The Mathematical Association of Victoria

SPECIALIST MATHEMATICS 2010

Trial Written Examination 1--SOLUTIONS

Question 1

$3x^2 + 4y^2 = 48$

$\frac{d}{dx}\left(3x^2\right) + \frac{d}{dx}\left(4y^2\right) = \frac{d}{dx}(48)$ [M1]

$6x + \frac{d}{dy}\left(4y^2\right)\frac{dy}{dx} = 0$

$6x + 8y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{3x}{4y}$ [A1]

When $x = 2$, $12 + 4y^2 = 48$

$y = \pm 3$

When $x = 2$, $y = 3$, $\frac{dy}{dx} = -\frac{6}{12} = -\frac{1}{2}$

When $x = 2$, $y = -3$, $\frac{dy}{dx} = -\frac{6}{-12} = \frac{1}{2}$ [A1]

Total 3 marks

Question 2

$y = \arcsin(2x)$

$y = \sin^{-1}(u)$ where $u = 2x$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \times 2$

$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}$ [A1]

$\frac{dy}{dx} = 2\left(1-4x^2\right)^{-\frac{1}{2}}$

$\frac{d^2y}{dx^2} = 2 \times -\frac{1}{2}\left(1-4x^2\right)^{-\frac{3}{2}} \times -8x$ [A1]

$$\frac{d^2y}{dx^2} = 8x\left(1-4x^2\right)^{-\frac{3}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{8x}{\sqrt{\left(1-4x^2\right)^3}}$$ [A1]

Total 3 marks

Question 3

$\tan(2\theta) = \sqrt{3}$, where $\theta \in \left(-\pi, -\frac{\pi}{2}\right)$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$$

$\sqrt{3}\left(1-\tan^2(\theta)\right) = 2\tan(\theta)$ [M1]

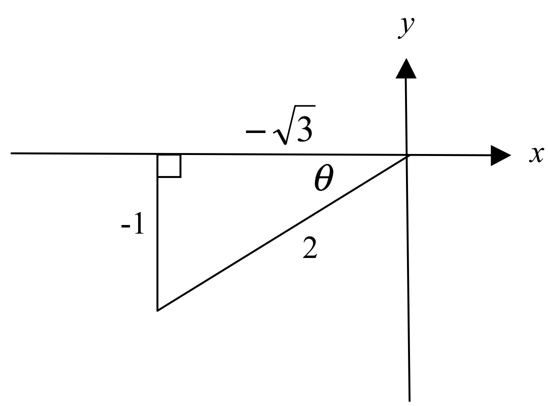
$$\sqrt{3}\tan^2(\theta) + 2\tan(\theta) - \sqrt{3} = 0$$

$\tan(\theta) = \frac{-2 \pm \sqrt{4-4\left(\sqrt{3}\right)\left(-\sqrt{3}\right)}}{2\sqrt{3}}$ quadratic formula [M1]

$$= \frac{-2 \pm 4}{2\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}, \frac{-3}{\sqrt{3}}$$

$= \frac{1}{\sqrt{3}}$, since $\theta \in \left(-\pi, -\frac{\pi}{2}\right)$



From diagram, $\sin(\theta) = -\frac{1}{2}$, $\cos(\theta) = -\frac{\sqrt{3}}{2}$ [M1]

$$\text{cis}(\theta) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$= -\frac{1}{2}\left(\sqrt{3}+i\right)$ [A1]

Total 4 marks

Question 4

a.

$$-2+2\sqrt{3}i = 4\text{cis}\left(\frac{2\pi}{3}\right)$$

Let $z = r\text{cis}(\theta)$ be a square root of $-2+2\sqrt{3}i$

Hence $z^2 = r^2\text{cis}(2\theta) = 4\text{cis}\left(\frac{2\pi}{3}+2k\pi\right)$, where $k \in J$ [M1]

$$r = 2,\ 2\theta = \frac{2\pi}{3}+2k\pi$$

Let $k = 0,\ \theta = \frac{\pi}{3}$

Let $k = 1,\ 2\theta = \frac{2\pi}{3}+2\pi$

$$\theta = \frac{4\pi}{3}$$

$z = 2\text{cis}\left(\frac{\pi}{3}\right)$, $z = 2\text{cis}\left(-\frac{2\pi}{3}\right)$ [A1]

