Year 2010

VCE

Specialist Mathematics Solutions Trial Examination 1

KILBAHA MULTIMEDIA PUBLISHING PO BOX 2227 KEW VIC 3101 AUSTRALIA	TEL: (03) 9817 5374 FAX: (03) 9817 4334 kilbaha@gmail.com http://kilbaha.com.au

Question 1

$x^2+2xy+3y^2=18$

taking $\frac{d}{dx}$ of each term (implicit differentiation)

$\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(2xy)+\frac{d}{dx}\left(3y^2\right)=\frac{d}{dx}(18)$

product rule in the second term

$2x+2y+2x\frac{dy}{dx}+6y\frac{dy}{dx}=0$ M1

$(2x+6y)\frac{dy}{dx}=-2x-2y$

$\frac{dy}{dx}=\frac{-(x+y)}{x+3y}$ A1

when the tangent is horizontal $\frac{dy}{dx}=0 \quad \Rightarrow \quad y=-x$

$x^2-2x^2+3x^2=2x^2=18 \quad \Rightarrow \quad x^2=9 \quad x=\pm 3 \quad \Rightarrow \quad y=\mp 3$

coordinates are $(3,-3)$ and $(-3,3)$ A1

Question 2

$\int \frac{4-3x}{\sqrt{16-9x^2}}\,dx$

$=\int \frac{4}{\sqrt{16-9x^2}}\,dx - \int \frac{3x}{\sqrt{16-9x^2}}\,dx$

let $u=3x \quad \frac{du}{dx}=3$ let $v=16-9x^2 \quad \frac{dv}{dx}=-18x$ M1

$=\frac{4}{3}\int \frac{1}{\sqrt{16-u^2}}\,du + \frac{1}{6}\int v^{-\frac{1}{2}}\,dv$

A1

$=\frac{4}{3}\sin^{-1}\left(\frac{u}{4}\right)+\frac{1}{3}v^{\frac{1}{2}}+c$

$=\frac{4}{3}\sin^{-1}\left(\frac{3x}{4}\right)+\frac{1}{3}\sqrt{16-9x^2}+c$ A1

since an antiderivative is required, any value of c, including zero is acceptable

Question 3

i. $(a+bi)^2 = 3+4i$ where $a,b \in R$

$a^2 + 2ab\,i + b^2 i^2 = (a^2 - b^2) + 2ab\,i = 3+4i$

equating real and imaginary parts

real (1) $a^2 - b^2 = 3$ M1

imaginary (2) $2ab = 4$ from (2) $b = \dfrac{2}{a}$ substitute into (1)

$a^2 - \left(\dfrac{2}{a}\right)^2 = 3$ M1

$a^2 - \dfrac{4}{a^2} - 3 = 0$ multiply by a^2

$a^4 - 3a^2 - 4 = 0$

$(a^2 - 4)(a^2 + 1) = 0$

$a = \pm 2$ $a = \pm i$ but $a \in R$ so $a = \pm 2$ only $b = \pm 1$ A1

$(\pm(2+i))^2 = 3+4i$

alternative methods such as using polar and DeMoivre's Theorem are acceptable.

ii. $z^2 + i\,z - 1 - i = 0$

using the quadratic formulae with $a = 1$ $b = i$ $c = -1-i$

$\Delta = b^2 - 4ac = i^2 - 4(-1-i) = -1+4+4i = 3+4i$ M1

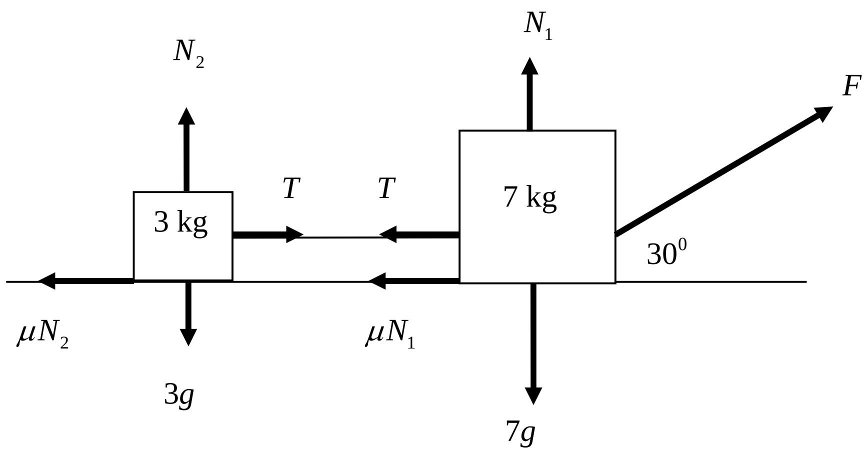
$z = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

