Year 2011

VCE

Specialist Mathematics Solutions Trial Examination 1

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Question 1

$z=\sqrt{5}+bi$

$|z|=\sqrt{\left(\sqrt{5}\right)^2+b^2}=\sqrt{5+b^2}=3 \qquad b^2+5=9 \qquad b^2=4$

$\Rightarrow b=\pm 2$

since $\quad b=\text{Im}(z)<0 \quad$ then $\quad b=-2$ A1

$\text{Arg}(z)=\theta=\tan^{-1}\left(-\frac{2}{\sqrt{5}}\right)$, so that $\tan(\theta)=-\frac{2}{\sqrt{5}}$

$\tan(2\theta)=\frac{2\tan(\theta)}{1-\tan^2(\theta)}=\frac{-\frac{4}{\sqrt{5}}}{1-\left(-\frac{2}{\sqrt{5}}\right)^2}=\frac{-\frac{4}{\sqrt{5}}}{1-\frac{4}{5}}=\frac{-\frac{4}{\sqrt{5}}}{\frac{1}{5}}$ M1

$\tan(2\theta)=-\frac{4}{\sqrt{5}}\times\frac{5}{1}\times\frac{\sqrt{5}}{\sqrt{5}}=-4\sqrt{5}$

$k=-4$ A1

Question 2

a. $P(z)=z^3+\left(3i-3\sqrt{3}\right)z^2+5z-15\sqrt{3}+15i$

$P\left(3\sqrt{3}-3i\right)=\left(3\sqrt{3}-3i\right)^3+\left(3i-3\sqrt{3}\right)\left(3\sqrt{3}-3i\right)^2+5\left(3\sqrt{3}-3i\right)-15\sqrt{3}+15i$

$=\left(3\sqrt{3}-3i\right)^3-\left(3\sqrt{3}-3i\right)^3+15\sqrt{3}-15i-15\sqrt{3}+15i$ M1

$=0 \quad$ shown

b. Hence $\left(z-3\sqrt{3}+3i\right)$ is a factor

$z^3+\left(3i-3\sqrt{3}\right)z^2+5z-15\sqrt{3}+15i=0$

$\left(z-3\sqrt{3}+3i\right)\left(z^2+5\right)=0$ A1

$\left(z-3\sqrt{3}+3i\right)\left(z^2-5i^2\right)=0$

$\left(z-3\sqrt{3}+3i\right)\left(z+\sqrt{5}\,i\right)\left(z-\sqrt{5}\,i\right)=0$

$z=3\sqrt{3}-3i$ and $\pm\sqrt{5}\,i$ A1

Question 3

$y = \dfrac{2x^3 - 4}{x} = 2x^2 - \dfrac{4}{x}$

$y = 2x^2$ and $x = 0$ are asymptotes A1

crosses x-axis $y = 0 \Rightarrow 2x^3 = 4 \quad x = \sqrt[3]{2} \quad \left(\sqrt[3]{2}, 0\right)$ A1

