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SPECIALIST MATHS TRIAL EXAMINATION 2 SOLUTIONS 2009

Section 1 – Multiple-choice answers

1.	**B**	**7.**	**D**	**13.**	**B**	**19.**	**A**
2.	**C**	**8.**	**E**	**14.**	**C**	**20.**	**E**
3.	**E**	**9.**	**C**	**15.**	**D**	**21.**	**D**
4.	**E**	**10.**	**C**	**16.**	**C**	**22.**	**B**
5.	**C**	**11.**	**E**	**17.**	**A**		
6.	**B**	**12.**	**D**	**18.**	**D**		

Section 1- Multiple-choice solutions

Question 1

The asymptotes are $y = ax$ and $x = 0$. Only option B offers these asymptotes since

$$y = \frac{ax^3 + 1}{x^2}$$

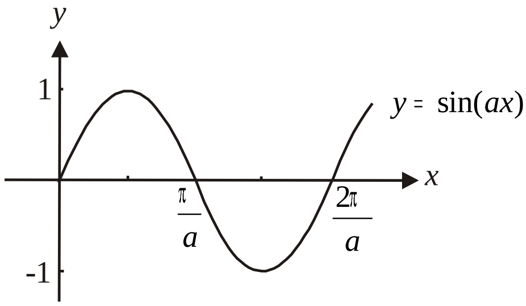
$$y = ax + \frac{1}{x^2},\ a > 0$$

The answer is B.

Question 2

$$y = \text{cosec}(ax)$$
$$= \frac{1}{\sin(ax)}$$

Now $\sin(ax) = 0$ for $x = 0, \frac{\pi}{a}, \frac{2\pi}{a}, \frac{3\pi}{a}, ...$



So, $y = \frac{1}{\sin(ax)}$ will have asymptotes at $x = 0, \frac{\pi}{a}, \frac{2\pi}{a}, \frac{3\pi}{a}, ...$

The answer is C.

Question 3

$y = 2\arcsin(2x+1) - \pi$

For the domain,
$-1 \le 2x+1 \le 1$
$-2 \le 2x \le 0$
$-1 \le x \le 0$

For the range,
$-\frac{\pi}{2} \le \arcsin(2x+1) \le \frac{\pi}{2}$
$-\pi \le 2\arcsin(2x+1) \le \pi$
$-2\pi \le 2\arcsin(2x+1) - \pi \le 0$
The answer is E.

Question 4

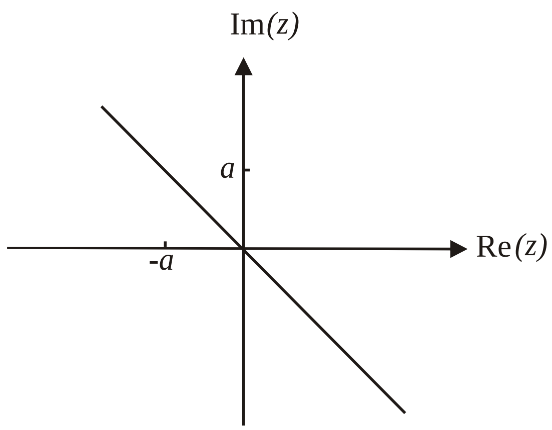
$z = \sqrt{3}\text{cis}\left(-\frac{\pi}{2}\right)$
$z^6 = \left(\sqrt{3}\right)^6 \text{cis}\left(-\frac{6\pi}{2}\right)$ (De Moivre)
$= 27\text{cis}(-3\pi)$
$= 27\text{cis}(\pi)$
$\text{Arg}(z^6) = \pi$
Note that $-\pi < \text{Arg}(z^6) \le \pi$.
The answer is E.

Question 5

The complex number $\bar{z}$, the conjugate of z can be represented by the point *T*.
The complex number $\bar{z}i$ is obtained by rotating point *T* by $\frac{\pi}{2}$ radians in an anticlockwise direction.
So $\bar{z}i$ is represented by the point *R*.
The answer is C.

Question 6

Method 1 – graphical approach.
The graph is made up of the set of complex numbers which are the same distance from the complex number *ai* and – *a*.



These lie on the line that passes through the point representing the complex number $-a+ai$ and has a gradient of – 1.
The answer is B.

Method 2 – algebraic approach

$$|z-ai|=|z+a|,\ a>0$$
$$|x+yi-ai|=|x+yi+a|$$
$$\sqrt{x^2+(y-a)^2}=\sqrt{(x+a)^2+y^2}$$
$$x^2+y^2-2ay+a^2=x^2+2ax+a^2+y^2$$
$$-2ay=2ax$$
$$y=-x$$

The line passes through the complex number $-a+ai$ and has a gradient of – 1.
The answer is B.

Question 7

Since the coefficients of the cubic polynomial are real, the conjugate root theorem applies and so $z=ai$ is another root as is $z=b$ where *b* is any real number.

$$\begin{aligned}p(z)&=(z-ai)(z+ai)(z-b)\\&=(z^2+a^2)(z-b)\\&=z^3-bz^2+a^2z-a^2b\end{aligned}$$

If $b=0$,

$$p(z)=z^3+a^2z$$

If $b=1$,

$$p(z)=z^3-z^2+a^2z-a^2$$

The answer is D.

Question 8

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos^3(x)dx$$

let $u = \cos(x)$

$$\frac{du}{dx} = -\sin(x)$$

$$x = \frac{\pi}{2},\ u = 0$$

$$x = 0,\ u = 1$$

$$= \int_1^0 -\frac{du}{dx}u^3 dx$$

$$= \int_0^1 u^3 du$$

The answer is E.

Question 9

$$f'(t) = \ln(t+1) \quad t > 0$$

$$\int_0^1 f'(t)\,dt = f(1) - f(0)$$

Since $f(0) = 2$,

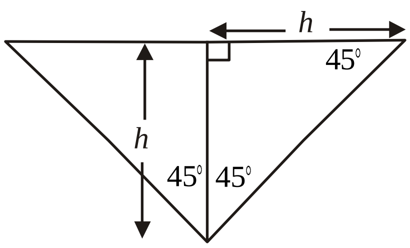
$$\int_0^1 f'(t)\,dt = f(1) - 2$$

so $f(1) = \int_0^1 f'(t)dt + 2$

$$= \int_0^1 \ln(t+1)dt + 2$$

The answer is C.

