SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1

2011 Trial Examination

SOLUTIONS

Question 1

The complex number $a+ib$ can be regarded as the rotation of the

complex number $-\sqrt{3}-3i$ anticlockwise by an angle of $\frac{\pi}{3}$ radians.......[M1]

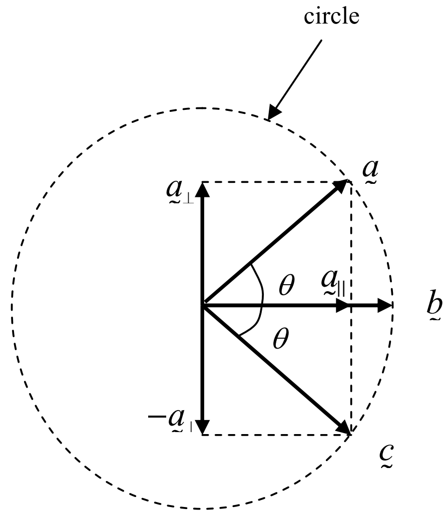
$$\left.\begin{aligned} a+ib &= (-\sqrt{3}-3i)\times cis\left(\frac{\pi}{3}\right) \\ &= (-\sqrt{3}-3i)\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right) \end{aligned}\right\} \text{.......[M1]}$$

$$\left.\begin{aligned} &= (-\sqrt{3}-3i)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}+\frac{3\sqrt{3}}{2}+i\left(-\frac{3}{2}-\frac{3}{2}\right) \\ &\therefore \quad a+ib=\sqrt{3}-3i \qquad \therefore \qquad a=\sqrt{3} \quad \text{and} \quad b=-3 \end{aligned}\right\} \text{.......[A1]}$$

Question 2

See the diagram

$\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$



$$\underset{\sim}{a}_{\parallel} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2} \underset{\sim}{b} = \frac{1-1+1}{1+1+1}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) = \frac{1}{3}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j} + \frac{1}{3}\underset{\sim}{k} \quad \text{........}[M1]$$

$$\underset{\sim}{a}_{\perp} = \underset{\sim}{a} - \underset{\sim}{a}_{\parallel} \quad , \quad \underset{\sim}{c} = \underset{\sim}{a}_{\parallel} + (-\underset{\sim}{a}_{\perp}) \quad \text{........}[M1]$$

$$\underset{\sim}{c} = 2\underset{\sim}{a}_{\parallel} - \underset{\sim}{a} \quad \therefore \quad \underset{\sim}{c} = 2\left(\frac{1}{3}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j} + \frac{1}{3}\underset{\sim}{k}\right) - (\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}) \quad \text{........}[A1]$$

$$\underset{\sim}{c} = -\frac{1}{3}\underset{\sim}{i} + \frac{5}{3}\underset{\sim}{j} - \frac{1}{3}\underset{\sim}{k} \quad \text{........}[A1]$$

Question 3

a.

$$\sin^{-1} x + \frac{\pi}{6} = \sin^{-1}\left(\frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{1-x^2}\right)$$

We have to show that $LHS = RHS \quad for \quad x = \frac{\sqrt{3}}{2}$

$$LHS = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6} = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2} \quad \text{........}[A1]$$

$$RHS = \sin^{-1}\left(\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2}\sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$= \sin^{-1}\left(\frac{3}{4} + \frac{1}{2} \times \frac{1}{2}\right) = \sin^{-1}(1) = \frac{\pi}{2} \quad \text{........} \quad \therefore \quad LHS = RHS \quad \text{........}[A1]$$

b.

Apply sine to both sides:

$$\sin\left(\sin^{-1}x+\frac{\pi}{6}\right)=\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}x+\frac{1}{2}\sqrt{1-x^2}\right)\right)$$

$$\therefore\quad \sin\left(\sin^{-1}x+\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}x+\frac{1}{2}\sqrt{1-x^2}$$

..........[M1]

Let $LHS=\sin\left(\sin^{-1}x+\frac{\pi}{6}\right)$ and $\text{RHS}=\frac{\sqrt{3}}{2}x+\frac{1}{2}\sqrt{1-x^2}$

$$LHS=\sin\left(\sin^{-1}x+\frac{\pi}{6}\right)=\sin\left(\sin^{-1}x\right)\times\cos\left(\frac{\pi}{6}\right)+\cos\left(\sin^{-1}x\right)\times\sin\left(\frac{\pi}{6}\right)$$

