INSIGHT

YEAR 12 ***Trial Exam Paper***

2012

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- correct solutions with full working
- mark allocations
- tips and guidelines

Question 1

a. Show that for $0 < x < 1$, $\dfrac{d}{dx}\left(\arcsin(2x-1)\right) = \dfrac{1}{\sqrt{x-x^2}}$.

Worked solution

$$\frac{d}{dx}\left(\arcsin(2x-1)\right)$$

$$= 2 \times \frac{1}{\sqrt{1-(2x-1)^2}}$$

$$= 2 \times \frac{1}{\sqrt{1-(4x^2-4x+1)}}$$

$$= 2 \times \frac{1}{\sqrt{4x-4x^2}}$$

$$= \frac{2}{2\sqrt{x-x^2}}$$

$$= \frac{1}{\sqrt{x-x^2}}$$

2 marks

Mark allocation

- 1 mark for correctly using the chain rule to differentiate $\left(\arcsin(2x-1)\right)$.
- 1 mark for simplifying.

Tip

The derivative of arcsin(u) is $\dfrac{u'}{\sqrt{1-u^2}}$

b. Hence, find the exact value of $\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{6}{\sqrt{x-x^2}}\,dx$.

Worked solution

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{6}{\sqrt{x-x^2}}\,dx$$

$$= 6\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{1}{\sqrt{x-x^2}}\,dx$$

$$= 6\left[\arcsin(2x-1)\right]_{\frac{1}{2}}^{\frac{3}{4}}$$

$$= 6\left[\arcsin\left(\frac{1}{2}\right) - \arcsin(0)\right]$$

$$= 6 \times \frac{\pi}{6}$$

$$= \pi$$

2 marks

Mark allocation

- 1 mark for the correct antiderivative.
- 1 mark for the correct answer.

END OF QUESTION 1
TURN OVER

Question 2

a. Solve the following equation over *C*.

$z^2 - 2iz + 5 = 0$

Worked solution

$z^2 - 2iz + 5 = 0$

$$\Rightarrow z = \frac{2i \pm \sqrt{(2i)^2 - 4 \times 1 \times 5}}{2 \times 1}$$

$$= \frac{2i \pm \sqrt{4i^2 + 20i^2}}{2}$$

$$= \frac{2i \pm \sqrt{24i^2}}{2}$$

$$= \frac{2i \pm \left(2\sqrt{6}\right) i}{2}$$

$$= \left(1 \pm \sqrt{6}\right) i$$

2 marks

Mark allocation

- 1 mark for the correct use of the quadratic formula.
- 1 mark for the correct answer.

Question 2 – continued

b. Let $z_1 = \sqrt{3} - i$.

Express z_1 in polar form, $r cis\, \theta$ where $\theta = Arg(z_1)$.

Worked solution

$$z_1 = \sqrt{3} - i$$

$$|z_1| = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}$$

$$\Rightarrow |z_1| = \sqrt{4} = 2$$

$$Arg(z_1) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

$$= -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= -\frac{\pi}{6}$$

$$\Rightarrow z_1 = 2cis\left(\frac{-\pi}{6}\right)$$

1 mark

Mark allocation

- 1 mark for the correct answer.

Tip

z_1 is in the fourth quadrant, so $\frac{-\pi}{2} < Arg(z_1) < 0$.