Question 10



Let V = volume of water in trough.

$$V = h^2 \times 150(\text{cm}^3)$$

$$\frac{dV}{dh} = 300h$$

$$\frac{dh}{dt} = \frac{dh}{dV}\cdot\frac{dV}{dt}$$

$$= \frac{1}{300h}\times 300$$

$$= \frac{1}{h}$$

When $h = 20$

$$\frac{dh}{dt} = \frac{1}{20}\text{ cm/min}$$

$$= 0\cdot 05\text{ cm/min}$$

The answer is C.

Question 11

$$\int_0^2 \frac{2x-1}{\sqrt{3-x}}dx$$
$$= -\int_3^1 (5-2u)\frac{1}{\sqrt{u}} \times \frac{du}{dx}\, dx$$
$$= -\int_3^1 (5u^{-\frac{1}{2}} - 2u^{\frac{1}{2}})du$$
$$= \int_1^3 (5u^{-\frac{1}{2}} - 2u^{\frac{1}{2}})du$$

let $u = 3 - x$

$\frac{du}{dx} = -1$

$x = 3 - u$

$2x - 1 = 2(3-u) - 1$

$= 6 - 2u - 1$

$= 5 - 2u$

Also $x = 2, u = 1$

$x = 0, u = 3$

The answer is E.

Question 12

$$\frac{dH}{dt} = \text{rate of inflow of } H \text{ (litres / minute)} - \text{rate of outflow of } H \text{ (litres / minute)}$$
$$= 5 - \frac{H}{200} \times 5$$
$$= 5 - \frac{H}{40}$$
$$= \frac{200 - H}{40}$$

Initially there is 40% of 200 = 80 litres of hydrogen in the cylinder.

So $\frac{dH}{dt} = \frac{200 - H}{40}$, $t = 0, H = 80$.

The answer is D.

Question 13

The particles are in the same position when

$t^2 = 7t - 10$ AND $3t - 4 = 11$

$t^2 - 7t + 10 = 0$ $3t = 15$

$(t-5)(t-2) = 0$ $t = 5$

$t = 5, 2$

At $t = 5$ the particles are in the same position.

The answer is B.

Question 14

$$\underset{\sim}{a}.\underset{\sim}{b} = (\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})(2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})$$

$$= 2 - 2 + 4 = 4$$

Also, $\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

So, $4 = \sqrt{1+4+4}.\sqrt{4+1+4}.\cos\theta$

So, $\cos\theta = \dfrac{4}{9}$, and therefore $\theta = 63°37'$

The answer is C.

Question 15

Since *PQRS* is a rhombus the diagonals cross at right angles so

$$(\underset{\sim}{a} + \underset{\sim}{b}) \bullet (\underset{\sim}{a} - \underset{\sim}{b}) = 0$$

$$\underset{\sim}{a} \bullet \underset{\sim}{a} - \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{b} - \underset{\sim}{b} \bullet \underset{\sim}{b} = 0$$

$$\underset{\sim}{a} \bullet \underset{\sim}{a} = \underset{\sim}{b} \bullet \underset{\sim}{b}$$

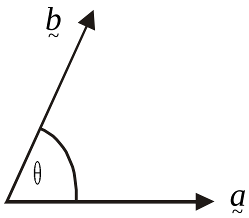
So A is true.

Since *PQRS* is a rhombus and $|\underset{\sim}{a}| = 1$ then $|\underset{\sim}{b}| = 1$ so $|\underset{\sim}{a}| = |\underset{\sim}{b}|$ so B is true.

$$\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

$$= \cos\theta$$

since $|\underset{\sim}{a}| = |\underset{\sim}{b}| = 1$



So option C is true but option D isn't because

$\theta \neq 180 - \theta$ since $\theta \neq 90^\circ$. So option D is false.

For option E, since $\theta \neq 90°$, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are never at right angles so. $\underset{\sim}{a} \bullet \underset{\sim}{b} \neq 0$

Option E is true.

The answer is D.

Question 16

$$\underset{\sim}{r} = 2\sqrt{t}\,\underset{\sim}{i} + (5 - t)\,\underset{\sim}{j}$$

$$\text{distance from origin } = \sqrt{(2\sqrt{t})^2 + (5-t)^2}$$

$$= \sqrt{4t + 25 - 10t + t^2}$$

$$= \sqrt{t^2 - 6t + 25}$$

<u>Method 1</u>

This is a minimum when $2t - 6 = 0$

$$t = 3$$

<u>Method 2</u>

$(t^2 - 6t + 9) - 9 + 25 = (t-3)^2 + 16$

Minimum occurs when $t = 3$.

The answer is C.

Question 17

The total force $\underset{\sim}{F}$ acting on the body is

$$\underset{\sim}{F} = \underset{\sim}{P} + \underset{\sim}{Q} + \underset{\sim}{R}$$
$$= 4\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{i} + 3\underset{\sim}{j}$$
$$= 3\underset{\sim}{i} + 4\underset{\sim}{j}$$

Now $\underset{\sim}{F} = m\underset{\sim}{a}$

$$3\underset{\sim}{i} + 4\underset{\sim}{j} = 8\underset{\sim}{a}$$
$$\underset{\sim}{a} = \frac{3}{8}\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j}$$
$$|\underset{\sim}{a}| = \sqrt{\frac{9}{64} + \frac{1}{4}}$$
$$= \sqrt{\frac{25}{64}}$$
$$= \frac{5}{8} \text{ m/s}^2$$

The answer is A.