$z = \dfrac{-i \pm (2+i)}{2} = \dfrac{-i+2+i}{2}$, $\dfrac{-i-2-i}{2}$

$z = 1$ and $-1-i$ A1

Question 4

i. all the forces correct A1



$F = 40$ newtons $\mu = \dfrac{\sqrt{3}}{4}$

ii. resolving around the 7 kg box

horizontally (1) $F\cos(30^0) - T - \mu N_1 = 7a$

vertically (2) $F\sin(30^0) + N_1 - 7g = 0$ M1

$(2) \Rightarrow N_1 = 7g - F\sin(30^0)$ substitute into (1)

$F\cos(30^0) - T - \mu\left(7g - F\sin(30^0)\right) = 7a$

$F\left(\cos(30^0) + \mu\sin(30^0)\right) - T - 7\mu g = 7a \quad (3)$

resolving around the 3 kg box

horizontally (4) $T - \mu N_2 = 3a$ M1

vertically (5) $N_2 - 3g = 0 \Rightarrow N_2 = 3g$ substitute into (4)

(4) $T - 3\mu g = 3a$ adding this to (3) to eliminate T

$F\left(\cos(30^0) + \mu\sin(30^0)\right) - 10\mu g = 10a$

$40\left(\dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{3}}{4} \text{x} \dfrac{1}{2}\right) - \dfrac{10\sqrt{3}\text{ x}9.8}{4} = 10a$

$\dfrac{\sqrt{3}}{4}(80 + 20 - 98) = \dfrac{\sqrt{3}}{2} = 10a$

$a = \dfrac{\sqrt{3}}{20}$ m/s^2 A1

Question 5

$A\left(-2+\sqrt{3},3\right)$, $B\left(2\sqrt{3}-2,0\right)$ and $C(-2,2)$

$\overrightarrow{OA}=\left(-2+\sqrt{3}\right)\underset{\sim}{i}+3\underset{\sim}{j}$ $\quad\overrightarrow{OB}=\left(2\sqrt{3}-2\right)\underset{\sim}{i}$ $\quad\overrightarrow{OC}=-2\underset{\sim}{i}+2\underset{\sim}{j}$

$\overrightarrow{CA}=\overrightarrow{OA}-\overrightarrow{OC}=\sqrt{3}\underset{\sim}{i}+\underset{\sim}{j}$ $\quad\left|\overrightarrow{CA}\right|=\sqrt{3+1}=2$

$\overrightarrow{CB}=\overrightarrow{OB}-\overrightarrow{OC}=2\sqrt{3}\underset{\sim}{i}-2\underset{\sim}{j}$ $\quad\left|\overrightarrow{CB}\right|=\sqrt{12+4}=4$

$\overrightarrow{CA}.\overrightarrow{CB}=6-2=4$ M1

$$\cos(\angle ACB)=\frac{\overrightarrow{CA}.\overrightarrow{CB}}{\left|\overrightarrow{CA}\right|\left|\overrightarrow{CB}\right|}=\frac{1}{2}$$

$\angle ACB=\cos^{-1}\left(\frac{1}{2}\right)=60^0\quad\left(\text{ or }\frac{\pi}{3}\right)$ A1

Question 6

a. using $\tan(2A)=\dfrac{2\tan(A)}{1-\tan^2(A)}$ with $A=\dfrac{\pi}{8}$ and $2A=\dfrac{\pi}{4}$

$\tan\left(\frac{\pi}{4}\right)=1=\dfrac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}\quad\Rightarrow\quad 1-\tan^2\left(\frac{\pi}{8}\right)=2\tan\left(\frac{\pi}{8}\right)$ M1

$\tan^2\left(\frac{\pi}{8}\right)+2\tan\left(\frac{\pi}{8}\right)=1\quad\Rightarrow\quad\tan^2\left(\frac{\pi}{8}\right)+2\tan\left(\frac{\pi}{8}\right)+1=2$

$\left(\tan\left(\frac{\pi}{8}\right)+1\right)^2=2\quad\Rightarrow\quad\tan\left(\frac{\pi}{8}\right)+1=\pm\sqrt{2}$