$= 2\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)$, $= 2\left(\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right)$

$= 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$, $= 2\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$

$= 1+\sqrt{3}i,\ -1-\sqrt{3}i$ [A1]

Alternative solution

Let $z = a+bi$ be a square root of $-2+2\sqrt{3}i$, where $a, b \in R$

$z^2 = a^2 + 2abi - b^2 = -2+2\sqrt{3}i$ [M1]

Equating real coefficients and equating complex coefficients

$a^2 - b^2 = -2 \quad \ldots(1)$ $\qquad 2ab = 2\sqrt{3}$

$$b = \frac{\sqrt{3}}{a} \qquad \ldots(2)$$

Substituting (2) into (1)

$a^2 - \frac{3}{a^2} = -2$

$a^4 - 3 = -2a^2$

$a^4 + 2a^2 - 3 = 0$

$a^4 - 3 = -2a^2$

$(a^2 + 3)(a^2 - 1) = 0$ [A1]

$a^2 = -3$ (no real solutions) $\quad a^2 = 1$

$a = \pm 1$

If $a = 1,\ b = \sqrt{3}$ and if $a = -1,\ b = -\sqrt{3}$

Hence $z = 1 + \sqrt{3}i,\ -1 - \sqrt{3}i$ [A1]

b.

$z^2 + (\sqrt{3} - i)z + (1 - \sqrt{3}i) = 0$

$z = \frac{-\sqrt{3} + i \pm \sqrt{(\sqrt{3} - i)^2 - 4(1 - \sqrt{3}i)}}{2}$ [M1]

$= \frac{-\sqrt{3} + i \pm \sqrt{3 - 2\sqrt{3}i - 1 - 4 + 4\sqrt{3}i}}{2}$

$= \frac{-\sqrt{3} + i \pm \sqrt{-2 + 2\sqrt{3}i}}{2}$ [A1]

$= \frac{-\sqrt{3} + i + (1 + \sqrt{3}i)}{2}, \frac{-\sqrt{3} + i - (1 + \sqrt{3}i)}{2}$, from part **a.**

$\frac{(1 - \sqrt{3}) + (1 + \sqrt{3})i}{2}, \frac{(-1 - \sqrt{3}) + (1 - \sqrt{3})i}{2}$ [A1]

Total 6 marks

Question 5

$y = \frac{x^3 - 1}{x}$

$y = x^2 - \frac{1}{x}$

Sketch $y_1 = x^2$ and $y_2 = -\frac{1}{x}$ and use addition of ordinates.

x-intercepts (y = 0) $\quad \frac{x^3 - 1}{x} = 0$

$x^3 - 1 = 0$

$x = 1$

x-intercept (1, 0) [A1]

Stationary points: $\frac{dy}{dx} = 0 \quad \frac{dy}{dx} = 2x + \frac{1}{x^2}$

$2x + \frac{1}{x^2} = 0$

$2x^3 + 1 = 0$

$x^3 = -\frac{1}{2}$

$x = -\frac{1}{\sqrt[3]{2}}, \; y = \frac{-\frac{3}{2}}{-\frac{1}{\sqrt[3]{2}}}$ [A1]

