INSIGHT

YEAR 12 *Trial Exam Paper*

2011

SPECIALIST MATHEMATICS
UNIT 4

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions.
- mark allocations

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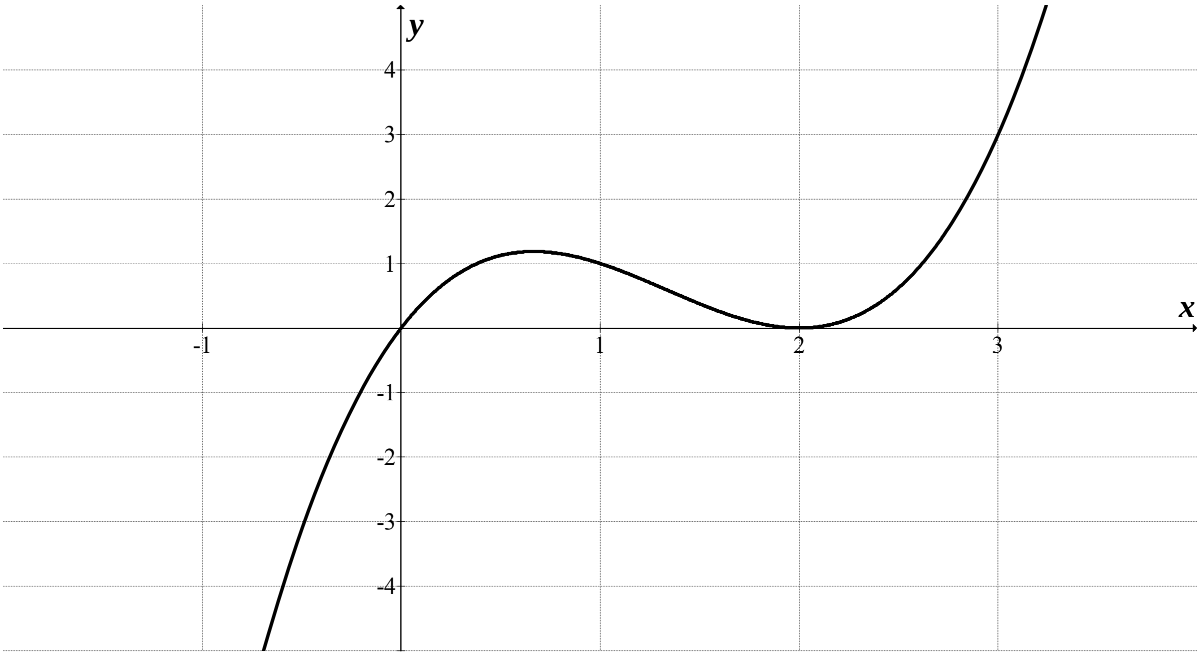
Section 1

Instructions for Section 1
Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1

The graph of $y = f(x)$ is shown below.



SECTION 1 – Question 1 – continued
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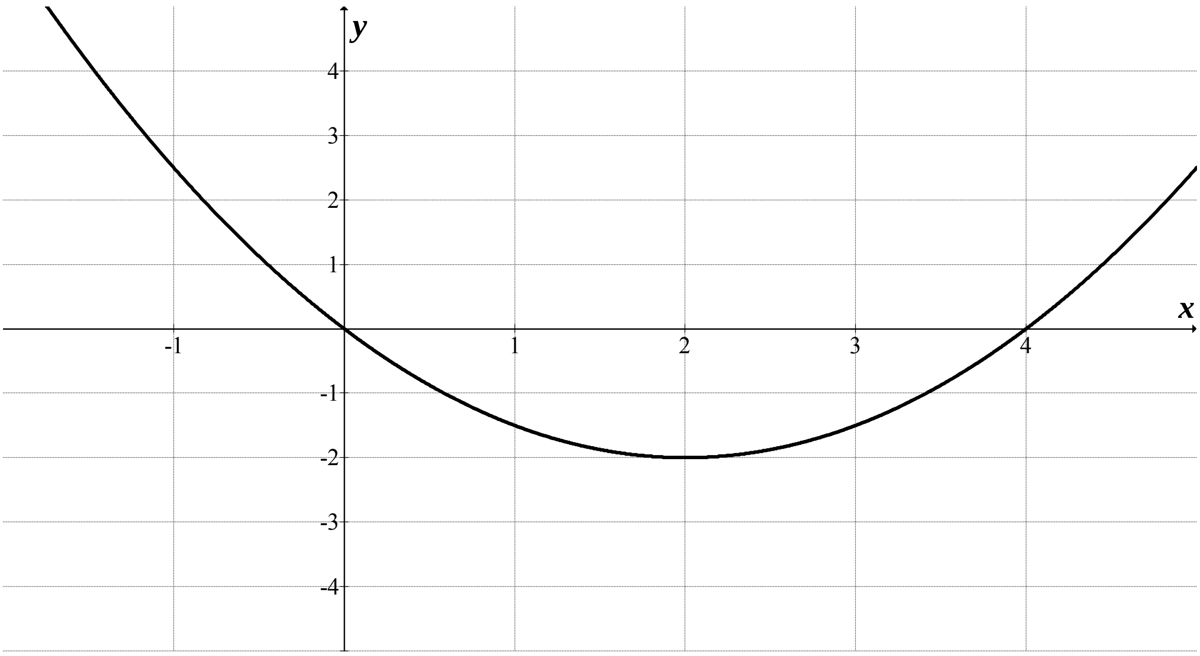
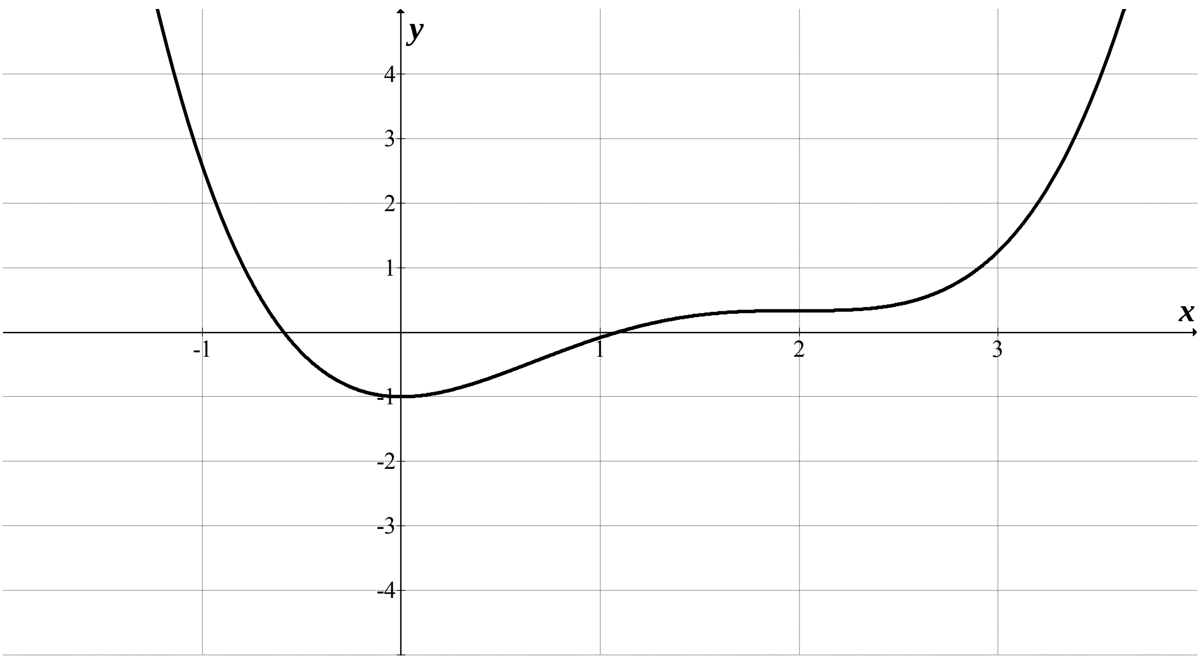
Which of the following graphs could be its antiderivative $F(x)$?

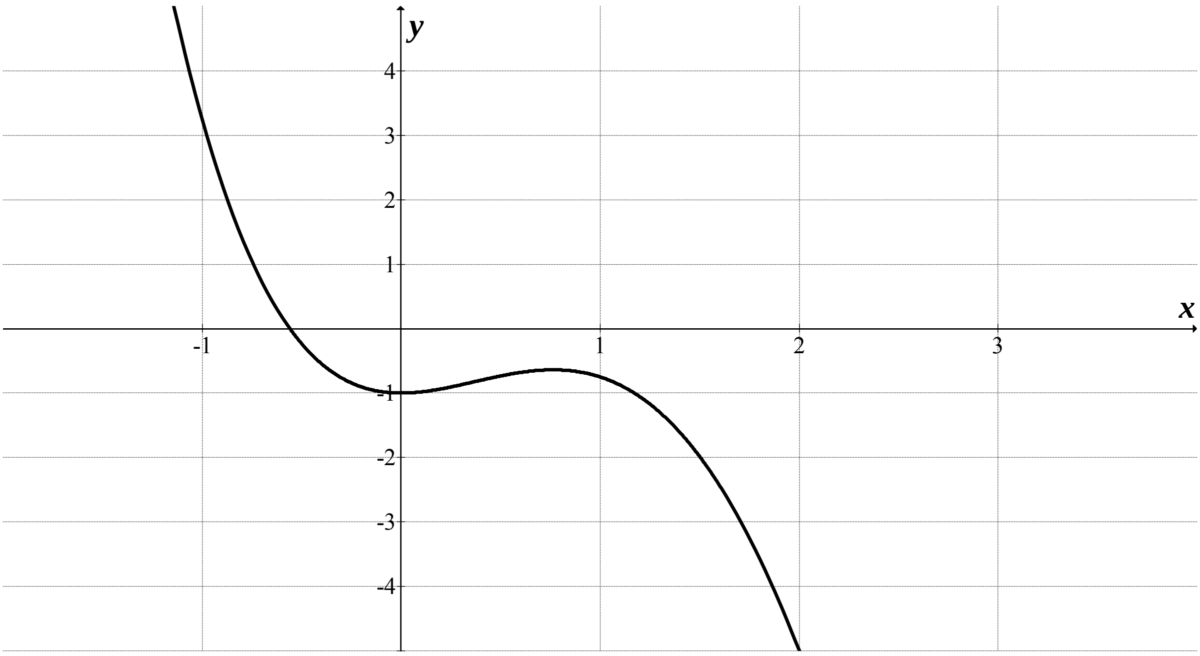
A.



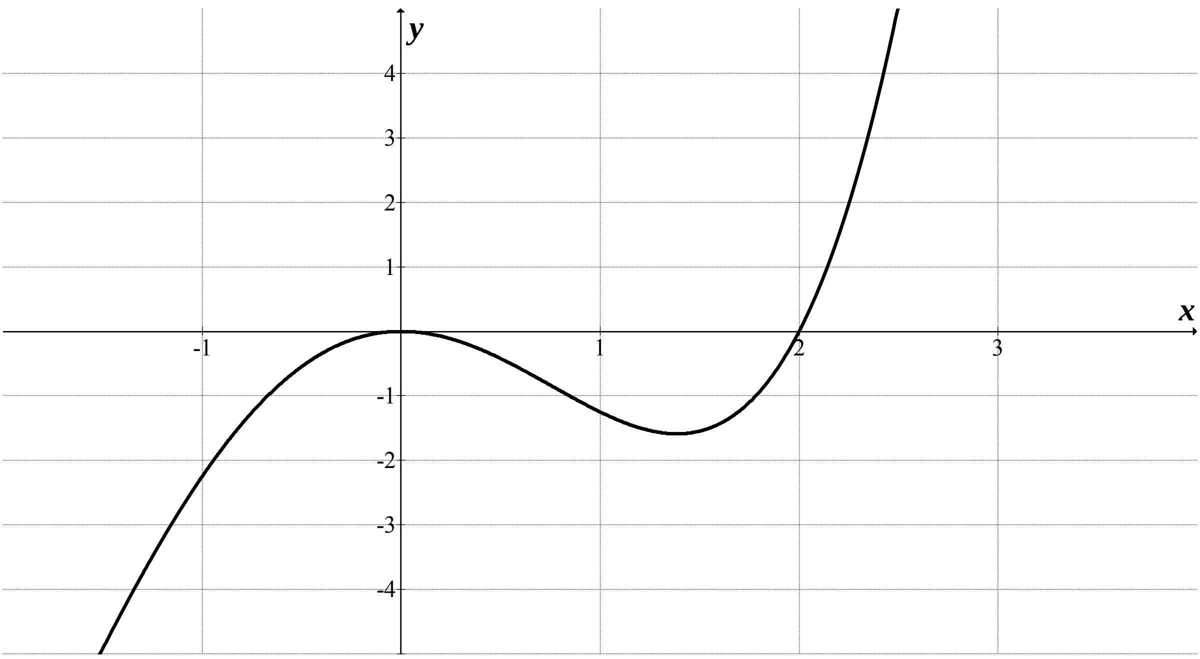
B.



