SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1

2009 Trial Examination

SOLUTIONS

Question 1

a. Vectors $\mathbf{u} = \mathbf{a} + k\,\mathbf{b}$ and $\mathbf{v} = \mathbf{a} - \mathbf{b}$ are perpendicular, therefore $\mathbf{u} \bullet \mathbf{v} = 0$

$(\mathbf{a} + k\mathbf{b}) \bullet (\mathbf{a} - \mathbf{b}) = 0$

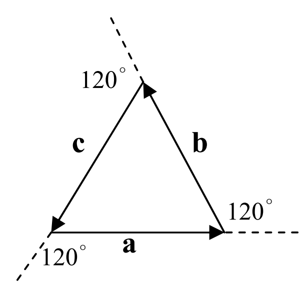
$|\mathbf{a}|^2 - \mathbf{a} \bullet \mathbf{b} + k\mathbf{a} \bullet \mathbf{b} - k|\mathbf{b}|^2 = 0 \qquad (1)$ M1

Also $\angle(\mathbf{a}, \mathbf{b}) = 120^\circ$, so $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 120^\circ = -|\mathbf{a}|^2$. M1

Substituting $\mathbf{a} \bullet \mathbf{b} = -|\mathbf{a}|^2$ and $|\mathbf{b}| = 2|\mathbf{a}|$ into the equation (1), gives $|\mathbf{a}|^2 + |\mathbf{a}|^2 - k|\mathbf{a}|^2 - 4k|\mathbf{a}|^2 = 0$

After simplifying we have $(2 - 5k)|\mathbf{a}|^2 = 0$ and $k = \frac{2}{5}$. A1

b. As $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ and $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$, vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ form an equilateral triangle.



M1

The angle between each two vectors is 120^0 and $\mathbf{a} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{c} = \mathbf{b} \bullet \mathbf{c} = -\frac{1}{2}$.

It follows $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} = -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{3}{2}$. A1

Question 2

a.

$$w = \frac{1+i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{1+2i+i^2}{1-i^2}$$ A1

$$= \frac{2i}{2}$$

$$= i$$

b. Let $z = x + iy$. Then

$$|x - iy - i(x+iy)| = 2\sqrt{2}$$ M1

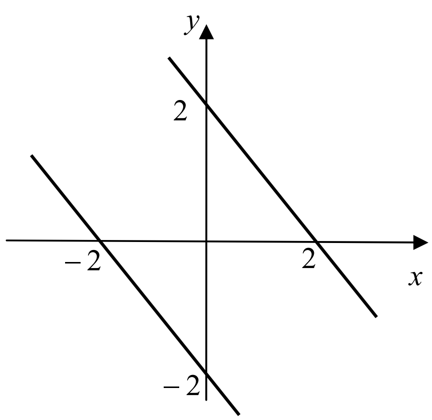
$$|x + y - i(x+y)| = 2\sqrt{2}$$

$$(x+y)^2 + (x+y)^2 = 8$$

$$(x+y)^2 = 4$$

$$x + y = \pm 2$$

This subset is represented by two parallel lines $x + y = 2$ and $x + y = -2$. A1



A1

c. $z_1 w = z_1 i$ which represents a rotation by $\frac{\pi}{2}$, therefore the required numbers are M1

$z_1 = 2, \ -2, \ 2i, \ -2i$ A1

Question 3

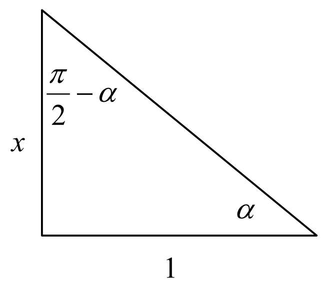
a. $\sin\left(\tan^{-1}\frac{1}{\sqrt{3}}\right)-\cos\left(\tan^{-1}\sqrt{3}\right)=\sin\frac{\pi}{6}-\cos\frac{\pi}{3}$

$$=\frac{1}{2}-\frac{1}{2}$$

$$=0$$ A1

b. Given that $x=\tan\alpha$ and $\alpha\in\left(0,\frac{\pi}{2}\right)$ using the right-angle triangle below, we have

$\tan\left(\frac{\pi}{2}-\alpha\right)=\frac{1}{x}$ and $\left(\frac{\pi}{2}-\alpha\right)=\tan^{-1}\frac{1}{x}$



M1

$\sin\left(\tan^{-1}x\right)-\cos\left(\tan^{-1}\frac{1}{x}\right)=\sin\alpha-\cos\left(\frac{\pi}{2}-\alpha\right)$

$=0$ as $\sin\alpha=\cos\left(\frac{\pi}{2}-\alpha\right)$ A1

Question 4

To find the points of intersection, substitute $x = 1$ into $y^2-2x^2+2xy-6=0$ which gives $y^2+2y-8=0 \Rightarrow y=2$ and $y=-4$. A1

To find the gradient, differentiate implicitly as follows:

$2y\frac{dy}{dx}-4x+2y+2x\frac{dy}{dx}=0$. When rearranged, $\frac{dy}{dx}=\frac{2x-y}{x+y}$.