Question 18

$$\underset{\sim}{p} = m\underset{\sim}{v}$$
$$20 = 4m$$
$$m = 5kg$$

Later when $p = 45$

$$45 = 5 \times v$$
$$v = 9\text{m/s}$$

Since acceleration is constant and $u = 4, a = 0 \cdot 5$ and $v = 9$,

$$v^2 = u^2 + 2as$$

so, $81 = 16 + 2 \times \frac{1}{2} \times s$

$$s = 81 - 16$$
$$= 65$$

The mass covers 65m.
The answer is D.

Question 19

$\int_0^{20} f(t)dt$ gives the displacement of particle B from the start.

$\int_0^{12} f(t)dt - \int_{12}^{20} f(t)dt$ gives the distance travelled.

Therefore the distance between the two particles is given by $\left|\int_0^{20} f(t)\, dt - 20g(20)\right|$.

The answer is A.

Question 20

$$a = \frac{1}{\sqrt{4-x^2}}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{\sqrt{4-x^2}}$$

$$\frac{1}{2}v^2 = \int \frac{1}{\sqrt{4-x^2}}\, dx$$

$$\frac{1}{2}v^2 = \sin^{-1}\left(\frac{x}{2}\right) + c$$

when $x = 0, v = 4$

$c = 8$

$$\frac{1}{2}v^2 = \sin^{-1}\left(\frac{x}{2}\right) + 8$$

$$v^2 = 2\sin^{-1}\left(\frac{x}{2}\right) + 16$$

$$\frac{v^2 - 16}{2} = \sin^{-1}\left(\frac{x}{2}\right)$$

$$\frac{x}{2} = \sin\left(\frac{v^2 - 16}{2}\right)$$

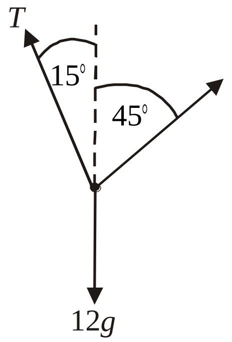
$$x = 2\sin\left(\frac{v^2 - 16}{2}\right)$$

The answer is E.

Question 21

<u>Method 1</u> – using Lami's Theorem

Let T be the tension in the shorter string.



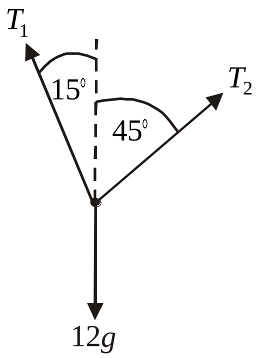
$$\frac{T}{\sin(180^\circ - 45^\circ)} = \frac{12g}{\sin(60^\circ)} \qquad \text{(Lami' s Theorem)}$$

$$\begin{aligned} T &= 12g \div \frac{\sqrt{3}}{2} \times \sin(135^\circ) \\ &= 12g \times \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{2}} \\ &= \frac{24g}{\sqrt{6}} \\ &= 4\sqrt{6}g \text{ newtons} \end{aligned}$$

The answer is D.

<u>Method 2</u> – resolving horizontally and vertically

Let T_1 be the tension in the shorter string and
let T_2 be the tension in the longer string.
T_1 is the tension required.



Resolving horizontally:
$T_1 \cos(75°) = T_2 \cos(45°)$ (1)

Resolving vertically:
$T_1 \cos(15°) + T_2 \cos(45°) = 12g$ (2)

(1) in (2) gives $T_1 \cos(15°) + T_1 \cos(75°) = 12g$

$$T_1 (\cos(15°) + \cos(75°)) = 12g$$

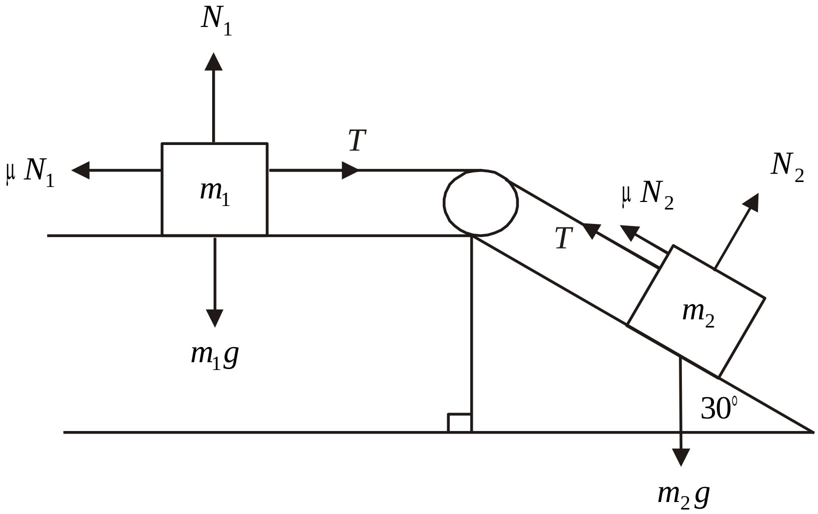
$$T_1 = \frac{12g}{\cos(15°) + \cos(75°)}$$

$$T_1 = 96.02 \text{ newtons}$$

The answer is D.

Question 22

Mark in the forces operating.



Around the m_1 kg mass

$T = \mu N_1$ and $N_1 = m_1 g$

so $T = \mu m_1 g$

Around the m_2 kg mass

$T + \mu N_2 = m_2 g \sin(30^\circ)$ and $N_2 = m_2 g \cos(30^\circ)$

$$T + \frac{\mu\sqrt{3}m_2 g}{2} = \frac{m_2 g}{2} \qquad = \frac{\sqrt{3}}{2} m_2 g$$

$$T = \frac{m_2 g - \mu\sqrt{3}m_2 g}{2}$$

$$\text{So } \mu m_1 g = \frac{m_2 g\left(1 - \sqrt{3}\mu\right)}{2}$$

$$\frac{m_1}{m_2} = \frac{1 - \sqrt{3}\mu}{2\mu}$$

The answer is B.