C.



D.



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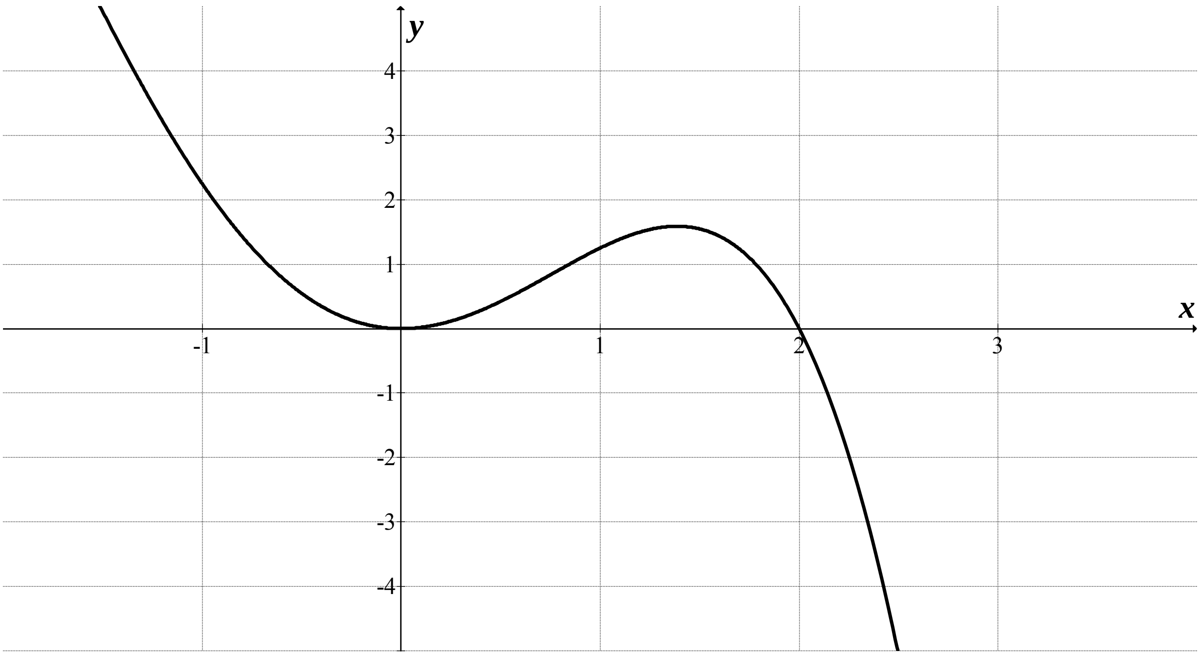
E.



Answer is B.

Worked solution

$f(x)$ is the gradient function of $F(x)$.

$f(x) = 0$ at $x = 0$ and $x = 2$.

For $x < 0,\ f(x) < 0$.

For $0 < x < 2,\ f(x) > 0$.

$\Rightarrow F(x)$ has a local minimum stationary point at $x = 0$.

For $x > 2,\ f(x) > 0$.

$\Rightarrow F(x)$ has a stationary point of inflection at $x = 2$.

Question 2

The graph of the function $f(x) = ax + \dfrac{b}{x^2}$, where $a < 0$ and $b > 0$, must have

A. one asymptote and a local minimum at $x = \left(\dfrac{b}{2a}\right)^{\frac{1}{3}}$.

B. one asymptote and a local maximum at $x = \left(\dfrac{b}{2a}\right)^{\frac{1}{3}}$.

C. two asymptotes and a local minimum at $x = \left(\dfrac{b}{2a}\right)^{\frac{1}{3}}$.

D. two asymptotes and a local minimum at $x = \left(\dfrac{2b}{a}\right)^{\frac{1}{3}}$.

E. two asymptotes and a local maximum at $x = \left(\dfrac{b}{2a}\right)^{\frac{1}{3}}$.

Answer is D.

Worked solution

$f(x)$ has a vertical asymptote at $x = 0$.
$f(x)$ has an oblique asymptote at $y = ax$.

$$f'(x) = a - 2bx^{-3} = 0$$

$$a = 2bx^{-3}$$

$$ax^3 = 2b$$

$$x^3 = \frac{2b}{a}$$

$$x = \left(\frac{2b}{a}\right)^{\frac{1}{3}} < 0 \text{ since } a < 0 \text{ and } b > 0.$$

$$f''(x) = 6bx^{-4}$$

$$= \frac{6b}{x^4} > 0 \text{ since } b > 0 \text{ and } x^4 = \left(\frac{2b}{a}\right)^{\frac{4}{3}} > 0.$$

$\therefore\ f(x)$ has a local minimum at $x = \left(\dfrac{2b}{a}\right)^{\frac{1}{3}}$.

Question 3

Which one of the following relations has a pair of asymptotes with an angle of $\frac{\pi}{3}$ between them?

A. $2(x-2)^2-(y+1)^2=1$

B. $3(x-2)^2-(y+1)^2=1$

C. $(x-2)^2-3(y+1)^2=3$

D. $3(x-2)^2+(y+1)^2=3$

E. $x^2-4x-2y^2+1=0$

Answer is C.

Worked solution

The relation with a pair of asymptotes is a hyperbola.

Since the angle between the asymptotes is $\frac{\pi}{3}$, the angle between the asymptote with a positive gradient of $\frac{b}{a}$ and the positive x-axis is $\frac{\pi}{3}\div 2=\frac{\pi}{6}$.

$$\therefore\ \frac{b}{a}=\tan^{-1}\frac{\pi}{6}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{b^2}{a^2}=\frac{1}{3}$$

A. $2(x-2)^2-(y+1)^2=1$

$$\Rightarrow \frac{(x-2)^2}{\frac{1}{2}}-\frac{(y+1)^2}{1}=1$$

which is a hyperbola with $a^2=\frac{1}{2}$ and $b^2=1$.

$$\Rightarrow \frac{b^2}{a^2}=\frac{1}{\frac{1}{2}}=2$$

B. $3(x-2)^2-(y+1)^2=1$

$$\Rightarrow \frac{(x-2)^2}{\frac{1}{3}}-\frac{(y+1)^2}{1}=1$$

which is a hyperbola with $a^2=\frac{1}{3}$ and $b^2=1$.

$$\Rightarrow \frac{b^2}{a^2}=\frac{1}{\frac{1}{3}}=3$$

SECTION 1 – Question 3 – continued

C. $3(x-2)^2-(y+1)^2=3$

$$\Rightarrow \frac{(x-2)^2}{1}-\frac{(y+1)^2}{3}=1$$

which is a hyperbola with $a^2=1$ and $b^2=3$.

$$\Rightarrow \frac{b^2}{a^2}=\frac{3}{1}=3$$

which is the required value of $\frac{b^2}{a^2}$.

Question 4

The position vectors of two particles, P and Q, are given by

$\underset{\sim}{p}=(5t-3)\underset{\sim}{i}+t^2\underset{\sim}{j}$ and $\underset{\sim}{q}=7\underset{\sim}{i}+(5t-6)\underset{\sim}{j},\ t\geq 0.$

The two particles will collide when

A. $t=3$
B. $t=6$
C. $t=2$ and $t=3$
D. $t=2$
E. $t=2$ and $t=6$

Answer is D.

Worked solution

Particles P and Q will collide when their position vectors are equal at the same time.

$x=5t-3=7$

$5t=10$

$t=2$

$y=t^2=5t-6$

$t^2-5t+6=0$

$(t-2)(t-3)=0$

$t=2$ or $t=3$

P and Q have the same position at $t=2$.

Question 5

The velocity of an object at time t seconds is given by $\underset{\sim}{v}(t) = 2t\underset{\sim}{i} + 5\underset{\sim}{j} - \frac{4}{(t+1)^2}\underset{\sim}{k}$.

When $t = 1$ the position of the particle is $\underset{\sim}{r}(t) = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$.

The initial position of the object is given by

A. $\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$

B. $2\underset{\sim}{i} + 3\underset{\sim}{j} - 6\underset{\sim}{k}$

C. $\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$

D. $2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$

E. $\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

Answer is E.

Worked solution

$$\underset{\sim}{v}(t) = 2t\underset{\sim}{i} + 5\underset{\sim}{j} - \frac{4}{(t+1)^2}\underset{\sim}{k}$$

$$\underset{\sim}{r}(t) = (t^2 + c_1)\underset{\sim}{i} + (5t + c_2)\underset{\sim}{j} + \left(\frac{4}{(t+1)} + c_3\right)\underset{\sim}{k}$$

$$\underset{\sim}{r}(1) = (1 + c_1)\underset{\sim}{i} + (5 + c_2)\underset{\sim}{j} + (2 + c_3)\underset{\sim}{k}$$

$$= 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$$

$$\Rightarrow c_1 = 1, c_2 = -2, c_3 = -1$$

$$\underset{\sim}{r}(t) = (t^2 + 1)\underset{\sim}{i} + (5t - 2)\underset{\sim}{j} + \left(\frac{4}{(t+1)} - 1\right)\underset{\sim}{k}$$

$$\Rightarrow \underset{\sim}{r}(0) = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$$

Question 6

The value of the constant p for which the vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{v} = -\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{w} = p\underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k}$ are linearly dependent is

A. □1
B. 1
C. 0
D. 2
E. □2

Answer is E.

Worked solution

For linear dependence $\underset{\sim}{w} = n\underset{\sim}{u} + m\underset{\sim}{v}$, where $m, n \in R$.

$\underset{\sim}{i} \Rightarrow p = 2n - m \quad$ eqn 1

$\underset{\sim}{j} \Rightarrow m - n = 3 \quad$ eqn 2

$\underset{\sim}{k} \Rightarrow 3n - 2m = -3 \quad$ eqn 3

From eqn 2:

$m = n + 3$

Substitute into eqn 3:

$$3n - 2(n+3) = -3$$
$$n - 6 = -3$$
$$\Rightarrow n = 3$$

Substitute into eqn 2:

$$m - 3 = 3$$
$$\Rightarrow m = 6$$

Substitute into eqn 1:

$$\therefore p = 2 \times 3 - 6 = 0$$

Tip

- Three vectors are linearly dependent if one of the vectors can be expressed as the sum of a multiple of the other two vectors.

Question 7

The vector resolute of $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ that is perpendicular to $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$ is

A. $\frac{1}{3}(2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k})$

B. $\frac{7}{17}(4\underset{\sim}{i} - 3\underset{\sim}{j} + 3\underset{\sim}{k})$

C. $\frac{2\sqrt{17}}{17}$

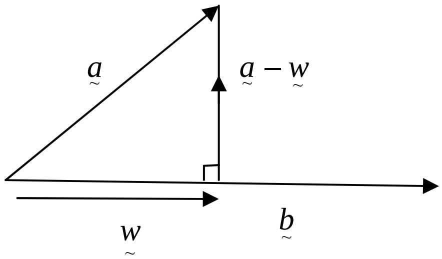
D. $\frac{2}{17}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k})$

E. $\frac{2}{\sqrt{17}}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k})$

Answer is B.

Worked solution

Let the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b} = \underset{\sim}{w} = (\underset{\sim}{a}.\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$.



$$\underset{\sim}{w} = (\underset{\sim}{a}.\underset{\sim}{b})\underset{\sim}{b} \div (b^2)$$

$$= \frac{(6-2-2)}{\left(\sqrt{\left(3^2+2^2+(-2)^2\right)}\right)^2} \times (3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k})$$

$$= \frac{2}{17} \times (3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k})$$

Then the vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is

$$= \underset{\sim}{a} - \underset{\sim}{w}$$

$$= 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} - \frac{2}{17}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k})$$

$$= \frac{1}{17}(28\underset{\sim}{i} - 21\underset{\sim}{j} + 21\underset{\sim}{k})$$

$$= \frac{7}{17}(4\underset{\sim}{i} - 3\underset{\sim}{j} + 3\underset{\sim}{k})$$

Question 8

The graph of the function $f:[0,\infty) \to R,\ f(x) = \sec(ax),\ a > 0$ has asymptotes located at

A. $x = 0, \frac{\pi}{a}, \frac{2\pi}{a}, \frac{3\pi}{a}, ...$

B. $x = 0, \frac{\pi}{2a}, \frac{3\pi}{2a}, \frac{5\pi}{2a}, ...$

C. $x = ..., \frac{-5\pi}{2a}, \frac{-3\pi}{2a}, \frac{-\pi}{2a}$

D. $x = \frac{\pi}{2a}, \frac{3\pi}{2a}, \frac{5\pi}{2a}, ...$

E. $x = \frac{\pi}{a}, \frac{2\pi}{a}, \frac{3\pi}{a}, ...$

Answer is D.

