



Trial Examination 2011

# **VCE Specialist Mathematics Units 3 & 4**

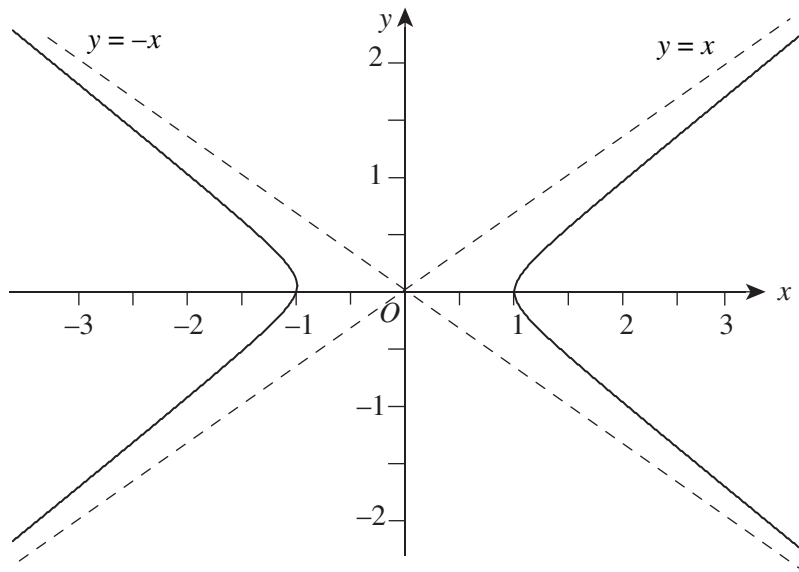
Written Examination 1

**Suggested Solutions**

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**Question 1**

a.



Two correct branches crossing the  $x$ -axis at  $(\pm 1, 0)$ .

A1

Asymptotes  $y = \pm x$ .

A1

b. Let the volume be  $V$ , where  $V = \pi \int_1^{\sqrt{3}} y^2 dx$  and  $y^2 = x^2 - 1$ .

$$= \pi \left[ \frac{x^3}{3} - x \right]_1^{\sqrt{3}}$$

A1

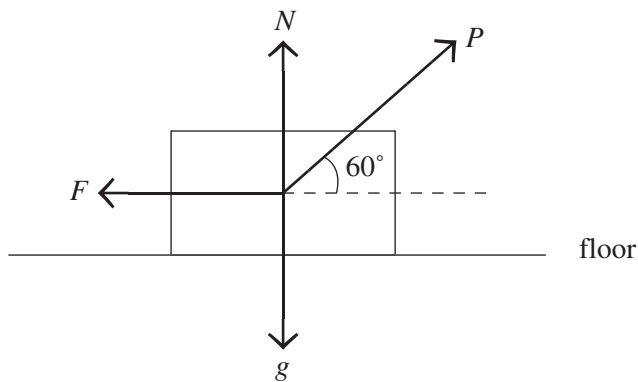
$$= \pi \left[ (\sqrt{3} - \sqrt{3}) - \left( \frac{1}{3} - 1 \right) \right]$$

$$= \frac{2\pi}{3} \text{ (cubic units)}$$

A1

**Question 2**

a.



A1

b. The body is moving, hence  $F = \frac{N}{\sqrt{2}}$ .

Vertical:  $N = g - P \sin(60^\circ)$ , i.e.  $N = g - \frac{\sqrt{3}P}{2}$ . A1

Horizontal:  $a = P \cos(60^\circ) - \frac{N}{\sqrt{2}}$ , i.e.  $a = \frac{P}{2} - \frac{N}{\sqrt{2}}$ . A1

Substituting  $N = g - \frac{\sqrt{3}P}{2}$  into  $a = \frac{P}{2} - \frac{N}{\sqrt{2}}$  gives  $a = \frac{P}{2} - \frac{1}{\sqrt{2}}\left(g - \frac{\sqrt{3}P}{2}\right)$ .

$$a = \frac{P}{2} - \frac{g}{\sqrt{2}} + \frac{\sqrt{3}P}{2\sqrt{2}} \text{ (or equivalent)}$$

So  $a = \frac{P - \sqrt{2}g + \sqrt{6}P}{2}$ . A1

Note:  $a = \frac{P}{2} - \frac{g}{\sqrt{2}} + \frac{\sqrt{3}P}{2\sqrt{2}}$  (or equivalent) is needed for the final A1.