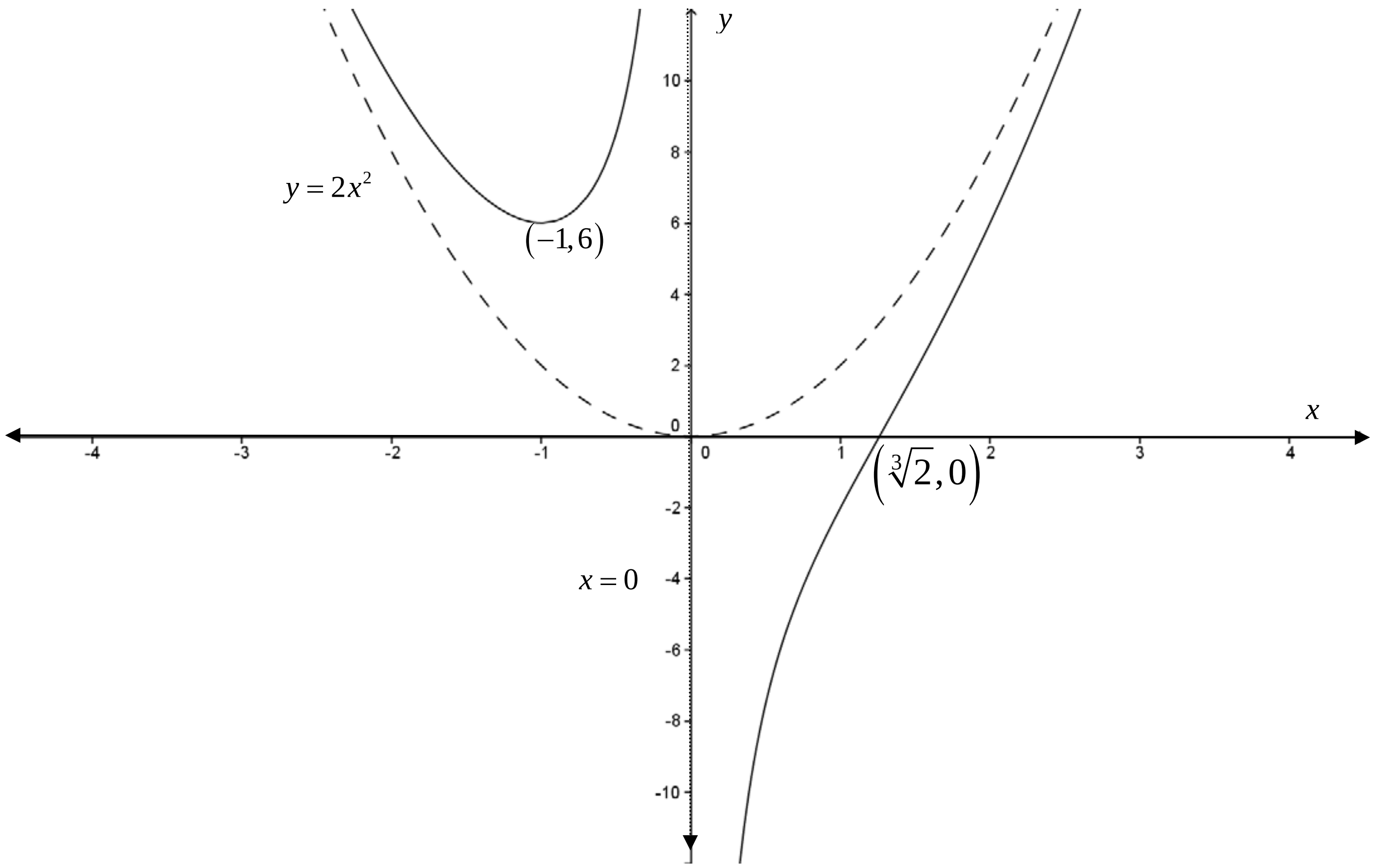
Note that $1 < \sqrt[3]{2} < 2$

$\dfrac{dy}{dx} = 4x + \dfrac{4}{x^2} = 0$ for turning points

$\Rightarrow x^3 = -1 \quad x = -1$ and $y = 2 + 4 = 6$

$(-1, 6)$ is a minimum turning point A1

correct graph, shape asymptotes, turning points A1



Question 4

a. $F = ma \qquad F = 4x^3 + 12x$ newtons $\quad m = 2\,\text{kg}$

$2a = 4x^3 + 12x$ since $a = a(x)$ use $a = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx}$

$a = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = 2x^3 + 6x$ M1

$\frac{1}{2}v^2 = \int \left(2x^3 + 6x\right) dx$

$\frac{1}{2}v^2 = \frac{1}{2}x^4 + 3x^2 + c$

Now when $x = 1$ $v = 4$

$8 = \frac{1}{2} + 3 + c \;\Rightarrow\; c = \frac{9}{2}$ A1

$v^2 = x^4 + 6x^2 + 9 = \left(x^2 + 3\right)^2$ since $v > 0$

$v = x^2 + 3$ shown

b. $v = \dfrac{dx}{dt} = x^2 + 3$

$\Rightarrow \dfrac{dt}{dx} = \dfrac{1}{x^2 + 3}$ M1

$t = \int \dfrac{1}{x^2 + 3} dx = \dfrac{1}{\sqrt{3}} \tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right) + c$

Now when $t = 0$ $x = 1$

$0 = \dfrac{1}{\sqrt{3}} \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) + c \;\Rightarrow c = -\dfrac{\pi}{6\sqrt{3}}$ A1

$t = \dfrac{1}{\sqrt{3}}\left(\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right) - \dfrac{\pi}{6}\right)$ now when $x = \sqrt{3}$

M1

$t = \dfrac{\sqrt{3}}{3}\left(\tan^{-1}(1) - \dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{3}\left(\dfrac{\pi}{4} - \dfrac{\pi}{6}\right)$