$\tan\left(\frac{\pi}{8}\right)=-1\pm\sqrt{2}$ since $\tan\left(\frac{\pi}{8}\right)>0$ A1

$\tan\left(\frac{\pi}{8}\right)=\sqrt{2}-1$ shown

b. let $z=1+\left(\sqrt{2}-1\right)i$ $\quad\text{Arg}(z)=\tan^{-1}\left(\sqrt{2}-1\right)=\dfrac{\pi}{8}$ A1

now $iz=i+\left(\sqrt{2}-1\right)i^2=1-\sqrt{2}+i$

$\text{Arg}(iz)=\text{Arg}(i)+\text{Arg}(z)=\dfrac{\pi}{2}+\text{Arg}(z)$

$\text{Arg}\left(1-\sqrt{2}+i\right)=\dfrac{\pi}{2}+\dfrac{\pi}{8}$ A1

$\text{Arg}\left(1-\sqrt{2}+i\right)=\dfrac{5\pi}{8}$

Question 7

a. $y = \dfrac{x^2+4}{2x} = \dfrac{x}{2} + \dfrac{2}{x}$

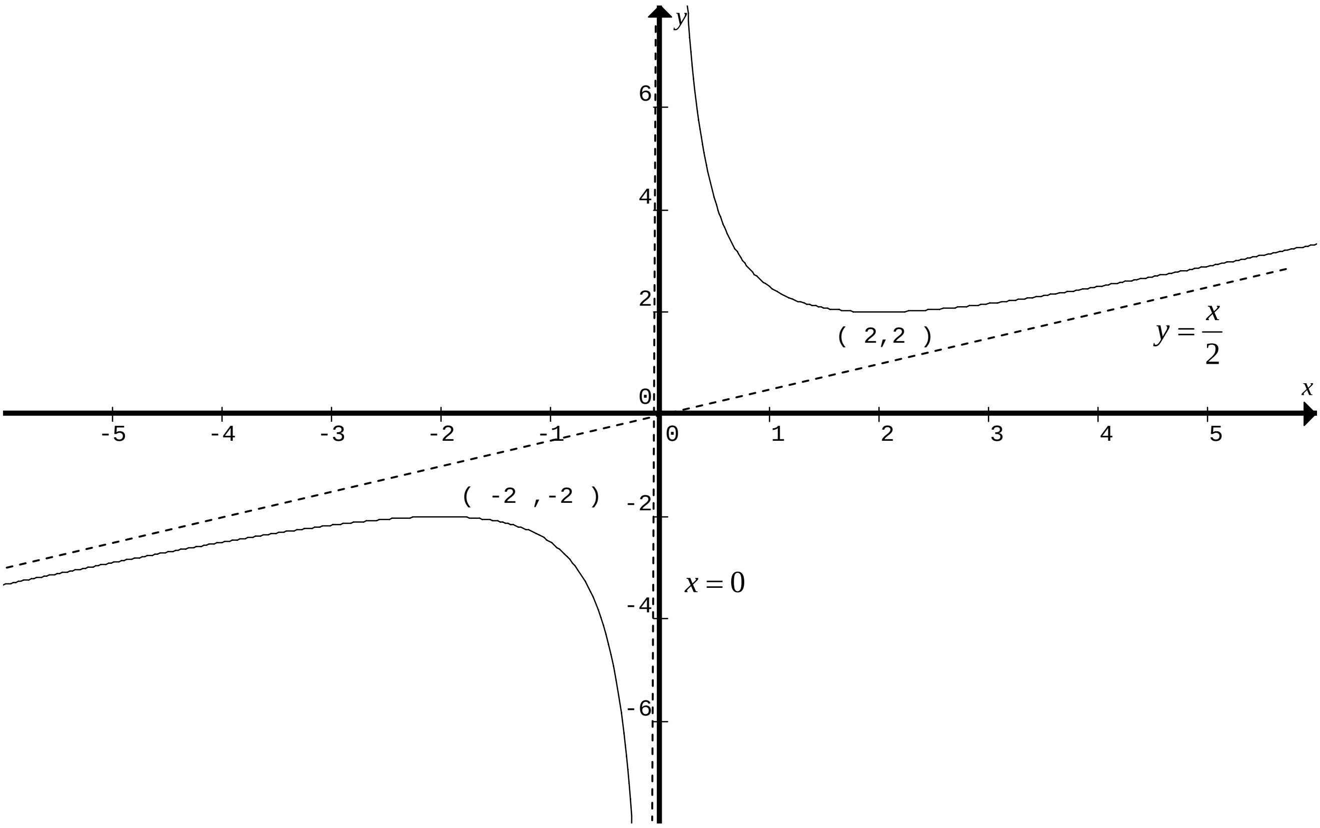
$y = \dfrac{x}{2}$ is an asymptote and $x = 0$, y-axis is a vertical asymptote A1

the graph does not cross the x or y-axis,

for turning points, $\dfrac{dy}{dx} = \dfrac{1}{2} - \dfrac{2}{x^2} = 0 \quad \Rightarrow x^2 = 4$ so $x = \pm 2$ M1

