$$\frac{\sqrt{3}}{3} = \frac{10}{h}$$

So, height of liquid fertiliser in filter when completely full is $10\sqrt{3}$ cm.

1 mark

Mark allocation

- 1 mark for exact answer $10\sqrt{3}$ cm.

c. When the filter is completely full, the tap is closed and the outlet valve opened. The liquid fertiliser drips out from the outlet valve at the same rate as before;

i.e. $\frac{\sqrt{h}}{5}$ cm^3/s.

i. Find $\frac{dh}{dt}$ and **use calculus to find *t* exactly in terms of *h*.**

Worked solution

$$\frac{dV}{dt} = -\frac{\sqrt{h}}{5}$$

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{5} \times \frac{3}{\pi h^2} \quad \text{(as before since } V = \frac{\pi h^3}{9}\text{)}$$

$$\frac{dh}{dt} = -\frac{3\sqrt{h}}{5\pi h^2}$$

$$\frac{dt}{dh} = -\frac{5\pi}{3} h^{\frac{3}{2}}$$

$$t = -\frac{5\pi}{3} \int h^{\frac{3}{2}}\, dh$$

$$t = -\frac{2\pi}{3} h^{\frac{5}{2}} + c$$

When $t = 0$, $h = 10\sqrt{3}$

So $0 = -\frac{2\pi}{3}(10\sqrt{3})^{\frac{5}{2}} + c$

$$c = \frac{2\pi}{3}(10\sqrt{3})^{\frac{5}{2}}$$

$$\therefore\ t = \frac{2\pi}{3}[(10\sqrt{3})^{\frac{5}{2}} - h^{\frac{5}{2}}]$$

5 marks

Mark allocation

- 1 mark for using chain rule to find $\frac{dh}{dt} = -\frac{3\sqrt{h}}{5\pi h^2}$.
- 1 mark for reciprocating and writing the integral necessary to find $t = -\frac{5\pi}{3}\int h^{\frac{3}{2}}\, dh$.
- 1 mark for integrating to find $t = -\frac{2\pi}{3} h^{\frac{5}{2}} + c$ (using calculus).
- 1 mark for $t = 0$, $h = 10\sqrt{3}$ so $c = \frac{2\pi}{3}(10\sqrt{3})^{\frac{5}{2}}$.
- 1 mark for finding t exactly in terms of h; i.e. $t = \frac{2\pi}{3}[(10\sqrt{3})^{\frac{5}{2}} - h^{\frac{5}{2}}]$.

ii. How long, to the nearest minute, does it take for the filter to empty?

Worked solution

When $h = 0$, $t = \frac{2\pi}{3}(10\sqrt{3})^{\frac{5}{2}}$ seconds.

Hence, $t = 44$ min (to the nearest minute).

1 mark

Mark allocation

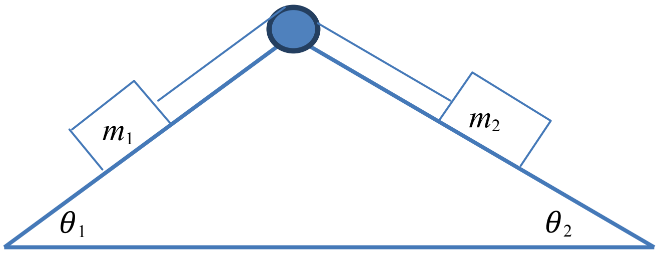
- 1 mark for $t = 44$ min.

5 + 1 = 6 marks

Total: 5 + 1 + 6 = 12 marks

Question 4

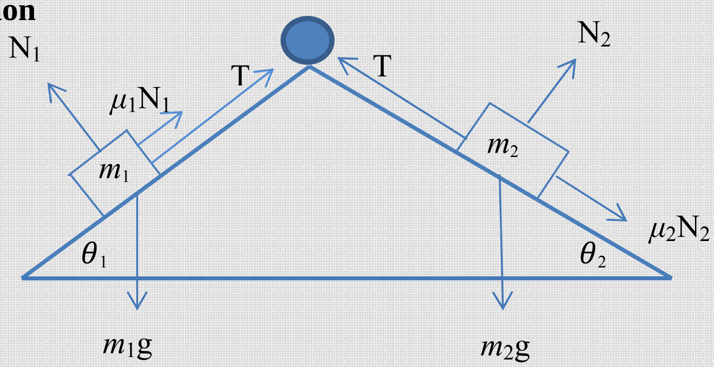
Two objects of mass m_1 and m_2 are connected by a light string passing over a smooth pulley, which is located at the apex of two inclined planes, as shown in the diagram below. The coefficients of friction of the two surfaces are μ_1 and μ_2 and the angles of inclination are θ_1 and θ_2, respectively.



