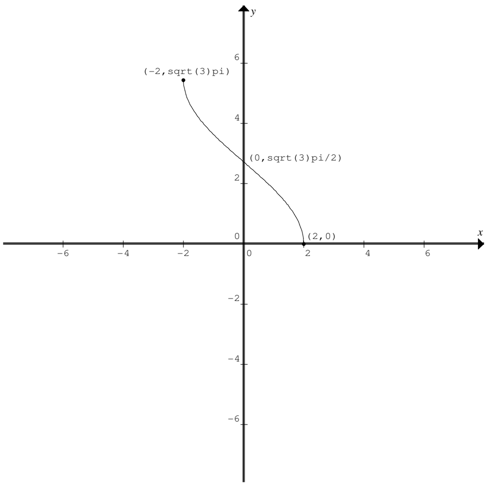
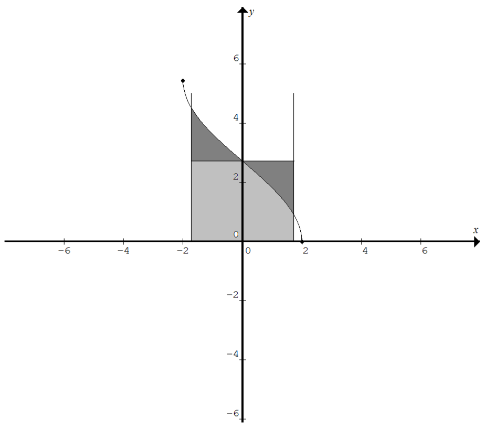
2011 Specialist Mathematics Trial Exam 1 Solutions

Q1a $y=\sqrt{3}\cos^{-1}\left(\frac{x}{2}\right)$



Q1b



From the graph, $\int_{-\sqrt{3}}^{\sqrt{3}}\sqrt{3}\cos^{-1}\left(\frac{x}{2}\right)dx$ = *area under graph*

= *area of rectangle* $=2\sqrt{3}\times\frac{\pi\sqrt{3}}{2}=3\pi$

Q2a $A(-2,1,0)$, $B(-1,2,-2)$ and $C(0,-3,4)$

$\overrightarrow{OA}=-2\tilde{i}+\tilde{j}$, $\overrightarrow{OB}=-\tilde{i}+2\tilde{j}-2\tilde{k}$, $\overrightarrow{OC}=-3\tilde{j}+4\tilde{k}$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\tilde{i}+\tilde{j}-2\tilde{k}$, $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\tilde{i}-5\tilde{j}+6\tilde{k}$

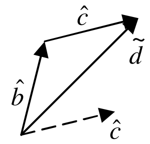
$\overrightarrow{BC}\neq m\overrightarrow{AB}$, .: A, B and C are not collinear.

Q2b $\overrightarrow{OA}=-2\tilde{i}+\tilde{j}$, $\overrightarrow{OB}=-\tilde{i}+2\tilde{j}-2\tilde{k}$, $\overrightarrow{OC}=-3\tilde{j}+4\tilde{k}$,

$\overrightarrow{OA}-2\overrightarrow{OB}-\overrightarrow{OC}=0$. .: $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are linearly dependent. Hence the three position vectors are coplanar.

Q2c Let $\hat{b}$ be a unit vector in the direction of $\overrightarrow{OB}$ and $\hat{c}$ a unit vector in the direction of $\overrightarrow{OC}$.

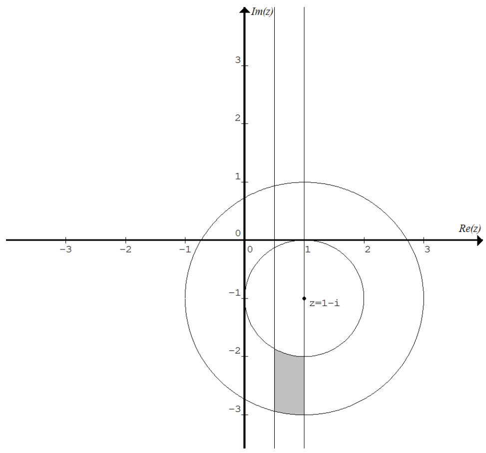
$\hat{b}=\frac{1}{3}\left(-\tilde{i}+2\tilde{j}-2\tilde{k}\right)$, $\hat{c}=\frac{1}{5}\left(-3\tilde{j}+4\tilde{k}\right)$