SECTION 2

Question 1

a. $\underset{\sim}{r}(t) = 2\cos(t)\underset{\sim}{i} + 3\sin(t)\underset{\sim}{j}$

$x = 2\cos(t) \qquad y = 3\sin(t)$

$x^2 = 4\cos^2(t) \qquad y^2 = 9\sin^2(t)$

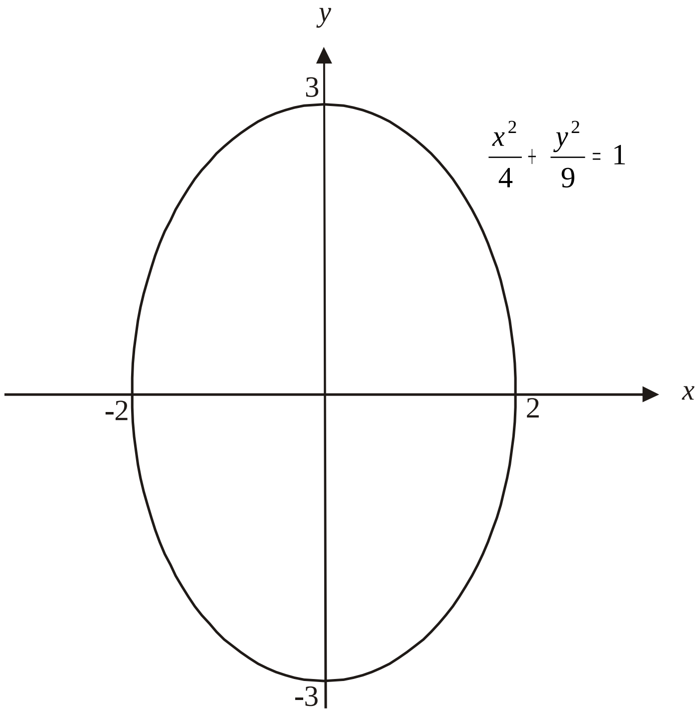
$\frac{x^2}{4} = \cos^2(t) \qquad \frac{y^2}{9} = \sin^2(t)$

$\frac{x^2}{4} + \frac{y^2}{9} = \cos^2(t) + \sin^2(t)$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

(1 mark)

b.



(1 mark) for correct shaped graph and intercepts

c. $\underset{\sim}{r}(0) = 2\cos(0)\underset{\sim}{i} + 3\sin(0)\underset{\sim}{j}$

$= 2\underset{\sim}{i} + 0\underset{\sim}{j}$

The red light starts at the point $(2,0)$ and moves in an anticlockwise direction around the ellipse.

(1 mark) correct starting point
(1 mark) correct direction

d. The period of the *x*-coordinate of motion is 2π and the period of the *y*-coordinate of motion is also 2π. So it takes 2π seconds to complete one circuit.

(1 mark)

e. Given $\underset{\sim}{r}(t) = 2\cos(t)\underset{\sim}{i} + 3\sin(t)\underset{\sim}{j}$

$\underset{\sim}{v}(t) = -2\sin(t)\underset{\sim}{i} + 3\cos(t)\underset{\sim}{j}$ **(1 mark)**

$\left|\underset{\sim}{v}(t)\right| = \sqrt{(-2\sin(t))^2 + (3\cos(t))^2}$

$\left|\underset{\sim}{v}(t)\right| = \sqrt{4\sin^2(t) + 9\cos^2(t)}$

(1 mark)

f. **i.** <u>Method 1</u> – "Hence"

speed is a min/max when

$-5\sin(t)\cos(t) = 0$ (Note $\sqrt{5\cos^2(t) + 4} \neq 0$)

$\frac{-5}{2} \times 2\sin(t)\cos(t) = 0$

$\frac{-5}{2} \times \sin(2t) = 0$

$\sin(2t) = 0$ $\quad 0 < t \leq 2\pi$

$2t = \pi, 2\pi, 3\pi, 4\pi$ $\quad 0 < 2t \leq 4\pi$

$t = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(1 mark)

$\text{speed} = \sqrt{4\sin^2(t) + 9\cos^2(t)}$

when $t = \frac{\pi}{2}$, $\text{speed} = \sqrt{4+0} = 2$

when $t = \pi$, $\text{speed} = \sqrt{0 + 9(-1)^2} = 3$

when $t = \frac{3\pi}{2}$, $\text{speed} = \sqrt{4 \times (-1)^2 + 0} = 2$

when $t = 2\pi$, $\text{speed} = \sqrt{0+9} = 3$

Speed is a maximum when $t = \pi$ sec and $t = 2\pi$ sec.

(1 mark)

<u>Method 2</u> – "otherwise"

$\left|\underset{\sim}{v}(t)\right| = \sqrt{4\sin^2(t) + 9\cos^2(t)}$ (from part **e.**)

$= \sqrt{4\sin^2(t) + 4\cos^2(t) + 5\cos^2(t)}$

$= \sqrt{4 + 5\cos^2(t)}$ **(1 mark)**

So speed is a max. when $\cos(t) = \pm 1$.

$t = \pi,\ 2\pi$ since $0 < t \leq 2\pi$

Speed is a maximum when $t = \pi$ sec and $t = 2\pi$ sec. **(1 mark)**

ii. From part **i.**, maximum speed is $\left|\underset{\sim}{v}(\pi)\right| = \sqrt{4\sin^2(\pi) + 9\cos^2(\pi)}$ = 3 m/s.