Worked solution

$\sec(ax) = \dfrac{1}{\cos(ax)}$ has vertical asymptotes wherever $\cos(ax) = 0$.

$$ax = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...$$

$$\Rightarrow x = \frac{\pi}{2a}, \frac{3\pi}{2a}, \frac{5\pi}{2a}, ...$$

Tip

- A reciprocal function has vertical asymptotes wherever the denominator is equal to zero.

Question 9

For $\frac{-\pi}{2} < x < \frac{\pi}{2}$, the graphs of the two curves given by $y = \text{cosec}^2(x)$ and $y = 2|\cot(x)|$ intersect at

A. the point $\left(\frac{\pi}{4}, 1\right)$ only.

B. the two points $\left(\frac{-\pi}{4}, 2\right)$ and $\left(\frac{\pi}{4}, 2\right)$.

C. the point $\left(\frac{-\pi}{4}, -2\right)$ only.

D. the point $\left(\frac{\pi}{4}, 2\right)$ only.

E. the two points $\left(\frac{-\pi}{6}, 1\right)$ and $\left(\frac{\pi}{6}, 1\right)$.

Answer is B.

Worked solution

$$\text{cosec}^2 x = 2|\cot x|$$

$$\frac{1}{\sin^2 x} = 2\left|\frac{\cos x}{\sin x}\right|$$

$$\frac{1}{\sin^2 x} = \frac{2\cos x}{\sin x} \quad \text{or} \quad \frac{1}{\sin^2 x} = \frac{-2\cos x}{\sin x}$$

$$\Rightarrow \frac{1}{\sin x} = 2\cos x \quad \text{or} \quad \frac{1}{\sin x} = -2\cos x$$

$$2\sin x \cos x = 1 \quad \text{or} \quad 2\sin x \cos x = -1$$

$$\sin 2x = 1 \quad \text{or} \quad \sin 2x = -1$$

$$2x = \frac{-\pi}{2}, \frac{\pi}{2}$$

$$\Rightarrow x = \frac{-\pi}{4}, \frac{\pi}{4} \quad \text{for} \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$$

$$\Rightarrow y = 2\left|\cot\frac{\pi}{4}\right| \quad \text{or} \quad 2\left|\cot\frac{-\pi}{4}\right|$$

$$\Rightarrow y = 2 \times 1 = 2 \quad \text{or} \quad 2 \times |-1| = 2$$

$\therefore$ The two curves intersect at $\left(\frac{-\pi}{4}, 2\right)$ and $\left(\frac{\pi}{4}, 2\right)$.

Tip

- The absolute value function can be either positive or negative.
- A graph of these two curves will also give these intersection points.

Question 10

Given that $z = 2\,\text{cis}\left(\frac{\pi}{3}\right)$ and $w = 5\,\text{cis}\left(\frac{-\pi}{6}\right)$, it follows that $\text{Arg}\left(z^3(\overline{w})^4\right)$ is

A. $-\pi$

B. $\frac{2\pi}{3}$

C. $\frac{5\pi}{3}$

D. $\frac{-\pi}{3}$

E. $\frac{\pi}{3}$

Answer is D.

Worked solution

$$\begin{aligned}\text{Arg}\left(z^3(\overline{w})^4\right) &= \arg\left(z^3\right) + \arg\left((\overline{w})^4\right)\\ &= 3\arg(z) + 4\arg(\overline{w})\\ &= 3\times\frac{\pi}{3} + 4\times\frac{\pi}{6}\\ &= \frac{5\pi}{3}\end{aligned}$$

$$\begin{aligned}\Rightarrow \text{Arg}\left(z^3(\overline{w})^4\right) &= \frac{5\pi}{3} - 2\pi\\ &= -\frac{\pi}{3}\end{aligned}$$

Tip

- *The argument of the product of two complex numbers is equal to the sum of the argument of each of the two complex numbers.*

Question 11

If $z_1 = \sqrt{3} + i$ and $z_2 = 1 - i$, then $\dfrac{z_1}{z_2}$ is equal to

A. $\sqrt{2}\,\text{cis}\left(\dfrac{5\pi}{12}\right)$

B. $\sqrt{2}\,\text{cis}\left(\dfrac{\pi}{12}\right)$

C. $\dfrac{1}{\sqrt{2}}\text{cis}\left(\dfrac{5\pi}{12}\right)$

D. $\dfrac{1}{\sqrt{2}}\text{cis}\left(\dfrac{\pi}{12}\right)$

Answer is A.

Worked solution

$$z_1 = \sqrt{3} + i$$

$$|z_1| = \sqrt{\left((\sqrt{3})^2 + 1^2\right)}$$

$$= \sqrt{4} = 2$$

$$\text{Arg}(z_1) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\Rightarrow z_1 = 2\,\text{cis}\frac{\pi}{6}$$

$$z_2 = 1 - i$$

$$|z_2| = \sqrt{\left(1^2 + (-1)^2\right)}$$

$$= \sqrt{2}$$

$$\text{Arg}(z_2) = \tan^{-1}\left(\frac{-1}{1}\right)$$

$$= -\frac{\pi}{4}$$

$$\Rightarrow z_2 = \sqrt{2}\,\text{cis}\frac{-\pi}{4}$$

$$\therefore \frac{z_1}{z_2} = \frac{2\,\text{cis}\frac{\pi}{6}}{\sqrt{2}\,\text{cis}\frac{-\pi}{4}}$$

$$= \sqrt{2}\,\text{cis}\frac{5\pi}{12}$$

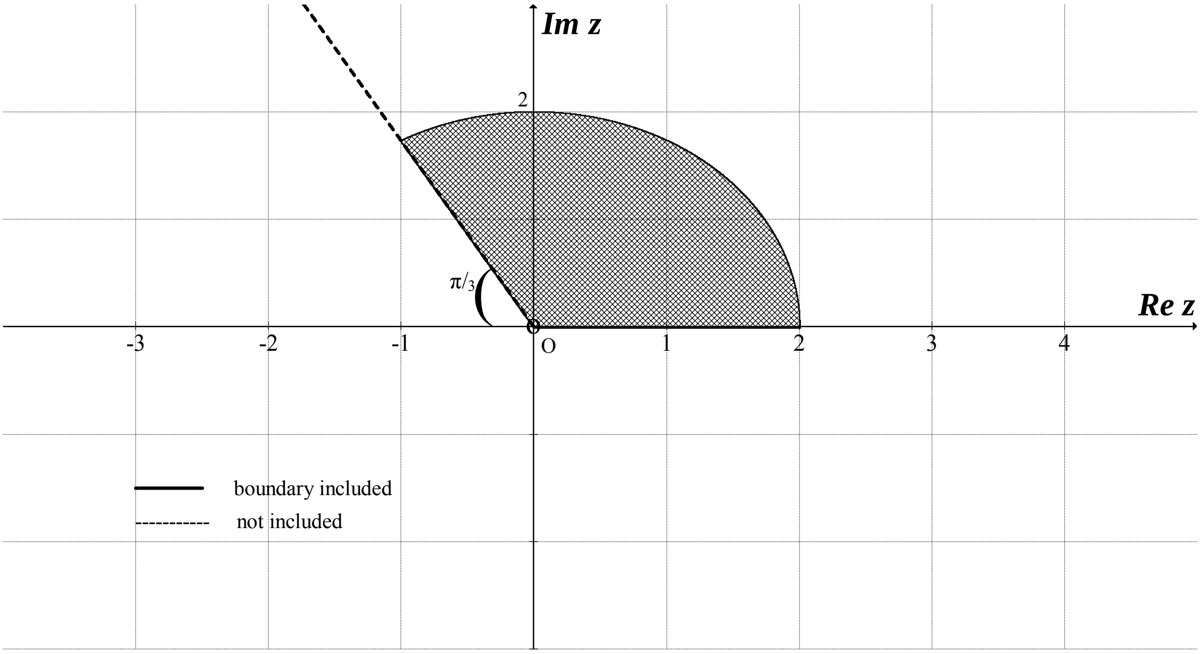
One is the area between the rays Arg 0 (included) and Arg $\dfrac{2\pi}{3}$ (not included).

The other area is the area inside a circle of centre (0, 0) and radius 2 (including the boundary).

$$\Rightarrow \left\{z : 0 \le \text{Arg}(z) < \frac{2\pi}{3}\right\} \cap \{z : |z| \le 2\}$$

SECTION 1 – continued
TURN OVER

Question 12



The shaded region on the Argand diagram above can be described by

A. $\left\{z : Arg(z) > \frac{\pi}{3}\right\} \cup \{z : |z| \leq 2\}$

B. $\left\{z : 0 < Arg(z) < \frac{2\pi}{3}\right\} \cup \{z : |z| \leq 2\}$

C. $\left\{z : 0 < Arg(z) < \frac{2\pi}{3}\right\} \cap \{z : |z| \leq 2\}$

D. $\left\{z : 0 \leq Arg(z) < \frac{2\pi}{3}\right\} \cap \{z : |z| < 2\}$

E. $\left\{z : 0 \leq Arg(z) < \frac{2\pi}{3}\right\} \cap \{z : |z| \leq 2\}$

Worked solution

The shaded area is the intersection of two regions.

One is the area between the rays Arg 0 (included) and Arg $\frac{2\pi}{3}$ (not included).

The other area is the area inside a circle of centre (0, 0) and radius 2 (including the boundary).

$\Rightarrow \left\{z : 0 \leq Arg(z) < \frac{2\pi}{3}\right\} \cap \{z : |z| \leq 2\}$

Question 13

On an Argand diagram, a set of points that lie on a circle of radius 4 centred at (1, 0) is

A. $\{z : (z-1)(\overline{z}-1) = 16\}$

B. $\{z : (z-1)^2 = 4\}$

C. $\{z : (z-1)(\overline{z}-1) = 4\}$

D. $\{z : z\overline{z} = 4\}$

E. $\{z : (\text{Re}(z))^2 + (\text{Im}(z))^2 = 4\}$

Answer is A.

Worked solution

A. $\{z : (z-1)(\overline{z}-1) = 16\}$

$$(x+yi-1)(x-yi-1) = 16$$
$$((x-1)+yi)((x-1)-yi) = 16$$
$$(x-1)^2 - (yi)^2 = 16$$
$$(x-1)^2 - y^2 i^2 = 16$$
$$(x-1)^2 + y^2 = 16$$

which is a circle of centre (1, 0) and radius 4.

Question 14

If $f'(t) = e^{t^2}$ and $f(0) = 3,$ then the solution to the differential equation when $t = 2$ can be found by evaluating

A. $\int_0^2 e^{t^2}.dt + 3$

B. $\int_0^2 e^{t^2}.dt - 3$

C. $\int_0^2 2te^{t^2}.dt + 3$

D. $\int_0^2 \left(e^{t^2} + 3\right).dt$

E. $\int_0^3 e^{t^2}.dt - 2$

Answer is A.

SECTION 1 – Question 14 – continued
TURN OVER

Worked solution

$$f(b) = \int_a^b f'(t).dt + f(a)$$

$$\Rightarrow f(2) = \int_0^2 f'(t).dt + f(0)$$

$$\Rightarrow f(2) = \int_0^2 e^{t^2}.dt + 3$$

Question 15

Given $\frac{dy}{dx} = \frac{2x}{y+2}$ and $(x_0, y_0) = (1, 2)$.

Using Euler's rule with a step size of 0.2, the value of y_2, correct to 3 decimal places, is

A. 2.100
B. 2.060
C. 2.200
D. 2.217
E. 2.211

Answer is D.

Worked solution

$$f'(x, y) = \frac{dy}{dx} = \frac{2x}{y+2} \text{ and } (x_0, y_0) = (1, 2)$$

$$y_1 = y_0 + h\,f'(x_0, y_0)$$

$$y_1 = 2 + 0.2 \times \frac{2 \times 1}{2+2} = 2.1$$

$$x_1 = 1.2$$

$$y_2 = 2.1 + 0.2 \times \frac{2 \times 1.2}{2.1+2}$$

$$y_2 = 2.1 + 0.2 \times \frac{2.4}{4.1}$$

$$y_2 = 2.21707$$

$$\Rightarrow y_2 \approx 2.217$$

SECTION 1 – continued

Question 16
The temperature inside a kitchen is 22°C. A cup of soup is removed from the microwave and stirred evenly when its temperature is 70°C. The rate at which the temperature of the soup drops is proportional to the excess of its temperature above its surrounding air temperature.