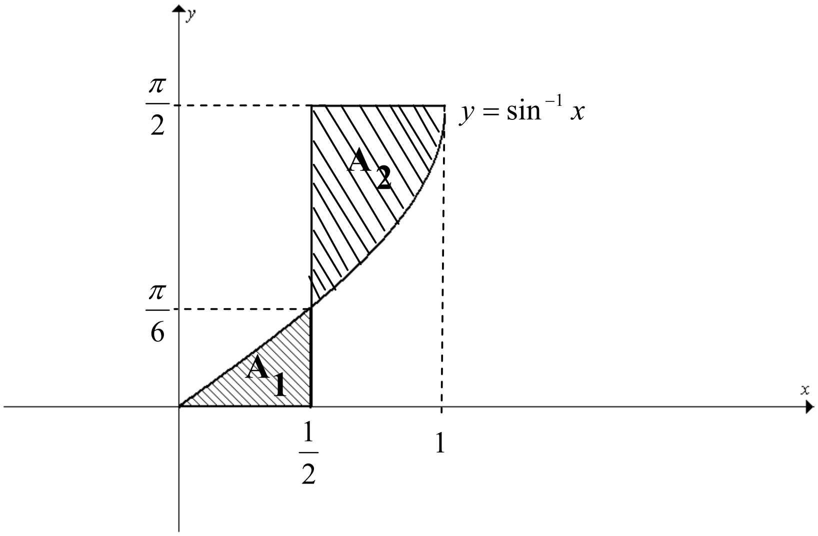
The gradient of the normal is given by the expression $\frac{dy}{dx}=\frac{x+y}{y-2x}$. M1

At the point (1, 2) the gradient of the normal is undefined, therefore its equation is $x = 1$. M1

At the point (1, -4), the gradient of the normal is $\frac{1}{2}$ and the equation of the normal is

$y+4=\frac{1}{2}(x-1)$. After simplifying, the equation is $x-2y-9=0$. A1

Question 5



The area can be found by integrating $x = \sin y$. M1

$$A_1 = \frac{\pi}{6} \times \frac{1}{2} - \int_0^{\frac{\pi}{6}} \sin y\,dy$$

$$= \frac{\pi}{12} + (\cos y)_0^{\frac{\pi}{6}}$$ M1

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

$$A_2 = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin y\,dy - \left(\frac{\pi}{2} - \frac{\pi}{6}\right) \times \frac{1}{2}$$

$$= (-\cos y)_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \frac{\pi}{6}$$ M1

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

Total area = $A_1 + A_2 = \sqrt{3} - \frac{\pi}{12} - 1$ A1

Question 6

Let $\ln x = u$. Then $\frac{1}{x}dx = du$.

For $x = e^{-3}$, $u = \ln e^{-3} = -3$ and for $x = e^{-2}$, $u = \ln e^{-2} = -2$. M1

$$\int_{e^{-3}}^{e^{-2}} \frac{1+\ln x}{x \ln x(1-\ln x)}dx = \int_{-3}^{-2} \frac{u+1}{u(1-u)}du$$ A1

$$\frac{u+1}{u(1-u)} = \frac{A}{u} + \frac{B}{1-u}$$ M1

$$A(1-u) + Bu = u+1 \Rightarrow A = 1,\ B = 2$$

$$\int_{-3}^{-2} \frac{u+1}{u(1-u)}du = \int_{-3}^{-2}\left(\frac{1}{u} + \frac{2}{1-u}\right)du$$

$$= \left(\ln|u| - 2\ln|1-u|\right)_{-3}^{-2}$$

$$= \left(\ln\left|\frac{u}{(1-u)^2}\right|\right)_{-3}^{-2}$$ M1, A1

$$= \ln\frac{2}{9} - \ln\frac{3}{16}$$

$$= \ln\frac{\frac{2}{9}}{\frac{3}{16}} = \ln\frac{32}{27}$$

Question 7

a. $x = 2\cos\frac{\pi}{10}t, \quad y = 3t, \quad t \geq 0$

$t = \frac{10}{\pi}\cos^{-1}\frac{x}{2}$, so the Cartesian equation is $y = \frac{30}{\pi}\cos^{-1}\frac{x}{2}$. A1

The domain of this function is $[-2,2]$, the range is $[0,30]$. A1

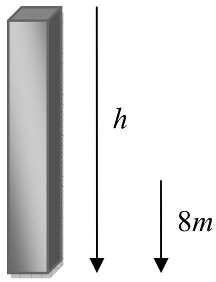
b. The speed of a particle can be found as the magnitude of its velocity vector. The position vector is $\mathbf{r}(t) = 2\cos\frac{\pi}{10}t\,\mathbf{i} + 3t\,\mathbf{j}$, its velocity vector is $\dot{\mathbf{r}}(t) = -\frac{\pi}{5}\sin\frac{\pi}{10}t\,\mathbf{i} + 3\,\mathbf{j}$.