$t = \dfrac{\sqrt{3}\,\pi}{36}$ sec A1

Question 5

a. $I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx$

$u = a+b-x \quad \Rightarrow \quad \frac{du}{dx} = -1$ and $x = a+b-u$ M1

terminals when $x = b \quad u = a$ and when $x = a \quad u = b$

$I = -\int_b^a \frac{f(a+b-u)}{f(a+b-u)+f(u)}\,du = \int_a^b \frac{f(a+b-u)}{f(a+b-u)+f(u)}\,du$

but u is a dummy variable, so $I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,dx$

$I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx$ adding M1

$2I = \int_a^b \frac{f(a+b-x)+f(x)}{f(a+b-x)+f(x)}\,dx = \int_a^b 1\,dx = [x]_a^b = b-a$

$I = \frac{1}{2}(b-a)$

b. $f(x) = \sqrt{x} \quad a = 1 \quad b = 4$

$A = \int_1^4 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$ from **a.**

$A = \frac{1}{2}(4-1) = \frac{3}{2} = 1.5$ A1

Question 6

$V = \pi \int_a^b x^2 dy$

$y = \dfrac{\sqrt{9x^2 - 1}}{2x} \qquad \Rightarrow 2xy = \sqrt{9x^2 - 1}$ M1

$4x^2y^2 = 9x^2 - 1 \quad \Rightarrow 1 = x^2\left(9 - 4y^2\right) \quad$ so $\quad x^2 = \dfrac{1}{9 - 4y^2}$

$V = \pi \int_0^{\sqrt{2}} x^2 dy = \pi \int_0^{\sqrt{2}} \dfrac{1}{9 - 4y^2} dy \quad$ partial fractions

$\dfrac{1}{9 - 4y^2} = \dfrac{A}{3 - 2y} + \dfrac{B}{3 + 2y} = \dfrac{A(3 + 2y) + B(3 - 2y)}{(3 - 2y)(3 + 2y)} = \dfrac{3(A + B) + 2y(A - B)}{9 - 4y^2}$ M1

$(1) \quad A - B = 0 \quad \Rightarrow \quad A = B$

$(2) \quad 3(A + B) = 1 \qquad A = B = \dfrac{1}{6}$

$V = \dfrac{\pi}{6} \int_0^{\sqrt{2}} \left(\dfrac{1}{3 - 2y} + \dfrac{1}{3 + 2y} \right) dy$ A1

$V = \dfrac{\pi}{6} \left[-\dfrac{1}{2}\log_e|3 - 2y| + \dfrac{1}{2}\log_e|3 + 2y| \right]_0^{\sqrt{2}}$

$V = \dfrac{\pi}{12} \left[\log_e \left| \dfrac{3 + 2y}{3 - 2y} \right| \right]_0^{\sqrt{2}}$

$V = \dfrac{\pi}{12} \log_e \left(\dfrac{3 + 2\sqrt{2}}{3 - 2\sqrt{2}} \right)$ units^3 A1

Question 7

a. $\dfrac{dy}{dx}=2x-y \quad y_0=y(0)=0 \quad h=\dfrac{1}{4} \quad x_0=0 \quad f(x,y)=2x-y$

$y_1=y_0+hf(x_0,y_0) \qquad y_1=0+\dfrac{1}{4}(2\times0-0)=0$

M1

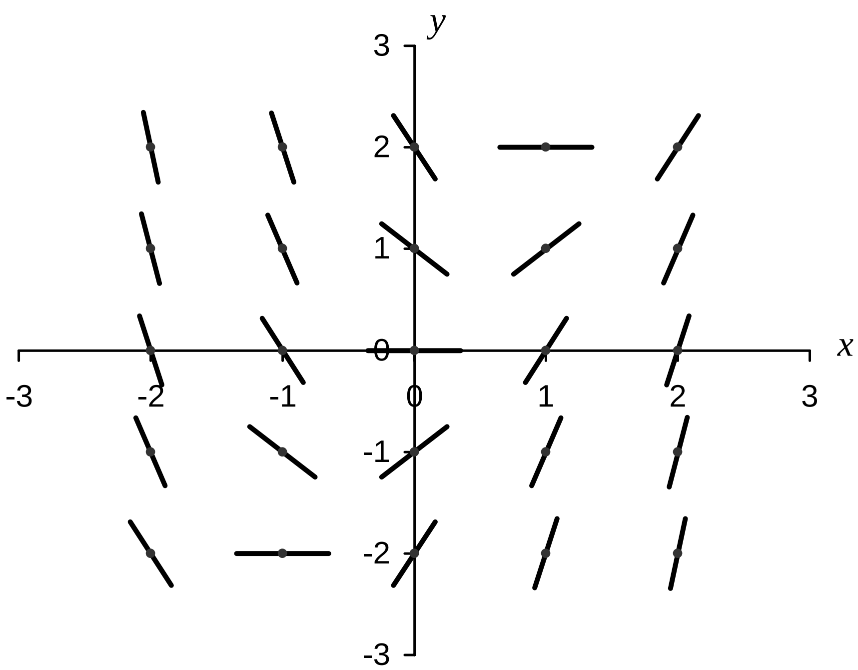
$y_2=y_1+hf(x_1,y_1) \quad x_1=x_0+h=\dfrac{1}{4} \quad y_2=0+\dfrac{1}{4}\left(2\times\dfrac{1}{4}-0\right)=\dfrac{1}{8}$

