# SPECIALIST MATHEMATICS

## Units 3 & 4 – Written examination 2

## 2008 Trial Examination

## ***SOLUTIONS***

**SECTION 1: Multiple-choice questions (1 mark each)**

**Question 1**

*Answer:* D

*Explanation:*

The asymptotes of the function $y=\dfrac{2x^3-7x^2+5x}{x^2-3x}$ can be found by simplifying first, then dividing the polynomials.

$$y=\frac{x(2x^2-7x+5)}{x(x-3)}=\frac{2x^2-7x+5}{x-3}=2x-1+\frac{2}{x-3}$$

The function has a hole at $x=0$

**Question 2**

*Answer:* D

*Explanation:*

The equation of a hyperbola with its centre at $(-1,2)$ is $\dfrac{(x+1)^2}{a^2}-\dfrac{(y-2)^2}{b^2}=1$. The equation of its asymptote is $5x-3y+11=0$, or $y=\dfrac{5}{3}x+\dfrac{11}{3}$. It follows that $\dfrac{b}{a}=\dfrac{5}{3}$, therefore $a=\dfrac{3b}{5}$.

From the given alternatives, it has to be $a=6, b=10$

**Question 3**

*Answer:* B

*Explanation:*

For the graph of $y = \frac{-2}{x^2 - (m-1)x - m}, m > -1$ to be positive, it is necessary that

$x^2 - (m-1)x - m$ is negative.

$x^2 - (m-1)x - m < 0$

$x^2 - mx + x - m < 0$

$x(x - m) + (x - m) < 0$

$(x - m)(x + 1) < 0$

$x \in (-1, m)$

**Question 4**

*Answer:* A

*Explanation:*

Rotating a complex number anticlockwise for an angle of $\frac{\pi}{2}$ gives the same result as multiplying the number by *i*. Because $|w| = |3z|$, it must be $w = 3zi$.

Proof:

Let $z = |z| cis\varphi$. Then $w = 3|z| cis\left(\varphi + \frac{\pi}{2}\right)$

$$= 3|z|\left(\cos\left(\varphi + \frac{\pi}{2}\right) + i\sin\left(\varphi + \frac{\pi}{2}\right)\right) = 3|z|(-\sin\varphi + i\cos\varphi)$$

$$= 3|z|(i^2\sin\varphi + i\cos\varphi) = 3|z|(\cos\varphi + i\sin\varphi)i = 3zi$$

**Question 5**

*Answer:* D

*Explanation:*

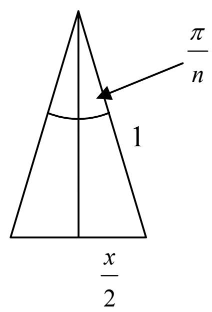
If $z = 2cis\left(-\frac{2\pi}{3}\right)$ then $z^{-2} = 2^{-2} cis\left(\frac{4\pi}{3}\right) = \frac{1}{4} cis\left(-\frac{2\pi}{3}\right)$

**Question 6**

*Answer:* E

*Explanation:*

If $z^n = 1$, then the complex roots $z = cis\frac{2k\pi}{n}$ are symetrically placed around the circle of radius 1. The regular polygon obtained from these vertices can be divided into *n* congruent isosceles triangles with side *x* and the top angle of $\frac{2\pi}{n}$.




From the diagram, $\sin\frac{\pi}{n} = \frac{\frac{x}{2}}{1}$, $x = 2\sin\frac{\pi}{n}$. The perimeter is $nx = 2n\sin\frac{\pi}{n}$

**Question 7**

*Answer:* C

*Explanation:*

For $y = x^2e^{-x}$, $\frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = e^{-x}(2x - x^2)$ and $\frac{d^2y}{dx^2} = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x)$

$= e^{-x}(x^2 - 4x + 2)$

$\frac{dy}{dx} > 0$ for $2x - x^2 > 0$ which is $x \in (0,2)$

$\frac{d^2y}{dx^2} < 0$ for $x^2 - 4x + 2 < 0$ which is $x \in (2-\sqrt{2}, 2+\sqrt{2})$. The intersection is $x \in (2-\sqrt{2}, 2)$

**Question 8**

*Answer:* A

*Explanation:*

By substituting points:

$a\sin^{-1}\left(-\frac{1}{2}\right)+b=-\frac{\pi}{12}$ and $a\sin^{-1}\left(\frac{1}{2}\right)+b=\frac{7\pi}{12}$

$-\frac{a\pi}{6}+b=-\frac{\pi}{12} \quad (1)$

$\frac{a\pi}{6}+b=\frac{7\pi}{12} \quad (2)$

Adding equations (1) and (2) gives $2b=\frac{6\pi}{12}$, $b=\frac{\pi}{4}$.