If T is the temperature of the soup at time t minutes after it is removed from the microwave and k is a positive constant, a differential equation relating T and t is

A. $\frac{dT}{dt} = k(T - 22),\ T(0) = 70$

B. $\frac{dT}{dt} = k(T - 70),\ T(0) = 22$

C. $\frac{dT}{dt} = -k(T - 70),\ T(0) = 22$

D. $\frac{dT}{dt} = -kT - 22,\ T(0) = 70$

E. $\frac{dT}{dt} = -k(T - 22),\ T(0) = 70$

Answer is E.

Worked solution

$\frac{dT}{dt} = k(T - T_s)$, where $T_s = 22$.

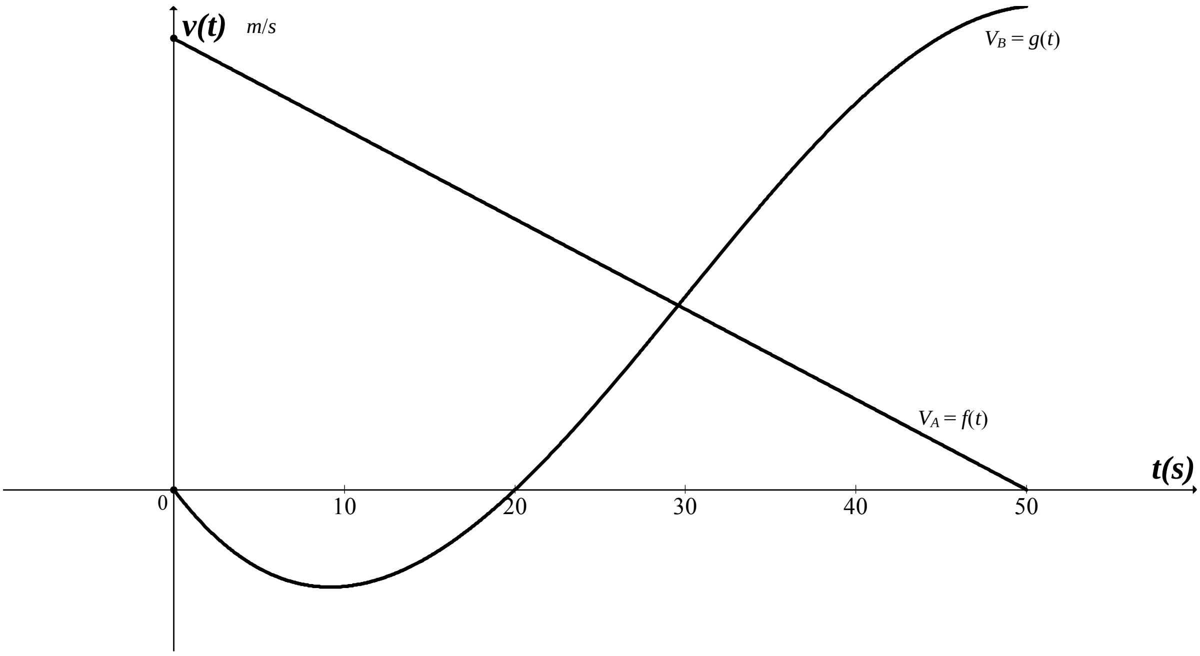
Initial temperature of the soup is $T(0) = 70$.

For cooling, $k < 0$.

Since k is to be a positive constant

$\Rightarrow \frac{dT}{dt} = -k(T - 22),\ T(0) = 70$

Question 17



Two particles, A and B, move from the same point in a straight line with their respective velocities $V_A = f(t)$ and $V_B = g(t)$ in m/s, where t is in seconds and $t \geq 0$.
The velocity□time graphs for the particles is shown above.

At $t = 50$ seconds, the distance, in metres, between the two particles is

A. $\left|\int_0^{50} g(t).dt - 50f(t)\right|$

B. $\left|\int_{20}^{50} g(t).dt - \int_0^{20} g(t).dt\right| - 50f(t)$

C. $\left|\int_{20}^{50} g(t).dt - \int_0^{20} g(t).dt - 50f(t)\right|$

D. $\left|\int_{30}^{50} g(t).dt + \int_0^{30} g(t).dt - 50f(t)\right|$

E. $\left|\int_0^{50} (g(t) - f(t)).dt\right|$

Answer is E.

Worked solution

The displacement of A after 50 seconds of motion is $\int_0^{50} f(t).dt.$

The displacement of B after 50 seconds of motion is $\int_0^{50} g(t).dt.$

So, the distance between A and B after 50 seconds of motion is $\left|\int_0^{50} (g(t) - f(t)).dt\right|$.

Tips

- The displacement of each particle is given by the integral of the velocity□time function over the time interval [0, 50].
- The distance between two moving particles over a period of time is the absolute value of the difference between their displacements.

Question 18

Using an appropriate substitution, $\int_0^{\frac{\pi}{3}} \cot(x)\sec^2(x).dx$ can be expressed as

A. $\int_0^{\frac{1}{2}} \frac{du}{u}$

B. $\int_0^{\sqrt{3}} \frac{du}{u}$

C. $\int_0^{\frac{\sqrt{3}}{2}} \frac{1-u^2}{u}.du$

D. $-\int_0^{\sqrt{3}} u.du$

E. $\int_0^{\sqrt{3}} u.du$

Answer is B.

Worked solution

$$\int_0^{\frac{\pi}{3}} \cot(x)\sec^2(x).dx$$

$$\int_0^{\frac{\pi}{3}} \frac{1}{\tan(x)}\sec^2(x).dx$$

Let $u = \tan(x)$

When $x = 0$, $u = \tan(0) = 0$

When $x = \frac{\pi}{3}$, $u = \tan\frac{\pi}{3} = \sqrt{3}$

$$\frac{du}{dx} = \sec^2(x)$$

$$\Rightarrow \int_0^{\frac{\pi}{3}} \frac{1}{\tan(x)} \sec^2(x).dx = \int_0^{\sqrt{3}} \frac{1}{u}.\frac{du}{dx}.dx$$
$$= \int_0^{\sqrt{3}} \frac{du}{u}$$

Tip

- The substitution u = tan x is made because its derivative $\sec^2 x$ is a factor of the integrand.

Question 19

A mass is travelling in a straight line at 6 m/s and has a momentum 24 kg m/s. A force acts on the mass in the same direction of motion so that the acceleration of the mass is 0.25 m/s^2. When it has a momentum of 32 kg m/s, the mass has travelled a distance of

A. 14 metres
B. 96 metres
C. 24 metres
D. 28 metres
E. 56 metres

Answer is E.

Worked solution

$p = mv$

When $p = 24$:

$6m = 24$

$\Rightarrow m = 4$ kg

When $p = 32$:

$4v = 32$

$\Rightarrow v = 8$

$u = 6, v = 8, a = 0.25, s = ?$

$v^2 = u^2 + 2as$

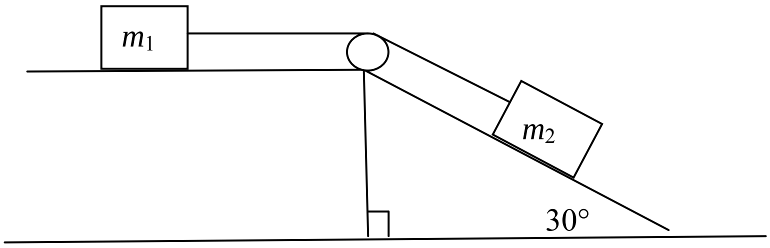
$8^2 = 6^2 + 2 \times 0.25 \times s$

$0.5s = 64 - 36$

$0.5s = 28$

$\Rightarrow s = 56$ metres

Question 20
A mass of m_1 kg rests on a rough horizontal surface and is connected to a second mass m_2 kg, which is twice the mass of m_1. The second mass, m_2, rests on a rough surface inclined at an angle of 30° to the horizontal. The two masses are connected by a light, inextensible string that passes over a smooth pulley. The coefficient of friction between each mass and their surface is the same. The mass m_2 is on the verge of sliding down the plane.



The value of the coefficient of friction μ is

A. $\dfrac{1}{\sqrt{3}-2}$

B. $\dfrac{1+\sqrt{3}}{2}$

C. $\dfrac{1}{1+\sqrt{3}}$

D. $\dfrac{2}{\sqrt{3}-1}$

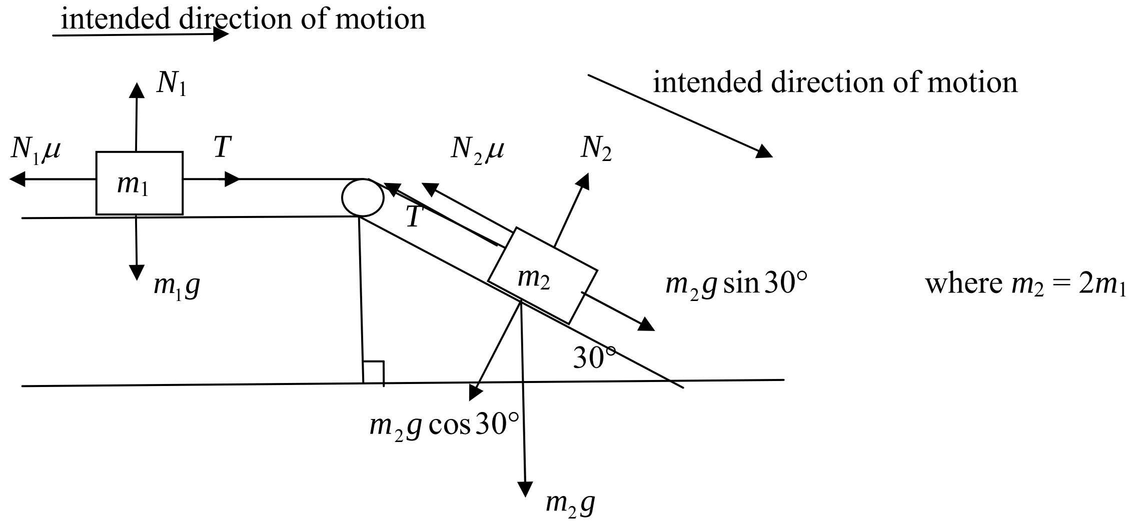
E. $\dfrac{1}{1-\sqrt{3}}$

Answer is C.

SECTION 1 – Question 20 – continued
TURN OVER

Worked solution

Show the forces acting on the two masses.



The system is on the verge of moving to the right and down the plane (as indicated on the diagram above.

$\Rightarrow a = 0$

For $m_1 : N_1 = m_1 g$

For $m_2 : N_2 = m_2 g\cos 30° = \dfrac{m_2 g\sqrt{3}}{2}$

$\Rightarrow N_2 = m_1 g\sqrt{3}$, since $m_2 = 2m_1$

$R = m_2 g\sin 30° - N_2\mu - T + T - N_1\mu = (m_1 + m_2)a$

$\Rightarrow R = 2m_1 g \times \dfrac{1}{2} - m_1 g\sqrt{3}\mu - m_1 g\mu = 0$, since the system is on the verge of moving.

$$m_1 g - m_1 g\sqrt{3}\mu - m_1 g\mu = 0$$

$$1 - \sqrt{3}\mu - \mu = 0$$

$$1 - \mu(1+\sqrt{3}) = 0$$

$$\mu(1+\sqrt{3}) = 1$$

$$\mu = \frac{1}{1+\sqrt{3}}$$

Tip

- For connected particles, the total resultant force acting in the direction of intended motion can be treated as the sum of the forces acting on each particle in the direction of intended motion.

Question 21

An object moves in a straight line with velocity, in m/s, $v = \dfrac{4}{t+1}, t \geq 0$.

Which one of the following statements about the object's motion is **false**?

A. The object's acceleration is always negative.
B. The object has an initial acceleration of -4 m/s^2.
C. The distance travelled by the object in the first 3 seconds of motion is 4 $\log_e$ 4 metres.
D. The object has an initial velocity of 4 m/s.
E. The object gradually increases its non-zero initial velocity.

Answer is E.

Worked solution

Since $t \geq 0, v = \dfrac{4}{t+1} \geq 0$ for all values of *t*.

