SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2

2009 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: C

Explanation:

The asymptotes of the hyperbola of the form $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ are of the form

$y-k = \pm\frac{b}{a}(x-h),\ a=2, b=4$

$y-3 = \pm 2(x+1)$

$2x - y + 5 = 0$

$2x + y - 1 = 0$

The asymptote $2x + y - 1 = 0$ is offered as alternative C.

Question 2

Answer: D

Explanation:

$$f(x) = \frac{x+a}{x^2 - 2ax - 3a^2} = \frac{x+a}{(x+a)(x-3a)}$$

$f(x)$ has a vertical asymptote $x = 3a$ and a hole for $x = -a$. Therefore the first three options are incorrect.

D is a correct option because: $\lim_{x \to -a} f(x) = \lim_{x \to -a} \frac{x+a}{(x+a)(x-3a)} = \lim_{x \to -a} \frac{1}{(x-3a)} = -\frac{1}{4a}$.

Question 3

Answer: A

Explanation:

$y = \cos\left(\sin^{-1} 2x\right)$ is a composite function $y = f \circ g$. The range of g must be a subset of the domain of f.
The range of $\sin^{-1} 2x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, while the domain of $\cos x$ is $[0, \pi]$. Therefore, $\sin^{-1} 2x$ must be restricted so that its range is $\left[0, \frac{\pi}{2}\right]$.

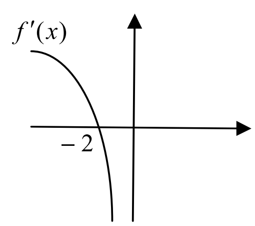
<u>Domain of *y*</u>	<u>Range of *y*</u>
$0 \le \sin^{-1} 2x \le \frac{\pi}{2}$	$\cos(\sin^{-1} 0) = \cos 0 = 1$
$0 \le 2x \le 1$	$\cos\left(\sin^{-1} 2 \times \frac{1}{2}\right) = \cos\frac{\pi}{2} = 0$
$0 \le x \le \frac{1}{2}$	
Domain is $\left[0, \frac{1}{2}\right]$	Range is $[0,1]$

Question 4

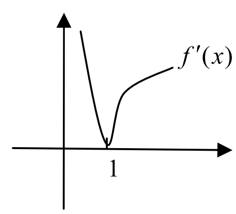
Answer: D

Explanation:

Option A can be correct: $f''(x)$ is discontinuous for $x = 0$, so $f(x)$ is also discontinuous for $x = 0$.
Option B can be correct: $f'(-2)$ can equal 0.
For example, the graph on the right shows $f'(x)$.
The derivative of this function ($f''(x)$) is negative for every $x < 0$.



Option C can be correct: $f'(x)$ can be positive for every $x > 0$.
The function ($f'(x)$) on the right has a negative gradient ($f''(x)$)for all $x < 1$, a gradient of zero for $x = 1$ and a positive gradient for $x > 1$.



Option D is incorrect: For $f(x)$ to have a local maximum at $x = 1$, according to second derivative test, $f''(1)$ would have to be less than zero, while in the given graph $f''(1) = 0$.

Option E can be correct: $f''(1) = 0$, so $f(x)$ can have a stationary point of inflexion for $x = 1$.

Question 5

Answer: B

Explanation:

Let $z = a + bi,\ \ a, b \geq 0$ and $a = 4b$. Then $z = 4b + bi.$

$z^2 = b^2(4+i)^2$

$\quad = b^2(15+8i)$

$\operatorname{Re} z^2 = 15b^2$

$\operatorname{Im} z^2 = 8b^2$

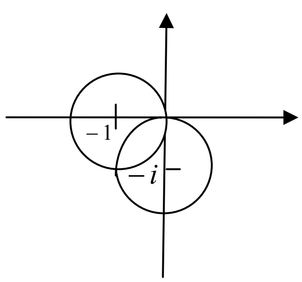
$\operatorname{Re} z^2 : \operatorname{Im} z^2 = 15:8$

Question 6

Answer: C

Explanation:

Complex number z which satisfies $|z+1| = |z+i| = 1$ is represented by the point of intersection of two circles: $|z+1| = 1$ and $|z+i| = 1$. From the diagram below, as z is a non-zero number, we have $z = -1 - i$ and $|z| = \sqrt{2},\ Arg\ z = -\frac{3\pi}{4}$.



Question 7

Answer: E

Explanation:

If $z = 2 - i\sqrt{3}$ is a solution of $z^4 + az^3 + bz^2 - 32z + 56 = 0,\ a, b \in R$, then $z = 2 + i\sqrt{3}$ is also a solution (conjugate root theorem). It follows that one quadratic factor is

$z^2 - (2 - i\sqrt{3} + 2 + i\sqrt{3})z + \left(2 - i\sqrt{3}\right)\left(2 + i\sqrt{3}\right) = 0$

$z^2 - 4z + 7 = 0$

$z^4 + az^3 + bz^2 - 32z + 56 = (z^2 - 4z + 7)(z^2 + kz + 8)$

By equating coefficients of z: $-32 + 7k = -32 \Rightarrow k = 0$, $z^2 + 8$ is the other quadratic factor.