Substitute back into (2) $\frac{a\pi}{6}+\frac{\pi}{4}=\frac{7\pi}{12}$, $a=2$

**Question 9**

*Answer:* C

*Explanation:*

The scalar resolute of $\mathbf{a}=2\mathbf{i}+x\mathbf{j}-\mathbf{k}$ in the direction of $\mathbf{b}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ is

$$\begin{aligned}\mathbf{a}\cdot\hat{\mathbf{b}}&=\frac{1}{3}(2\mathbf{i}+x\mathbf{j}-\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\\&=\frac{1}{3}(2+2x-2)\end{aligned}$$

From $\frac{1}{3}(2+2x-2)=2$ the value of $x$ is 3.

**Question 10**

*Answer:* D

*Explanation:*

$$\cos\theta=\frac{\mathbf{m}\cdot\mathbf{n}}{|\mathbf{m}||\mathbf{n}|}=\frac{2-2-1}{\sqrt{6}\sqrt{6}}=-\frac{1}{6}$$

$$\cos 2\theta=2\cos^2\theta-1=\frac{2}{36}-1=-\frac{17}{18}$$

**Question 11**

*Answer:* B

*Explanation:*

Let $u = 2x - 3$. Then $\frac{du}{dx} = 2$ and $x = \frac{u+3}{2}$

$$\int x(2x-3)^3\,dx = \frac{1}{2}\int \frac{u+3}{2}u^3\,du = \frac{1}{4}\int \left(u^4 + 3u^3\right)du$$

$$= \frac{(2x-3)^5}{20} + \frac{3(2x-3)^4}{16}$$

**Question 12**

*Answer:* C

*Explanation:*

$$\frac{dx}{dt} = 5 \times 8 - \frac{10x}{100-2t}$$

$$= 40 - \frac{5x}{50-t}$$

**Question 13**

*Answer:* B

*Explanation:*

We are given $\frac{dV}{dt} = \frac{\pi}{3}$ and need to find $\frac{dS}{dt}$ when $r = 5$, where $S$ is the surface area of the balloon.

Using the chain rule $\frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dV}\frac{dV}{dt}$ and $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$, we have

$\frac{dV}{dr} = 4\pi r^2$ and $\frac{dS}{dr} = 8\pi r$

$\frac{dS}{dt} = 8\pi r \times \frac{1}{4\pi r^2} \times \frac{\pi}{3} = \frac{2\pi}{3r}$. For $r = 5$, $\frac{dS}{dt} = \frac{2\pi}{15}$

**Question 14**

*Answer:* A

*Explanation:*

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n, y_n), \; x_0 = 4, y_0 = 1, h = 0.04, \quad f(x,y) = y - x^2$$
$$x_1 = 4.04, y_1 = 1 + 0.04(1 - 4^2) = 0.4$$
$$x_2 = 4.08, y_2 = 0.4 + 0.04(0.4 - 4.04^2) = -0.23864$$
$$x_3 = 4.12, y_3 = -0.23864 + 0.04(-0.23864 - 4.08^2) = -0.9122$$

**Question 15**

*Answer:* A

*Explanation:*

$$\int \sin 2x \sec^2 2x \; dx = \int \frac{\sin 2x}{\cos^2 2x} dx$$

Using substitution $u = \cos 2x$, $\int \frac{\sin 2x}{\cos^2 2x} dx = -\frac{1}{2}\int \frac{1}{u^2} du = -\frac{1}{2} \times -\frac{1}{u}$

$$= \frac{1}{2\cos 2x} = \frac{1}{2}\sec 2x$$

**Question 16**

*Answer:* E

*Explanation:*

$$\int_0^2 \frac{x^2}{4+x^2} dx = \int_0^2 \left(1 - \frac{4}{4+x^2}\right) dx = \left[x - 2\tan^{-1}\frac{x}{2}\right]_0^2 = 2 - \frac{\pi}{2}$$