$(2,2)$ local min $(-2,-2)$ local max A1

correct graph, with asymptotes G1



b. $\underset{\sim}{r}(t) = 2\tan(t)\,\underset{\sim}{i} + 2\text{cosec}(2t)\,\underset{\sim}{j}$

$x = 2\tan(t)$

$\dfrac{x^2+4}{2x} = \dfrac{4\tan^2(t)+4}{4\tan(t)}$

$\dfrac{x^2+4}{2x} = \dfrac{4\left(1+\tan^2(t)\right)}{4\tan(t)}$ M1

$\dfrac{x^2+4}{2x} = \dfrac{\sec^2(t)}{\tan(t)} = \dfrac{1}{\cos^2(t)} \text{ x } \dfrac{\cos(t)}{\sin(t)}$

$\dfrac{x^2+4}{2x} = \dfrac{1}{\sin(t)\cos(t)} = \dfrac{2}{2\sin(t)\cos(t)} = \dfrac{2}{\sin(2t)}$ A1

$y = 2\text{cosec}(2t)$ shown

Question 8

i. $\ddot{x} = 18x^3 + 18x^2 + 4x$ use $\ddot{x} = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx}$

$\dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = 18x^3 + 18x^2 + 4x$

$\frac{1}{2}v^2 = \int\left(18x^3 + 18x^2 + 4x\right)dx$ M1

$\frac{1}{2}v^2 = \frac{9}{2}x^4 + 6x^3 + 2x^2 + C_1$

now when $x = 1$ $v = -5$

$\frac{25}{2} = \frac{9}{2} + 6 + 2 + C_1 \quad \Rightarrow C_1 = 0$

$v^2 = 9x^4 + 12x^3 + 4x^2$

$v^2 = x^2\left(9x^2 + 12x + 4\right)$ A1

$v^2 = x^2(3x+2)^2$

ii. $v = \pm x(3x+2)$ since when $t=0$, $x=1$, $v=-5$ take the negative since $t>0$ and $v<0$

$$v = \frac{dx}{dt} = -x(3x+2)$$

$$\frac{dt}{dx} = \frac{-1}{x(3x+2)}$$

by partial fractions

$$\frac{A}{x} + \frac{B}{3x+2} = \frac{A(3x+2)+Bx}{x(3x+2)} = \frac{x(B+3A)+2A}{x(3x+2)}$$ M1

$$B+3A=0 \quad \text{and} \quad 2A=-1 \;\Rightarrow\; A=-\frac{1}{2} \quad B=-3A=\frac{3}{2}$$

$$t = \frac{1}{2}\int\left(\frac{3}{3x+2}-\frac{1}{x}\right)dx$$

$$2t = \log_e(3x+2) - \log_e(x) + C_2$$ A1

now when $t=0$ $x=1$

$$0 = \log_e(5) + C_2 \quad \Rightarrow \quad C_2 = -\log_e(5)$$

$$2t = \log_e\left(\frac{3x+2}{5x}\right)$$

$$\frac{3x+2}{5x} = e^{2t}$$

$$\frac{3x+2}{x} = 5e^{2t}$$ M1

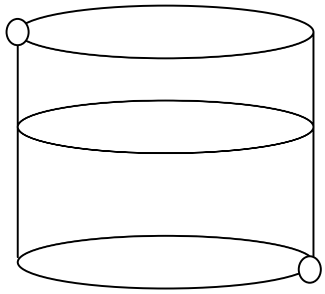
$$3 + \frac{2}{x} = 5e^{2t}$$

$$\frac{2}{x} = 5e^{2t} - 3$$

$$x = x(t) = \frac{2}{5e^{2t}-3}$$ A1

Question 9

i.