$y_3=y_2+hf(x_2,y_2) \quad x_2=x_1+h=\dfrac{1}{2}$

$y_3=\dfrac{1}{8}+\dfrac{1}{4}\left(2\times\dfrac{1}{2}-\dfrac{1}{8}\right)=\dfrac{1}{8}+\dfrac{1}{4}\times\dfrac{7}{8}=\dfrac{11}{32}$ A1

b. correct slopes in the table A1
correct slopes in the graph below A1

	$x=-2$	$x=-1$	$x=0$	$x=1$	$x=2$
$y=-2$	–2	0	2	4	6
$y=-1$	–3	–1	1	3	5
$y=0$	–4	–2	0	2	4
$y=1$	–5	–3	–1	1	3
$y=2$	–6	–4	–2	0	2



Question 8

a. $y = x\sin^{-1}\left(\frac{2x}{3}\right)$ using product rule

$$\frac{dy}{dx} = \sin^{-1}\left(\frac{2x}{3}\right)\frac{d}{dx}(x) + x\frac{d}{dx}\left(\sin^{-1}\left(\frac{2x}{3}\right)\right)$$ M1

$$\frac{dy}{dx} = \sin^{-1}\left(\frac{2x}{3}\right) + \frac{2x}{\sqrt{9-4x^2}}$$ A1

b. $\frac{dy}{dx} = \arcsin\left(\frac{2x}{3}\right)$

$$y = \int \sin^{-1}\left(\frac{2x}{3}\right)dx$$

since $\frac{d}{dx}\left[x\sin^{-1}\left(\frac{2x}{3}\right)\right] = \sin^{-1}\left(\frac{2x}{3}\right) + \frac{2x}{\sqrt{9-4x^2}}$

$$y = x\sin^{-1}\left(\frac{2x}{3}\right) - \int \frac{2x}{\sqrt{9-4x^2}}dx$$

let $u = 9-4x^2$ $\quad \frac{du}{dx} = -8x$

M1

$$y = x\sin^{-1}\left(\frac{2x}{3}\right) + \frac{1}{4}\int u^{-\frac{1}{2}}du$$

$$y = x\sin^{-1}\left(\frac{2x}{3}\right) + \frac{1}{2}u^{\frac{1}{2}} + c$$

A1

$$y = x\sin^{-1}\left(\frac{2x}{3}\right) + \frac{1}{2}\sqrt{9-4x^2} + c$$

Now when $x = \frac{3}{2}$ $\quad y = 0$

$$0 = \frac{3}{2}\sin^{-1}(1) + 0 + c \quad \Rightarrow \quad c = -\frac{3\pi}{4}$$

$$y = x\sin^{-1}\left(\frac{2x}{3}\right) + \frac{1}{2}\sqrt{9-4x^2} - \frac{3\pi}{4}$$ A1

Question 9

a. $\underset{\sim}{r}(t) = (t - \sin(t))\underset{\sim}{i} + (1 + \cos(t))\underset{\sim}{j}$ for $0 \le t \le 2\pi$

$\dot{\underset{\sim}{r}}(t) = (1 - \cos(t))\underset{\sim}{i} - \sin(t)\underset{\sim}{j}$

$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{(1 - \cos(t))^2 + (-\sin(t))^2}$ A1

$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)}$

$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{2 - 2\cos(t)}$

$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{2(1 - \cos(t))}$ using half-angle formulae M1

$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{2\left(2\sin^2\left(\frac{t}{2}\right)\right)} = \left|2\sin\left(\frac{t}{2}\right)\right|$ since $0 \le \frac{t}{2} \le \pi$