Note: The system is on the verge of moving to the left.

a. Label all the forces acting on the two objects using the symbols **T**, $\mathbf{N_1}$, $\mathbf{N_2}$, $\boldsymbol{\mu_1}\mathbf{N_1}$, $\boldsymbol{\mu_2}\mathbf{N_2}$, $\boldsymbol{m_1 g}$ and $\boldsymbol{m_2 g}$.

Worked solution



Eight forces in total:

T, $\mathbf{N_1}$, $\boldsymbol{\mu_1}\mathbf{N_1}$, $\boldsymbol{m_1 g}$ all act on m_1 with **T** (tension) and $\boldsymbol{\mu_1}\mathbf{N_1}$ (friction) acting upwards on plane 1.

T, $\mathbf{N_2}$, $\boldsymbol{\mu_2}\mathbf{N_2}$, $\boldsymbol{m_2 g}$ all act on m_2 with **T** (tension) acting upwards on plane 2 and $\boldsymbol{\mu_2}\mathbf{N_2}$ (friction) acting downwards on plane 2.

2 marks

Mark allocation

- 1 mark for describing four forces acting on each object.
- 1 mark for the correct direction of all eight forces.

b. The planes are now lubricated and can be considered to be smooth.

i. If the system is in equilibrium, find $\frac{m_1}{m_2}$ in terms of θ_1 and θ_2.

Worked solution

Resolving forces parallel to planes:

$T = m_1 g \sin\theta_1$

$T = m_2 g \sin\theta_2$

$\therefore\ m_1 g \sin\theta_1 = m_2 g \sin\theta_2$

$$\frac{m_1}{m_2} = \frac{\sin\theta_2}{\sin\theta_1}$$

3 marks

Mark allocation

- 1 mark for resolving forces parallel to planes.
- 1 mark for $T = m_1 g \sin\theta_1$ and $T = m_2 g \sin\theta_2$.
- 1 mark for $\frac{m_1}{m_2} = \frac{\sin\theta_2}{\sin\theta_1}$.

ii. The object m_1 is now replaced by an object with a mass that is double that of m_2 and both angles of inclination are the same (θ).
Find the acceleration of the system, if it is moving to the left.

Worked solution

Let the acceleration to the left = a.
Equations of motion parallel to planes are:
$2m_2 g \sin\theta - T = 2m_2 a$ (1)
$T - m_2 g \sin\theta = m_2 a$ (2)

Adding equations (1) and (2) gives:
$m_2 g \sin\theta = 3m_2 a$

$$a = \frac{g\sin\theta}{3}$$

3 marks

Mark allocation

- 1 mark for the equation of motion for object with mass $2m^2$ (1).
- 1 mark for the equation of motion for object with mass m^2 (2).
- 1 mark for adding equations (1) + (2) to give $a = \frac{g\sin\theta}{3}$.

3 + 3 = 6 marks

Total: 2 + 6 = 8 marks

Question 5

a. Consider $u = a - a\sqrt{3}\,i$, where $a < 0$, and $w = b\,\text{cis}\left(\frac{\pi}{4}\right)$.

i. Express u in polar form.

Worked solution

$|u| = \sqrt{a^2 + \left(-a\sqrt{3}\right)^2} = \sqrt{4a^2} = -2a$, since $a < 0$.

$\tan\theta = -\frac{a\sqrt{3}}{a}$, so $\theta = \frac{2\pi}{3}$, since u is in the second quadrant.

$u = -2a\,\text{cis}\left(\frac{2\pi}{3}\right)$

2 marks

Mark allocation

- 1 mark for $-2a$.
- 1 mark for $\theta = \frac{2\pi}{3}$.

ii. Express w in Cartesian form.

Worked solution

$w = b\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

$w = \frac{b\sqrt{2}}{2} + \frac{ib\sqrt{2}}{2}$

1 mark

Mark allocation

- 1 mark for answer in correct Cartesian form.