$(z^2 - 4z + 7)(z^2 + 8) = z^4 - 4z^3 + 15z^2 - 32z + 56 \Rightarrow a = -4, b = 15$

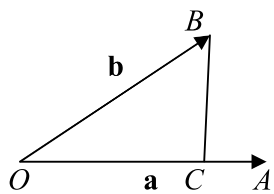
Question 8

Answer: A

Explanation:

The shortest distance of point B from the line OA is the magnitude of vector $\overrightarrow{BC}$, which is the perpendicular component of vector $\overrightarrow{OB}$ in the direction of $\overrightarrow{OA}$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

Then $\overrightarrow{OC} = (\mathrm{b} \bullet \hat{\mathrm{a}})\hat{\mathrm{a}}$, $\hat{\mathbf{a}} = \frac{1}{\sqrt{10}}(3\mathbf{i} + \mathbf{j})$



$$\overrightarrow{OC} = \frac{1}{10} \times 15(3\mathbf{i} + \mathbf{j}) = \frac{3}{2}(3\mathbf{i} + \mathbf{j})$$

$$\left|\overrightarrow{OC}\right| = \sqrt{\frac{90}{4}}, \quad \left|\overrightarrow{OB}\right| = 5$$

$$\left|\overrightarrow{BC}\right| = \sqrt{25 - \frac{90}{4}} = \sqrt{\frac{10}{4}} = \frac{\sqrt{10}}{2}$$

Question 9

Answer: C

Explanation:

$$|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b})$$

$$= |\mathbf{a}|^2 + 2\mathbf{a} \bullet \mathbf{b} + |\mathbf{b}|^2$$

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\frac{2\pi}{3} = 4 \times 3 \times \frac{-1}{2} = -6$$

$$|\mathbf{a} + \mathbf{b}|^2 = 16 - 12 + 9 = 13.$$

Thus, $|\mathbf{a} + \mathbf{b}| = \sqrt{13}$

Question 10

Answer: E

Explanation

The equation $a\tan^2 x + b\cot^2 x = a,\ a, b \in R^+$ can be written as

$$a\tan^2 x + \frac{b}{\tan^2 x} = a \quad (1)$$

$$a\tan^4 x - a\tan^2 x + b = 0$$

Let $t = \tan^2 x$. The equation (1) becomes $at^2 - at + b = 0$. It has a solution if the discriminate is greater than or equal to zero.

Discriminant $= a^2 - 4ab \geq 0$ for a $\in (-\infty, 0] \cup [4b, \infty)$. Because $a \in R^+$, the smallest positive value of a from the interval $(-\infty, 0] \cup [4b, \infty)$ is $4b$. Therefore, $a = 4b$ is the required value.

Question 11

Answer: E

Explanation:

The gradient of the tangent to the curve $y = \arcsin\frac{k}{x}$ at the point $\left(2k, \frac{\pi}{6}\right)$ is $y'(2k)$. First,

finding the derivative: $y' = \dfrac{1}{\sqrt{1 - \frac{k^2}{x^2}}} \times \left(-\dfrac{k}{x^2}\right) = -\dfrac{k}{x\sqrt{x^2 - k^2}}$. The gradient when $x = 2k$ is

$y'(2k) = -\dfrac{k}{2k\sqrt{3k^2}} = -\dfrac{1}{2k\sqrt{3}} = -\dfrac{\sqrt{3}}{6k}$. The angle with the x – axis = $\arctan\left(-\dfrac{\sqrt{3}}{6k}\right)$.

Question 12

Answer: B

Explanation:

Let $\cos^{-1}\dfrac{2}{\sqrt{m^2+4}} = x$. Then $\cos x = \dfrac{2}{\sqrt{m^2+4}}$.

Using the identity $\sec^2 x = 1 + \tan^2 x$, $\tan^2 x = \dfrac{m^2+4}{4} - 1 = \dfrac{m^2}{4}$.

As inverse cosine is restricted to $[0, \pi]$, x must be from the first quadrant. Thus, $\tan x = \dfrac{m}{2}$.

$$\tan\left(2\cos^{-1}\frac{2}{\sqrt{m^2+4}}\right) = \tan 2x = \frac{2 \times \frac{m}{2}}{1 - \frac{m^2}{4}} = \frac{4m}{4 - m^2}$$

Question 13

Answer: E

Explanation:

Option A is correct as $\dfrac{\sin 2x}{1+\cos 2x} = \dfrac{2\sin x\cos x}{1+\cos^2 x - \sin^2 x} = \dfrac{2\sin x\cos x}{2\cos^2 x} = \tan x$

Option B: If $u = 1+\cos 2x$, then $du = -2\sin 2x dx$ and $\sin x dx = -\dfrac{1}{2}du$. Substitution into the integral gives option B.