(or $\left|\underset{\sim}{v}(2\pi)\right| = \sqrt{4\sin^2(2\pi) + 9\cos^2(2\pi)} = 3$ m/s)

(1 mark)
Total 10 marks

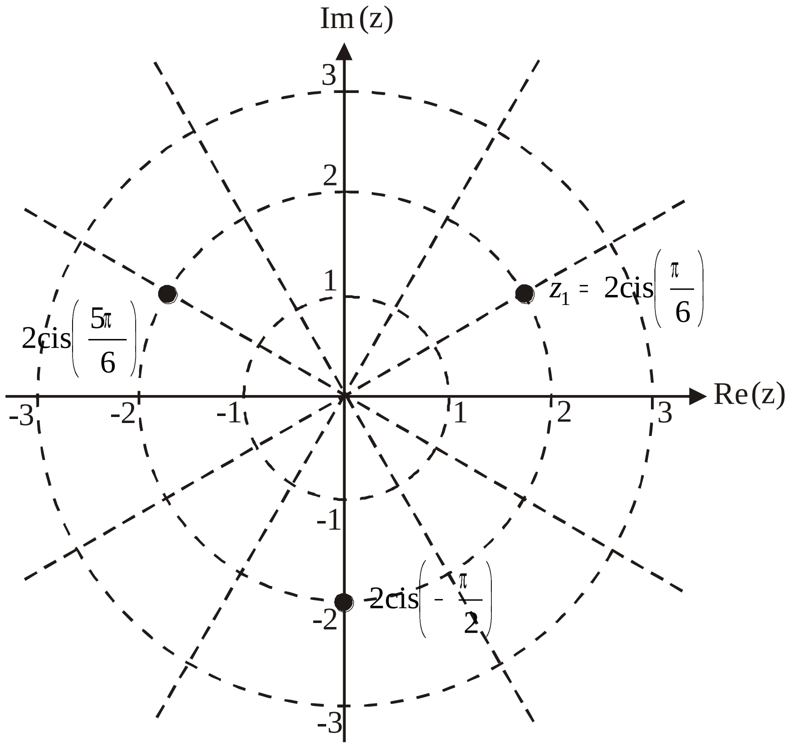
Question 2

a.

$$z_1 = \sqrt{3} + i$$
$$r = \sqrt{3+1}$$
$$= 2$$
$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{\pi}{6}$$
$$z_1 = 2\text{cis}\left(\frac{\pi}{6}\right)$$

(1 mark)

b.



(1 mark) - z_1

(1 mark) - $2\text{cis}\left(\frac{5\pi}{6}\right)$

(1 mark) - $2\text{cis}\left(\frac{-\pi}{2}\right)$

c. $z\bar{z}+|z_1|\times \text{Re}(i^2 z)-2\text{Im}(z)=-1$

Now $i^2 z=-1(x+iy)$
$=-x-iy$

so $\text{Re}(i^2 z)=-x$

Also $2\text{Im}(z)=2y$

So we have

$(x+iy)(x-iy)+2\times -x-2y=-1$ **(1 mark)**

$x^2+y^2-2x-2y=-1$

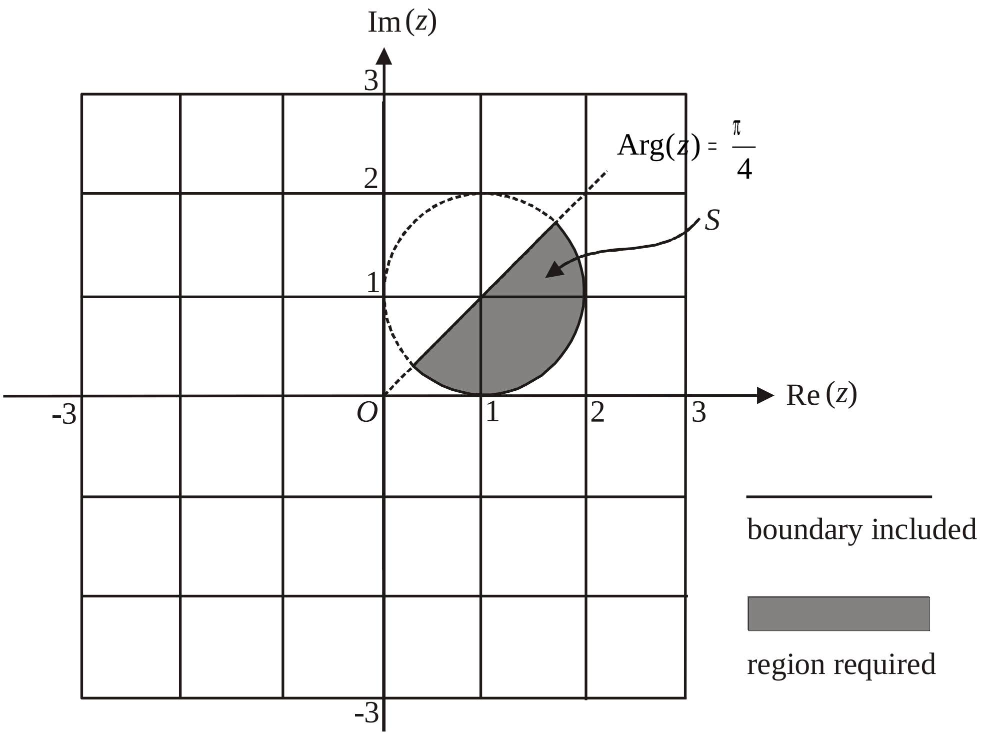
$x^2-2x+1+y^2-2y+1=-1+2$

$(x-1)^2+(y-1)^2=1$

as required

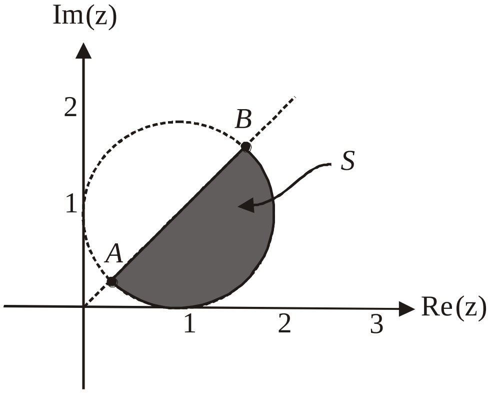
(1 mark) – completing the squares

d.