.........[M1]

As $\sin^{-1}x$ is an angle in the first quadrant, $\cos\left(\sin^{-1}x\right)=\sqrt{1-x^2}$

$$\therefore\quad LHS=\frac{\sqrt{3}}{2}x+\frac{1}{2}\sqrt{1-x^2}\quad\therefore\quad LHS=RHS$$

.........[A1]

Question 4

$$\left.\begin{aligned}&3x^2-2xy+2y^3=7 \qquad y=1,\ x<0\\&3x^2-2x+2=7 \quad \therefore \quad 3x^2-2x-5=0 \quad \therefore \quad (x+1)(3x-5)=0\\&\therefore \quad x=-1 \quad \text{as} \quad x<0\end{aligned}\right\}\text{.....[M1]}$$

$$\left.\begin{aligned}&\frac{d}{dx}\left(3x^2-2xy+2y^3\right)=\frac{d}{dx}(7)\\&6x-2y-2x\frac{dy}{dx}+6y^2\frac{dy}{dx}=0\\&-2x\frac{dy}{dx}+6y^2\frac{dy}{dx}=-6x+2y\end{aligned}\right\}\qquad\text{.....[M1]}$$

$$\left.\begin{aligned}&\text{Find } \frac{dy}{dx} \text{ at } (-1,1)\\&-2\times(-1)\frac{dy}{dx}+6\times1\times\frac{dy}{dx}=6\times(-1)+2\times1\\&8\times\frac{dy}{dx}=8 \quad \therefore \quad \frac{dy}{dx}=1\end{aligned}\right\}\quad\text{.....[A1]}$$

Equation of the normal : $y-1=-(x+1) \quad \therefore \quad y=-x$ ....[A1]

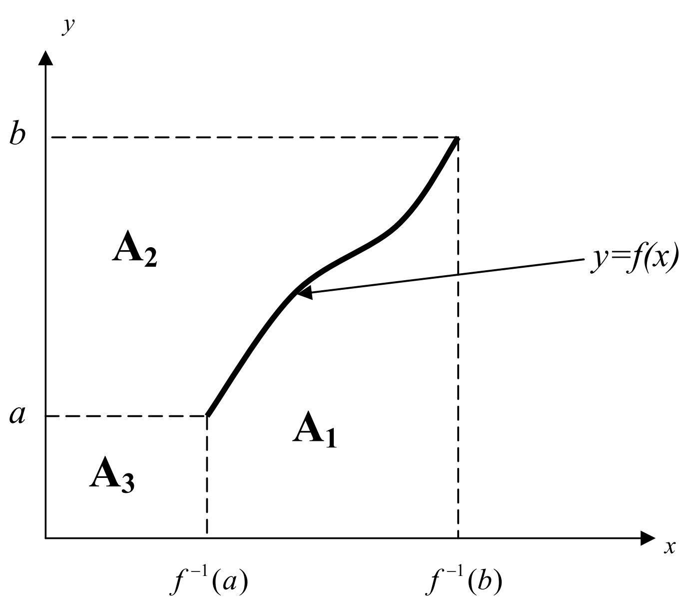
Question 5

See the diagram below

Let A be the area of the big rectangle , $A = b\,f^{-1}(b)$
As seen from the diagram, this area A can be
written as the sum of three areas:
$A = A_1 + A_2 + A_3$ ....[M1]

$A_1 = \int_{f^{-1}(a)}^{f^{-1}(b)} y\,dx$, $y = f(x)$ $\therefore$ $A_1 = \int_{f^{-1}(a)}^{f^{-1}(b)} f(x)dx$
$A_2 = \int_a^b x\,dy$, $x = f^{-1}(y)$ $\therefore$ $A_2 = \int_a^b f^{-1}(y)dy$ ....[M1]