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Question 2 ï continued

c. On the argand diagram below, plot and clearly label

i. z_1

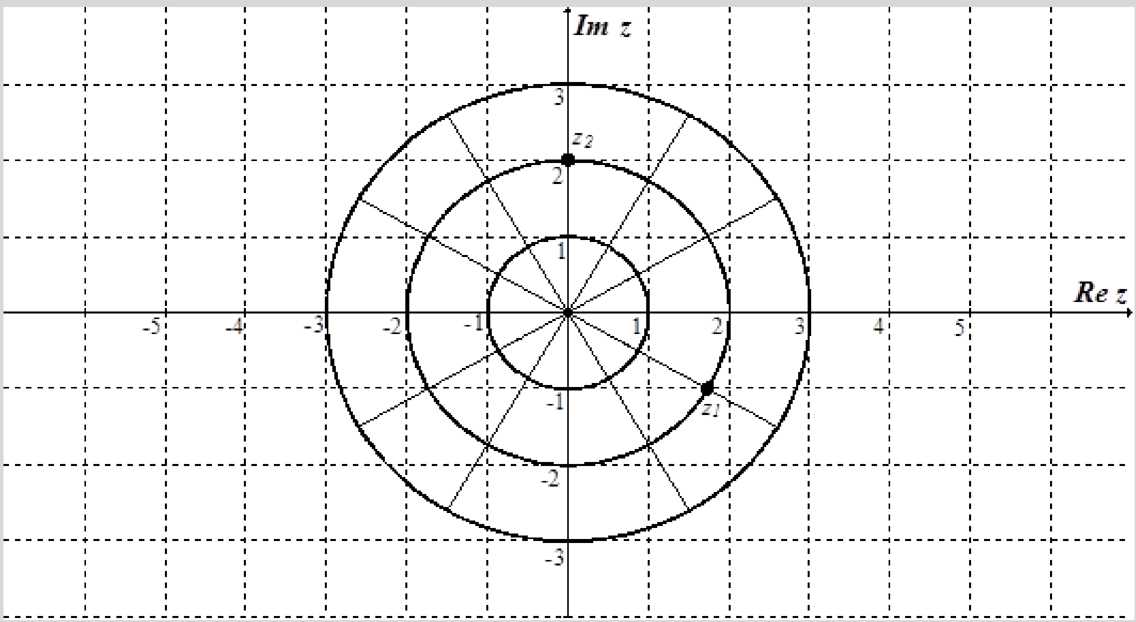
ii. $z_2 = \left|\overline{z_1}\right| i$

Worked solution

$z_1 = \sqrt{3} - i$

$z_2 = \left|\overline{z_1}\right| i = \left|z_1\right| i$

$z_2 = 2i$



2 marks

Mark allocation

- 1 mark for the correct position of z_1.
- 1 mark for the correct position of z_2.

END OF QUESTION 2
TURN OVER

Question 3

A particle moves in a straight line with an acceleration of a m/s^2, as given by

$$a = \frac{v^2 + v}{1 + \log_e(v+1)} \ , \quad v > 0.$$

At time t seconds, its displacement is x metres from a fixed point and its velocity is v m/s.
What is the displacement of the particle as it moves from its position where v = (e ï 1) m/s to its position where v = (e^2 ï 1) m/s?

Worked solution

$$a = v.\frac{dv}{dx} = \frac{v^2 + v}{1 + \log_e(v+1)}$$

$$\frac{dv}{dx} = \frac{v+1}{1 + \log_e(v+1)}$$

$$\frac{dx}{dv} = \frac{1 + \log_e(v+1)}{v+1}$$

$$x = \int_{e-1}^{e^2-1} \frac{1 + \log_e(v+1)}{v+1}.dv$$

Let $u = 1 + \log_e(v+1)$

$$\frac{du}{dv} = \frac{1}{v+1}$$

$$x = \int_2^3 u.\frac{du}{dv}.dv$$

$$= \int_2^3 u.du$$

$$= \left[\frac{u^2}{2}\right]_2^3$$

$$= \frac{1}{2}\left[3^2 - 2^2\right]$$

$$= \frac{5}{2}$$

$\therefore$ The displacement $\frac{5}{2}$ or 2.5 is metres.

3 marks

Question 3 ï continued

Mark allocation

- 1 mark for the correct differential equation for $\frac{dx}{dv}$.
- 1 mark for antidifferentiating correctly.
- 1 mark for the correct answer.

Tip

Since x is required as a function of v, and the acceleration is given as a function of v, then a is replaced by $v.\frac{dv}{dx}$.

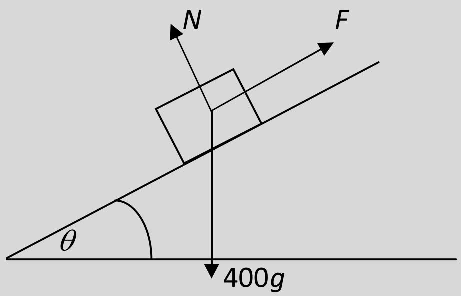
END OF QUESTION 3
TURN OVER

Question 4

A container of mass 400 kg rests on the rough surface of an inclined tray truck. The tray is inclined at an angle of $\theta°$ to the horizontal.

a. On the diagram below, clearly label the three forces, including the normal force, N, and the friction force, F, acting on the container.