2 + 1 = 3 marks

b. Express each of the following as a complex number.

i. $u + w$ in Cartesian form

Worked solution

$$u + w = \left(a + \frac{b\sqrt{2}}{2}\right) + i\left(\frac{b\sqrt{2}}{2} - a\sqrt{3}\right)$$

1 mark

Mark allocation

- 1 mark for correct answer in Cartesian form.

ii. uw in polar form

Worked solution

$$uw = (-2a \times b)\ \text{cis}\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$$

$$uw = -2ab\ \text{cis}\left(\frac{11\pi}{12}\right)$$

1 mark

Mark allocation

- 1 mark for correct answer in polar form.

iii. u^6 in Cartesian form

Worked solution

$$u^6 = \left(-2a\ \text{cis}\left(\frac{2\pi}{3}\right)\right)^6$$

$$u^6 = 64a^6\ \text{cis}(4\pi)$$

$$u^6 = 64a^6$$

1 mark

Mark allocation

- 1 mark for correct answer in Cartesian form.

1 + 1 + 1 = 3 marks

c. Find the square roots of w in polar form.

Worked solution

Let $z = \sqrt{w}$.

Hence, $z^2 = w$.

$(r \text{ cis } \theta)^2 = b \text{ cis}\left(\frac{\pi}{4}\right)$

$r^2 = b$ and $2\theta = \frac{\pi}{4} + 2k\pi, \quad k \in J$

$r = \sqrt{b}$ and $\theta = \frac{\pi}{8}$ and $-\frac{7\pi}{8}$

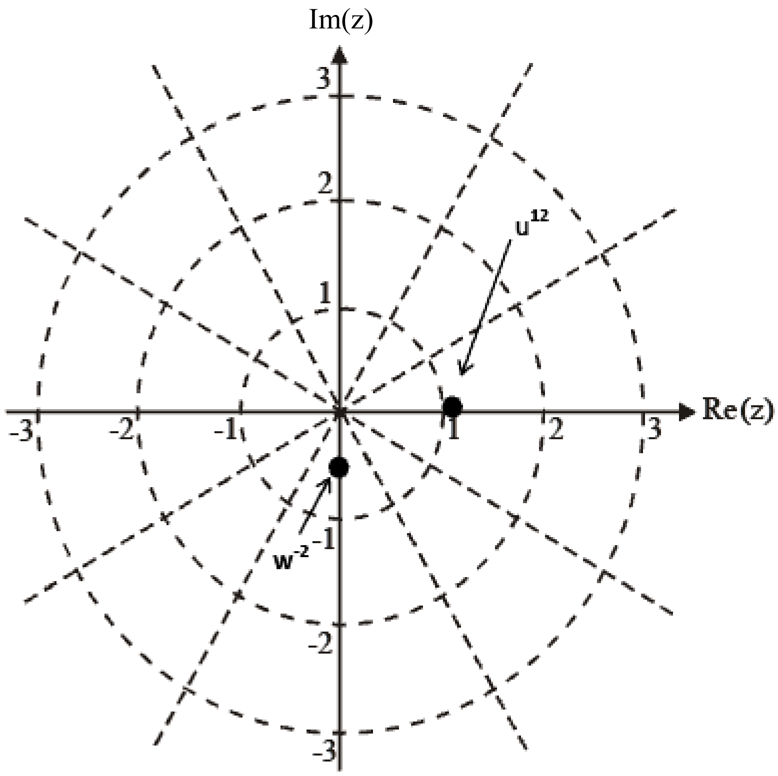
The square roots of w in polar form are $\sqrt{b} \text{ cis}\left(\frac{\pi}{8}\right)$ and $\sqrt{b} \text{ cis}\left(-\frac{7\pi}{8}\right)$.

2 marks

Mark allocation

- 1 mark for $\sqrt{b} \text{ cis}\left(\frac{\pi}{8}\right)$.
- 1 mark for $\sqrt{b} \text{ cis}\left(-\frac{7\pi}{8}\right)$.

d. If $a = -\frac{1}{2}$ and $b = \sqrt{2}$, plot u^{12} and w^{-2} on the Argand diagram below.



Worked solution

$a = -\frac{1}{2}$, so $u^{12} = \left[\text{cis}\left(\frac{2\pi}{3}\right)\right]^{12} = \text{cis}\left(\frac{24\pi}{3}\right) = \text{cis}\,(0)$, which is Cartesian point (1, 0).

$b = \sqrt{2}$, so $w^{-2} = \left[\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right]^{-2} = \frac{1}{2}\,\text{cis}\left(-\frac{\pi}{2}\right)$, which is Cartesian point $\left(0, -\frac{1}{2}\right)$.

2 marks

Mark allocation

- 1 mark for each point plotted correctly.

Total: 3 + 3 + 2 + 2 = 10 marks

END OF SECTION 2

END OF SOLUTIONS BOOK