(1 mark) – correct circular boundary
(1 mark) – correct linear boundary
(1 mark) - correct boundary marking and shading

e. Method 1



At point A on the diagram above $|z_2|$ is a minimum and at point B, $|z_2|$ is a maximum.
Find the point of intersection of the Cartesian equations

$(x-1)^2+(y-1)^2=1$ and $y=x$

$$(x-1)^2+(x-1)^2=1$$
$$2(x-1)^2=1$$
$$(x-1)^2=\frac{1}{2}$$
$$x-1=\pm\frac{1}{\sqrt{2}}$$
$$x=1\pm\frac{1}{\sqrt{2}}$$

(1 mark)

So A is point $\left(1-\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}\right)$ since $y=x$

So minimum value of $|z_2|$ is

$$\sqrt{\left(1-\frac{1}{\sqrt{2}}\right)^2+\left(1-\frac{1}{\sqrt{2}}\right)^2}$$
$$=\sqrt{2\left(1-\frac{2}{\sqrt{2}}+\frac{1}{2}\right)}$$
$$=\sqrt{2\left(\frac{3}{2}-\sqrt{2}\right)}$$
$$=\sqrt{3-2\sqrt{2}}$$

(1 mark)

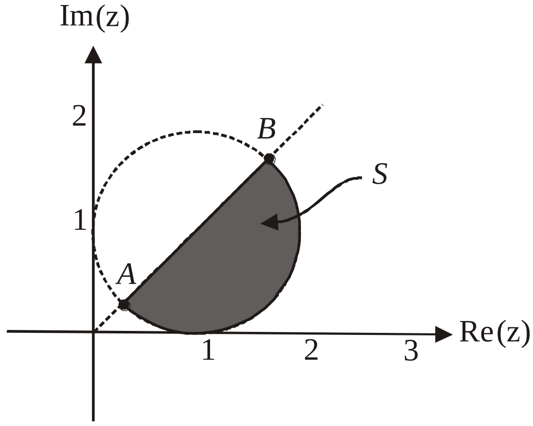
B is the point $\left(1+\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$

So, similarly, maximum value of $|z_2|$ is

$$\sqrt{\left(1+\frac{1}{\sqrt{2}}\right)^2+\left(1+\frac{1}{\sqrt{2}}\right)^2}$$
$$=\sqrt{3+2\sqrt{2}}$$

(1 mark)

<u>Method 2</u>



At point *A* on the diagram above $|z_2|$ is a minimum and at point *B*, $|z_2|$ is a maximum.
The centre of the circle is a distance of $\sqrt{2}$ units from the origin so **(1 mark)**

A is $\sqrt{2}-1$ units from $(0,0)$ and so the minimum value of $|z_2|$ is $\sqrt{2}-1$.
(1 mark)

Similarly, *B* is $\sqrt{2}+1$ units from $(0,0)$ and so the maximum value of $|z_2|$ is $\sqrt{2}+1$.
(1 mark)

(Note that the answers obtained using Method 1 and Method 2 are equivalent since

$\left(\sqrt{2}-1\right)^2 = 2-2\sqrt{2}+1 = 3-2\sqrt{2}$ and $\left(\sqrt{2}+1\right)^2 = 2+2\sqrt{2}+1 = 3+2\sqrt{2}$

So $\sqrt{2}-1 = \sqrt{3-2\sqrt{2}}$ and $\sqrt{2}+1 = \sqrt{3+2\sqrt{2}}$)

Total 12 marks

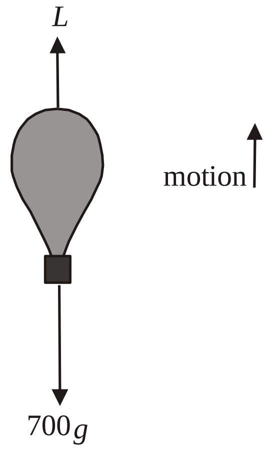
Question 3

a. $L - 700g = 700a$

Since $a = 0 \cdot 5$,

$L = 700 \times 0 \cdot 5 + 700g$

$= 7210$ newtons



(1 mark)

b. Since acceleration is constant, use

$s = ut + \frac{1}{2}at^2$

$50 = 0 + \frac{1}{2} \times 0 \cdot 5t^2$

$t^2 = 200$

$t = 10\sqrt{2}$ secs since $t \geq 0$

(1 mark)

c. Method 1

$v = u + at$

$= 0 + 0 \cdot 5 \times 10\sqrt{2}$

$= 5\sqrt{2}$ m/s

(1 mark)

Method 2

$v^2 = u^2 + 2as$

$v^2 = 0 + 2 \times 0 \cdot 5 \times 50$

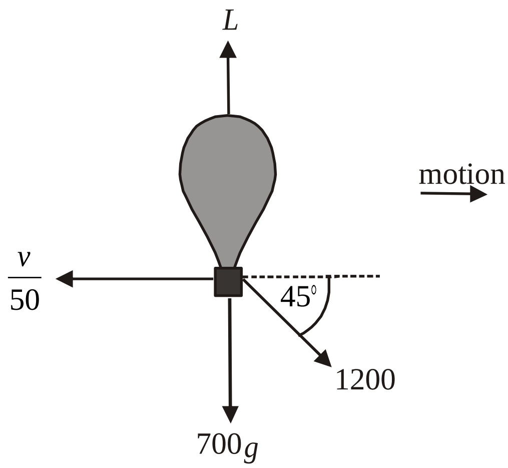
$= 50$

$v = \sqrt{50}$

$= 5\sqrt{2}$ m/s

(1 mark)

d.

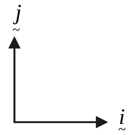


(1 mark)

e. The equation of motion is given by $\underset{\sim}{R} = m\underset{\sim}{a}$.

$$(1200\cos(45°) - \frac{v}{50})\underset{\sim}{i} + (L - 700g - 1200\sin(45°))\underset{\sim}{j} = 700a\underset{\sim}{i}$$ **(1 mark)**