Worked solution



1 mark

Mark allocation

- 1 mark for correctly labelling the three forces.

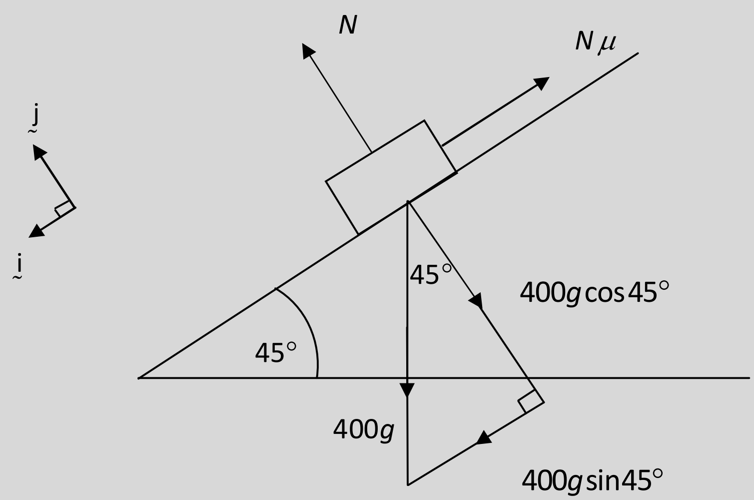
Question 4 ï continued

When the tray is raised to an angle of 45° to the horizontal, the container accelerates down the tray at $\frac{g\sqrt{2}}{20}$ m/s^2.

b. What is the coefficient of friction between the container and the surface of the tray?

Worked solution

Label the forces acting on the container, parallel and perpendicular to the tray.



$$\underset{\sim}{R} = (400g\sin 45° - N\mu)\ \underset{\sim}{i} + (N - 400g\cos 45°)\ \underset{\sim}{j} = 400 \times \frac{g\sqrt{2}}{20}\ \underset{\sim}{i}$$

$$\underset{\sim}{R} = (200g\sqrt{2} - N\mu)\ \underset{\sim}{i} + (N - 200g\sqrt{2})\ \underset{\sim}{j} = 20g\sqrt{2}\ \underset{\sim}{i}$$

$$N - 200g\sqrt{2} = 0$$

$$N = 200g\sqrt{2}$$

$$200g\sqrt{2} - N\mu = 20g\sqrt{2}$$

$$200g\sqrt{2} - 200g\sqrt{2}\mu = 20g\sqrt{2}$$

$$200 - 200\mu = 20$$

$$200\mu = 180$$

$$\Rightarrow \mu = \frac{180}{200} = 0.9$$

3 marks

Mark allocation

- 1 mark for the correct equation of motion.
- 1 mark for correctly evaluating the normal, N.
- 1 mark for the correct answer.

END OF QUESTION 4
TURN OVER

Question 5

a. Use a compound angle formula to show that $\cos\left(\frac{\pi}{12}\right)=\frac{\sqrt{6}+\sqrt{2}}{4}$.

Worked solution

$$\cos\left(\frac{\pi}{12}\right)=\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$
$$=\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$
$$=\frac{1}{2}\times\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}$$
$$=\frac{\sqrt{6}+\sqrt{2}}{4}$$

1 mark

Mark allocation

- 1 mark for correctly evaluating the appropriate compound angle formula.

Tip

$\cos\left(\frac{\pi}{12}\right)=\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$ *could also be used to obtain the correct answer.*

Question 5 ï continued

b. Hence, evaluate $\int_{\frac{\pi}{12}}^{\frac{\pi}{2}} 4\sin x\cos^3 x.dx$

Express the answer in the form $\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^n$, where n is an integer.

Worked solution

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{2}} 4\sin x\cos^3 x.dx$$

Let $u = \cos x$

$$\Rightarrow \frac{du}{dx} = -\sin x$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{2}} 4\sin x\cos^3 x.dx = \int_{\cos\left(\frac{\pi}{12}\right)}^{\cos\left(\frac{\pi}{2}\right)} -4\frac{du}{dx}u^3.dx$$

$$= \int_{\cos\left(\frac{\pi}{12}\right)}^{\cos\left(\frac{\pi}{2}\right)} -4u^3.du$$

$$= \left[-u^4\right]_{\cos\frac{\pi}{12}}^{\cos\frac{\pi}{2}}$$

$$= -\cos^4\left(\frac{\pi}{2}\right) + \cos^4\left(\frac{\pi}{12}\right)$$

$$= 0 + \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^4$$

$$= \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^4$$

2 marks

Mark allocation

- 1 mark for the correct antiderivative.
- 1 mark for the correct answer.

