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SPECIALIST MATHS TRIAL EXAMINATION 1 SOLUTIONS 2010

Question 1

$z = 1 - 2i$ is a solution so $z = 1 + 2i$ is also a solution (conjugate root theorem).

(1 mark)

Now, $(z - 1 + 2i)(z - 1 - 2i)$

$$= z^2 - z - 2iz - z + 1 + 2i + 2iz - 2i + 4$$
$$= z^2 - 2z + 5$$

Method 1

$$\begin{array}{r|l} & \qquad\quad z + 4 \\ \hline z^2 - 2z + 5 & z^3 + 2z^2 - 3z + 20 \\ & z^3 - 2z^2 + 5z \\ \hline & \quad 4z^2 - 8z + 20 \\ & \quad 4z^2 - 8z + 20 \\ \hline \end{array}$$

So $z^3 + 2z^2 - 3z + 20 = (z^2 - 2z + 5)(z + 4) = 0$.
The other solution is $z = -4$.

(1 mark)

Method 2

$$\begin{aligned} z^3 + 2z^2 - 3z + 20 &= \_(z^2 - 2z + 5)\,\_(z^2 - 2z + 5) \\ &= z(z^2 - 2z + 5)\,\_(z^2 - 2z + 5) \\ &= z(z^2 - 2z + 5) + 4(z^2 - 2z + 5) \\ &= (z + 4)(z^2 - 2z + 5) \end{aligned}$$

So $z^3 + 2z^2 - 3z + 20 = (z^2 - 2z + 5)(z + 4) = 0$.
The other solution is $z = -4$.

(1 mark)

Question 2

a. $A(2, 1, \sqrt{15}),\ B(2, -4, 0),\ O(0, 0, 0)$

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$
$$= -(\overrightarrow{OA}) + (\overrightarrow{OB})$$
$$= -2\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{15}\,\underset{\sim}{k} + 2\underset{\sim}{i} - 4\underset{\sim}{j}$$
$$= -5\underset{\sim}{j} - \sqrt{15}\,\underset{\sim}{k}$$

(1 mark)

b. To prove: ΔABO contains a right angle and has two sides of equal length.

$$\overrightarrow{AB} \bullet \overrightarrow{AO} = (-5\underset{\sim}{j} - \sqrt{15}\,\underset{\sim}{k}) \bullet (-2\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{15}\,\underset{\sim}{k})$$
$$= 0 + 5 + 15 \neq 0$$
$$\overrightarrow{AB} \bullet \overrightarrow{BO} = (-5\underset{\sim}{j} - \sqrt{15}\,\underset{\sim}{k}) \bullet (-2\underset{\sim}{i} + 4\underset{\sim}{j})$$
$$= 0 - 20 + 0 \neq 0$$
$$\overrightarrow{OA} \bullet \overrightarrow{OB} = (2\underset{\sim}{i} + \underset{\sim}{j} + \sqrt{15}\,\underset{\sim}{k}) \bullet (2\underset{\sim}{i} - 4\underset{\sim}{j})$$
$$= 4 - 4 = 0$$

So $\overrightarrow{OA}$ is at right angles to $\overrightarrow{OB}$ so ΔABO contains a right angle.

(1 mark)

$$|\overrightarrow{AB}| = \sqrt{25 + 15} = \sqrt{40}$$
$$|\overrightarrow{AO}| = \sqrt{4 + 1 + 15} = \sqrt{20}$$
$$|\overrightarrow{BO}| = \sqrt{4 + 16} = \sqrt{20}$$

Since $|\overrightarrow{AO}| = |\overrightarrow{BO}|$, ΔABO contains two sides of equal length.

So ΔABO is a right-angled, isosceles triangle.

(1 mark)

Question 3

a.

$$3y^2 + 4x - 2x^2 y = 5$$

$$6y\frac{dy}{dx} + 4 - 2x^2\frac{dy}{dx} - 4xy = 0$$

(1 mark)

$$\frac{dy}{dx}(6y - 2x^2) = 4xy - 4$$

$$\frac{dy}{dx} = \frac{4xy - 4}{6y - 2x^2}$$

$$\frac{dy}{dx} = \frac{2xy - 2}{3y - x^2}$$

(1 mark)

b. When $x = 1$,

$$3y^2 + 4x - 2x^2 y = 5$$

becomes $3y^2 + 4 - 2y = 5$

$$3y^2 - 2y - 1 = 0$$

$$(3y + 1)(y - 1) = 0$$

$$y = -\frac{1}{3} \text{ or } y = 1$$

The first quadrant point is $(1,1)$.