Resolving horizontally,



$$1200\cos 45^\circ - \frac{v}{50} = 700a$$

$$700a = \frac{1200}{\sqrt{2}} - \frac{v}{50}$$

$$700a = \frac{60\,000 - \sqrt{2}v}{50\sqrt{2}}$$

$$a = \frac{60\,000 - \sqrt{2}v}{35000\sqrt{2}}$$

$$= \frac{60\,000\sqrt{2} - 2v}{70\,000}$$

$$a = \frac{30\,000\sqrt{2} - v}{35\,000}$$ **(1 mark)**

f. $a = \dfrac{30\,000\sqrt{2} - v}{35000}$ from part **e.**

$$v\frac{dv}{dx} = \frac{30\,000\sqrt{2} - v}{35000}$$

$$\frac{dv}{dx} = \frac{30\,000\sqrt{2} - v}{35\,000v}$$

$$\frac{dx}{dv} = \frac{35\,000v}{30\,000\sqrt{2} - v}$$

$$x = \int_0^5 \frac{35\,000v}{30\,000\sqrt{2} - v}\,dv$$ **(1 mark)**

$$= 10 \cdot 3128$$

Distance covered by balloon is 10.31m (correct to 2 decimal places). **(1 mark)**

g. The distance travelled by the balloon is given by

$$x = \int_0^{15\cdot 5685} \frac{35000\,v}{30000\,\sqrt{2} - v}\,dv$$

$= 100$ (to the nearest whole number) as required **(1 mark)**

h. From part **e.**,

$$a = \frac{30\,000\sqrt{2} - v}{35000}$$

$$\frac{dv}{dt} = \frac{30\,000\sqrt{2} - v}{35\,000}$$

$$\frac{dt}{dv} = \frac{35\,000}{30\,000\sqrt{2} - v}$$

$$t = 35\,000 \int_0^{15\cdot5685} \frac{1}{30\,000\sqrt{2} - v}\, dv$$

(1 mark) – correct integrand
(1 mark) – correct terminals

$$= 12 \cdot 85 \text{ secs (correct to 2 decimal places)}$$

(1 mark) – correct answer

Total 12 marks

Question 4

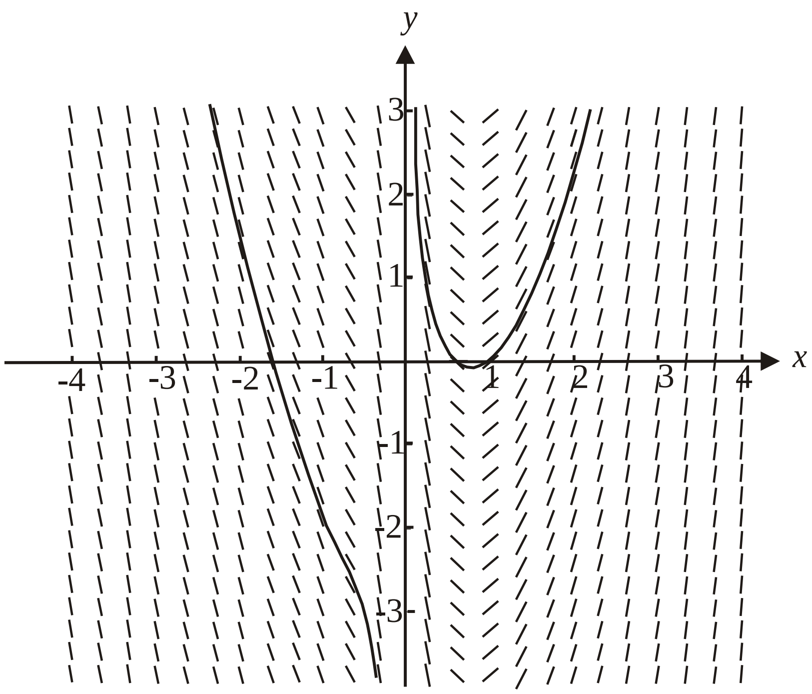
a. The left hand branch can be drawn using the direction field and passing through the point $(-1,-2)$.

$$\text{Now, } \frac{dy}{dx} = 2x - \frac{1}{x^2}$$
$$y = \int\left(2x - \frac{1}{x^2}\right)dx$$
$$= x^2 + \frac{1}{x} + c$$

For this particular solution, $x = -1$ and $y = -2$

$$-2 = 1 - 1 + c \text{ so } c = -2$$

So this solution is $y = x^2 + \frac{1}{x} - 2$ which passes through the point $(1,0)$ for example.
Use this point and the direction field to sketch the second branch.



(1 mark) – correct left branch
(1 mark) – correct value of *c* (correct right branch)
(1 mark) – correct right branch

b. From the formulae sheet,

If $\frac{dy}{dx}=f(x),\ x_0=a$ and $y_0=b,$ then $x_{n+1}=x_n+h$ and $y_{n+1}=y_n+hf(x_n)$

So $x_0=-1$ and $y_0=-2$

and $x_1=-1+0\cdot 25$ and $y_1=-2+0\cdot 25\times\left(2\times -1-\frac{1}{(-1)^2}\right)$

$=-0\cdot 75 \qquad =-2-0\cdot 75$

$=-2\cdot 75$

(1 mark)

So $x_2=-0\cdot 75+0\cdot 25 \qquad y_2=-2\cdot 75+0\cdot 25\times\left(2\times -0\cdot 75-\frac{1}{(-0\cdot 75)^2}\right)$

$=-0\cdot 5 \qquad =-3.57$ (to 2 dec. places)

(1 mark)

c. $\frac{dy}{dx}=2x-\frac{1}{x^2}$

$\frac{d^2y}{dx^2}=2+\frac{2}{x^3}$

$=\frac{2x^3+2}{x^3}$

$\frac{d^2y}{dx^2}=0$ when $2x^3+2=0$

$x^3=-1$

$x=-1$

A point of inflection occurs when $\frac{d^2y}{dx^2}=0$ AND $\frac{d^2y}{dx^2}$ changes sign on either side of $x=-1$.

When $x=-2, \frac{d^2y}{dx^2}=\frac{-16+2}{-8}=\frac{7}{4}>0$

When $x=-\frac{1}{2}, \frac{d^2y}{dx^2}=\frac{2\times\frac{-1}{8}+2}{-\frac{1}{8}}=-14<0$

So a point of inflection occurs at $x=-1$.