END OF QUESTION 5
TURN OVER

Question 6

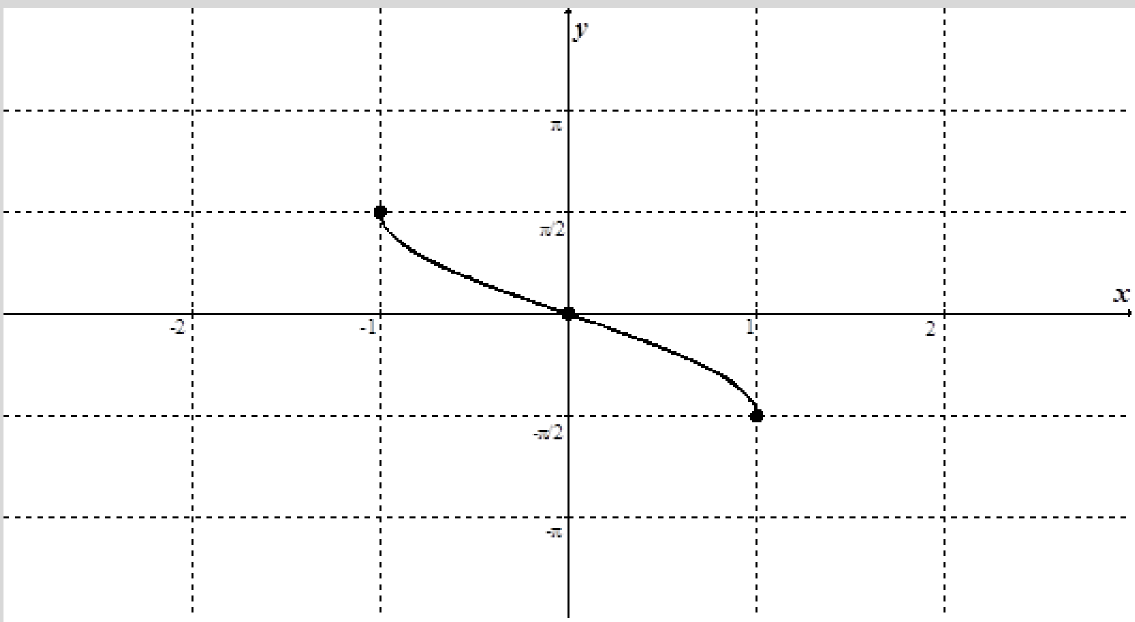
a. Sketch the graph of the curve with equation $y = \cos^{-1}(x) - \frac{\pi}{2}$. on the set of axes below.

Worked solution

$x = -1, y = \cos^{-1}(-1) - \frac{\pi}{2} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

$x = 0, y = \cos^{-1}(0) - \frac{\pi}{2} = \frac{\pi}{2} - \frac{\pi}{2} = 0$

$x = 1, y = \cos^{-1}(1) - \frac{\pi}{2} = 0 - \frac{\pi}{2} = -\frac{\pi}{2}$



1 mark

Mark allocation

- 1 mark for the correct graph.

Question 6 ï continued

b. Find the volume generated when the region enclosed by the curve with equation $y = \cos^{-1}(x) - \frac{\pi}{2}$, the y-axis and the lines $y = 0$ and $y = \frac{\pi}{2}$ is rotated about the y-axis to form a solid of revolution.