### Question 3

Let  $u = \sin(x)$  and so  $\frac{du}{dx} = \cos(x)$ . When  $x = 0$ ,  $u = 0$  and when  $x = \frac{\pi}{2}$ ,  $u = 1$ .

So  $\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{1 + \sin^2(x)} dx = \int_0^1 \frac{1}{1 + u^2} du$  A1

$$\begin{aligned} &= [\tan^{-1}(u)]_0^1 \\ &= \tan^{-1}(1) - \tan^{-1}(0) \\ &= \frac{\pi}{4} \end{aligned}$$

A1

### Question 4

Using the scalar product, i.e.  $\cos(\theta) = \frac{(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (5\mathbf{i} - \mathbf{j} - \mathbf{k})}{\sqrt{3} \cdot \sqrt{27}}$  and so  $\cos(\theta) = \frac{1}{3}$ . M1

Using  $\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$ , we obtain  $2\cos^2\left(\frac{\theta}{2}\right) - 1 = \frac{1}{3}$ . M1

Rearranging  $2\cos^2\left(\frac{\theta}{2}\right) - 1 = \frac{1}{3}$ , we obtain  $\cos^2\left(\frac{\theta}{2}\right) = \frac{2}{3}$ . A1

As  $\theta$  is an acute angle, we reject  $\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{2}{3}}$  and so  $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{2}{3}}$ . A1

**Question 5**

Using implicit differentiation to differentiate  $x^2 + xy = e^y$ : M1

$$2x + y + x \frac{dy}{dx} = e^y \frac{dy}{dx} \text{ (or equivalent)} \quad \text{A1}$$

Let the gradient of the normal be  $m_N$ .

At  $(-1, 0)$ ,  $\frac{dy}{dx} = -1$  and so  $m_N$  is 1. A1

So the equation of the normal is  $y = x + 1$ . A1

**Question 6**

Attempting to use  $a = \frac{d}{dx} \left( \frac{1}{2} v^2 \right)$ . M1

$$\frac{d}{dx} \left( \frac{1}{2} v^2 \right) = \frac{3}{x^3} - \frac{1}{x^2}$$

$$\frac{1}{2} v^2 = \int \left( \frac{3}{x^3} - \frac{1}{x^2} \right) dx$$

$$= \frac{-3}{2x^2} + \frac{1}{x} + c \text{ where } c \text{ is an arbitrary constant} \quad \text{A1}$$

So  $v^2 = \frac{-3}{x^2} + \frac{2}{x} + k$ , where  $k$  is an arbitrary constant.

Given that  $v = 0$  at  $x = 1$ , we find that  $0 = -3 + 2 + k$ , i.e.  $k = 1$ .

Hence,  $v^2 = \frac{-3}{x^2} + \frac{2}{x} + 1$ . M1

Writing as a single fraction, we obtain  $v^2 = \frac{x^2 + 2x - 3}{x^2}$ .

Taking the square root of both sides we obtain  $v = \pm \sqrt{\frac{x^2 + 2x - 3}{x^2}}$ .