Option C is correct (similar to option B)

Option D: $\dfrac{\sin 2x}{1+\cos 2x} = \dfrac{2\sin x\cos x}{1+1-2\sin^2 x} = \dfrac{2\sin x\cos x}{2(1-\sin^2 x)} = \dfrac{\sin x\cos x}{(1-\sin^2 x)}$

Let $u = \sin x$. Then $du = \cos x dx$ and $I = \int \dfrac{u}{1-u^2}du = \dfrac{1}{2}\int \dfrac{1}{1-u} - \dfrac{1}{1+u}du$

Therefore, option D is correct.

Option E is incorrect: $\dfrac{(\sin x+\cos x)^2}{1+\sin 2x} = \dfrac{\sin^2 x + 2\sin x\cos x + \cos^2 x}{1+2\sin x\cos x} = \dfrac{1+2\sin x\cos x}{1+2\sin x\cos x} = 1$

Question 14

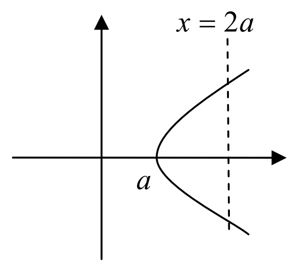
Answer: D

Explanation:

$V = \pi\int_a^{2a} y^2 dx$, where $y^2 = \dfrac{b^2}{a^2}\left(x^2 - a^2\right)$

$V = \dfrac{b^2\pi}{a^2}\int_a^{2a}\left(x^2 - a^2\right)dx = \dfrac{b^2\pi}{a^2}\left[\dfrac{x^3}{3} - a^2 x\right]_a^{2a}$

$V = \dfrac{b^2\pi}{a^2}\left[\dfrac{8a^3}{3} - 2a^3 - \dfrac{a^3}{3} + a^3\right] = \dfrac{b^2\pi}{a^2}\times\dfrac{4a^3}{3} = \dfrac{4ab^2\pi}{3}$



Question 15

Answer: A

Explanation:

The differential equation for the melting block is $\dfrac{dm}{dt} = \dfrac{k}{m}$. Solving: $\dfrac{dt}{dm} = \dfrac{m}{k} \Rightarrow t = \dfrac{m^2}{2k} + c$

When $t = 0$, $m = 100$ and $\dfrac{10000}{2k} + c = 0 \Rightarrow c = -\dfrac{5000}{k}$. When $t = 2$, $m = 50$ and after substituting *c*, we have: $\dfrac{2500}{2k} - \dfrac{5000}{k} = 2 \Rightarrow k = -1875$ and thus $c = \dfrac{8}{3} = 2.67$.

For $m = 0$, $t = c$, therefore the block will melt in 2.67 days.

Question 16

Answer: B

Explanation:

From the slope field given, it can be concluded that $\frac{dy}{dx} = 0$ when $x = -y$. Also $\frac{dy}{dx}$ is undefined when $x = y$.

$\frac{dy}{dx} = \frac{x+y}{x-y}$ is the only differential equation among offered options that satisfies both conditions.

Question 17

Answer: D

Explanation:

$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n, y_n), \; x_0 = 0, y_0 = 1, h = 0.2, \quad f(x,y) = y + e^{-2x} - 1$

$x_0 = 0 \qquad y_1 = 1 + 0.2(1 + e^0 - 1) = 1.2$

$$\begin{aligned} x_1 = 0.2, \quad y_2 &= 1.2 + 0.2(1.2 + e^{-0.4} - 1) \\ &= 1.2 + 0.2(0.2 + e^{-0.4}) \\ &= 1.2 + 0.04 + 0.2e^{-0.4} \\ &= 1.24 + 0.2e^{-0.4} \end{aligned}$$

Question 18

Answer: C

Explanation:

$x_{up} = 5gt - \frac{1}{2}gt^2, \quad x_{down} = \frac{1}{2}gt^2$

$v_{up} = 5g - gt, \qquad v_{down} = gt$

When the balls meet: $v_{down} = 4v_{up}$

$gt = 4(5g - gt) \Rightarrow t = 4$

Distance traveled by the falling ball $\frac{1}{2}g \times 4^2 = 8g = 8 \times 9.6 = 78.4m$

Question 19

Answer: C

Explanation:

$$\dot{x}(t) = 30 + \frac{10}{(1+10t)^2}, \quad \ddot{x}(t) = -\frac{200}{(1+10t)^3}$$

When $t = 2$, $\ddot{x}(t) = -\frac{200}{(1+20)^3} = -0.0215959...$ and when $t \to \infty$, $\dot{x} \to 30$ as $\frac{10}{(1+10t)^2} \to 0$

Question 20

Answer: A

Explanation:

$\dot{r}(t) = 2t\,\text{i} - 2e^{-2t}\,\text{j}$

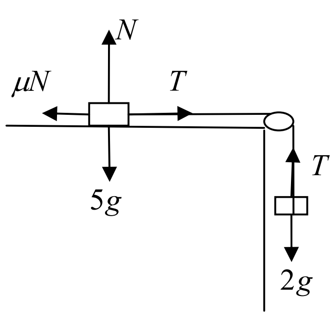
$|\dot{r}(t)| = \sqrt{4t^2 + 4e^{-4t}} = 4.2$

Using calculator: $\Rightarrow t = 2.0999...$

Question 21

Answer: D

Explanation:



$s = ut + \frac{1}{2}at^2, \quad s = 1.5, t = 2, u = 0$

$1.5 = \frac{1}{2}a \times 4 \Rightarrow a = \frac{3}{4}$

$2g - T = 2 \times \frac{3}{4} \Rightarrow T = 2g - \frac{3}{2}$

$N = 5g$

$T - \mu N = \frac{3}{4} \times 5$

$5g\mu = T - \frac{15}{4} = 2g - \frac{3}{2} - \frac{15}{4} = 14.35$

$\mu = \frac{14.35}{5 \times 9.8} = 0.29$

Question 22

Answer: B

Explanation:

$a = \frac{1+v^2}{v}, \quad v\frac{dv}{dx} = \frac{1+v^2}{v}$

$\frac{dv}{dx} = \frac{1+v^2}{v^2} \Rightarrow \frac{dx}{dv} = \frac{v^2}{1+v^2}$

$x = \int \frac{v^2}{1+v^2}\,dv = \int \left(1 - \frac{1}{1+v^2}\right)dv = v - \tan^{-1} v + c$

When $x = 0, v = 1$.

$\tan^{-1} 1 - 1 = c$

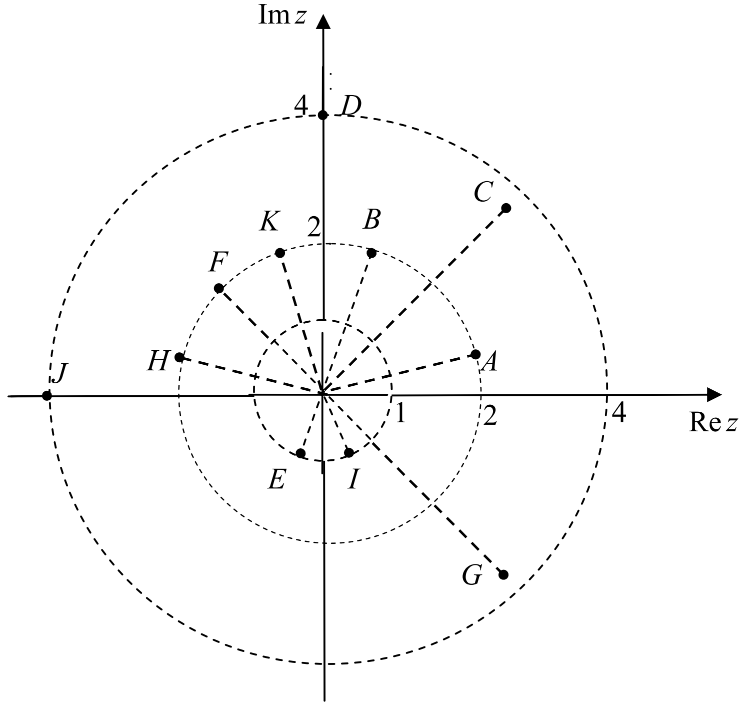
$c = \frac{\pi}{4} - 1$

When $v = 3, x = 3 - \tan^{-1} 3 + \frac{\pi}{4} - 1 = 1.53635....$

SECTION 2

Question 1

a.



$\frac{u}{v} = \frac{2cis\frac{\pi}{12}}{2cis\frac{5\pi}{12}} = cis\left(\frac{\pi}{12} - \frac{5\pi}{12}\right) = cis\left(-\frac{\pi}{3}\right)$, represented by point *I*. A1

$uv = 4cis\left(\frac{\pi}{12} + \frac{5\pi}{12}\right) = 4cis\frac{\pi}{2}$, represented by point *D*. A1

The sum $u + v$ is represented by point *C*. Because $\bar{u} + \bar{v} = \left(\overline{u + v}\right)$, the required point is *G*.
A1

$\frac{16}{v^3} = \frac{16}{8cis\frac{5\pi}{4}} = 2cis\frac{-5\pi}{4} = 2cis\frac{3\pi}{4}$, represented by point *F*. A1

b. i Rearrangement of $x + \frac{1}{x} = 2\cos\alpha$ gives $x^2 - 2\cos\alpha\, x + 1 = 0$.

