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SPECIALIST MATHS
TRIAL EXAMINATION 1
SOLUTIONS
2009

Question 1

a. $f(x)=\cos^{-1}(3x-2)$

$$-1\le 3x-2\le 1$$
$$1\le 3x\le 3$$
$$\frac{1}{3}\le x\le 1$$
$$d_f=\left[\frac{1}{3},1\right]$$

(1 mark)

b. $f(x)=\cos^{-1}(3x-2)$

<u>Method 1</u> – short way – observation

$$f'(x)=\frac{-1}{\sqrt{1-(3x-2)^2}}\times 3$$
$$=\frac{-3}{\sqrt{1-(3x-2)^2}}$$

(1 mark)

<u>Method 2</u> – long way – chain rule

$y=\cos^{-1}(3x-2)$

Let $u=3x-2$

$\frac{du}{dx}=3$

$y=\cos^{-1}(u)$

$$\frac{dy}{du}=\frac{-1}{\sqrt{1-u^2}}$$
$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}\quad\text{(Chain rule)}$$
$$=\frac{-1}{\sqrt{1-u^2}}\cdot 3$$
$$=\frac{-3}{\sqrt{1-(3x-2)^2}}$$

(1 mark)

Question 2

$$6x^2 - 3y^3 + 2x^2y^2 = 5$$

$$12x - 9y^2\frac{dy}{dx} + 4xy^2 + 2x^2 \times 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(4x^2y - 9y^2) = -4xy^2 - 12x$$

$$\frac{dy}{dx} = \frac{-4xy^2 - 12x}{4x^2y - 9y^2}$$

(1 mark)

When $y = 1$

$$6x^2 - 3y^3 + 2x^2y^2 = 5$$

becomes $6x^2 - 3 + 2x^2 = 5$

$$8x^2 = 8$$

$$x^2 = 1$$

$$x = \pm 1$$

Reject $x = 1$ since the point is in the second quadrant.

(1 mark)

At $(-1,1)$, $\frac{dy}{dx} = \frac{4+12}{4-9}$

$$= -\frac{16}{5}$$

(1 mark)

Question 3

$$\frac{(x-1)^2}{4}-\frac{y^2}{25}=1$$

This is the equation of an hyperbola.
The centre is at $(1,0)$.

The equations of the asymptotes are $y=\pm\frac{5}{2}(x-1)$.

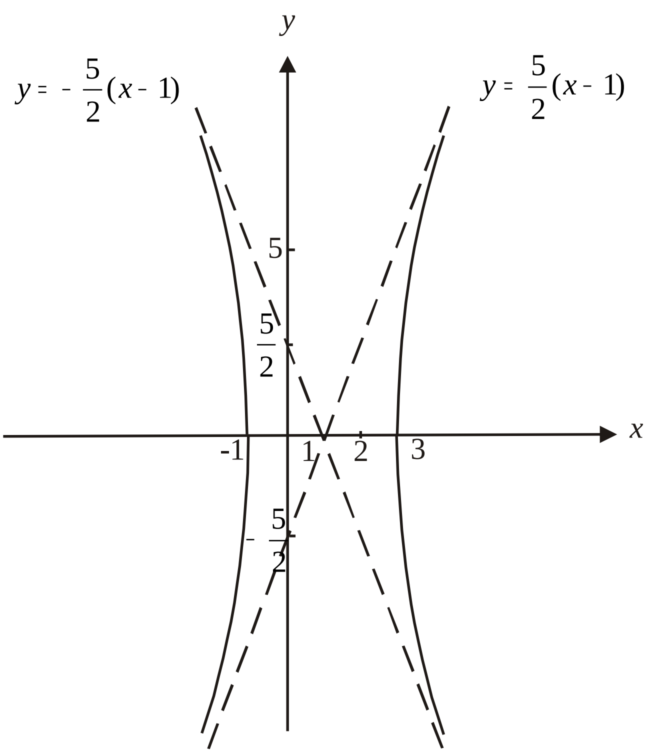
<u>x-intercepts</u> occur when $y=0$

$$\frac{(x-1)^2}{4}=1$$
$$(x-1)^2=4$$
$$x-1=\pm 2$$
$$x=-1,3$$

<u>y-intercepts</u> occur when $x=0$

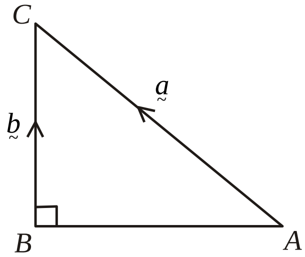
$$\frac{1}{4}-\frac{y^2}{25}=1$$
$$\frac{y^2}{25}=-\frac{3}{4}$$

There are no y-intercepts.