Worked solution

$$y = \cos^{-1}(x) - \frac{\pi}{2}$$

$$\cos^{-1}(x) = y + \frac{\pi}{2}$$

$$\Rightarrow x = \cos\left(y + \frac{\pi}{2}\right)$$

$$x^2 = \cos^2\left(y + \frac{\pi}{2}\right)$$

$$\text{Volume } = \pi\int_0^{\frac{\pi}{2}} \cos^2\left(y + \frac{\pi}{2}\right).dy$$

$$= \pi\int_0^{\frac{\pi}{2}} \frac{1 + \cos(2y + \pi)}{2}.dy$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} (1 + \cos(2y + \pi)).dy$$

$$= \frac{\pi}{2}\left[y + \frac{1}{2}\sin(2y + \pi)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2} \times \sin(2\pi)\right) - \left(0 + \frac{1}{2}\sin(\pi)\right)\right]$$

$$= \frac{\pi}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi^2}{4}$$

4 marks

Mark allocation

- 1 mark for correctly expressing x as a function of y.
- 1 mark for the correct definite integral representing the volume.
- 1 mark for correctly using the double angle formula in the integrand.
- 1 mark for the correct answer.

END OF QUESTION 6
TURN OVER

Question 7

For the relation $\log_e(xy) = x^2y^2$, show that $\frac{dy}{dx} = \frac{\text{ï}\, y}{x}$.

Worked solution

$$\log_e(xy) = x^2y^2$$

$$\log_e(x) + \log_e(y) = x^2y^2$$

$$\frac{d}{dx}\left(\log_e(x) + \log_e(y)\right) = \frac{d}{dx}\left(x^2y^2\right)$$

$$\frac{1}{x} + \frac{d}{dy}\left(\log_e(y)\right).\frac{dy}{dx} = 2x.y^2 + x^2.\frac{d}{dy}\left(y^2\right).\frac{dy}{dx}$$

$$\frac{1}{x} + \frac{1}{y}.\frac{dy}{dx} = 2xy^2 + 2yx^2.\frac{dy}{dx}$$

$$\frac{1}{x} - 2xy^2 = \frac{dy}{dx}\left(2yx^2 - \frac{1}{y}\right)$$

$$\frac{1 - 2x^2y^2}{x} = \frac{dy}{dx}\left(\frac{2x^2y^2 - 1}{y}\right)$$

$$\frac{dy}{dx} = \frac{1 - 2x^2y^2}{x} \times \frac{y}{2x^2y^2 - 1}$$

$$\frac{dy}{dx} = \frac{-\left(2x^2y^2 - 1\right)}{x} \times \frac{y}{2x^2y^2 - 1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

3 marks

Mark allocation

- 1 mark for correctly differentiating the relation using implicit differentiation.
- 1 mark for correctly expressing the relation as a function of $\frac{dy}{dx}$.
- 1 mark for simplifying correctly.

Tip

The relation cannot be expressed explicitly as a function of x, so implicit differentiation is required to obtain $\frac{dy}{dx}$.

END OF QUESTION 7

Question 8

The position vector of a moving particle, $\underset{\sim}{r}(t)$ metres, at any time, t seconds, is given by

$$\underset{\sim}{r}(t) = 2\tan(t)\,\underset{\sim}{i} + \sec^2(t)\,\underset{\sim}{j}, \quad t \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

a. Determine the Cartesian equation for the path of the particle. State the domain and range.

Worked solution

$$\underset{\sim}{r}(t) = 2\tan(t)\underset{\sim}{i} + \sec^2(t)\underset{\sim}{j}$$

$$x = 2\tan(t)$$

$$\Rightarrow \tan(t) = \frac{x}{2}$$

$$y = \sec^2(t)$$

$$= 1 + \tan^2(t)$$

$$\Rightarrow y = 1 + \frac{x^2}{4}$$

Domain, $x = 2\tan(t)$, $t \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow x \in R$

Range, $y = 1 + \frac{x^2}{4}$

$\Rightarrow y \in [1, \infty)$ since $x \in R$

3 marks

Mark allocation

- 1 mark for the correct Cartesian equation.
- 1 mark for the correct domain.
- 1 mark for the correct range.

Tip

Use the identity $\sec^2(t) = 1 + \tan^2(t)$.

Question 8 ï continued
TURN OVER

b. Find the minimum speed of the particle.