$A_3 =$ the area of the small rectangle , $A_3 = a\,f^{-1}(a)$
$b\,f^{-1}(b) = \int_{f^{-1}(a)}^{f^{-1}(b)} f(x)dx + \int_a^b f^{-1}(y)dy + a\,f^{-1}(a)$
$\int_{f^{-1}(a)}^{f^{-1}(b)} f(x)dx + \int_a^b f^{-1}(y)dy = b\,f^{-1}(b) - a\,f^{-1}(a)$
as required ....[A1]



Question 6

Refer to the graph below

$\int_{-3}^{-1} \log_e |x| \, dx$ is equal to the area A

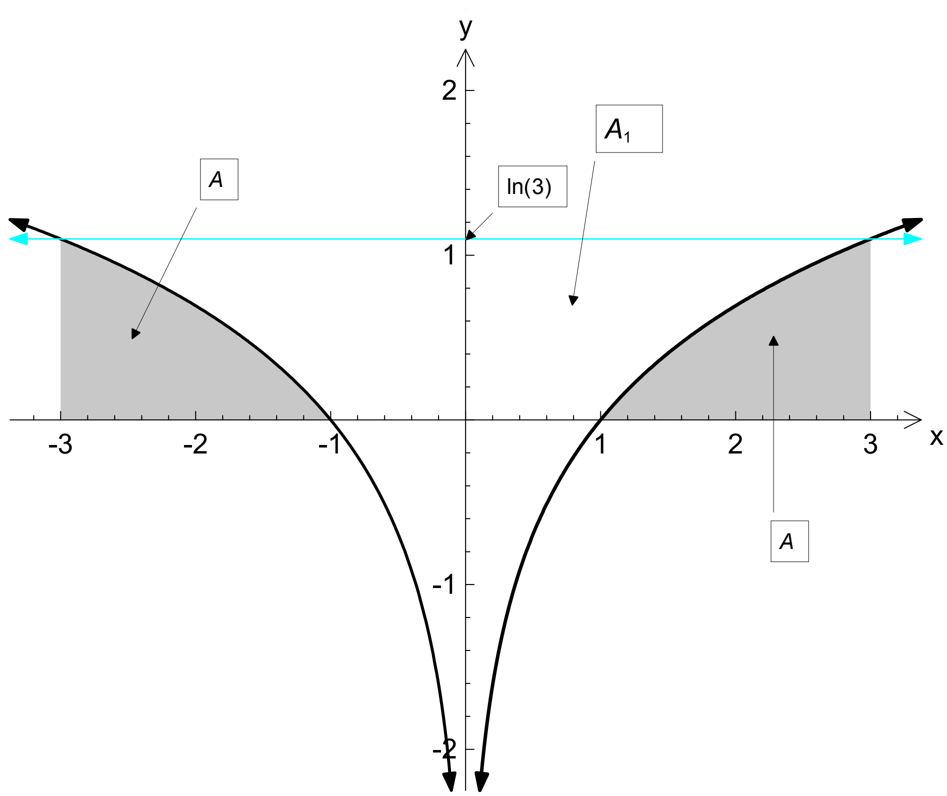
But also, from symmetry,

$A = \int_{1}^{3} \log_e |x| \, dx = \int_{1}^{3} \log_e(x) dx \quad \text{as} \quad |x| = x$ ....[M1]

$A_1 = \int_{0}^{\log_e 3} x dy \quad , \quad y = \log_e(x) \quad \therefore \quad x = e^y$

$\therefore \quad A_1 = \int_{0}^{\log_e 3} e^y dy = \left[e^y\right]_0^{\log_e 3} = 2$ ....[M1]

$A = 3\log_e 3 - 2$ ....[A1]



Question 7

a.

$$\left.\begin{array}{l} \underset{\sim}{r}_A(t)=\int(-2\underset{\sim}{i}+3\underset{\sim}{j})dt \quad \therefore \quad \underset{\sim}{r}_A(t)=(-2\underset{\sim}{i}+3\underset{\sim}{j})t+\underset{\sim}{c}_1 \\ \text{At} \quad t=0 \qquad \underset{\sim}{r}_A(0)=4\underset{\sim}{i}-3\underset{\sim}{j} \quad \therefore \quad \underset{\sim}{c}_1=4\underset{\sim}{i}-3\underset{\sim}{j} \\ \therefore \quad \underset{\sim}{r}_A(t)=(-2\underset{\sim}{i}+3\underset{\sim}{j})t+4\underset{\sim}{i}-3\underset{\sim}{j} \quad \therefore \quad \underset{\sim}{r}_A(t)=(-2t+4)\underset{\sim}{i}+(3t-3)\underset{\sim}{j} \end{array}\right\} \quad \ldots\ldots[\text{M1}]$$