$v(0) = 4$

$$x = \int_0^3 \frac{4}{t+1}.dt = 4\left[\log_e(t+1)\right]_0^3$$

$$= 4\log_e 4$$

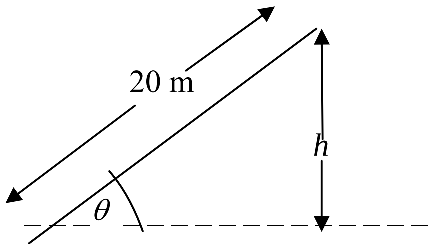
$$a = \frac{-4}{(t+1)^2} \leq 0 \text{ for all } t \geq 0$$

$a(0) = -4$

The object's acceleration is always negative; the object has an initial acceleration of -4 m/s^2; the distance travelled by the object in the first 3 seconds of motion is 4 $\log_e$ 4 metres; and the object has an initial velocity of 4 m/s.
The object **does not** gradually increase its non-zero initial velocity since the acceleration is always negative.

SECTION 1 – Question 21 – continued
TURN OVER

Question 22
A crane of length 20 metres makes an angle of θ radians with the horizontal ground as it is being raised. The height of the end of the crane above the ground is h metres.



The crane is being raised at a rate of 0.08 radians/second. The rate at which the height of the crane is increasing when $\theta = \frac{\pi}{4}$ is

A. $\frac{4\sqrt{2}}{5}$ m/s

B. $\frac{\sqrt{2}}{5}$ m/s

C. $\frac{8\sqrt{2}}{5}$ m/s

D. $\frac{2\sqrt{2}}{5}$ m/s

E. $\frac{8}{5}$ m/s

Answer is A.

Worked solution

$$h = 20\sin\theta$$

$$\frac{dh}{d\theta} = 20\cos\theta$$

$$\theta = \frac{\pi}{4}$$

So $\frac{dh}{d\theta} = 20\cos\frac{\pi}{4}$

$$= 20 \times \frac{\sqrt{2}}{2}$$

$$= 10\sqrt{2}$$

$$\frac{d\theta}{dt} = 0.08$$

So $$\frac{dh}{dt} = \frac{dh}{d\theta} \times \frac{d\theta}{dt}$$
$$= 10\sqrt{2} \times 0.08$$
$$= 0.8\sqrt{2}$$
$$= \frac{4\sqrt{2}}{5}$$

END OF SECTION 1

END OF SECTION 1

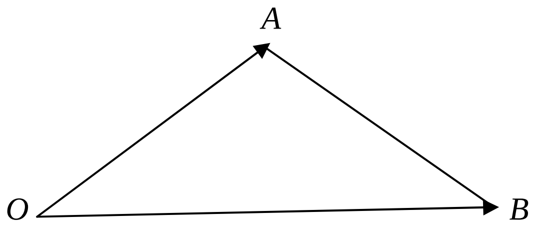
TURN OVER

Section 2

Instructions for Section 2
Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1

In the triangle below $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $|\underset{\sim}{b}| = 2|\underset{\sim}{a}|$.



a. If $\underset{\sim}{a}.\underset{\sim}{b} = a^2$, find the size of $\angle AOB$.

2 marks

Worked solution

$\underset{\sim}{a}.\underset{\sim}{b} = ab\cos\theta$

1M

$a^2 = a \times 2a\cos\theta$

$a^2 = 2a^2\cos\theta$

$\Rightarrow \cos\theta = \dfrac{a^2}{2a^2} = \dfrac{1}{2}$

$\Rightarrow \theta = \dfrac{\pi}{3}$

$\therefore \angle AOB = \dfrac{\pi}{3}$ or $60°$. 1A

Mark allocation

- 1 mark for using the dot product.
- 1 mark for the correct answer.

b. Express $\overrightarrow{AB}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Hence use a vector method to show that $\angle OAB$ is equal to 90°.

2 marks

Worked solution

$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{OA}.\overrightarrow{AB} = \underset{\sim}{a}.(\underset{\sim}{b} - \underset{\sim}{a})$ 1M

$\overrightarrow{OA}.\overrightarrow{AB} = \underset{\sim}{a}.\underset{\sim}{b} - \underset{\sim}{a}.\underset{\sim}{a}$

$= a^2 - a^2$

$= 0$ 1M

$\Rightarrow \overrightarrow{OA}$ is perpendicular to $\overrightarrow{AB}$.

$\therefore \angle OAB$ is a right angle.

Mark allocation

- 1 mark for the dot product of $\overrightarrow{OA}$ and $\overrightarrow{AB}$.
- 1 mark for showing the dot product is zero.

Tip

- $\underset{\sim}{a}.\underset{\sim}{a} = a^2$
- When a question says 'Use a vector method', then you must do so.

c. If M is the midpoint of $\overrightarrow{OB}$, express $\overrightarrow{MA}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **Hence** use a vector method to show that triangle OAM is equilateral.

2 marks

Worked solution

$\overrightarrow{MA} = \underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}$

$\left|\overrightarrow{MA}\right|^2 = \overrightarrow{MA}.\overrightarrow{MA}$

$\overrightarrow{MA}.\overrightarrow{MA} = (\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}).(\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b})$

$= a^2 - \underset{\sim}{a}.\underset{\sim}{b} + \frac{1}{4}b^2$

$= a^2 - a^2 + \frac{1}{4}(2a)^2$

$= a^2$

$\Rightarrow \left|\overrightarrow{MA}\right| = a$ 1M

$\left|\overrightarrow{OA}\right| = a$

$\left|\overrightarrow{OM}\right| = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}(2a) = a$ 1M

$$\Rightarrow \left|\overrightarrow{MA}\right| = \left|\overrightarrow{OA}\right| = \left|\overrightarrow{OM}\right|$$

Therefore, triangle *OAM* is equilateral.

Mark allocation

- 1 mark for expressing $\overrightarrow{MA}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and for showing that $\left|\overrightarrow{MA}\right| = a$.
- 1 mark for showing that the three sides of triangle *OAM* are equal.

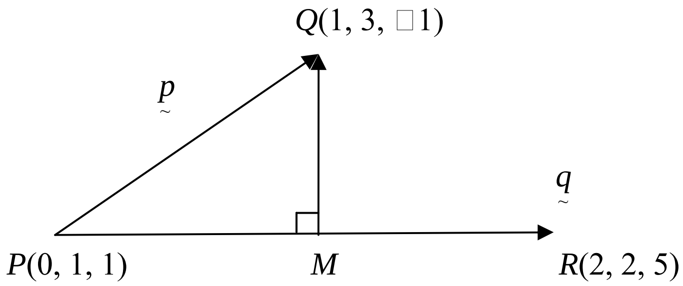
Three points, *P*, *Q* and *R*, have their respective coordinates given by (0, 1, 1), (1, 3, □1) and (2, 2, 5).

d. Find the coordinates of a point *M* on $\overrightarrow{PR}$ that is closest to the point *Q*.

4 marks

Total = 2 + 2 + 2 + 4 = 10 marks

Worked solution



Let *O* be the origin (0, 0, 0), $\overrightarrow{PQ} = \underset{\sim}{p}$, $\overrightarrow{PR} = \underset{\sim}{q}$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k} - (\underset{\sim}{j} + \underset{\sim}{k})$$

$$\Rightarrow \underset{\sim}{p} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k} - (\underset{\sim}{j} + \underset{\sim}{k})$$

$$\Rightarrow \underset{\sim}{q} = 2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$$

1M

$$\overrightarrow{PM} = (\underset{\sim}{p}.\ \underset{\sim}{\hat{q}})\ \underset{\sim}{\hat{q}} = (\underset{\sim}{p}.\ \underset{\sim}{q})\ \underset{\sim}{q} \div q^2$$

$$= (\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}).\ (2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k})\ (2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}) \div (2^2 + 1^2 + 4^2)$$

$$= \frac{-4}{21}\ (2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k})$$

1M

$$\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}$$

$$= (\underset{\sim}{j} + \underset{\sim}{k}) - \frac{4}{21}\ (2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k})$$

$$= \frac{-8}{21}\underset{\sim}{i} + \frac{17}{21}\underset{\sim}{j} + \frac{5}{21}\underset{\sim}{k}$$

1M

$\Rightarrow M$ is the point $\left(\frac{-8}{21}, \frac{17}{21}, \frac{5}{21}\right)$. 1A

Mark allocation

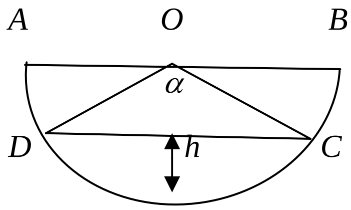
- 1 mark for correctly expressing $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ in terms of $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$.
- 1 mark for correctly expressing $\overrightarrow{PM}$ in terms of $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$.
- 1 mark for correctly expressing $\overrightarrow{OM}$ in terms of $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$.
- 1 mark for the correct answer.

Tip

- If A and B have respective coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2), then vector $\overrightarrow{AB} = (x_2 - x_1)\underset{\sim}{i} + (y_2 - y_1)\underset{\sim}{j} + (z_2 - z_1)\underset{\sim}{k}$.

Question 2

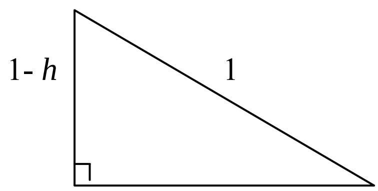
A relaxation bath is 1 metre long and has a semi-circular cross-section of radius 1 metre, as shown below.
AB is the diameter of the semi-circle, the angle $DOC = \alpha$ radians and h is the depth of water in the bath.



a. Show that the height of water in the bath is $h = 1 - \cos\left(\frac{\alpha}{2}\right)$.

1 mark

Worked solution



$\cos\frac{\alpha}{2} = \frac{1-h}{1}$

$\cos\frac{\alpha}{2} = 1 - h$ 1M

$\Rightarrow h = 1 - \cos\frac{\alpha}{2}$

Mark allocation

- 1 mark for correctly expressing $\cos\frac{\alpha}{2}$ in terms of h.

b. Show that the volume of water in the tank is $V = 0.5(\alpha - \sin\alpha)$.

1 mark

Worked solution

$$V = 1 \times \frac{1}{2} \times r^2(\alpha - \sin\alpha)$$

$$= \frac{1}{2}(1)^2(\alpha - \sin\alpha) \qquad \text{1M}$$

$$= 0.5(\alpha - \sin\alpha)$$

Mark allocation

- 1 mark for the correct expression for the volume as $1\times$(the area of the sector OCD subtract the area of the triangle OCD).

c. Water is pumped into the bath at a rate of 0.5 m^3 per minute. Find the exact rate at which the water level is rising when the depth of water in the bath is 0.5 metres.

4 marks

Worked solution

$$\frac{dV}{dt} = 0.5$$

$$h = 0.5$$

$$\cos\frac{\alpha}{2} = 1 - h = 1 - 0.5 = 0.5$$

$$\Rightarrow \frac{\alpha}{2} = \cos^{-1} 0.5 = \frac{\pi}{3}$$

$$\Rightarrow \alpha = \frac{2\pi}{3}$$

1M

$$h = 1 - \cos\frac{\alpha}{2}$$

$$\Rightarrow \frac{dh}{d\alpha} = 0.5\sin\frac{\alpha}{2} = 0.5\sin\frac{\pi}{3}$$

$$= 0.5 \times \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{4}$$

$$V = 0.5(\alpha - \sin\alpha)$$

$$\Rightarrow \frac{dV}{d\alpha} = 0.5 - 0.5\cos\alpha = 0.5 - 0.5\cos\frac{2\pi}{3}$$

$$= 0.5 - 0.5 \times \frac{-1}{2}$$

$$= 0.75$$ 1M

$$\frac{dh}{dV} = \frac{dh}{d\alpha} \div \frac{dV}{d\alpha} = \frac{\sqrt{3}}{4} \div 0.75$$

$$= \frac{\sqrt{3}}{4} \times \frac{4}{3} = \frac{\sqrt{3}}{3}$$ 1M

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$\frac{dh}{dt} = 0.5 \times \frac{\sqrt{3}}{3}$$

$$\Rightarrow \frac{dh}{dt} = \frac{\sqrt{3}}{6} \text{ m/min}$$ 1A

Mark allocation

- 1 mark for the correct values of $\frac{dh}{d\alpha}$ and $\frac{dV}{d\alpha}$ when $h = 0.5$.
- 1 mark for the correct value of $\frac{dh}{d\alpha}$ when $h = 0.5$.
- 1 mark for the correct value of $\frac{dh}{dV}$.
- 1 mark for the correct answer.

d. When there is 2000 litres of pure water in the bath, an Epsom salt solution of concentration 0.08 kg per litre is pumped in at a rate of 50 litres per minute. The mixture is kept uniform and at the same time the bath is drained at a rate of 50 litres per minute.

i. What is the limit to the amount of Epsom salt in the bath water?

1 mark

Worked solution

Limit for $x = 2000 \times 0.08 = 160$ kg ($x < 160$) 1M

Mark allocation

- 1 mark for the correct answer.

ii. Show that $\frac{dx}{dt} = \frac{160 - x}{40}$, where x is the amount of Epsom salt in the bath after t minutes.