As  $a > 0$  when  $x = 1$ ,  $v$  is initially positive. A1

Hence,  $v = \frac{\sqrt{x^2 + 2x - 3}}{x}$  for  $x \geq 1$ . A1

**Question 7**

$$\text{a. } |\underline{r}(t)| = \sqrt{\sin^2(2t) + 4\cos^2(t)} \quad \text{M1}$$

$$= \sqrt{4\sin^2(t)\cos^2(t) + 4\cos^2(t)} \quad (\text{using } \sin(2t) = 2\sin(t)\cos(t)) \quad \text{A1}$$

$$= \sqrt{4\cos^2(t)(1 + \sin^2(t))}$$

$$= 2\sqrt{(1 - \sin^2(t))(1 + \sin^2(t))} \quad (\text{using } \cos^2(t) = 1 - \sin^2(t)) \quad \text{A1}$$

$$\text{Hence } |\underline{r}(t)| = 2\sqrt{1 - \sin^4(t)}. \quad \text{A1}$$

*Note: Only award the last A1 if the previous line of work is present.*

$$\text{b. } \text{The minimum value of } \sin^4(t) \text{ is zero, and so } |\underline{r}(t)|_{\max} = 2 \quad \text{A1}$$

$$|\underline{r}(t)|_{\max} \text{ occurs at } t = n\pi, \text{ where } n = 0, 1, 2, 3, \dots \quad \text{A1}$$

**Question 8**

$$(z^4 - \text{cis}(\theta)) = 0 \text{ or } (z^4 - \text{cis}(-\theta)) = 0 \text{ where } 2\cos(\theta) = 1.$$

$$2\cos(\theta) = 1 \text{ and so } \theta = \frac{\pi}{3}. \quad \text{A1}$$

$$z^4 = \text{cis}\left(\frac{\pi}{3}\right) \text{ or } z^4 = \text{cis}\left(-\frac{\pi}{3}\right). \quad \text{A1}$$

$$z = \text{cis}\left(\frac{\pi}{12} + \frac{2k\pi}{4}\right) \text{ or } z = \text{cis}\left(-\frac{\pi}{12} + \frac{2k\pi}{4}\right), \text{ where } k \in Z. \quad \text{M1}$$

$$\text{Hence, } z = \text{cis}\left(\pm\frac{\pi}{12}\right), \text{cis}\left(\pm\frac{5\pi}{12}\right), \text{cis}\left(\pm\frac{7\pi}{12}\right), \text{cis}\left(\pm\frac{11\pi}{12}\right). \quad \text{A1 A1}$$

$$\text{Note: Award A1 for } \text{cis}\left(\pm\frac{\pi}{12}\right) \text{ and } \text{cis}\left(\pm\frac{5\pi}{12}\right), \text{ and A1 for } \text{cis}\left(\pm\frac{7\pi}{12}\right) \text{ and } \text{cis}\left(\pm\frac{11\pi}{12}\right).$$

**Question 9**

$$\text{a. } \frac{\tan\left(\tan^{-1}\left(\frac{1}{m}\right)\right) + \tan\left(\tan^{-1}\left(\frac{1}{n}\right)\right)}{1 - \tan\left(\tan^{-1}\left(\frac{1}{m}\right)\right)\tan\left(\tan^{-1}\left(\frac{1}{n}\right)\right)} = \tan\left(\frac{\pi}{4}\right) \quad (\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)})$$

$$\text{So } \frac{\frac{1}{m} + \frac{1}{n}}{1 - \frac{1}{m} \times \frac{1}{n}} = 1 \quad \text{M1}$$

$$\frac{m+n}{mn-1} = 1 \quad (\text{multiplying numerator and denominator of the LHS by } mn) \quad \text{A1}$$

$$mn - m - n - 1 = 0$$

$$m(n-1) - n - 1 = 0$$

$$m(n-1) - n + 1 = 2 \quad (\text{adding 2 to both sides}) \quad \text{M1}$$

$$m(n-1) - 1(n-1) = 2$$

$$(m-1)(n-1) = 2 \quad \text{A1}$$

$$\text{b. } \text{From } (m-1)(n-1) = 2, \text{ we obtain } n = \frac{2}{m-1} + 1, \text{ i.e. } n = \frac{m+1}{m-1}. \quad \text{A1}$$

$$\text{Given that } m = k, \frac{1}{n} = \frac{k-1}{k+1}, \text{ we obtain } \tan^{-1}\left(\frac{1}{k}\right) + \tan^{-1}\left(\frac{k-1}{k+1}\right) = \frac{\pi}{4}. \quad \text{A1}$$