Neap:

Trial Examination 2011

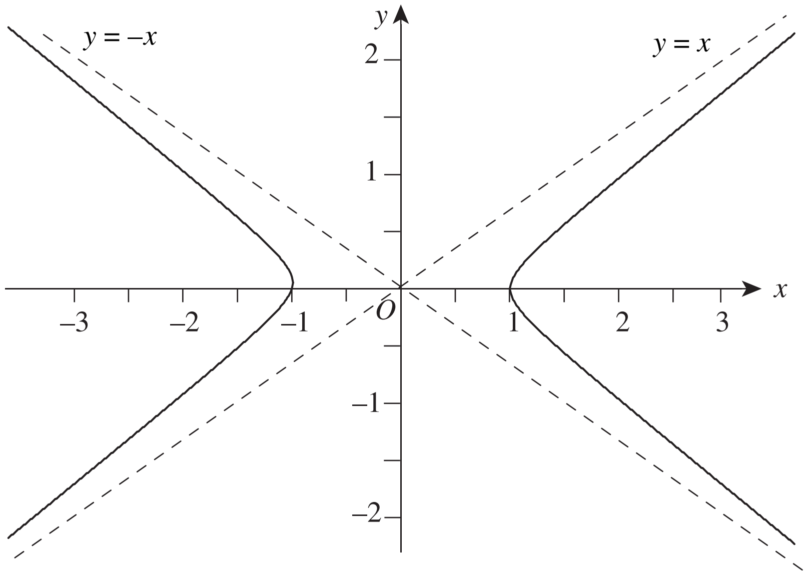
VCE Specialist Mathematics Units 3 & 4

Written Examination 1

Suggested Solutions

Question 1

a.



Two correct branches crossing the x-axis at $(\pm 1, 0)$. A1

Asymptotes $y = \pm x$. A1

b. Let the volume be V, where $V = \pi \int_{1}^{\sqrt{3}} y^2 dx$ and $y^2 = x^2 - 1$.

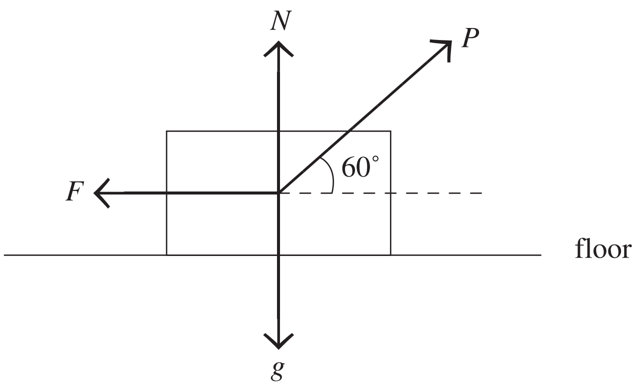
$$= \pi \left[\frac{x^3}{3} - x \right]_1^{\sqrt{3}}$$ A1

$$= \pi \left[(\sqrt{3} - \sqrt{3}) - \left(\frac{1}{3} - 1 \right) \right]$$

$$= \frac{2\pi}{3} \text{ (cubic units)}$$ A1

Question 2

a.



A1

b. The body is moving, hence $F = \frac{N}{\sqrt{2}}$.

Vertical: $N = g - P\sin(60°)$, i.e. $N = g - \frac{\sqrt{3}P}{2}$. A1

Horizontal: $a = P\cos(60°) - \frac{N}{\sqrt{2}}$, i.e. $a = \frac{P}{2} - \frac{N}{\sqrt{2}}$. A1

Substituting $N = g - \frac{\sqrt{3}P}{2}$ into $a = \frac{P}{2} - \frac{N}{\sqrt{2}}$ gives $a = \frac{P}{2} - \frac{1}{\sqrt{2}}\left(g - \frac{\sqrt{3}P}{2}\right)$.

$a = \frac{P}{2} - \frac{g}{\sqrt{2}} + \frac{\sqrt{3}P}{2\sqrt{2}}$ (or equivalent)

So $a = \frac{P - \sqrt{2}g}{2} + \frac{\sqrt{6}P}{4}$. A1

Note: $a = \frac{P}{2} - \frac{g}{\sqrt{2}} + \frac{\sqrt{3}P}{2\sqrt{2}}$ *(or equivalent) is needed for the final A1.*

Question 3

Let $u = \sin(x)$ and so $\frac{du}{dx} = \cos(x)$. When $x = 0, u = 0$ and when $x = \frac{\pi}{2}, u = 1$.

So $\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{1 + \sin^2(x)}dx = \int_0^1 \frac{1}{1 + u^2}du$ A1

$= [\tan^{-1}(u)]_0^1$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{\pi}{4}$ A1

Question 4

Using the scalar product, i.e. $\cos(\theta) = \frac{(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) \cdot (5\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k})}{\sqrt{3}\sqrt{27}}$ and so $\cos(\theta) = \frac{1}{3}$. M1

Using $\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$, we obtain $2\cos^2\left(\frac{\theta}{2}\right) - 1 = \frac{1}{3}$. M1

Rearranging $2\cos^2\left(\frac{\theta}{2}\right) - 1 = \frac{1}{3}$, we obtain $\cos^2\left(\frac{\theta}{2}\right) = \frac{2}{3}$. A1

As θ is an acute angle, we reject $\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{2}{3}}$ and so $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{2}{3}}$. A1

Question 5

Using implicit differentiation to differentiate $x^2 + xy = e^y$: M1

$2x + y + x\dfrac{dy}{dx} = e^y\dfrac{dy}{dx}$ (or equivalent) A1

Let the gradient of the normal be m_N.