1 mark

Worked solution

$$\frac{dx}{dt} = \text{Input rate of salt} - \text{output rate of salt}$$

$$= 4 - \frac{x}{2000} \times 50$$ 1M

$$= 4 - \frac{x}{40}$$

$$= \frac{160 - x}{40}$$

Mark allocation

- 1 mark for the correct set up of the input/output equation.

iii. **Hence**, show that $x = 160\left(1 - e^{\frac{-t}{40}}\right)$.

3 marks

Worked solution

$$\frac{dx}{dt} = \frac{160 - x}{40}$$

$$\Rightarrow \frac{dt}{dx} = \frac{40}{160 - x}$$

$$t = \int \frac{40}{160 - x}.dx$$ 1M

$$= -40\log_e k|160 - x|$$

When $t = 0$, $x = 0$:

$$\Rightarrow -40\log_e k|160| = 0$$

$$160k = 1$$

$$\Rightarrow k = \frac{1}{160}$$

$$\therefore\ t = -40\log_e \left|\frac{160 - x}{160}\right|$$ 1M

$$\log_e \left|\frac{160 - x}{160}\right| = \frac{-t}{40}$$

$$\left|\frac{160 - x}{160}\right| = e^{\frac{-t}{40}}$$

$$|160 - x| = 160e^{\frac{-t}{40}}$$

$$160 - x = 160e^{\frac{-t}{40}} \quad \text{or} \quad 160 - x = -160e^{\frac{-t}{40}}$$

$$\Rightarrow x = 160 - 160e^{\frac{-t}{40}} \quad \text{or} \quad x = 160 + 160e^{\frac{-t}{40}}$$

$$x = 160(1 - 160e^{\frac{-t}{40}}) \quad \text{or} \quad x = 160(1 + 160e^{\frac{-t}{40}})$$

But $x < 2000 \times 0.08 = 160$ kg (as determined from part **i**)

$\therefore\ x = 160(1 - 160e^{\frac{-t}{40}})$ 1M

Mark allocation

- 1 mark for correctly setting up $t = \int \frac{dt}{dx}.dx$.
- 1 mark for correctly expressing t as a function of x.
- 1 mark for correctly rearranging $t(x)$ as $x(t)$.

e. From the time the mixing began, how long will it take for the concentration of Epsom salt in the bathwater to reach 0.05 kg per litre? Give your answer to the nearest tenth of a minute.

1 mark

Total = 1 + 1 + 4 + 5 + 1 = 12 marks

Worked solution

$x = 2000 \times 0.05 = 100$ kg

$$t = -40\log_e\left|\frac{160 - x}{160}\right|$$

$$\Rightarrow t = -40\log_e\left(\frac{60}{160}\right)$$

$$= 39.233$$

$\therefore\ t \approx 39.2$ minutes

Mark allocation

- 1 mark for the correct answer.

Question 3

The position vector of a particle at any time $t \geq 0$ is $\underset{\sim}{r}(t) = t\underset{\sim}{i} + \tan^{-1}\left(\frac{t}{2}\right)\underset{\sim}{j}$.

a. Determine the Cartesian equation of the path of the particle.

1 mark

Worked solution

$$\underset{\sim}{r}(t) = t\underset{\sim}{i} + \tan^{-1}\left(\frac{t}{2}\right)\underset{\sim}{j}, \quad t \geq 0$$

$$x = t \text{ and } y = \tan^{-1}\left(\frac{t}{2}\right)$$

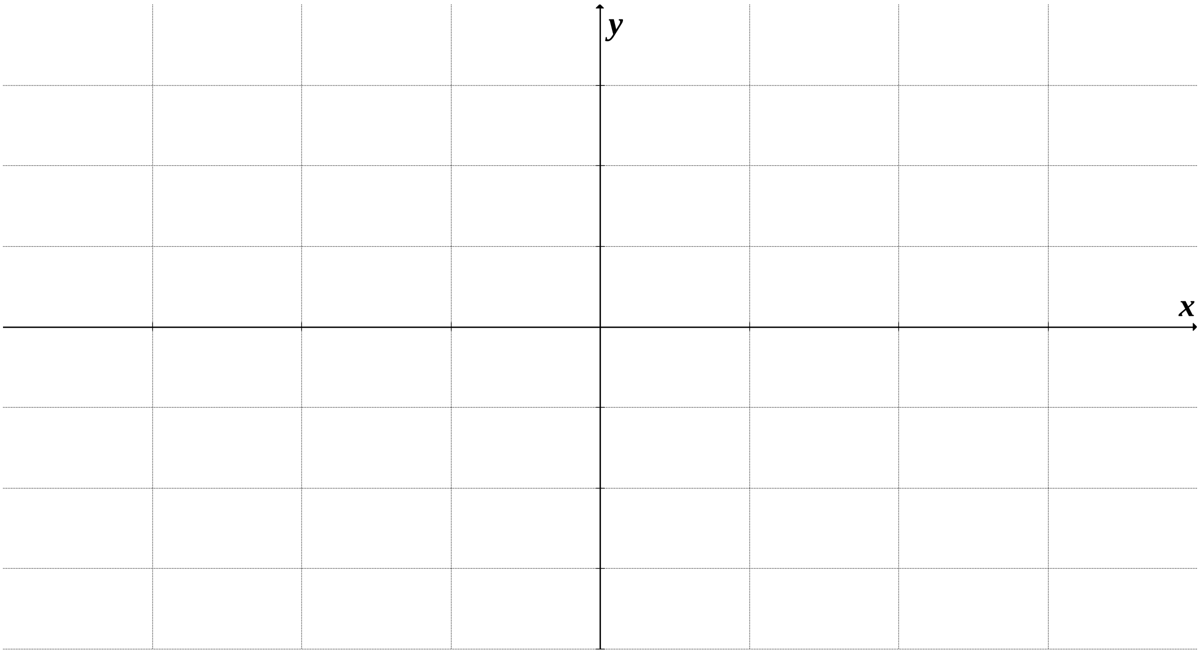
$$\Rightarrow y = \tan^{-1}\left(\frac{x}{2}\right) \quad x, y \geq 0$$

1M

Mark allocation

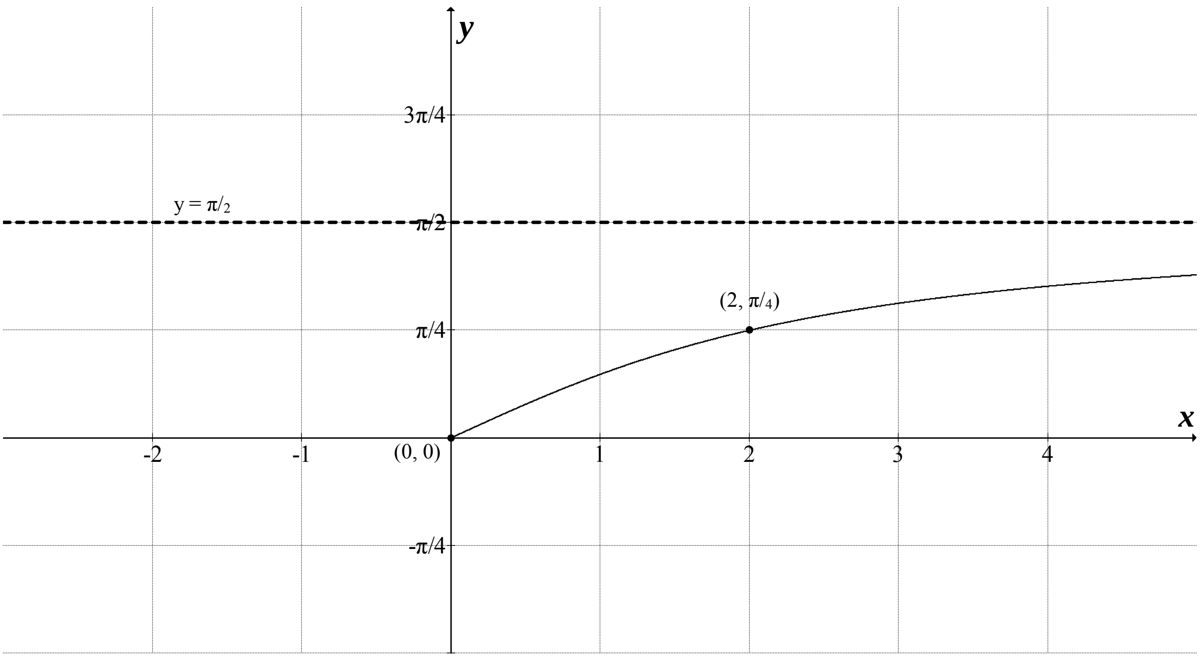
- 1 mark for the correct answer.

b. Sketch the graph on the set of axes below, labelling any asymptotes and the coordinates of any end points.



2 marks

Worked solution



Mark allocation

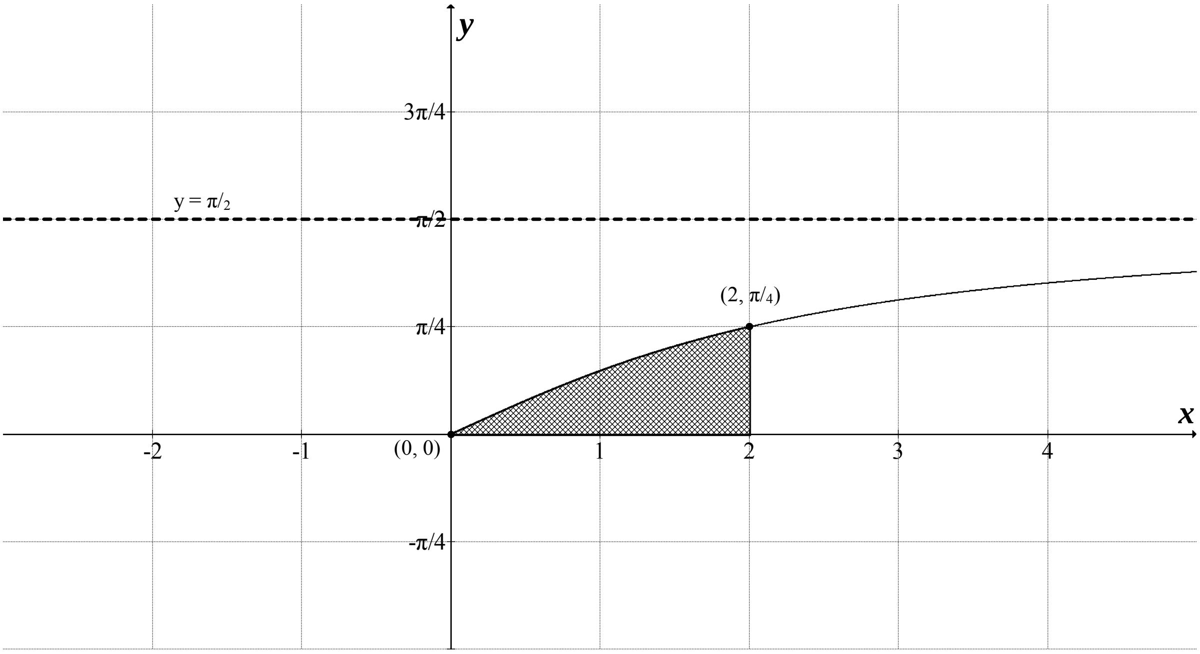
- 1 mark for the correct curve.
- 1 mark for correctly labelling end point and horizontal asymptote.

c. **i.** On the same set of axes given in part **b**, shade the area between the graph, the x-axis and the line $x = 2$.

.

1 mark

Worked solution



Mark allocation

- 1 mark for shading the correct area.

ii. Use Calculus to determine the exact area between the graph, the x-axis and the line
$x = 2$. Express your answer in the form $a + \log_e b$, where $a, b \in R$.

4 marks

Worked solution

The area required is the area of the rectangle – the area between the curve and the y-axis.