By using the quadratic formula or by completing the square

$x = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2} = \frac{2\cos\alpha \pm \sqrt{-4\sin^2\alpha}}{2} = \cos\alpha \pm i\sin\alpha$. A1

As $\alpha \in [0, \pi]$, $\sin\alpha > 0$, thus $x = \cos\alpha + i\sin\alpha$ A1

ii $x^n = cis(n\alpha)$, and $\frac{1}{x^n} = cis(-n\alpha)$. M1

$x^n - \frac{1}{x^n} = (\cos(n\alpha) + i\sin(n\alpha)) - (\cos(n\alpha) - \sin(n\alpha)) = 2i\sin(n\alpha)$ A1

iii.

$$\left(\sqrt{3}+i\right)^7-\left(\sqrt{3}-i\right)^7=2^7\,cis\frac{7\pi}{6}-2^7\,cis\left(-\frac{7\pi}{6}\right)$$

$$=2^7\left(cis\frac{7\pi}{6}-cis\left(-\frac{7\pi}{6}\right)\right)$$ M 1

Using the result from **ii**, $\left(\sqrt{3}+i\right)^7-\left(\sqrt{3}-i\right)^7=2^7\times 2i\sin\left(\frac{7\pi}{6}\right)$ M1

$$=2^7\times 2i\left(-\frac{1}{2}\right)=-2^7 i$$ A1

Question 2

a. $\mathbf{a}=2p\mathbf{i}+\mathbf{j}+(1-p)\mathbf{k}$, $\mathbf{b}=-\mathbf{i}+3\mathbf{j}$, $\mathbf{c}=5\mathbf{i}-\mathbf{j}+8\mathbf{k}$

$\angle(\mathbf{a},\mathbf{b})=\angle(\mathbf{a},\mathbf{c})$

$$\frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\frac{\mathbf{a}\bullet\mathbf{c}}{|\mathbf{a}||\mathbf{c}|},$$

$|\mathbf{b}|=\sqrt{10}$, $|\mathbf{c}|=\sqrt{90}$, $(\mathbf{a}\bullet\mathbf{b})=-2p+3$, $(\mathbf{a}\bullet\mathbf{c})=10p-1+8(1-p)$ M1

$$\frac{-2p+3}{\sqrt{10}}=\frac{10p-1+8(1-p)}{3\sqrt{10}}$$

$3(-2p+3)=10p-1+8p-8p$ M1

$-6p+9=2p+7$

$p=\frac{1}{4}$ A1

b. Vectors **a**, **b** and **c** are linearly dependent if one of them can be written as a linear combination of the other two.

$\mathbf{a}=m\mathbf{b}+n\mathbf{c}$

$2p\mathbf{i}+\mathbf{j}+(1-p)\mathbf{k}=m(-\mathbf{i}+3\mathbf{j})+n(5\mathbf{i}-\mathbf{j}+8\mathbf{k})$. M1

By equating coefficients of **i**, **j** and **k**:

$-m+5n=2p$ (1)

$3m-n=1$

$8n=1-p\Rightarrow n=\frac{1-p}{8}$ (2)

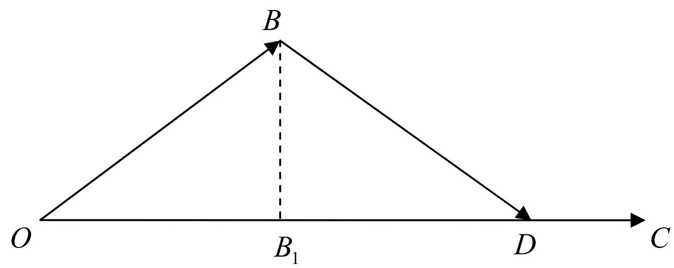
$3m=n+1=\frac{1-p}{8}+1\Rightarrow m=\frac{9-p}{24}$ (3) M1

Substituting (2) and (3) into the equation (1) gives

$-\frac{9-p}{24}+\frac{5-5p}{8}=2p$, $-9+p+15-15p=48p\Rightarrow p=\frac{3}{31}$ A1

c. Method 1

In the diagram below, $\left|\overrightarrow{BD}\right| = \left|\overrightarrow{OB}\right|$ and B_1 is the midpoint of OD.



M1

Vector $\overrightarrow{OB_1}$ is a vector resolute of $\overrightarrow{OB}$ in the direction of $\overrightarrow{OC}$.

$$\overrightarrow{OB_1} = (\mathrm{b} \bullet \hat{\mathrm{c}})\hat{\mathrm{c}} = \frac{1}{90}(-5-3)(5\mathrm{i} - \mathrm{j} + 8\mathrm{k})$$

$$= \frac{-4}{45}(5\mathrm{i} - \mathrm{j} + 8\mathrm{k})$$ A1

$$= -\frac{4}{9}\mathrm{i} + \frac{4}{45}\mathrm{j} - \frac{32}{45}\mathrm{k}$$

From $\overrightarrow{OD} = 2\overrightarrow{OB_1}$, the position vector of point D is $\overrightarrow{OD} = -\frac{8}{9}\mathbf{i} + \frac{8}{45}\mathbf{j} - \frac{64}{45}\mathbf{k}$ A1