(1 mark) – correct centre
(1 mark) – correct asymptotes
(1 mark) – correct x-intercepts
(1 mark) – correctly drawn curves with no curling away from the asymptotes

Question 4



Let $\underset{\sim}{a} = \overrightarrow{AC}$

$= -4\underset{\sim}{i} + \underset{\sim}{j}$

and $\underset{\sim}{b} = \overrightarrow{BC}$

The component of $\overrightarrow{AC}$ perpendicular to $\overrightarrow{BC}$ is $\overrightarrow{AB}$.

So $\overrightarrow{AB} = \underset{\sim}{a} - (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$ **(1 mark)**

Since $\overrightarrow{BC}$, and hence $\underset{\sim}{b}$, is parallel to $-3\underset{\sim}{i} + 2\underset{\sim}{j}$ then

$$\hat{\underset{\sim}{b}} = \frac{1}{\sqrt{13}}(-3\underset{\sim}{i} + 2\underset{\sim}{j})$$ **(1 mark)**

$$\text{So } \overrightarrow{AB} = -4\underset{\sim}{i} + \underset{\sim}{j} - \frac{1}{\sqrt{13}}(12+2) \times \frac{1}{\sqrt{13}}(-3\underset{\sim}{i} + 2\underset{\sim}{j})$$

$$= -4\underset{\sim}{i} + \underset{\sim}{j} - \frac{14}{13}(-3\underset{\sim}{i} + 2\underset{\sim}{j})$$

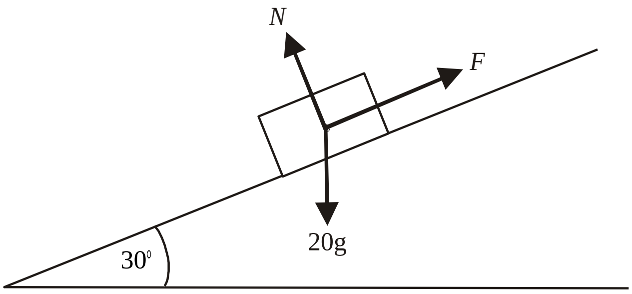
$$= -4\underset{\sim}{i} + \underset{\sim}{j} + \frac{42}{13}\underset{\sim}{i} - \frac{28}{13}\underset{\sim}{j}$$

$$= -\frac{10}{13}\underset{\sim}{i} - \frac{15}{13}\underset{\sim}{j}$$

(1 mark)

Question 5

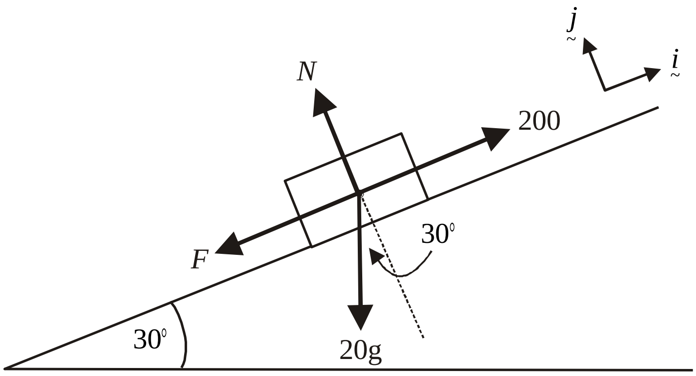
a.



(1 mark)

Note: because the inclination of the container is to slide down the slope the friction force is directed up the slope.

b.



Note that the friction is now directed down the slope.