Since $y = \tan^{-1}\left(\frac{x}{2}\right) \quad x, y \geq 0$

When $x = 2 \Rightarrow y = \tan^{-1}(1) = \frac{\pi}{4}$

$y = \tan^{-1}\left(\frac{x}{2}\right)$

$\Rightarrow \frac{x}{2} = \tan y$

$\Rightarrow x = 2\tan y$ 1M

$$\text{Area} = 2\times\frac{\pi}{4}-\int_0^{\frac{\pi}{4}} x.dy$$

$$= \frac{\pi}{2}-\int_0^{\frac{\pi}{4}} 2\tan y.dy$$

$$= \frac{\pi}{2}-\int_0^{\frac{\pi}{4}} \frac{2\sin y}{\cos y}.dy$$ 1M

For $\int_0^{\frac{\pi}{4}} \frac{2\sin y}{\cos y}.dy$

Let $u=\cos y \Rightarrow \frac{du}{dy}=-\sin y$

$y=\frac{\pi}{4} \quad \Rightarrow u=\frac{\sqrt{2}}{2}$ and $y=0 \quad \Rightarrow u=1$

$$\Rightarrow \int_1^{\frac{\pi}{4}} \frac{2\sin y}{\cos y}.dy = \int_1^{\frac{\sqrt{2}}{2}} \frac{-2}{u}.\frac{du}{dy}.dy$$

$$= \int_1^{\frac{\sqrt{2}}{2}} \frac{-2}{u}.du$$ 1M

$$= \left[-2\log_e|u|\right]_1^{\frac{\sqrt{2}}{2}}$$

$$= \left[-2\log_e\left|\frac{\sqrt{2}}{2}\right| - \left(-2\log_e 1\right)\right]$$

$$= -2\log_e 2^{\frac{-1}{2}} - 0$$

$$= \log_e 2$$

Area required $= \frac{\pi}{2}-\log_e 2 = \frac{\pi}{2}+\log_e\left(\frac{1}{2}\right)$ square units. 1A

Mark allocation

- 1 mark for expressing x as a function of y.
- 1 mark for setting up the correct calculation for the required area.
- 1 mark for using the correct substitution to evaluate the integral.
- 1 mark for the correct answer.

d. Use Calculus to determine the exact volume generated when the area between the graph, the x-axis and the line $x = 2$ is rotated around the y-axis.

3 marks

Worked solution

Volume required, $V = \pi \int_0^{\frac{\pi}{4}} \left(x_1^2 - x_2^2\right).dy$, where $x_1 = 2$ and $x_2 = \tan y$.

$$\Rightarrow V = \pi \int_0^{\frac{\pi}{4}} \left(2^2 - (2\tan y)^2\right).dy \qquad \text{1M}$$

$$= \pi \int_0^{\frac{\pi}{4}} \left(4 - 4\tan^2 y\right).dy$$

$$= 4\pi \int_0^{\frac{\pi}{4}} (1 - \tan^2 y).dy$$

$$= 4\pi \int_0^{\frac{\pi}{4}} (1 - (\sec^2 y - 1)).dy = 4\pi \int_0^{\frac{\pi}{4}} (2 - \sec^2 y).dy \qquad \text{1M}$$

$$= 4\pi \left[2y - \tan y\right]_0^{\frac{\pi}{4}}$$

$$= 4\pi \left[\frac{\pi}{2} - \tan\frac{\pi}{4} - (0 - \tan 0)\right]$$

$$= 4\pi \left(\frac{\pi}{2} - 1\right)$$

$$= 2\pi^2 - 4\pi = 2\pi(\pi - 2) \text{ cubic units} \qquad \text{1A}$$

Mark allocation
- 1 mark for setting up the correct integral to evaluate the required volume.
- 1 mark for using the correct identity so that the integrand can be antidifferentiated.
- 1 mark for the correct answer.

e. A normal line to the curve intersects with the x-axis at $x = 2\sqrt{3} + \frac{\pi}{24}$. Find the coordinates of the point on the curve through which this normal line passes.
Give your answer correct to 3 decimal places.

4 marks
Total = 1 + 2 + 5 + 3 + 4 = 15 marks

Worked solution

Let $P\left(x_1, \tan^{-1}\dfrac{x_1}{2}\right)$ be the point on the curve through which the normal line passes.

Let $Q\left(2\sqrt{3}+\dfrac{\pi}{24}, 0\right)$ be the point where the normal line intersects the x-axis.

$$y = \tan^{-1}\left(\frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{4+x^2} = \text{gradient of a tangent} \qquad \text{1M}$$

The gradient of a normal = $-\dfrac{4+x^2}{2}$ 1M

$$\text{Gradient of } \overline{PQ} = \frac{\tan^{-1}\left(\frac{x_1}{2}\right) - 0}{x_1 - \left(2\sqrt{3}+\frac{\pi}{24}\right)}$$

$$\therefore -\frac{4+x_1^{\ 2}}{2} = \frac{\tan^{-1}\left(\frac{x_1}{2}\right)}{\frac{24x-48\sqrt{3}-\pi}{24}} = \frac{24\tan^{-1}\left(\frac{x_1}{2}\right)}{24x_1-48\sqrt{3}-\pi}$$

$$-(4+x_1^{\ 2})(24x_1-48\sqrt{3}-\pi) = 48\tan^{-1}\left(\frac{x_1}{2}\right)$$

$$48\tan^{-1}\left(\frac{x_1}{2}\right) - (4+x_1^{\ 2})(24x_1 - -48\sqrt{3}-\pi) = 0 \qquad \text{1M}$$

Using a graphics calculator to solve this equation, the solution is (3.464, 1.047) 1A

Mark allocation

- 1 mark for correctly differentiating.
- 1 mark for the correct expression for the normal.
- 1 mark for simplifying the equation.
- 1 mark for the correct answer.

Question 4

A curve in the complex plane has its rule given by

$T = \{z : |z-5| - |z-1| = 2,\ z \in C\}$

a. Show that the points $z = 2$ and $z = 1+3i$ belong to T.

2 marks

Worked solution

$\{z : |z-5|-|z-1|=2,\ z \in C\}$

If $z = 2$, then:

$|2-5|-|2-1| = 3-1 = 2$ 1M

If $z = 1+3i$, then:

$$\begin{aligned}|1+3i-5|-|1+3i-1| &= |-4+3i|-|3i| \\ &= \sqrt{(-4)^2+3^2}-\sqrt{3^2} \\ &= \sqrt{25}-\sqrt{9} \\ &= 5-3 \\ &= 2\end{aligned}$$ 1M

$\Rightarrow z = 2$ and $z = 1+3i$ belong to T.

Mark allocation

- 1 mark for showing $z = 2$ belongs to *T*.
- 1 mark for showing $z = 1+3i$ belongs to *T*.

b. Find the exact values of b if $z = bi$ also belongs to *T*.

2 marks

Worked solution

$\{z : |z-5|-|z-1|=2,\ z \in C\}$

If $z = bi$, then:

$$|bi-5|-|bi-1| = 2$$

$$\sqrt{b^2+25}-\sqrt{b^2+1} = 2$$

$$\sqrt{b^2+25} = 2+\sqrt{b^2+1}$$

$$b^2+25 = 4+4\sqrt{b^2+1}+b^2+1$$

$$4\sqrt{b^2+1} = 20$$ 1M

$$\sqrt{b^2+1} = 5$$

$$b^2+1 = 25$$

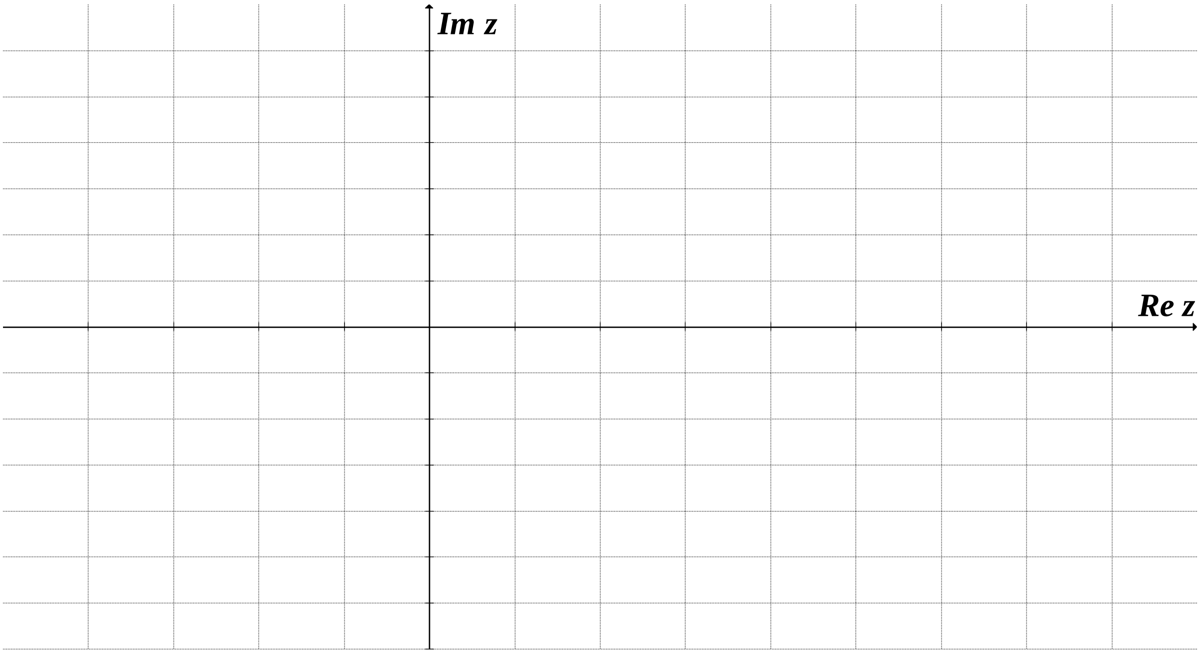
$$b^2 = 24$$

$$\Rightarrow b = \pm 2\sqrt{6}$$ 1A

Mark allocation

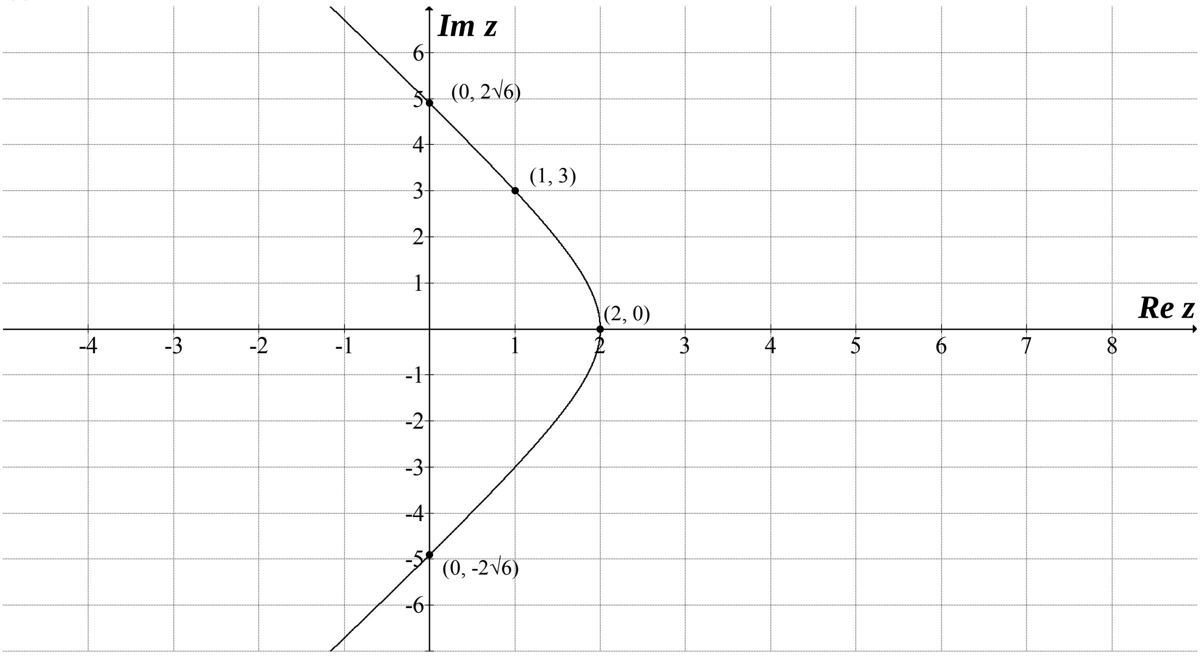
- 1 mark for setting up the correct equation for *b*.
- 1 mark for the correct answer.

c. Sketch the graph of T on the complex plane below. Clearly show all axes intercepts. (Do not show any asymptotes.)



2 marks

Worked solution



Mark allocation

- 1 mark for showing all axes intercepts correctly.
- 1 mark for the correct branch of the curve.

Tip

- T describes the locus of a set of points in the complex plane where the distance from $z = 5$ less the distance from $z = 1$ is always +2. This is the left branch of a hyperbola.
- Parts **a** and **b** include four points on this graph.

d. Express the relation defined by T in Cartesian form.

3 marks

Total = 2 + 2 + 2 + 3 = 9 marks

Worked solution

$T = \{z : |z-5| - |z-1| = 2,\ z \in C\}$

The locus of the curve described by T is the left-hand branch of a hyperbola with centre (3, 0).