At $(-1, 0)$, $\dfrac{dy}{dx} = -1$ and so m_N is 1. A1

So the equation of the normal is $y = x + 1$. A1

Question 6

Attempting to use $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$. M1

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{3}{x^3} - \frac{1}{x^2}$$

$$\frac{1}{2}v^2 = \int\left(\frac{3}{x^3} - \frac{1}{x^2}\right)dx$$

$$= \frac{-3}{2x^2} + \frac{1}{x} + c \text{ where } c \text{ is an arbitrary constant}$$ A1

So $v^2 = \dfrac{-3}{x^2} + \dfrac{2}{x} + k$, where k is an arbitrary constant.

Given that $v = 0$ at $x = 1$, we find that $0 = -3 + 2 + k$, i.e. $k = 1$.

Hence, $v^2 = \dfrac{-3}{x^2} + \dfrac{2}{x} + 1$. M1

Writing as a single fraction, we obtain $v^2 = \dfrac{x^2 + 2x - 3}{x^2}$.

Taking the square root of both sides we obtain $v = \pm\sqrt{\dfrac{x^2 + 2x - 3}{x^2}}$.

As $a > 0$ when $x = 1$, v is initally positive. A1

Hence, $v = \dfrac{\sqrt{x^2 + 2x - 3}}{x}$ for $x \geq 1$. A1

Question 7

a. $\left|\underset{\sim}{r}(t)\right| = \sqrt{\sin^2(2t) + 4\cos^2(t)}$ M1

$= \sqrt{4\sin^2(t)\cos^2(t) + 4\cos^2(t)}$ (using $\sin(2t) = 2\sin(t)\cos(t)$) A1

$= \sqrt{4\cos^2(t)(1 + \sin^2(t))}$

$= 2\sqrt{(1 - \sin^2(t))(1 + \sin^2(t))}$ (using $\cos^2(t) = 1 - \sin^2(t)$) A1

Hence $\left|\underset{\sim}{r}(t)\right| = 2\sqrt{1 - \sin^4(t)}$. A1

Note: Only award the last A1 if the previous line of work is present.

b. The minimum value of $\sin^4(t)$ is zero, and so $\left|\underset{\sim}{r}(t)\right|_{\max} = 2$ A1

$\left|\underset{\sim}{r}(t)\right|_{\max}$ occurs at $t = n\pi$, where $n = 0, 1, 2, 3, \ldots$ A1

Question 8

$(z^4 - \text{cis}(\theta)) = 0$ or $(z^4 - \text{cis}(-\theta)) = 0$ where $2\cos(\theta) = 1$.

$2\cos(\theta) = 1$ and so $\theta = \frac{\pi}{3}$. A1

$z^4 = \text{cis}\left(\frac{\pi}{3}\right)$ or $z^4 = \text{cis}\left(-\frac{\pi}{3}\right)$. A1

$z = \text{cis}\left(\frac{\pi}{12} + \frac{2k\pi}{4}\right)$ or $z = \text{cis}\left(-\frac{\pi}{12} + \frac{2k\pi}{4}\right)$, where $k \in Z$. M1

Hence, $z = \text{cis}\left(\pm\frac{\pi}{12}\right), \text{cis}\left(\pm\frac{5\pi}{12}\right), \text{cis}\left(\pm\frac{7\pi}{12}\right), \text{cis}\left(\pm\frac{11\pi}{12}\right)$. A1 A1

Note: Award A1 for $\text{cis}\left(\pm\frac{\pi}{12}\right)$ and $\text{cis}\left(\pm\frac{5\pi}{12}\right)$*, and A1 for* $\text{cis}\left(\pm\frac{7\pi}{12}\right)$ and $\text{cis}\left(\pm\frac{11\pi}{12}\right)$.

Question 9

a. $$\frac{\tan\left(\tan^{-1}\left(\frac{1}{m}\right)\right)+\tan\left(\tan^{-1}\left(\frac{1}{n}\right)\right)}{1-\tan\left(\tan^{-1}\left(\frac{1}{m}\right)\right)\tan\left(\tan^{-1}\left(\frac{1}{n}\right)\right)}=\tan\left(\frac{\pi}{4}\right)\quad\left(\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}\right)$$

So $\dfrac{\frac{1}{m}+\frac{1}{n}}{1-\frac{1}{m}\times\frac{1}{n}}=1$ M1

$\dfrac{m+n}{mn-1}=1$ (multiplying numerator and denominator of the LHS by *mn*) A1

$$mn-m-n-1=0$$
$$m(n-1)-n-1=0$$
$m(n-1)-n+1=2$ (adding 2 to both sides) M1
$$m(n-1)-1(n-1)=2$$
$(m-1)(n-1)=2$ A1

b. From $(m-1)(n-1)=2$, we obtain $n=\dfrac{2}{m-1}+1$, i.e. $n=\dfrac{m+1}{m-1}$. A1

Given that $m=k$, $\dfrac{1}{n}=\dfrac{k-1}{k+1}$, we obtain $\tan^{-1}\left(\dfrac{1}{k}\right)+\tan^{-1}\left(\dfrac{k-1}{k+1}\right)=\dfrac{\pi}{4}$. A1