(1 mark)

$$\frac{dy}{dx} = \frac{2xy - 2}{3y - x^2}$$

$$= \frac{2 - 2}{3 - 1}$$

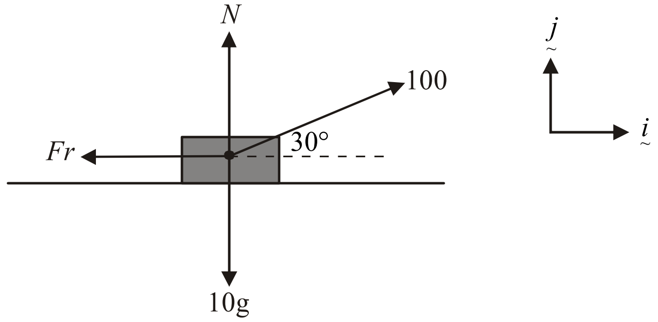
$$= 0$$

The gradient at the point where $x = 1$ is zero.

(1 mark)

Question 4

a.



(1 mark)

b.

$$\underset{\sim}{R} = m\underset{\sim}{a}$$

$$(100\cos 30° - Fr)\underset{\sim}{i} + (N + 100\sin 30° - 10g)\underset{\sim}{j} = ma\underset{\sim}{i}$$

Resolving horizontally:

$$100\cos(30^\circ) - Fr = 10 \times 8$$

$$\frac{100\sqrt{3}}{2} - \mu N = 80$$

$$\mu N = 50\sqrt{3} - 80$$

$$\mu = \frac{50\sqrt{3} - 80}{N} \qquad -(1)$$

(1 mark)

Resolving vertically:

$$N + 100\sin(30^\circ) = 10g$$

$$N = 98 - 50$$

$$= 48$$

In (1) $\mu = \frac{50\sqrt{3} - 80}{48}$

$$= \frac{25\sqrt{3} - 40}{24}$$

(1 mark)

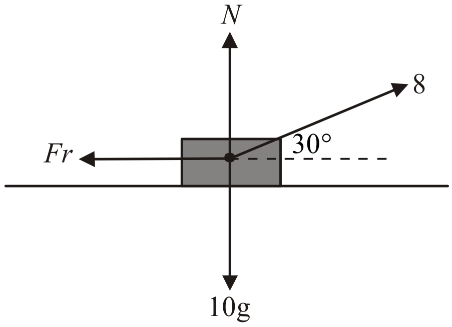
c. At the point at which the crate begins to move $Fr = \mu N$.

Resolving vertically:

$$N + 8\sin(30^\circ) = 10g$$

$$N = 98 - 4$$

$$= 94$$



So at the point of moving $Fr = \frac{1}{10} \times 94$

$$= 9.4$$

Resolving horizontally:

$$Fr = 8\cos(30°)$$

$$= \frac{8\sqrt{3}}{2}$$

$$= 4\sqrt{3}$$

(1 mark)

Since $4\sqrt{3} < 9.4$, the crate does not move. **(1 mark)**

Question 5

$$\frac{dy}{dx} = 3\sqrt{4 - y^2}$$

$$\frac{dx}{dy} = \frac{1}{3\sqrt{4 - y^2}}$$

$$x = \frac{1}{3}\int \frac{1}{\sqrt{4 - y^2}}dy$$

$$x = \frac{1}{3}\arcsin\left(\frac{y}{2}\right) + c$$ **(1 mark)**

Given $y(0) = 2$

$$0 = \frac{1}{3}\arcsin(1) + c$$

$$c = -\frac{1}{3} \times \frac{\pi}{2}$$

$$c = -\frac{\pi}{6}$$

$$x = \frac{1}{3}\arcsin\left(\frac{y}{2}\right) - \frac{\pi}{6}$$ **(1 mark)**

$$3\left(x + \frac{\pi}{6}\right) = \arcsin\left(\frac{y}{2}\right)$$

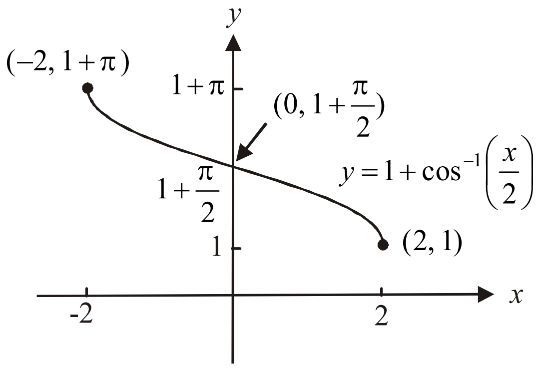
$$\sin\left(3\left(x + \frac{\pi}{6}\right)\right) = \frac{y}{2}$$

$$y = 2\sin\left(3\left(x + \frac{\pi}{6}\right)\right)$$

(1 mark)

Question 6

a.



(1 mark) – correct endpoints
(1 mark) – correct y-intercept and shape

b. **i.** $d_f = [-2,2]$

(1 mark)

ii. $r_f = [1, 1+\pi]$

(1 mark)

c. $f(x) = 1 + \cos^{-1}\left(\frac{x}{2}\right)$

$f'(x) = \frac{-1}{\sqrt{4-x^2}}$

When $x = \sqrt{3}$

$f'(x) = \frac{-1}{1} = -1$

The gradient of the normal is therefore 1.

(1 mark)

$f\left(\sqrt{3}\right) = 1 + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= 1 + \frac{\pi}{6}$

At the point $\left(\sqrt{3}, 1 + \frac{\pi}{6}\right)$,

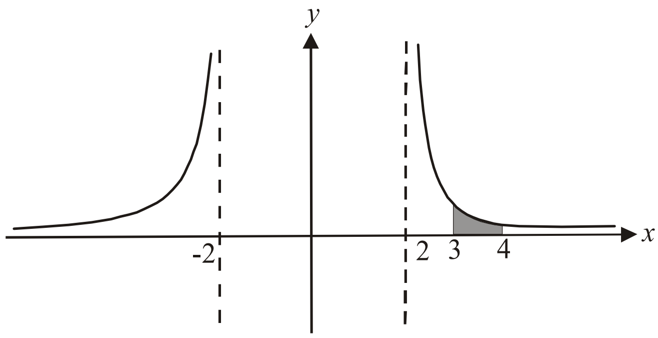
$y - y_1 = m(x - x_1)$ becomes

$y - \left(1 + \frac{\pi}{6}\right) = 1\left(x - \sqrt{3}\right)$

$y = x - \sqrt{3} + 1 + \frac{\pi}{6}$ is the equation of the normal.

(1 mark)

Question 7



$$y = \frac{2}{\sqrt{x^2 - 4}}$$

$$\text{Volume} = \pi \int_3^4 y^2 dx$$

$$= \pi \int_3^4 \frac{4}{x^2 - 4} dx$$

$$= 4\pi \int_3^4 \frac{1}{x^2 - 4} dx$$

(1 mark)

$$\text{Let } \frac{1}{x^2 - 4} \equiv \frac{A}{x-2} + \frac{B}{x+2}$$

$$\equiv \frac{A(x+2) + B(x-2)}{(x-2)(x+2)}$$

True iff $1 \equiv A(x+2) + B(x-2)$

Put $x = -2$, $1 = -4B$ $\quad B = -\frac{1}{4}$

Put $x = 2$, $1 = 4A$ $\quad A = \frac{1}{4}$

So $\frac{1}{x^2 - 4} = \frac{1}{4(x-2)} - \frac{1}{4(x+2)}$

(1 mark)

$$\text{Volume} = 4\pi \int_3^4 \left(\frac{1}{4(x-2)} - \frac{1}{4(x+2)} \right) dx$$

$$= \pi \int_3^4 \left(\frac{1}{x-2} - \frac{1}{x+2} \right) dx$$

(1 mark)

$$= \pi \left[\log_e |x-2| - \log_e |x+2| \right]_3^4$$

$$= \pi\{(\log_e(2) - \log_e(6)) - (\log_e(1) - \log_e(5))\}$$

$$= \pi(\log_e(2) - \log_e(6) - 0 + \log_e(5))$$

$$= \pi\left\{\log_e\left(\frac{2 \times 5}{6}\right)\right\}$$

$$= \pi \log_e\left(\frac{5}{3}\right) \text{cubic units}$$

(1 mark)

Question 8

$$a = v - 5$$

$$v\frac{dv}{dx} = v - 5$$ **(1 mark)**

$$\frac{dv}{dx} = \frac{v-5}{v}$$

$$\frac{dx}{dv} = \frac{v}{v-5}$$

Method 1

$x = \int \frac{v}{v-5}dv$ let $u = v - 5$

$= \int \frac{1}{u} \times (u+5)\frac{du}{dv}dv$ so $\frac{du}{dv} = 1$ and $v = u + 5$

$$= \int \left(1 + \frac{5}{u}\right)du$$

$$x = u + 5\log_e|u| + c$$

$$x = v - 5 + 5\log_e|v-5| + c$$ **(1 mark)**

When $x = 2,\ v = 6$

$$2 = 6 - 5 + 5\log_e(1) + c$$

$$c = 1$$

So $x = v - 4 + 5\log_e|v-5|$ **(1 mark)**

Method 2

Since $\frac{v}{v-5}$ is an improper fraction we divide.

$$\begin{array}{r} 1 \\ v-5\overline{)\,v} \\ \underline{v-5} \\ 5 \end{array}$$

So $\frac{v}{v-5} = 1 + \frac{5}{v-5}$

So $\frac{dx}{dv} = 1 + \frac{5}{v-5}$

$$x = \int (1 + \frac{5}{v-5})dv$$

$$= v + 5\log_e|v-5| + c$$ **(1 mark)**

When $x = 2,\ v = 6$

$$2 = 6 + 5\log_e(1) + c$$

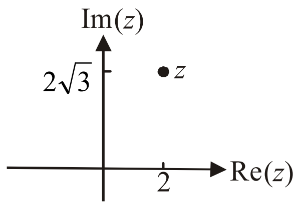
$$c = -4$$

So $x = v - 4 + 5\log_e|v-5|$

(1 mark)

Question 9

a. Let $z = 2 + 2\sqrt{3}\,i$



$$r = \sqrt{4 + 4 \times 3} = \sqrt{16} = 4$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \tan^{-1}\left(\sqrt{3}\right)$$

$= \frac{\pi}{3}$ since z is a first quadrant angle.

So $2 + 2\sqrt{3}\,i = 4\text{cis}\left(\frac{\pi}{3}\right)$

(1 mark)

b. $\sqrt{3}z^2 + \sqrt{2}z - \frac{i}{2} = 0$

This is a quadratic equation in z.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-\sqrt{2} \pm \sqrt{2 - 4 \times \sqrt{3} \times -\frac{i}{2}}}{2\sqrt{3}}$$

$$= \frac{-\sqrt{2} \pm \sqrt{2 + 2\sqrt{3}\,i}}{2\sqrt{3}}$$ **(1 mark)**

Now $\sqrt{2 + 2\sqrt{3}\,i} = \sqrt{4\text{cis}\left(\frac{\pi}{3}\right)}$ from part a.

$= \sqrt{4}\text{cis}\left(\frac{\pi}{3} \times \frac{1}{2}\right)$ De Moivre

$$= 2\text{cis}\left(\frac{\pi}{6}\right)$$

$$= 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

$$= 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$= \sqrt{3} + i$ **(1 mark)**

So $z = \dfrac{-\sqrt{2} \pm \left(\sqrt{3} + i\right)}{2\sqrt{3}}$

$z = \dfrac{-\sqrt{2} + \sqrt{3} + i}{2\sqrt{3}}$ or $z = \dfrac{-\sqrt{2} - \sqrt{3} - i}{2\sqrt{3}}$ **(1 mark)** – correct answers

Question 10

$$\text{Area} = \int_0^1 (f(x) - g(x))dx$$

$$= \int_0^1 \left(\frac{2}{4+x^2} - \frac{1-x}{\sqrt{4-x^2}} \right) dx$$

(1 mark)

$$= \int_0^1 \frac{2}{4+x^2} dx - \int_0^1 \frac{1}{\sqrt{4-x^2}} dx + \int_0^1 \frac{x}{\sqrt{4-x^2}} dx$$

let $u = 4 - x^2$

$\frac{du}{dx} = -2x$

$x = 1, \quad u = 3$

$x = 0, \quad u = 4$

$$= \left[\tan^{-1}\left(\frac{x}{2}\right) - \sin^{-1}\left(\frac{x}{2}\right) \right]_0^1 + \int_4^3 -\frac{1}{2} \frac{du}{dx} u^{-\frac{1}{2}} dx$$

$$= \left\{ \left(\tan^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right) \right) - (\tan^{-1}(0) - \sin^{-1}(0)) \right\} - \frac{1}{2} \int_4^3 u^{-\frac{1}{2}} du$$

$$= \tan^{-1}\left(\frac{1}{2}\right) - \frac{\pi}{6} - 0 - 0 - \frac{1}{2}\left[2u^{\frac{1}{2}} \right]_4^3$$

$$= \tan^{-1}\left(\frac{1}{2}\right) - \frac{\pi}{6} - \frac{1}{2}\left\{ 2\sqrt{3} - 2\sqrt{4} \right\}$$

$$= \tan^{-1}\left(\frac{1}{2}\right) - \frac{\pi}{6} - \sqrt{3} + 2 \text{ square units}$$

(1 mark) - $\tan^{-1}\left(\frac{x}{2}\right)$

(1 mark) - $\sin^{-1}\left(\frac{x}{2}\right)$

(1 mark) for $2u^{\frac{1}{2}}$ and correct terminals

(1 mark) for correct substitution of terminals

(1 mark) for correct answer