$\Rightarrow \dfrac{(x-3)^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Substituting (2, 0) we get:

$\Rightarrow \dfrac{1}{a^2} = 1$

$\Rightarrow a^2 = 1$ 1M

Substituting (1, 3) we get:

$\Rightarrow \dfrac{4}{a^2} - \dfrac{9}{b^2} = 1$

$\Rightarrow 4 - \dfrac{9}{b^2} = 1$, since $a^2 = 1$

$\Rightarrow \dfrac{9}{b^2} = 3$

$\Rightarrow b^2 = 3$ 1M

$\therefore$ T is $\dfrac{(x-3)^2}{1} - \dfrac{y^2}{3} = 1$ 1M

Mark allocation

- 1 mark for the correct value of a.
- 1 mark for the correct value of b.
- 1 mark for the correct Cartesian equation for T.

Tip

- The centre of the hyperbola is the midpoint of $z = 1$ and $z = 5$, which is $z = 3$ or the point (3, 0).
- An alternative solution is to express the equation in Cartesian form:

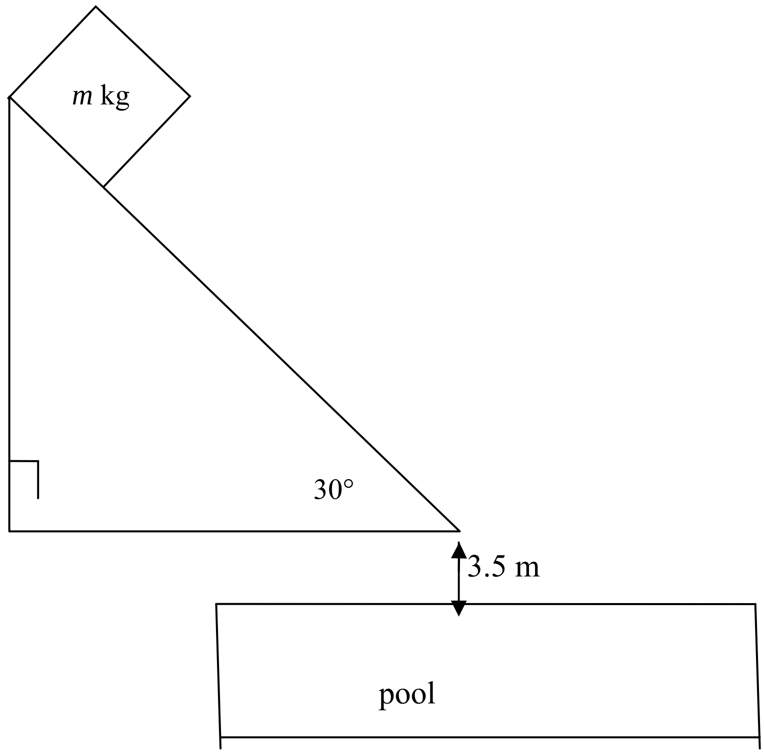
  $|z-5| - |z-1| = 2$

  $\Rightarrow \sqrt{(x-5)^2 + y^2} = 2 + \sqrt{(x-1)^2 + y^2}$

  Then simplify this (after several steps) into the form $\dfrac{(x-3)^2}{1} - \dfrac{y^2}{3} = 1$.

Question 5
Daredevil Park has a waterslide into a deep pool. The 12.5 metre slide is inclined at 30° to the horizontal and the bottom of the slide is 3.5 metres above the water surface of the pool.

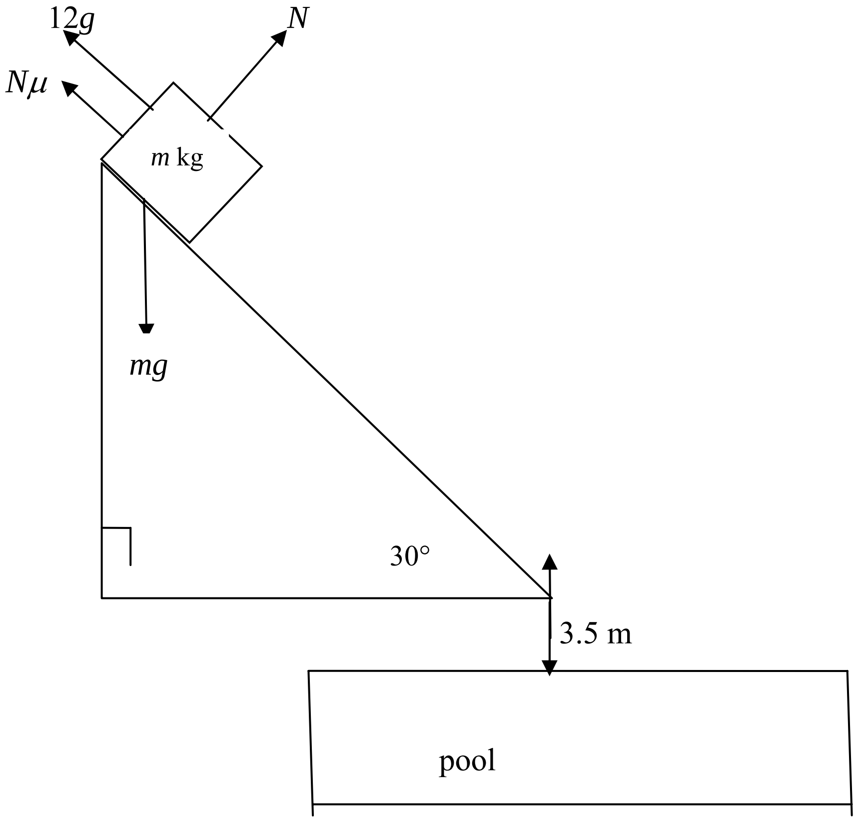
Albert, who is of mass *m* kg, positions himself on the slide at the top and is supported by his friend Charlie. When the applied force acting directly up the slide is 12*g* newtons, Albert is on the verge of moving down the slide. The coefficient of friction between Albert and the slide is $\frac{\sqrt{3}}{5}$. The situation is shown on the diagram below.



a. **i.** On the diagram above, label all the forces acting on Albert while he is being held at the top of the slide.

1 mark

Worked solution



Mark allocation

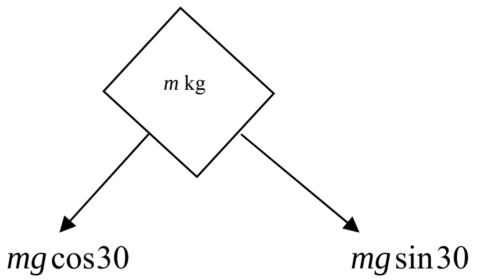
- 1 mark for correctly labelling all the forces.

ii. **Hence**, calculate the value of *m*.

3 marks

Worked solution

The perpendicular components of Albert's *m* kg weight force are:



$\Rightarrow N = mg\cos 30$

$= \dfrac{mg\sqrt{3}}{2}$

Down the slide, the resultant force is:

$R = mg\sin 30 - N\mu - 12g = 0$ 1M

$R = \dfrac{mg}{2} - \dfrac{mg\sqrt{3}}{2} \times \dfrac{\sqrt{3}}{5} - 12g = 0$ 1M

$\Rightarrow \dfrac{mg}{2} - \dfrac{3mg}{10} - 12g = 0$

$\dfrac{mg}{5} - 12g = 0$

$\dfrac{m}{5} = 12$

$m = 60$

$\therefore$ Albert has a mass of 60 kg. 1A

Mark allocation

- 1 mark for the correct equation of motion.
- 1 mark for simplifying the equation.
- 1 mark for the correct answer.

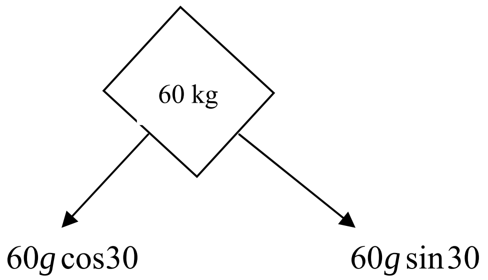
Suddenly, Charlie releases his hold on Albert, who moves from rest down the slide, reaching the end of the slide after travelling a distance of 12.5 metres.

b. Show that the acceleration of Albert while he is on the slide is $\frac{g}{5}$ m/s^2.

3 marks

Worked solution

The perpendicular components of Albert's 60 kg weight force are:



$$N = 60g\cos 30$$
$$= 30g\sqrt{3}$$

Down the slide, the resultant force is now:

$$R = 60g\sin 30 - N\mu = 60a \qquad \text{1M}$$

$$R = 60g\sin 30 - 30g\sqrt{3}\times\frac{\sqrt{3}}{5} = 60a \qquad \text{1M}$$

$$\Rightarrow 30g - 18g = 60a$$

$$12g = 60a \qquad \text{1M}$$

$$\Rightarrow a = \frac{g}{5}\ \text{m/s}^2$$

Mark allocation

- 1 mark for the correct equation of motion.
- 1 mark for correctly substituting the value of the normal into the equation.
- 1 mark for the correct evaluation of a.

c. Show that Albert reaches the end of the slide with a speed of 7 m/s.

1 mark

Worked solution

$$u = 0,\ a = \frac{g}{5},\ s = 12.5$$

$$v^2 = u^2 + 2as$$

$$v^2 = 0^2 + 2\times\frac{g}{5}\times 12.5 \qquad \text{1M}$$

$$v^2 = 5g = 49$$

$$\Rightarrow v = 7\ \text{m/s}$$

SECTION 2 – Question 5 – continued

TURN OVER

Mark allocation

- 1 mark for using the correct formula to evaluate *v*.

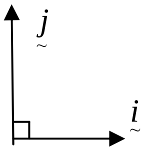
Take the bottom end of the slide as the origin, $\underset{\sim}{i}$ as the unit vector horizontally to the right and $\underset{\sim}{j}$ as the unit vector vertically up.

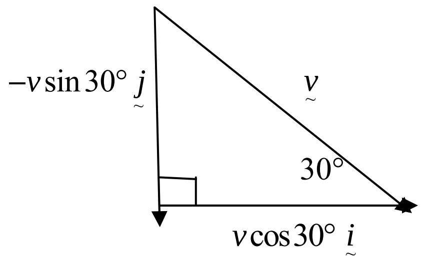
d. Find the velocity vector $\underset{\sim}{v}(t)$ which represents the velocity of Albert *t* seconds after he leaves the slide.

2 marks

Worked solution

Using





$$\underset{\sim}{a}(t) = -g\,\underset{\sim}{j}$$

$$\Rightarrow \underset{\sim}{v}(t) = c_1\,\underset{\sim}{i} + (c_2 - gt)\,\underset{\sim}{j}$$

$$\underset{\sim}{v}(0) = c_1\,\underset{\sim}{i} + c_2\,\underset{\sim}{j} = v\cos 30°\,\underset{\sim}{i} - v\sin 30°\,\underset{\sim}{j}, \quad v = 7$$

1M

$$\Rightarrow c_1 = v\cos 30° = 7 \times \frac{\sqrt{3}}{2} = 3.5\sqrt{3}$$

$$\Rightarrow c_2 = -v\sin 30° = -7 \times \frac{1}{2} = -3.5$$

$$\therefore\ \underset{\sim}{v}(t) = 3.5\sqrt{3}\,\underset{\sim}{i} - (3.5 + gt)\,\underset{\sim}{j}$$

1A

Mark allocation

- 1 mark for the correct initial velocity.
- 1 mark for the correct answer.

e. Calculate Albert's speed 0.5 seconds after he reaches the bottom of the slide.
Give your answer to the nearest tenth of a second.

2 marks
Total = 4 + 3 + 1 + 2 + 2 = 12 marks

Worked solution

$$\therefore\ \underset{\sim}{v}(t) = 3.5\sqrt{3}\,\underset{\sim}{i} - (3.5 + gt)\,\underset{\sim}{j}$$

$$\underset{\sim}{v}(0.5) = 3.5\sqrt{3}\,\underset{\sim}{i} - (3.5 + 0.5g)\,\underset{\sim}{j}$$

$$= 3.5\sqrt{3}\,\underset{\sim}{i} - 8.4\,\underset{\sim}{j}$$

1M

$$v(0.5) = \sqrt{\left(\left(3.5\sqrt{3}\right)^2 + (8.4)^2\right)}$$

$$v(0.5) = 10.359$$

$$\Rightarrow v(0.5) \approx 10.4 \text{ m/s}$$

1A

Mark allocation

- 1 mark for the correct velocity vector $\underset{\sim}{v}(0.5)$.
- 1 mark for the correct answer.

END OF SECTION 2

END OF WORKED SOLUTIONS BOOK