$$\left.\begin{array}{l} \underset{\sim}{r}_B(t)=\int \underset{\sim}{v}_B dt \quad \therefore \quad \underset{\sim}{r}_B(t)=\underset{\sim}{v}_B t+\underset{\sim}{c}_2 \\ \text{At} \quad t=0 \qquad \underset{\sim}{r}_B(0)=-\underset{\sim}{i}+\underset{\sim}{j} \quad \therefore \quad \underset{\sim}{c}_2=-\underset{\sim}{i}+\underset{\sim}{j} \quad \therefore \quad \underset{\sim}{r}_B(t)=\underset{\sim}{v}_B t-\underset{\sim}{i}+\underset{\sim}{j} \end{array}\right\} \quad \ldots\ldots[\text{M1}]$$

$$\left.\begin{array}{l} \text{When collision occurs} \quad \underset{\sim}{r}_A(1)=\underset{\sim}{r}_B(1) \\ (-2\times1+4)\underset{\sim}{i}+(3\times1-3)\underset{\sim}{j}=\underset{\sim}{v}_B\times1-\underset{\sim}{i}+\underset{\sim}{j} \\ 2\underset{\sim}{i}=\underset{\sim}{v}_B-\underset{\sim}{i}+\underset{\sim}{j} \quad \therefore \quad \underset{\sim}{v}_B=3\underset{\sim}{i}-\underset{\sim}{j} \end{array}\right\} \quad \ldots\ldots[\text{A1}]$$

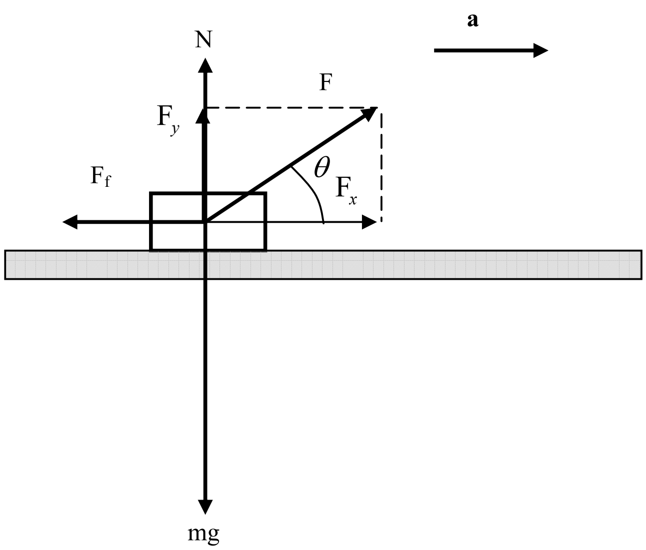
b.

$$\left.\begin{array}{l} d=\left|\underset{\sim}{r}_B(1)-\underset{\sim}{r}_B(0)\right|=\left|\underset{\sim}{v}_B-\underset{\sim}{i}+\underset{\sim}{j}-(-\underset{\sim}{i}+\underset{\sim}{j})\right| \\ d=\left|\underset{\sim}{v}_B\right|=\sqrt{3^2+1} \quad \therefore \quad d=\sqrt{10} \end{array}\right\} \quad \ldots\ldots[\text{A1}]$$

Question 8

a.

See the diagram below
Draw in the weight, the normal reaction force and the friction force........[A1]



b.

$$\left.\begin{array}{l} F_x - F_f = ma \\ N + F_y - mg = 0 \\ F_f = \mu N \end{array}\right\} \quad ....[\text{M1}]$$

$$\Rightarrow \quad \left.\begin{array}{l} F\cos\theta - F_f = ma \\ N + F\sin\theta - mg = 0 \\ F_f = \mu N \end{array}\right\} \quad ....[\text{M1}]$$

$$\Rightarrow F\cos\theta - \mu(mg - F\sin\theta) = ma \quad ....[\text{M1}]$$

$$\left.\begin{array}{l} F(\cos\theta + \mu\sin\theta) - \mu mg = ma \\ \therefore \quad a = \dfrac{F}{m}(\cos\theta + \mu\sin\theta) - \mu g \end{array}\right\} ....[\text{A1}]$$