Now $\frac{dQ}{dt} = \text{inflow} - \text{outflow}$

and the volume $V(t)$ of the tank at a time t, $V(t) = 100 + (5-3)t = 100 + 2t$

$$\frac{dQ}{dt} = 5\text{x}0.8 - \frac{3Q}{100+2t} = 4 - \frac{3Q}{100+2t}$$ A1

ii. $Q = \frac{4}{5}(100+2t) + C(100+2t)^n$

differentiating $LHS \ \frac{dQ}{dt} = \frac{8}{5} + 2nC(100+2t)^{n-1}$ M1

$RHS \quad 4 - \frac{3Q}{100+2t} = 4 - \frac{3}{100+2t}\left(\frac{4}{5}(100+2t) + C(100+2t)^n\right)$

$= 4 - \frac{12}{5} - 3C(100+2t)^{n-1}$

$= \frac{8}{5} - 3C(200+2t)^{n-1}$

therefore $n = -\frac{3}{2}$ A1

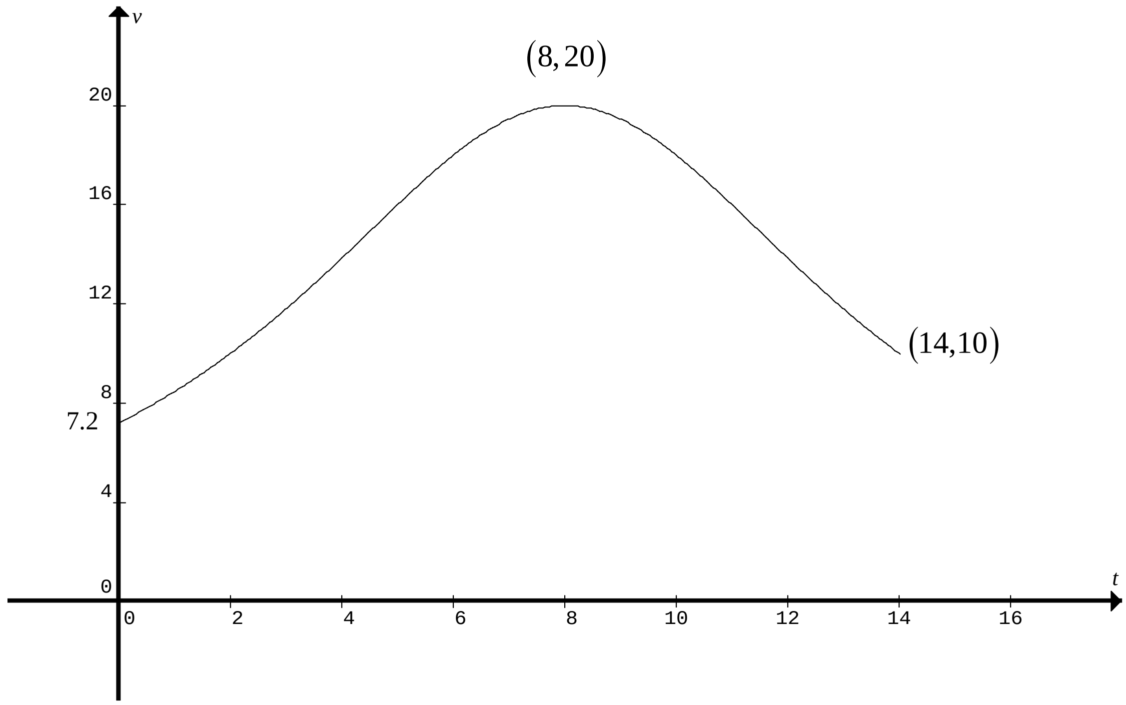
Question 10

i. $v(t)=\dfrac{720}{t^2-16t+100}=\dfrac{720}{t^2-16t+64+36}=\dfrac{720}{(t-8)^2+36}$

$v(0)=7.2 \quad v(8)=20 \quad v(14)=10$

The maximum velocity occurs when $t=8$ and is 20 m/s,

graph over $t\in[0,14]$ A1



ii. $v=\dfrac{dx}{dt}=\dfrac{720}{(t-8)^2+36}$

$x=\displaystyle\int_8^{14}\left(\frac{720}{(t-8)^2+36}\right)dt$ A1

$x=\left[\dfrac{720}{6}\tan^{-1}\left(\dfrac{t-8}{6}\right)\right]_8^{14}$ A1

$x=\dfrac{720}{6}\left(\tan^{-1}(1)-\tan^{-1}(0)\right)=120\left(\dfrac{\pi}{4}-0\right)$

$x=30\pi$ m A1

END OF SUGGESTED SOLUTIONS