**Question 17**

*Answer:* E

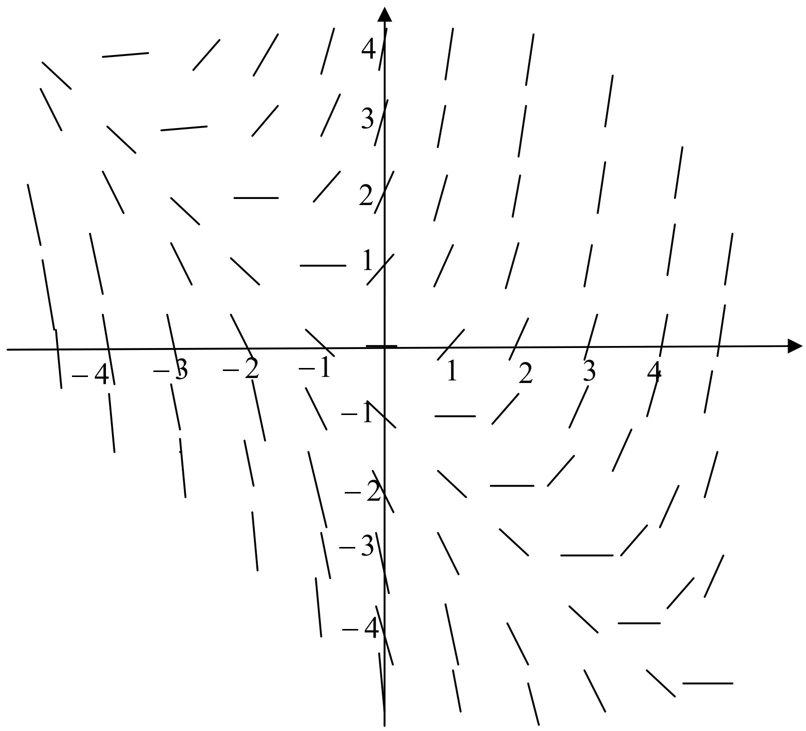
*Explanation:*

$$a = v\frac{dv}{dx} = (4 - x^2)(-2x)$$
$$= -8x + 2x^3$$

**Question 18**

*Answer:* D

*Explanation:*




From the direction field, $\frac{dy}{dx} = 0$ at $(1,-1),(2,-2),(3,-3)$..... which is correct only for the differential equation $\frac{dy}{dx} = x + y$.

**Question 19**

*Answer:* C

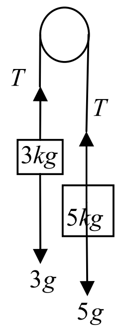
*Explanation:*

$$a = \frac{v_2 - v_1}{5} = \frac{-10}{5} = -2ms^{-2}$$
$$m = 1000kg, F = 1000 \times 2 = 2000N$$

**Question 20**

*Answer:* B

*Explanation:*




$$T - 3g = 3a$$
$$-T + 5g = 5a$$

By adding the equations $2g = 8a, a = \frac{g}{4}$. Substituting back into $T - 3g = 3a$ gives

$$T = 3g + 3a = \frac{15g}{4}$$

**Question 21**

*Answer:* A

*Explanation:*

$$a = -g$$
$$v = -gt + 29.4$$
$$s = -4.9t^2 + 29.4t + 20$$
$$s = 20 \text{ when } -4.9t^2 + 29.4t = 0$$
$$t = 0, t = 6$$

The ball is above the top of the building for 6 seconds.