(1 mark) – correctly giving $\frac{d^2y}{dx^2}=0$

(1 mark) –for $x=-1$

(1 mark) showing the change of sign

d. From the slope field for $x > 0$, we see a family of curves with a minimum between $x = 0$ and $x = 1$.
That minimum occurs when

$$\frac{dy}{dx} = 2x - \frac{1}{x^2} = 0$$

$$2x = \frac{1}{x^2}$$

$$x^3 = \frac{1}{2}$$

$$x = 2^{-\frac{1}{3}}$$

(1 mark)

Now, $y = x^2 + \frac{1}{x} + c$

When $x = 2^{-\frac{1}{3}}$,

$$y = 2^{-\frac{2}{3}} + 2^{\frac{1}{3}} + c$$

$$y = 1 \cdot 88988... + c$$

The graph touches the *x*-axis when $y = 0$ so we require $c = -1 \cdot 8899$ (correct to 4 decimal places).

(1 mark)
Total 10 marks

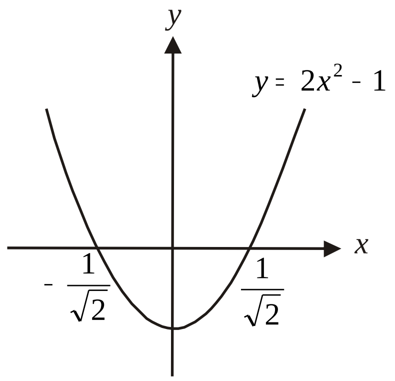
Question 5

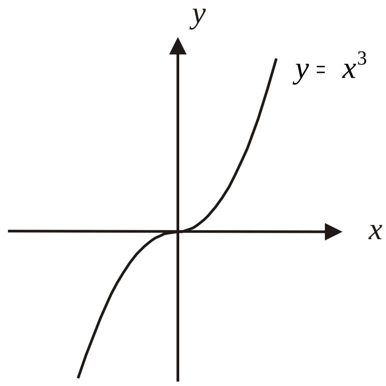
a. $f(x)=\sqrt{\dfrac{x^3}{2x^2-1}}$

For a maximal domain, $\dfrac{x^3}{2x^2-1}>0$.

$f(x)$ is defined when numerator and denominator are both positive or both negative.

Exclude where $2x^2-1=0$.





From the graphs, this occurs when $x\in\left(-\dfrac{1}{\sqrt{2}},0\right]\cup\left(\dfrac{1}{\sqrt{2}},\infty\right)$. **(1 mark)**

b. **i.** $V=\pi\displaystyle\int_1^{200} y^2\,dx$

$V=\pi\displaystyle\int_1^{200}\frac{x^3}{2x^2-1}\,dx$ **(1 mark)**

ii. Let $u=2x^2-1$ and so $x^2=\dfrac{u+1}{2}$ $x=1$ so $u=1$ and $x=200$ so $u=79\,999$

$\dfrac{du}{dx}=4x$

so $V=\pi\displaystyle\int_1^{79\,999}\frac{1}{u}\times\frac{1}{4}\frac{du}{dx}\times\frac{u+1}{2}\,dx$

$=\dfrac{\pi}{8}\displaystyle\int_1^{79\,999}\left(1+\frac{1}{u}\right)du$

(1 mark) – correct integrand
(1 mark) – correct terminals

iii. $V=\dfrac{\pi}{8}\displaystyle\int_1^{79\,999}\left(1+\frac{1}{u}\right)du$

$=\dfrac{\pi}{8}\left[u+\log_e|u|\right]_1^{79\,999}$ **(1 mark)**

$=\dfrac{\pi}{8}\{79\,999+\log_e(79\,999)-(1+\log_e(1))\}$

$=\dfrac{\pi}{8}(79\,998+\log_e(79\,999))\text{m}^3$ **(1 mark)**

c. height $=200-1=199\,\text{m}$ **(1 mark)**

d. At the base, $f(200)=10.0000625$, so the diameter is 20m (to the nearest whole metre)

(1 mark)

e. $f'(x)=\dfrac{x^2(2x^2-3)}{2(2x^2-1)^2\sqrt{\dfrac{x^3}{2x^2-1}}}$

For min/max $f'(x)=0$

$x^2(2x^2-3)=0$

$x=0,\ x=\pm\sqrt{\dfrac{3}{2}}$

The domain of f that describes a pylon is $x\in[1,200]$ so $x=\sqrt{\dfrac{3}{2}}$

(1 mark)

From the graph given, this is consistent with a minimum point.

Now $f\left(\sqrt{\dfrac{3}{2}}\right)=0\cdot 958415\ ....$

The minimum radius is therefore 0.96m (to 2 decimal places).

(1 mark)

f. For a point of inflection $f''(x)=0$

$-x(4x^4-20x^2-3)=0$

$x=0$ or $x=2\cdot 26842\ ...$

(1 mark)

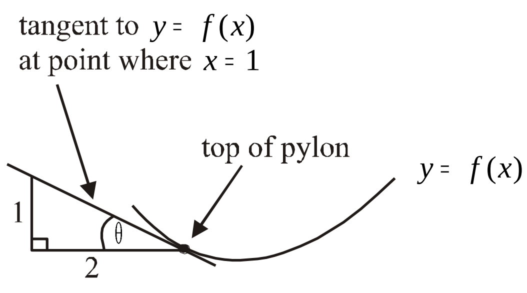
Now $x=0$ is outside the required domain of $x\in[1,200]$.
The top of the pylon coincides with $x=1$ so the lights are 1.27m (to 2 decimal places) below the top of the pylons.

(1 mark)

(Note that from the earlier given sketch it is evident that a point of inflection occurs around the point where $x=2\cdot 27$. If asked to verify that a point is a point of inflection it needs to be shown that $\dfrac{d^2y}{dx^2}=0$ AND that the sign of $\dfrac{d^2y}{dx^2}$ is different on either side of the point.)

g. At the top of the pylon $x=1$.

Now $f'(1)=-\dfrac{1}{2}$ **(1 mark)**



Angle required $=\theta$

$\theta=\tan^{-1}\left(\dfrac{1}{2}\right)$

$\theta=26\cdot 6^\circ$ (correct to 1 decimal place)

(1 mark)
Total 14 marks