The expression for the speed is $|\dot{\mathbf{r}}(t)| = \sqrt{\frac{\pi^2}{25}\sin^2\frac{\pi}{10}t + 9}$. M1

The minimum speed is 3 and it occurs when $\sin^2\frac{\pi}{10}t = 0$, which is for $\frac{\pi}{10}t = 0, \pi$ or $t = 0, 10$ A1

Question 8

At the time t when the object hits the ground: $a = g,\ v = gt,\ a = \frac{1}{2}gt^2$



When the object is 8 m above the ground, the time elapsed is $(t - 0.4)$ seconds and the velocity is $v_1 = g(t - 0.4)$. Therefore considering the last 0.4 seconds of travel M1

$8 = g(t - 0.4) \times 0.4 + \frac{1}{2} g \times 0.4^2$

Solving for *t:*

$$g\left(t - \frac{2}{5}\right) \times \frac{2}{5} + \frac{1}{2} g \times \frac{4}{25} = 8$$

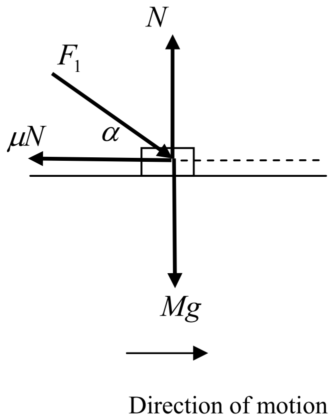
$$\frac{2g}{5}t - \frac{4g}{25} + \frac{2g}{25} = 8$$

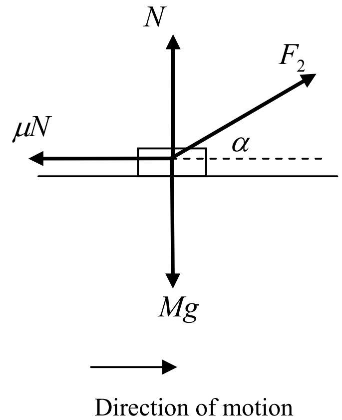
$$\frac{2gt}{5} = 8 + \frac{2g}{25} \Rightarrow t = \frac{20}{g} + \frac{1}{5} = \frac{100 + g}{5g}$$ A1

Substituting back into $h = \frac{1}{2}gt^2$ and simplifying gives $h = \frac{(100 + g)^2}{50g}$. A1

Question 9

a. Pushing force: Pulling force:





The velocity of the object is constant, therefore the acceleration is zero. By resolving the forces vertically and horizontally we have:

Pushing force:

$N - Mg - F_1 \sin\alpha = 0,\ N = Mg + F_1 \sin\alpha$

$-\mu N + F_1 \cos\alpha = 0$

$\mu Mg + \mu F_1 \sin\alpha - F_1 \cos\alpha = 0$

$F_1 = \dfrac{\mu Mg}{\cos\alpha - \mu \sin\alpha}$

Pulling force:

$N - Mg + F_2 \sin\alpha = 0,\ N = Mg - F_2 \sin\alpha$

$-\mu N + F_2 \cos\alpha = 0$

$-\mu Mg + \mu F_2 \sin\alpha + F_2 \cos\alpha = 0$

$F_2 = \dfrac{\mu Mg}{\cos\alpha + \mu \sin\alpha}$ M3

$\cos\alpha - \mu\sin\alpha < \cos\alpha + \mu\sin\alpha$ for every $\alpha \in \left(0, \dfrac{\pi}{2}\right)$ A1

Therefore $F_1 > F_2$

b.

$$\frac{F_1}{F_2} = \frac{\mu Mg}{\cos\alpha - \mu\sin\alpha} \times \frac{\cos\alpha + \mu\sin\alpha}{\mu Mg}$$

$$= \frac{\cos\alpha + \mu\sin\alpha}{\cos\alpha - \mu\sin\alpha}$$ M1

Dividing each term by $\cos\alpha$ gives $\dfrac{F_1}{F_2} = \dfrac{1 + \mu\tan\alpha}{1 - \mu\tan\alpha}$ A1