The equation of motion is $\quad \underset{\sim}{R} = m\underset{\sim}{a}$

$$(200 - 20g\sin 30^\circ - F)\underset{\sim}{i} + (N - 20g\cos 30^\circ)\underset{\sim}{j} = 20a\underset{\sim}{i}$$

$200 - 10g - F = 20a \qquad N = 20g \times \dfrac{\sqrt{3}}{2}$ **(1 mark)**

$200 - 98 - \mu N = 2 \qquad\quad = 10\sqrt{3}g$

$102 - 10\sqrt{3}g\mu = 2$

$-10\sqrt{3}g\mu = -100$

$$\mu = \frac{10}{\sqrt{3}g}$$

(1 mark)

c. At the point where the container is about to move up the slope,

$P - 20g\sin 30^\circ - F = 0$ and $N = 10\sqrt{3}g$ **(1 mark)**

$$P = 10g + \mu \times 10\sqrt{3}g$$

$$= 10g + \frac{10}{\sqrt{3}g} \times 10\sqrt{3}g$$

$$= 198\text{newtons}$$

(1 mark)

Question 6

a.

$$\int \sin^2\left(\frac{3x}{2}\right)dx$$

$$=\frac{1}{2}\int(1-\cos(3x))dx$$

$$=\frac{1}{2}\left(x-\frac{1}{3}\sin(3x)\right)+c$$

$$=\frac{x}{2}-\frac{1}{6}\sin(3x) \qquad \text{(put } c=0 \text{ for "an" antiderivative)}$$

(1 mark)

b.

$$\int_0^1 \frac{5x}{\sqrt{1+x^2}}dx$$

let $u=1+x^2$

$$\frac{du}{dx}=2x$$

$x=1, u=2$

$x=0, u=1$

$$=5\int_1^2 \frac{1}{2}\frac{du}{dx}u^{-\frac{1}{2}}dx$$

$$=\frac{5}{2}\int_1^2 u^{-\frac{1}{2}}du$$

(1 mark) - correct integrand
(1 mark) - correct terminals

$$=\frac{5}{2}\left[2u^{\frac{1}{2}}\right]_1^2$$

$$=\frac{5}{2}\left(2\sqrt{2}-2\sqrt{1}\right)$$

$$=5\left(\sqrt{2}-1\right)$$

(1 mark)

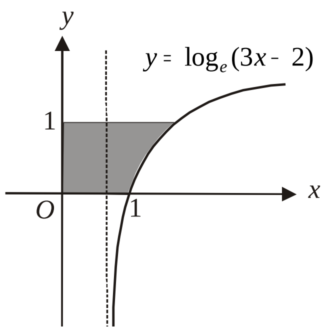
Question 7

$$V=\pi\int_0^1 x^2dy$$

$y=\log_e(3x-2)$

$e^y=3x-2$

$$x=\frac{e^y+2}{3}$$



$$=\pi\int_0^1\left(\frac{e^y+2}{3}\right)^2dy \qquad \textbf{(1 mark)}$$

$$=\frac{\pi}{9}\int_0^1(e^{2y}+4e^y+4)dy$$

$$=\frac{\pi}{9}\left[\frac{e^{2y}}{2}+4e^y+4y\right]_0^1 \qquad \textbf{(1 mark)}$$

$$=\frac{\pi}{9}\left\{\left(\frac{e^2}{2}+4e+4\right)-\left(\frac{e^0}{2}+4e^0+0\right)\right\}$$

$$=\frac{\pi}{9}\left(\frac{e^2}{2}+4e+4-\frac{1}{2}-4\right)$$

$$=\frac{\pi}{18}(e^2+8e-1)\text{ units}^3$$

(1 mark)

Question 8

b.

$$u=\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$$
$$=\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$$
$$=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$$
$$=1+i$$

(1 mark)

b.

$$\frac{u}{w}=\frac{\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)}{2\text{cis}\left(\frac{\pi}{6}\right)}$$
$$=\frac{1}{\sqrt{2}}\text{cis}\left(\frac{3\pi}{12}-\frac{2\pi}{12}\right)$$
$$=\frac{1}{\sqrt{2}}\text{cis}\left(\frac{\pi}{12}\right)$$

(1 mark)

c. From **b.**

$$\frac{u}{w}=\frac{1}{\sqrt{2}}\text{cis}\left(\frac{\pi}{12}\right)$$
$$=\frac{1}{\sqrt{2}}\left(\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)\right)$$

Also $\frac{u}{w}=\frac{1+i}{\sqrt{3}+i}$ since $w=2\text{cis}\left(\frac{\pi}{6}\right)$

$$=2\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$$
$$=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$
$$=\sqrt{3}+i$$

$$\frac{u}{w}=\frac{1+i}{\sqrt{3}+i}\times\frac{\sqrt{3}-i}{\sqrt{3}-i}$$
$$=\frac{\sqrt{3}-i+\sqrt{3}i+1}{4}$$
$$=\frac{\sqrt{3}+1}{4}+\frac{\left(\sqrt{3}-1\right)}{4}i$$ **(1 mark)**

So equating imaginary points,

$$\frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{3}-1}{4}$$
$$\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}$$

(1 mark)

d. From **a.**, $u = 1 + i$.