Worked solution

$$\underset{\sim}{v}(t) = \frac{d}{dt}[2\tan(t)\underset{\sim}{i} + \sec^2(t)\underset{\sim}{j}]$$

$$\underset{\sim}{v}(t) = \frac{d}{dt}[2\tan(t)\underset{\sim}{i} + \left(\cos(t)\right)^{-2}\underset{\sim}{j}]$$

$$\underset{\sim}{v}(t) = 2\sec^2(t)\underset{\sim}{i} + (-2)(-\sin(t))(\cos(t))^{-3}\underset{\sim}{j}$$

$$\underset{\sim}{v}(t) = 2\sec^2(t)\underset{\sim}{i} + \frac{2\sin(t)}{\cos^3(t)}\underset{\sim}{j}$$

$$\underset{\sim}{v}(t) = 2\sec^2(t)\underset{\sim}{i} + 2\tan(t)\sec^2(t)\underset{\sim}{j}$$

$$\Rightarrow v = \sqrt{4\sec^4(t) + 4\tan^2(t)\sec^4(t)}$$

$$= \sqrt{4\sec^4(t)\left(1 + \tan^2(t)\right)}$$

$$= \sqrt{4\sec^4(t)\left(\sec^2(t)\right)}$$

$$= \sqrt{4\sec^6(t)}$$

$$= 2\left|\sec^3(t)\right|$$

Minimum value of $\left|\sec(t)\right|$ is 1.

$\Rightarrow$ The minimum value of $\left|\sec^3(t)\right|$ is 1.

$\therefore$ The minimum speed is 2 m/s.

3 marks

Mark allocation

- 1 mark for the correct velocity vector.
- 1 mark for the correct equation of the speed as a function of *t*.
- 1 mark for the correct answer.

Tip

Use the identity $\sec^2(t) = 1 + \tan^2(t)$.

End of Question 8

Question 9

Three points, A, B and O, are given by $A(2,1,2)$, $B(2,2,0)$ and $O(0,0,0)$.

a. Find the vector $\overrightarrow{AB}$ expressed in the form $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$.

Worked solution

$\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{OB} = 2\underset{\sim}{i} + 2\underset{\sim}{j}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 2\underset{\sim}{i} + 2\underset{\sim}{j} - (2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k})$

$= \underset{\sim}{j} - 2\underset{\sim}{k}$

1 mark

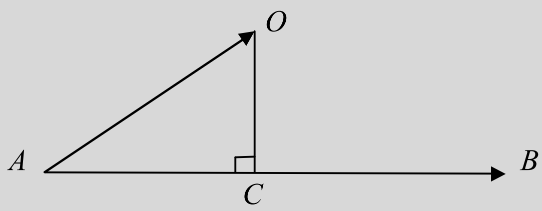
Mark allocation

- 1 mark for the correct answer.

Question 9 ï continued
TURN OVER

b. A point, C, on vector $\overrightarrow{AB}$ is closest to O. Find the coordinates of point C.

Worked solution



Let C be the point on $\overrightarrow{AB}$ where $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$.

$$\overrightarrow{AC} = \frac{\overrightarrow{AO}.\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} \times \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}$$

$$= \frac{1}{\left|\overrightarrow{AB}\right|^2}\left(\overrightarrow{AO}.\overrightarrow{AB}\right)\overrightarrow{AB}$$

$$= \frac{1}{\left(\sqrt{1^2+(-2)^2}\right)^2}(-2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}).\ (\underset{\sim}{j} - 2\underset{\sim}{k})\ (\underset{\sim}{j} - 2\underset{\sim}{k})$$

$$= \frac{1}{5}(3)\ (\underset{\sim}{j} - 2\underset{\sim}{k})$$

$$= \frac{3}{5}\underset{\sim}{j} - \frac{6}{5}\underset{\sim}{k}$$

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$

$$= 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k} + \frac{3}{5}\underset{\sim}{j} - \frac{6}{5}\underset{\sim}{k}$$

$$= 2\underset{\sim}{i} + \frac{8}{5}\underset{\sim}{j} + \frac{4}{5}\underset{\sim}{k}$$

$$\Rightarrow C \text{ is } \left(2, \frac{8}{5}, \frac{4}{5}\right)$$

3 marks

Question 9 ï continued

Mark allocation

- 1 mark for the correct vector $\overrightarrow{AC}$.
- 1 mark for the correct vector $\overrightarrow{OC}$.
- 1 mark for the correct answer.

Tip

The vector resolute of $\underset{\sim}{a}$ *onto* $\underset{\sim}{b}$ *is* $(\underset{\sim}{a}.\underset{\sim}{b})\,\underset{\sim}{b} \div b^2$

The vector $\overrightarrow{OC}$ *could also have been found by finding the vector resolute of* $\overrightarrow{BO}$ *onto* $\overrightarrow{BA}$.