Minimum: $\left(-\frac{1}{\sqrt[3]{2}}, \frac{3\sqrt[3]{2}}{2} \right)$

Point of inflexion: $\frac{d^2y}{dx^2} = 0 \quad \frac{dy}{dx} = 2x + x^{-2}$

$\frac{d^2y}{dx^2} = 2 - 2x^{-3}$

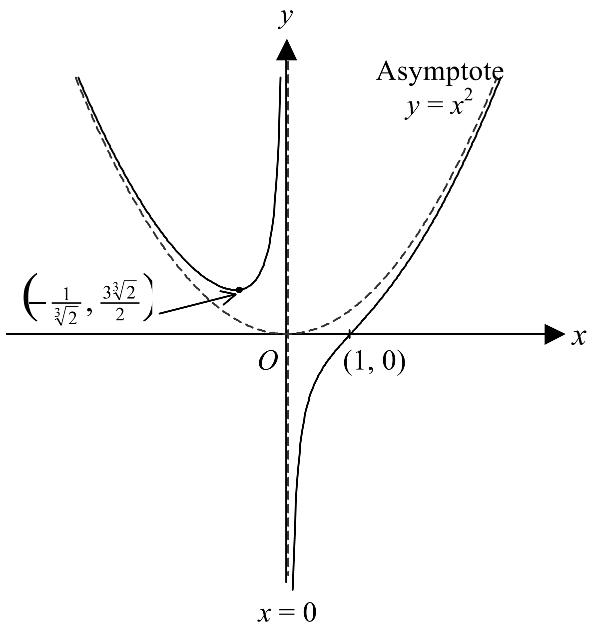
$2 - \frac{2}{x^3} = 0$

$2x^3 - 2 = 0$

$x^3 = 1$

Point of inflexion (1, 0) [A1]

Asymptotes: $y = x^2$ and $x = 0$ (y-axis) [A1]



Correct graph [A1]

Total 5 marks

Question 6

$$\int_m^{\frac{\pi}{2}} \sec^2\left(\frac{x}{2}\right)dx = \frac{2}{3}\left(3-\sqrt{3}\right)$$

$$\left[2\tan\left(\frac{x}{2}\right)\right]_m^{\frac{\pi}{2}} = \frac{2}{3}\left(3-\sqrt{3}\right)$$ [M1]

$$2\left(\tan\left(\frac{\pi}{4}\right)-\tan\left(\frac{m}{2}\right)\right) = \frac{2}{3}\left(3-\sqrt{3}\right)$$

$$1-\tan\left(\frac{m}{2}\right) = \frac{3-\sqrt{3}}{3}$$ [A1]

$$1-\tan\left(\frac{m}{2}\right) = 1-\frac{\sqrt{3}}{3}$$

$$\tan\left(\frac{m}{2}\right) = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\frac{m}{2} = \frac{\pi}{6}$$

$$m = \frac{\pi}{3}$$ [A1]

Total 3 marks

Question 7

$\int \frac{x}{\sqrt[3]{3x^2+1}}\,dx$ Let $u = 3x^2 + 1$

$= \frac{1}{6}\int \frac{6x}{(3x^2+1)^{\frac{1}{3}}}\,dx$ $\frac{du}{dx} = 6x$ [M1]

$\Rightarrow \quad du = 6x\,dx$

$= \frac{1}{6}\int u^{-\frac{1}{3}}\,du$ [A1]

$= \frac{1}{6} \times \frac{3}{2} u^{\frac{2}{3}} + c$

$= \frac{1}{4}(3x^2+1)^{\frac{2}{3}} + c$ [A1]

Total 3 marks

Question 8

$a = v^3 + \pi^2 v$

$v\frac{dv}{dx} = v^3 + \pi^2 v$

$\frac{dv}{dx} = v^2 + \pi^2$

$\frac{dx}{dv} = \frac{1}{v^2+\pi^2}$ [A1]

$x = \frac{1}{\pi}\int \frac{\pi}{v^2+\pi^2}\,dv$

$x = \frac{1}{\pi}\arctan\left(\frac{v}{\pi}\right) + c$ [A1]

Particle starts from rest at the origin:

At $x = 0,\ v = 0$ $0 = \frac{1}{\pi}\arctan\left(\frac{0}{\pi}\right) + c$

$\Rightarrow \quad c = 0$

$x = \frac{1}{\pi}\arctan\left(\frac{v}{\pi}\right)$

$v = \pi \tan(\pi x)$ [A1]

At $x = 0.25\,\text{m}$ $v = \pi \tan(0.25\pi)$

$v = \pi \tan\left(\frac{\pi}{4}\right)$

$v = \pi$ m/s [A1]

Total 4 marks

Question 9

The paths of the particles will meet when the particles are in the same position, but this may be at different times. Let t_A and t_B be the time variables for each particle.