Question 9

a.

$$\left.\begin{array}{l} x = 2\cos(nt+\varepsilon) \quad , \quad y = 3\sin(nt+\varepsilon) \\ t=0 \quad x=-2 \quad y=0 \quad \therefore \quad -2 = 2\cos\varepsilon \quad and \quad 0 = 3\sin\varepsilon \\ \therefore \quad \varepsilon = \pi \quad \therefore \quad x = 2\cos(nt+\pi) \quad \text{and} \quad y = 3\sin(nt+\pi) \\ \therefore \quad x = -2\cos(nt) \quad \text{and} \quad y = -3\sin(nt) \end{array}\right\} \quad ....[\text{M1}]$$

$$\left.\begin{array}{l} \underset{\sim}{r}(t) = -2\cos(nt)\,\underset{\sim}{i} - 3\sin(nt)\,\underset{\sim}{j} \quad \therefore \quad \underset{\sim}{v}(t) = 2n\sin(nt)\,\underset{\sim}{i} - 3n\cos(nt)\,\underset{\sim}{j} \\ \underset{\sim}{v}(0) = 2n\sin(n\times 0)\,\underset{\sim}{i} - 3n\cos(n\times 0)\,\underset{\sim}{j} \quad \therefore \quad \underset{\sim}{v}(0) = -3n \end{array}\right\} \quad ....[\text{M1}]$$

$$speed = |\underset{\sim}{v}(0)| \quad \therefore \quad 3n = 6 \quad \therefore \quad n = 2 \therefore \quad x = -2\cos(2t) \quad y = -3\sin(2t) \quad ....[\text{A1}]$$

b.

$$\underset{\sim}{r}(t) = -2\cos(2t)\,\underset{\sim}{i} - 3\sin(2t)\,\underset{\sim}{j} \quad \therefore \quad \underset{\sim}{v}(t) = 4\sin(2t)\,\underset{\sim}{i} - 6\cos(2t)\,\underset{\sim}{j} \quad ....[\text{M1}]$$

$$\left.\begin{array}{l} T = \pi \quad \therefore \quad \underset{\sim}{v}\left(\frac{\pi}{4}\right) = 4\sin\left(2\times\frac{\pi}{4}\right)\underset{\sim}{i} - 6\cos\left(2\times\frac{\pi}{4}\right)\underset{\sim}{j} = 4\underset{\sim}{i} \\ \therefore \quad speed = 4ms^{-1} \end{array}\right\} ....[\text{A1}]$$

Question 10

a.

$$a = -kv \quad \therefore \quad \frac{dv}{dt} = -kv \quad \therefore \quad \frac{dt}{dv} = -\frac{1}{kv}$$

$$\therefore \quad t = -\frac{1}{k}\int \frac{dv}{v} \quad \therefore \quad t = -\frac{1}{k}\log_e v + c$$

$$\text{At } t = 0 \quad v = u \quad \therefore \quad 0 = -\frac{1}{k}\log_e u + c \quad \therefore \quad c = \frac{1}{k}\log_e u$$

........[M1]

$$t = -\frac{1}{k}\log_e v + \frac{1}{k}\log_e u \quad \therefore \quad t = \frac{1}{k}\log_e\left(\frac{u}{v}\right) \quad \therefore \quad \frac{u}{v} = e^{kt} \quad \therefore \quad v = ue^{-kt}$$

$$\frac{u}{e} = ue^{-2k} \quad \therefore \quad e^{-2k} = e^{-1} \quad \therefore \quad 2k = 1 \quad \therefore \quad k = \frac{1}{2} \text{ and } v(t) = ue^{-\frac{1}{2}t}$$

........[A1]

b.

$$d = \int_0^2 ue^{-\frac{1}{2}t}\,dt = -2u\left[e^{-\frac{1}{2}t}\right]_0^2$$ ....[M1]

$$d = -2u(e^{-1} - 1) = 2u\left(1 - \frac{1}{e}\right)$$ ....[A1]