Since u is a root then $\bar{u}$ must also be a root (conjugate root theorem).

$$(z - 1 - i)(z - 1 + i) = z^2 - z + iz - z + 1 - i - iz + i + 1$$
$$= z^2 - 2z + 2 \text{ is a factor}$$

(1 mark)

$$\begin{array}{r|l} & \quad\quad\quad\quad z + 3 \\ z^2 - 2z + 2 & z^3 + z^2 - 4z + 6 \\ & \underline{z^3 - 2z^2 + 2z} \\ & \quad\quad 3z^2 - 6z + 6 \\ & \quad\quad \underline{3z^2 - 6z + 6} \end{array}$$

So $z^3 + z^2 - 4z + 6 = (z - 1 - i)(z - 1 + i)(z + 3) = 0$

The other roots are $1 - i$ and -3.

(1 mark)

Question 9

a. $a = v^2 + 3v + 2$

Because the particle moves "from rest" we have $t = 0, v = 0$.

So when $t = 0,\ a = 0 + 0 + 2$

$= 2\text{m/s}^2$

(1 mark)

b.

$$a = v^2 + 3v + 2,\ v \geq 0$$

$$\frac{dv}{dt} = v^2 + 3v + 2$$

$$\frac{dt}{dv} = \frac{1}{v^2 + 3v + 2}$$

$$t = \int \frac{1}{(v+2)(v+1)}\,dv$$

(1 mark)

Let $\dfrac{1}{(v+2)(v+1)} \equiv \dfrac{A}{v+2} + \dfrac{B}{v+1}$

$$\frac{1}{(v+2)(v+1)} \equiv \frac{A(v+1) + B(v+2)}{(v+2)(v+1)}$$

True iff $\quad 1 \equiv A(v+1) + B(v+2)$

Put $v = -1 \qquad 1 = B$

Put $v = -2 \qquad 1 = -A \qquad$ so $A = -1$

$$\frac{1}{(v+2)(v+1)} = \frac{-1}{v+2} + \frac{1}{v+1}$$

$$t = \int \left(\frac{-1}{v+2} + \frac{1}{v+1} \right) dv$$

$$t = -\log_e|v+2| + \log_e|v+1| + c$$

(1 mark)

When $t = 0,\ v = 0$ so

$$c = \log_e(2) - \log_e(1)$$

$$= \log_e(2)$$

$$t = -\log_e|v+2| + \log_e|v+1| + \log_e(2)$$

$$= \log_e \frac{2|v+1|}{|v+2|}$$

Since $v \geq 0$,

$$t = \log_e\left(\frac{2(v+1)}{v+2} \right)$$

(1 mark)

$$t = \log_e\left(\frac{2(v+1)}{v+2}\right)$$

$$e^t = \frac{2(v+1)}{v+2}$$

$$\frac{e^t}{2} = \frac{v+1}{v+2}$$

$$e^t(v+2) = 2(v+1)$$

$$v.e^t + 2e^t = 2v + 2$$

$$v.e^t - 2v = 2 - 2e^t$$

$$v(e^t - 2) = 2(1 - e^t)$$

$$v = \frac{2(1-e^t)}{e^t - 2}$$

(1 mark)

Question 10

a.

$$\cos(2x) = 2\cos^2(x) - 1 \qquad \text{(formula sheet)}$$

$$\cos\left(\frac{\pi}{4}\right) = 2\cos^2\left(\frac{\pi}{8}\right) - 1 \qquad \textbf{(1 mark)}$$

$$\frac{1}{\sqrt{2}} = 2\cos^2\left(\frac{\pi}{8}\right) - 1$$

$$\frac{\left(\frac{1}{\sqrt{2}} + 1\right)}{2} = \cos^2\left(\frac{\pi}{8}\right)$$

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{1+\sqrt{2}}{2\sqrt{2}} \qquad \textbf{(1 mark)}$$