END OF QUESTION 9
TURN OVER

Question 10

On the axes supplied, sketch the graph of $f:[0, 2\pi] \to R,\ f(x) = \cot\left(\frac{x}{2}\right) - 1$, clearly indicating the location of any asymptotes and intercepts with the axes.

Worked solution

$f:[0, 2\pi] \to R,\ f(x) = \cot\left(\frac{x}{2}\right) - 1$

Asymptotes:

$$\cot\left(\frac{x}{2}\right) = \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

$$\sin\left(\frac{x}{2}\right) = 0$$

$$\Rightarrow \frac{x}{2} = 0, \pi, 2\pi, \ldots$$

$$x = 0, 2\pi, 4\pi, \ldots$$

$\therefore$ The asymptotes are $x = 0$ and $x = 2\pi$, for $x \in [0, 2\pi]$.

$\therefore$ There is no *y*-intercept.

***x*-intercept:**

$$\cot\left(\frac{x}{2}\right) - 1 = 0$$

$$\cot\left(\frac{x}{2}\right) = 1$$

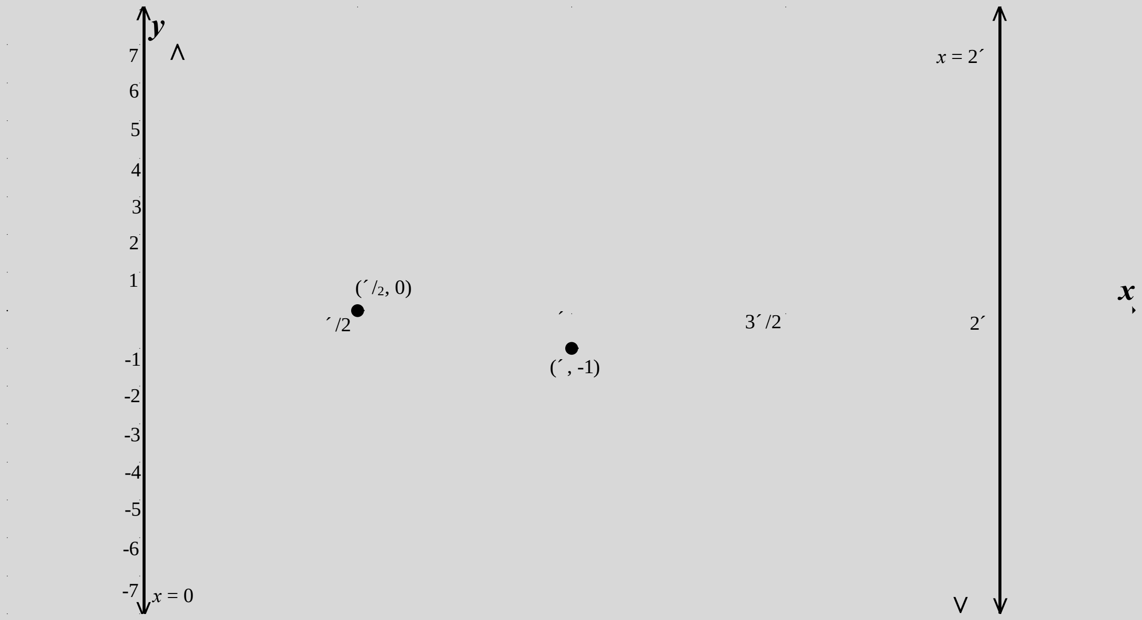
$$\Rightarrow \tan\left(\frac{x}{2}\right) = 1$$

$$\frac{x}{2} = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$$

$$x = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$$

$\therefore$ The *x*-intercept is $x = \frac{\pi}{2}$, for $x \in [0, 2\pi]$.

Question 10 – continued



3 marks

Mark allocation

- 1 mark for correctly labelling the *x*-intercept.
- 1 mark for correctly labelling both of the vertical asymptotes.
- 1 mark for the correct curve passing through $(\pi, -1)$.

Tip

Vertical asymptotes occur where the denominator of a rational function equals zero.

END OF SOLUTIONS BOOK

END OF SOLUTIONS BOOK