$\left|\dot{\underset{\sim}{r}}(t)\right| = 2\sin\left(\frac{t}{2}\right)$

b. $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$\frac{dy}{dt} = \dot{y} = -\sin(t)$ $\frac{dx}{dt} = \dot{x} = 1 - \cos(t)$

$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-\sin(t)}{1 - \cos(t)}$ M1

$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{-\sin\left(\frac{\pi}{3}\right)}{1 - \cos\left(\frac{\pi}{3}\right)} = \frac{-\frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} = -\sqrt{3}$ A1

Question 10

a. $9x^2 + 36x + 4y^2 - 8y + 4 = 0$

Using implicit differentiation

$18x + 36 + 8y\frac{dy}{dx} - 8\frac{dy}{dx} = 0$ M1

$\frac{dy}{dx}(8 - 8y) = 18x + 36x = 18(x + 2)$

$\frac{dy}{dx} = \frac{9(x + 2)}{4(1 - y)}$

$\frac{dy}{dx}=0 \quad \Rightarrow x=-2$

$36-72+4y^2-8y+4=0$

$4y^2-8y-32=0$

$y^2-2y-8=(y-4)(y+2)=0$

the points are $(-2,4)$ and $(-2,-2)$ A1

b. $9x^2+36x+4y^2-8y+4=0$

$9(x^2+4x \quad)+4(y^2-2y \quad)=-4$

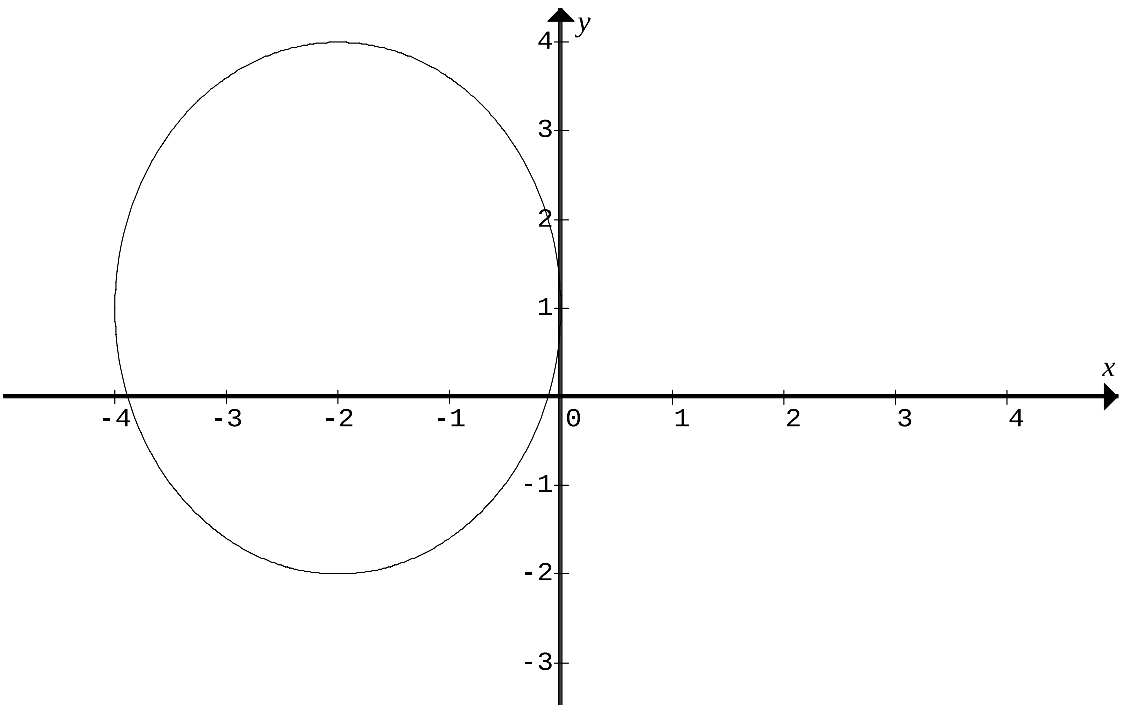
$9(x^2+4x+4)+4(y^2-2y+1)=-4+36+4$

$9(x+2)^2+4(y-1)^2=36$

$\frac{(x+2)^2}{4}+\frac{(y-1)^2}{9}=1$

ellipse centre $(-2,1)$, domain $[-4,0]$ range $[-2,4]$ A1

graph, correct shape, scale A1



END OF SUGGESTED SOLUTIONS