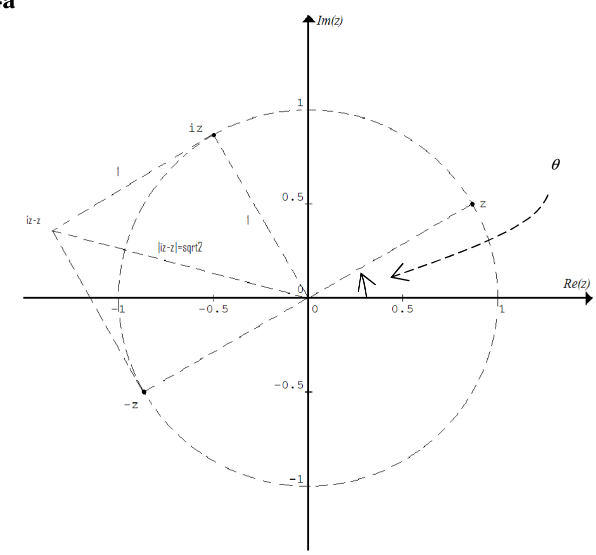
A vector that bisects the angle between $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is

$\tilde{d}=\hat{b}+\hat{c}=-\frac{1}{3}\tilde{i}+\frac{1}{15}\tilde{j}+\frac{2}{15}\tilde{k}$.

Q3 $\{z:1\le z+\bar{z}\le 2\}\cap\{z:1\le|z-1+i|\le 2\}$.



Q4a



Refer to the diagram above, $|iz-z|=\sqrt{2}$.

Q4b $\arg(iz-z)=\theta+\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}+\theta$

Q4c $z=\cos\theta+i\sin\theta$, $iz=-\sin\theta+i\cos\theta$

$\therefore\ iz-z=(-\sin\theta-\cos\theta)+i(\cos\theta-\sin\theta)$

From parts **a** and **b**,

$iz-z=\sqrt{2}cis\left(\frac{3\pi}{4}+\theta\right)=\sqrt{2}\cos\left(\frac{3\pi}{4}+\theta\right)+i\sqrt{2}\sin\left(\frac{3\pi}{4}+\theta\right)$

$\therefore\ \cos\theta-\sin\theta=\sqrt{2}\sin\left(\frac{3\pi}{4}+\theta\right)$

Q5a $\tilde{r}=\frac{t}{2}\tilde{i}+(49t-4.9t^2)\tilde{j}$, $\dot{\tilde{r}}=\frac{1}{2}\tilde{i}+(49-9.8t)\tilde{j}$

At $t=0$, $\dot{\tilde{r}}=\frac{1}{2}\tilde{i}+49\tilde{j}$

$\therefore$ initial speed $=\left|\dot{\tilde{r}}\right|=\sqrt{0.5^2+49^2}\approx 49$ ms^{-1}

Q5b $\tilde{a}=\ddot{\tilde{r}}=-9.8\tilde{j}$ a constant vector

Q5c $\tilde{r}=\frac{t}{2}\tilde{i}+(49t-4.9t^2)\tilde{j}$

At $t=0$, $\tilde{r}=\tilde{0}$ and the $\tilde{j}$ -component of $\tilde{r}$ is 0.

Let $49t-4.9t^2=0$ and $t>0$, $\therefore\ t=10$ and $\tilde{r}=5\tilde{i}$.

$\therefore$ displacement $=5\tilde{i}-\tilde{0}=5\tilde{i}$

Q5d $x=\frac{t}{2}$, $\therefore\ t=2x$

$y=49t-4.9t^2=49(2x)-4.9(2x)^2$

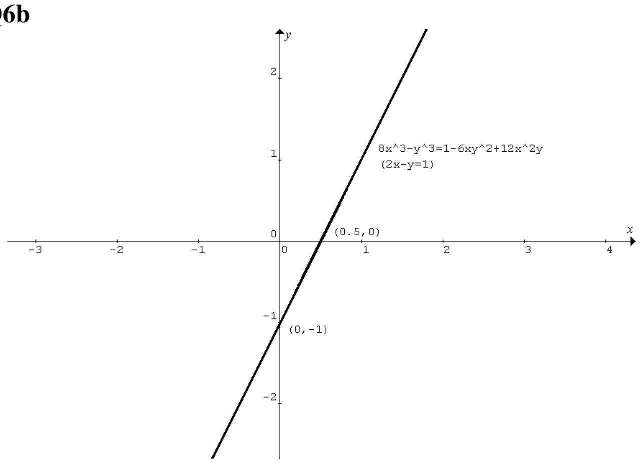
$\therefore\ y=98x-19.6x^2$, $x\ge 0$

Q6a $8x^3-y^3=1-6xy^2+12x^2y$

$8x^3-12x^2y+6xy^2-y^3=1$, $(2x-y)^3=1$

$\therefore\ 2x-y=1$, $\frac{dy}{dx}=2$

Q6b



Intercepts are $(0.5,0)$ and $(0,-1)$.