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}+2}{4}$$

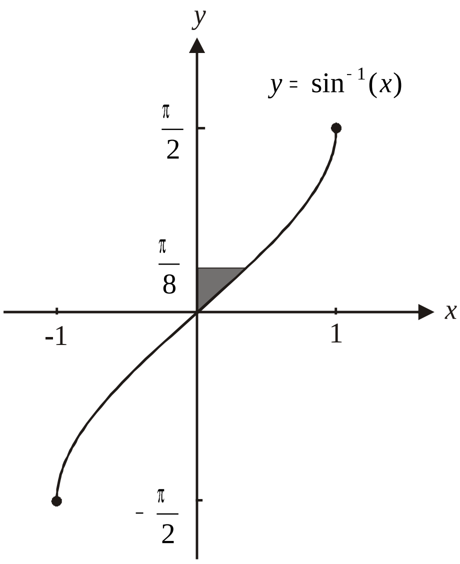
$$\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2} \qquad \text{since } \frac{\pi}{8} \text{ is a first quadrant angle}$$

as required

(1 mark)

b. Method 1

Do a quick sketch of $y = \sin^{-1}(x)$.



$$\text{Area required } = \int_0^{\frac{\pi}{8}} x\,dy \qquad \textbf{(1 mark)}$$

$$= \int_0^{\frac{\pi}{8}} \sin(y)\,dy \qquad \text{since } y = \sin^{-1}(x),\ \ x = \sin(y)$$

$$= -\Big[\cos(y)\Big]_0^{\frac{\pi}{8}}$$

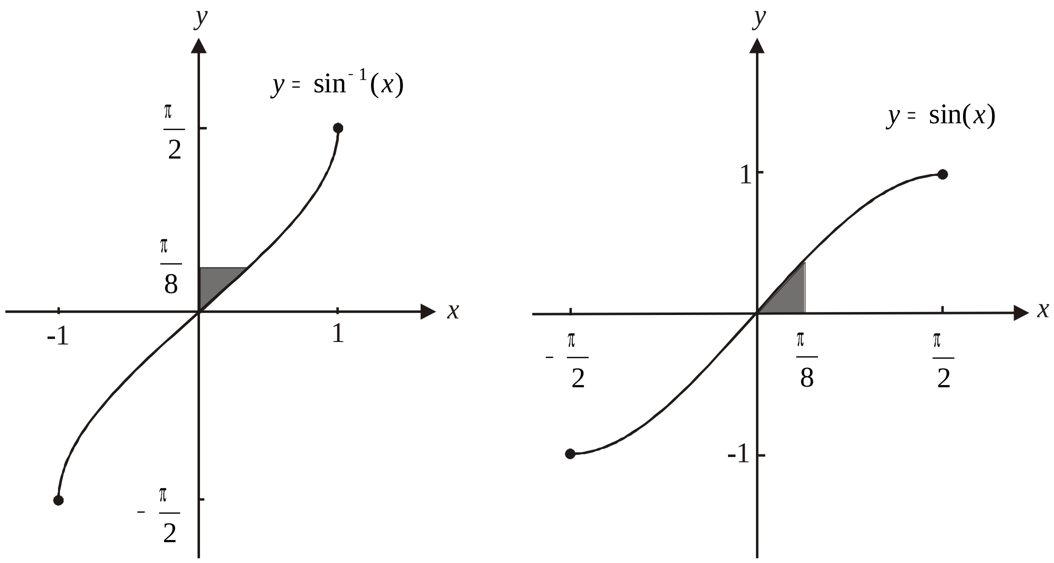
$$= -\left\{\cos\left(\frac{\pi}{8}\right) - \cos(0)\right\}$$

$$= \frac{-\sqrt{2+\sqrt{2}}}{2} + 1 \text{ square units.}$$

(1 mark)

Method 2

Do a quick sketch of $y = \sin^{-1}(x)$ and $y = \sin(x)$. Because these two graphs are reflections of each other in the line $y = x$, the two shaded areas are equivalent.



Area required $= \int_0^{\frac{\pi}{8}} \sin(x)\,dx$ **(1 mark)**

$$= -\left[\cos(x)\right]_0^{\frac{\pi}{8}}$$

$$= -\left\{\cos\left(\frac{\pi}{8}\right) - \cos(0)\right\}$$

$$= \frac{-\sqrt{2+\sqrt{2}}}{2} + 1 \text{ square units.}$$

(1 mark)

Total 40 marks