Method 2

Let $\overrightarrow{OD} = m\overrightarrow{OC}$. Then $\overrightarrow{OD} = 5m\mathbf{i} - m\mathbf{j} + 8m\mathbf{k}$

$$\overrightarrow{BD} = \overrightarrow{BO} + \overrightarrow{OD} = (1+5m)\mathrm{i} + (-3-m)\mathrm{j} + 8m\mathrm{k}$$

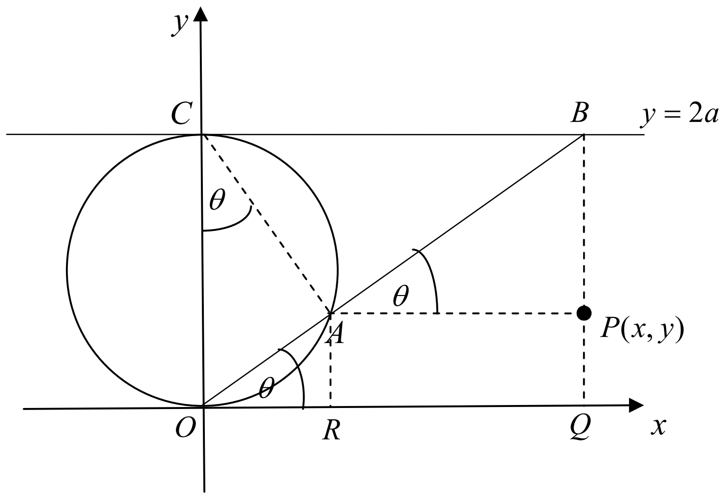
$$\left|\overrightarrow{OB}\right| = \sqrt{10}, \quad \left|\overrightarrow{\mathrm{BD}}\right| = \sqrt{(1+5m)^2 + (3+m)^2 + 64m^2} = \sqrt{90m^2 + 16m + 10}$$

$$90m^2 + 16m + 10 = 10 \Rightarrow m = -\frac{8}{45}$$

Therefore $\overrightarrow{OD} = -\frac{8}{9}\mathbf{i} + \frac{8}{45}\mathbf{j} - \frac{64}{45}\mathbf{k}$

Question 3

a.



From the triangle *OBQ*, $\frac{BQ}{OQ} = \tan\theta$, $OQ = \frac{BQ}{\tan\theta}$ and BQ $= 2a$, $OQ = x$.

Thus $x = \frac{2a}{\tan\theta} = 2a\cot\theta$. A1

To find y, note that $\angle OAC = \frac{\pi}{2}$ (angle subtended by diameter).

From the triangle *OAC*, $OA = 2a\sin\theta$ and from the triangle *OAR*, $AR = y = OA\sin\theta$.

After substituting $OA = 2a\sin\theta$ into $AR = y = OA\sin\theta$, it follows $y = 2a\sin^2\theta$. A1

b. From $\cot\theta = \frac{x}{2a}$, using the identity $\cot^2\theta + 1 = \frac{1}{\sin^2\theta}$, it follows $\sin^2\theta = \frac{4a^2}{x^2 + 4a^2}$

$$y = 2a\sin^2\theta$$

$$= 2a \times \frac{4a^2}{x^2 + 4a^2} \qquad \text{M2}$$

$$= \frac{8a^3}{x^2 + 4a^2}$$

c. $y' = \frac{-16a^3x}{\left(x^2 + 4a^2\right)^2}$, $y'' = \frac{16a^3\left(3x^2 - 4a^2\right)}{\left(x^2 + 4a^2\right)^3}$ M1

$y'' = 0$ when $3x^2 - 4a^2 = 0 \Rightarrow x = \pm\frac{2a}{\sqrt{3}}$. A1

After substituting into $y = \frac{8a^3}{x^2 + 4a^2}$, $y = \frac{8a^3}{\left(\frac{2a}{\sqrt{3}}\right)^2 + 4a^2} \Rightarrow y = \frac{3a}{2}$

Thus, the points of inflexion are $\left(-\frac{2a}{\sqrt{3}}, \frac{3a}{2}\right)$ and $\left(\frac{2a}{\sqrt{3}}, \frac{3a}{2}\right)$ A1

d. **i**

$$A = 2\int_0^{\frac{2a}{\sqrt{3}}} \frac{8a^3}{x^2+4a^2}\,dx = 8a^2\left(\tan^{-1}\frac{x}{2a}\right)_0^{\frac{2a}{\sqrt{3}}}$$

$$= 8a^2\tan^{-1}\frac{1}{\sqrt{3}}$$ M1, A1

$$= \frac{4a^2\pi}{3}$$

ii

$$V = \pi\int_a^{2a} x^2\,dy = \pi\int_a^{20}\left(\frac{8a^3}{y} - 4a^2\right)dy$$