$\underset{\sim}{r}_A(t) = (t_A^2 + 1)\underset{\sim}{i} + 2t_A\underset{\sim}{j}$

$\underset{\sim}{r}_B(t) = (7t_B - 5)\underset{\sim}{i} + (t_B + 6)\underset{\sim}{j}$

Equating $\underset{\sim}{i}$ components:

$t_A^2 + 1 = 7t_B - 5$ .... (1)

Equating $\underset{\sim}{j}$ components:

$2t_A = t_B + 6$

$t_B = 2t_A - 6$ .... (2)

Substitute (2) into (1)

$t_A^2 + 1 = 7(2t_A - 6) - 5$ [M1]

$t_A^2 + 1 = 14t_A - 42 - 5$

$t_A^2 - 14t_A + 48 = 0$

$(t_A - 6)(t_A - 8) = 0$

$t_A = 6, 8$ seconds [A1]

When $t_A = 6$ seconds, $t_B = 2\times 6 - 6 = 6$ seconds

The particles will be in the same position at the same time and so will collide.

Finding this position: $\underset{\sim}{r}_A(6) = (6^2 + 1)\underset{\sim}{i} + 2\times 6\underset{\sim}{j} = 37\underset{\sim}{i} + 12\underset{\sim}{j}$

$\underset{\sim}{r}_B((6) = (7\times 6 - 5)\underset{\sim}{i} + (6 + 6)\underset{\sim}{j} = 37\underset{\sim}{i} + 12\underset{\sim}{j}$

The particles collide at the point (37, 12) [A1]

When $t_A = 8$ seconds, $t_B = 2\times 8 - 6 = 10$ seconds.

$\underset{\sim}{r}_A(8) = (8^2 + 1)\underset{\sim}{i} + 2\times 8\underset{\sim}{j} = 65\underset{\sim}{i} + 16\underset{\sim}{j}$

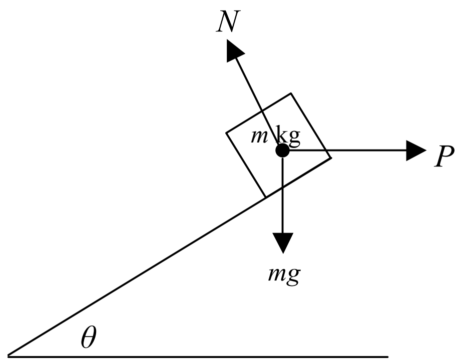
$\underset{\sim}{r}_B(8) = (7\times 10 - 5)\underset{\sim}{i} + (10 + 6)\underset{\sim}{j} = 65\underset{\sim}{i} + 16\underset{\sim}{j}$ [A1]

The paths meet at the point (65, 16), but this is not a collision because the particles will be in this position at different times. [A1]

Total 5 marks

Question 10

Let N newtons be the reaction force of the inclined plane on the box.
The weight force acting on the stationary box is mg newtons.
The plane is smooth.



N = Normal reaction
mg = Weight force

Resolving forces parallel to the plane

$mg\sin(\theta) - P\cos(\theta) = 0$

$P = mg\tan(\theta)$ ….. (1) [A1]

Resolving forces perpendicular to the plane

$N - P\sin(\theta) - mg\cos(\theta) = 0$

$N = P\sin(\theta) + mg\cos(\theta)$ …..(2) [A1]

Substitute (1) into (2) to eliminate P

$N = mg\tan(\theta)\sin(\theta) + mg\cos(\theta)$ newtons [A1]

$$N = mg\left(\frac{\sin^2(\theta)}{\cos(\theta)} + \cos(\theta)\right)$$

$$N = mg\left(\frac{\sin^2(\theta) + \cos^2(\theta)}{\cos(\theta)}\right)$$

$$N = mg\left(\frac{1}{\cos(\theta)}\right)$$ [A1]

$N = mg\sec(\theta)$ newtons, as required.

Total 4 marks
TOTAL MARKS: 40