Q7a $f(x)=\frac{\log_e(x^2)}{|x|}$,

$$f'(x)=\frac{|x|\left(\frac{2}{x}\right)-\log_e(x^2)\times\frac{d|x|}{dx}}{x^2}=\begin{cases}\frac{-2+\log_e(x^2)}{x^2}, & x<0\\ \frac{2-\log_e(x^2)}{x^2} & x>0\end{cases}$$

$\therefore\ f'(-e)=\frac{-2+\log_e(-e)^2}{(-e)^2}=0$

Q7b For $x<0$, $f(x)=\frac{\log_e(x^2)}{|x|}=\frac{2\log_e|x|}{|x|}=-\frac{2\log_e|x|}{x}$

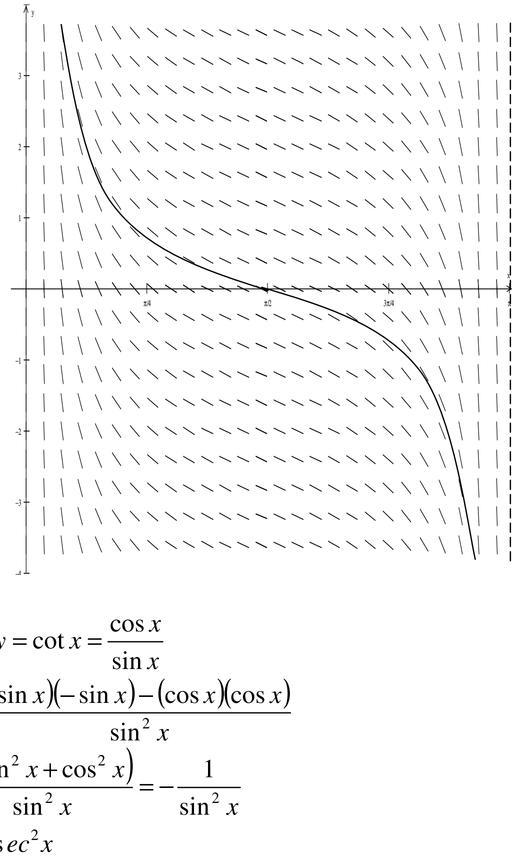
$\therefore\ \int_{-e}^{-1}f(x)dx=\int_{-e}^{-1}-\frac{2\log_e|x|}{x}dx$

Let $u=\log_e|x|$, $\frac{du}{dx}=\frac{1}{x}$

$=\int_{-e}^{-1}-2u\frac{du}{dx}dx=\int_{1}^{0}-2udu$

$=\int_{0}^{1}2udu=\left[u^2\right]_0^1=1$

Q8a



Q8b $y=\cot x=\frac{\cos x}{\sin x}$

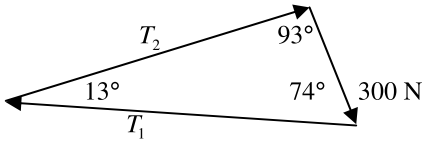
$\frac{dy}{dx}=\frac{(\sin x)(-\sin x)-(\cos x)(\cos x)}{\sin^2 x}$

$=\frac{-(\sin^2 x+\cos^2 x)}{\sin^2 x}=-\frac{1}{\sin^2 x}$

$=-\cos ec^2 x$

$\therefore\ f(x)=-\cos ec^2 x$

Q9a



Q9b The sine rule: $\frac{T_1}{\sin 93°} = \frac{300}{\sin 13°}$

$T_1 = \frac{300 \sin 93°}{\sin 13°} \approx 1332$ N

Q10a $v = \pm\sqrt{10 - 8x - 2x^2}$, .: $\frac{1}{2}v^2 = 5 - 4x - x^2$

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4 - 2x$

At $x = 0$, $a = -4$

Q10b Resultant force $F = ma = 0.2 \times {}^{-}4 = -0.8$ N

Q10c Maximum speed occurs when $a = 0$, i.e. $-4 - 2x = 0$

$x = -2$

.: maximum speed $= \sqrt{10 + 16 - 8} = 3\sqrt{2}$ ms^{-1}.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors