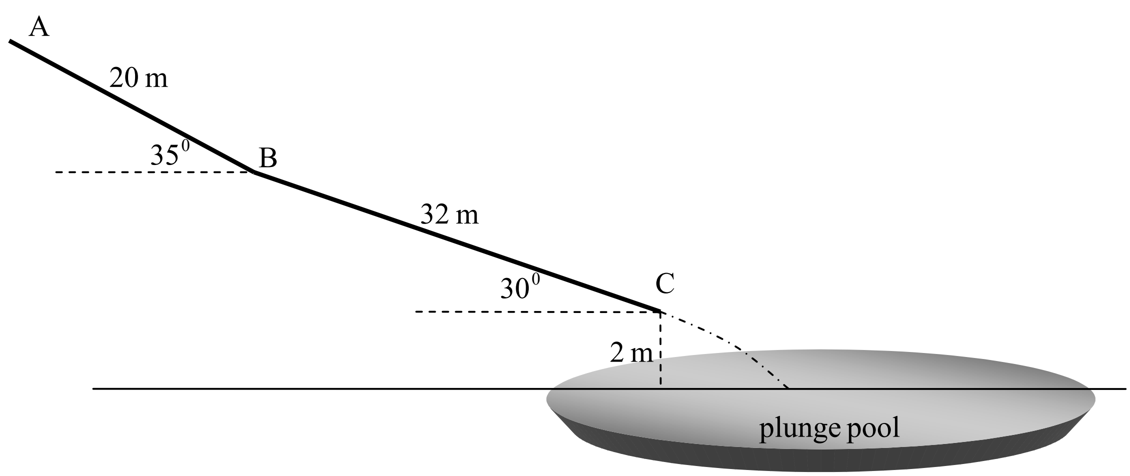
$$= \pi\left[8a^3\log_e y - 4a^2 y\right]_a^{2a}$$

$$= \pi(8a^3\log_e 2 - 4a^3)$$ M1, A1

$$= 4a^3\pi(2\log_e 2 - 1)$$

iii By using a graphics calculator, the point of intersection of the area and volume functions occurs when $a = 0.863$ A1

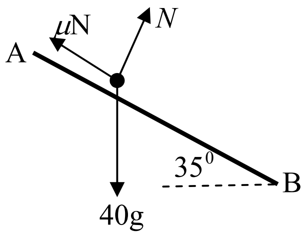
Question 4



a. **i** Given that $u = 0,\ v = 6,\ s = 20$, using constant acceleration formula $v^2 = u^2 + 2as$,

$36 = 40a \Rightarrow a = \frac{9}{10}$ ms^{-2}. A1

ii



From the diagram: $40g\sin 35^0 - \mu N = 40\times\frac{9}{10}$ and $N - 40g\cos 35^0 = 0$. It follows

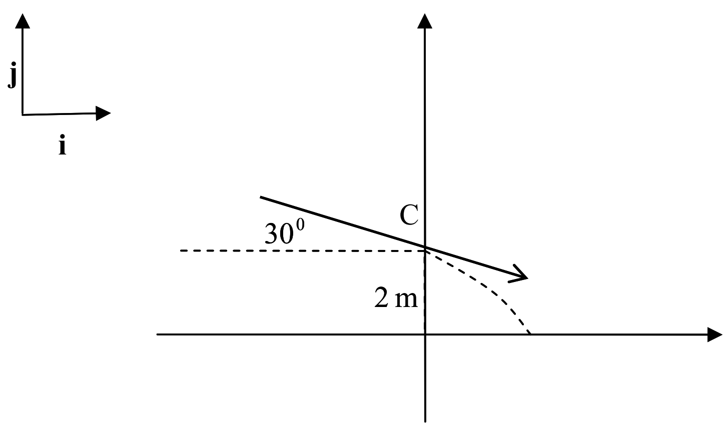
$40g\sin 35^0 - \mu\times 40g\cos 35^0 = 40\times\frac{9}{10}$ M1

$\mu = \frac{g\sin 35^0 - 0.9}{g\cos 35^0} = 0.575$ A1

b. It is known that $u = 6,\ \ s = 32$ and $a = -0.5$. Therefore $v^2 = 36 - 32 \Rightarrow v = 2$.
M1

From $s = \frac{u+v}{2}t,\ \ t = \frac{2s}{u+v} = 8$ seconds A1

c.



i By choosing the directions **i** and **j** as in the diagram above and using $v = 2$ at point C, we have $\dot{\mathbf{r}}(0) = 2\cos 30^0\,\mathbf{i} - 2\sin 30^0\,\mathbf{j}$, thus $\dot{\mathbf{r}}(0) = \sqrt{3}\mathbf{i} - \mathbf{j}$. A1

ii From $\ddot{\mathbf{r}}(t) = -g\mathbf{j}$, $\dot{\mathbf{r}}(t) = -gt\mathbf{j} + \sqrt{3}\mathbf{i} - \mathbf{j}$

$\dot{\mathbf{r}}(t) = \sqrt{3}\mathbf{i} - (gt+1)\mathbf{j}$ A1

After integrating the velocity vector: $\mathbf{r}(t) = \sqrt{3}\,t\,\mathbf{i} - \left(\frac{gt^2}{2} + t\right)\mathbf{j} + 2\mathbf{j}$

$\mathbf{r}(t) = \sqrt{3}\,t\,\mathbf{i} - \left(\frac{gt^2}{2} + t - 2\right)\mathbf{j}$ A1

d. **i** Before the speed can be found, we need to know the time when Adam hits the water. This is when the **j** component of the position vector is zero.

Solving $\frac{gt^2}{2}+t-2=0$ gives $t = 0.5449$. A1

The velocity vector at $t = 0.5449$ is $\dot{\mathbf{r}}(t)=\sqrt{3}\mathbf{i}-(9.8\times 0.5449+1)\mathbf{j}=1.732\mathbf{i}-6.34\mathbf{j}$. M1

The speed is the magnitude of the velocity vector.

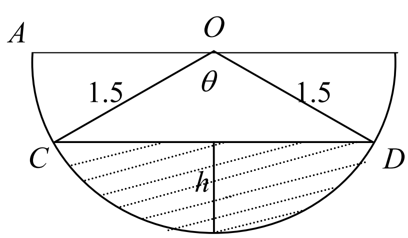
Speed = $\sqrt{3+6.34^2}=6.57\,\text{ms}^{-1}$ A1

ii The horizontal distance from the bottom of the slide is the **i** component of the position vector at $t = 0.5449$, which is $\sqrt{3}\times 0.5449=0.9438$.

The required distance is 0.94 m. A1

Question 5

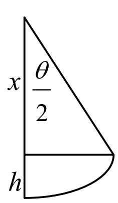
AB is the diameter of the semi-circle, $\angle\text{COD}=\theta$ and h is the depth



a. **i** The shaded area (segment) = $\frac{1}{2}\times 1.5^2(\theta-\sin\theta)=\frac{9}{8}(\theta-\sin\theta)$.

Therefore $V=8\times\frac{9}{8}(\theta-\sin\theta)=9(\theta-\sin\theta)$. M1

ii



$\frac{x}{1.5}=\cos\frac{\theta}{2}\Rightarrow x=\frac{3}{2}\cos\frac{\theta}{2}$

$h=\frac{3}{2}-\frac{3}{2}\cos\frac{\theta}{2}=\frac{3}{2}\left(1-\cos\frac{\theta}{2}\right)$ M1

b. $\frac{dh}{dt}=\frac{dh}{d\theta}\frac{d\theta}{dV}\frac{dV}{dt}$ (1)

From $h=\frac{3}{2}\left(1-\cos\frac{\theta}{2}\right)$, $\frac{dh}{d\theta}=\frac{3}{4}\sin\frac{\theta}{2}$.

Also, from $V=9(\theta-\sin\theta)$, $\frac{dV}{d\theta}=9(1-\cos\theta)$ and $\frac{dV}{dt}=0.2$ (given).

By substituting into (1), we have $\frac{dh}{dt}=\frac{3}{4}\sin\frac{\theta}{2}\times\frac{1}{9(1-\cos\theta)}\times 0.2$ M1

$$=\frac{\sin\frac{\theta}{2}}{60(1-\cos\theta)} \quad (2)$$ A1

For $h=0.8$, $\frac{3}{2}\left(1-\cos\frac{\theta}{2}\right)=0.8$

Solving this equation for θ gives $\theta=2.17^c$ or $\theta=124.36^0$ M1

After substitution into equation (2): $\frac{dh}{dt}=0.009\,\text{ms}^{-1}$ (3 decimal places)

A1

c. i $\frac{dx}{dt}=Rate\ \ in-Rate\ \ out$,

$\frac{dx}{dt}=0.05\times 100+0.04\times 50-\frac{150x}{20000}$. M1

After simplifying, $\frac{dx}{dt}=7-\frac{3x}{400}$

ii $\frac{dx}{dt}=\frac{2800-3x}{400}$

$\frac{dt}{dx}=\frac{400}{2800-3x}$ A1

$t=\int\frac{400}{2800-3x}dx=-\frac{400}{3}\log_e(2800-3x)+c \qquad 0\le x<\frac{2800}{3}$

$\log_e(2800-3x)=-\frac{3t}{400}+\frac{3c}{400}$ M1

$2800-3x=Ae^{-\frac{3t}{400}}$, where $A=e^{\frac{3c}{400}}$ M1

When $x=0$, $t=0\Rightarrow A=\frac{2800}{3}$ and therefore $x=\frac{2800}{3}(1-e^{-\frac{3t}{400}})$.

d. i The concentration of 0.03kg / L gives $x=20000\times 0.03=600$ kg of salt. M1

After solving the equation (algebraically or using graphics calculator)

$\frac{2800}{3}(1-e^{-\frac{3t}{400}})=600\Rightarrow t=137.28$ minutes A1

ii When $t \to \infty$, $e^{-\frac{3t}{400}} \to 0$ and $x \to \frac{2800}{3} = 933.33$ kg of salt. A1

The limiting concentration is $\frac{\frac{2800}{3}}{20000} = 0.047$ kg/L A1