**OR**

As the ball passes the top of the building on its way down its velocity will be equal and opposite to its initial velocity i.e. -29.4 $ms^{-1}$

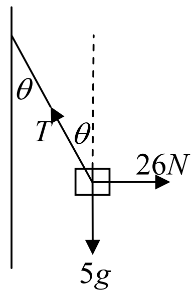
$$a = -g, u = 29.4, v = -29.4$$
$$v = u + at$$
$$t = \frac{v-u}{a} = \frac{-29.4 - 29.4}{-9.8} = 6$$

The ball is above the top of the building for 6 seconds

**Question 22**

*Answer:* C

*Explanation:*




$$T\cos\theta - 5g = 0 \Rightarrow T\cos\theta = 5g \qquad (1)$$
$$T\sin\theta - 26 = 0 \Rightarrow T\sin\theta = 26 \qquad (2)$$

Dividing equation (2) by equation (1)

$$\frac{\sin\theta}{\cos\theta} = \frac{26}{5g}, \theta = \tan^{-1}\left(\frac{26}{5g}\right) = 27.95^\circ.$$

## SECTION 2

### Question 1

**a.** If w is a solution of *p(z)* then $\overline{w}$ is also a solution. Therefore $z-w$ and $z-\overline{w}$ are factors of *p(z)*. Thus $(z-w)(z-\overline{w})$ is a quadratic factor of *p(z)*.

$(z-w)(z-\overline{w}) = z^2-(w+\overline{w})z+w\overline{w} = z^2-4z+8$

Therefore $z^2-4z+8$ is a quadratic factor of *p(z)*.

This could also be done by first solving $w+\overline{w}=4$ and $w\overline{w}=8$ simultaneously.

A1

**b.** $(z^2-5z-6)(z^2-4z+8) = z^4-9z^3+az^2-16z-48$ .:

M1

By equating quadratic terms, $8-6+20=a$ , $a=22$

A1

**c.** Solving $z^2-4z+8=0$ gives $z=\dfrac{4\pm\sqrt{-16}}{2}=2\pm 2i$ . Therefore, $w=2+2i$ .

This could also be done by solving $w+\overline{w}=4$ and $w\overline{w}=8$ simultaneously.

M1 + A1

**d.** Find, in polar form

**i.** $|w|=\sqrt{4+4}=2\sqrt{2}$ , $Arg w=\dfrac{\pi}{4}$ $\quad w=2\sqrt{2}cis\dfrac{\pi}{4}$

A1

**ii.** $\overline{w}^5=\left(2\sqrt{2}cis\left(-\dfrac{\pi}{4}\right)\right)^5=128\sqrt{2}cis\dfrac{3\pi}{4}$

A1

**iii.** $\sqrt[3]{w}=\left(2\sqrt{2}\right)^{\frac{1}{3}}cis\dfrac{\frac{\pi}{4}+2k\pi}{3}, k=-1,0,1 \quad =\sqrt{2}cis\left(-\dfrac{7\pi}{12}\right), \sqrt{2}cis\dfrac{\pi}{12}, \sqrt{2}cis\dfrac{3\pi}{4}$

M1+ A1

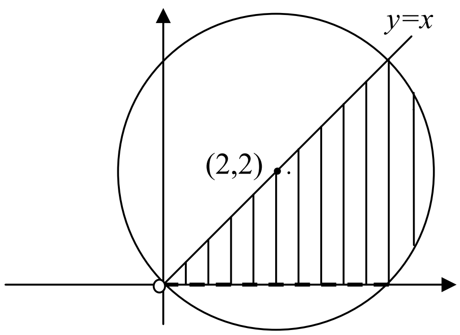
**e.** $S=\left\{z:\left|z-(2+2i)\right|\le 2\sqrt{2}\right\}$ Correct circle drawn

A1

$T=\left\{z:0<Arg z\le\dfrac{\pi}{4}\right\}$ Correct lines shown with a dotted line along the *x*-axis

A1




Correct region shaded with the origin excluded

A1

Total 12 marks

**Question 2**

**a.** Let $u = x^n$. Then $\dfrac{du}{dx} = nx^{n-1}$

M1

$$\int x^{n-1}e^{-x^n}\,dx = \frac{1}{n}\int e^{-u}\,du \quad = -\frac{1}{n}e^{-u} + c \quad = -\frac{1}{n}e^{-x^n} + c$$

M1 + A1

**b.** Using the result from **a.** or by substitution $\displaystyle\int_0^2 xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^2$

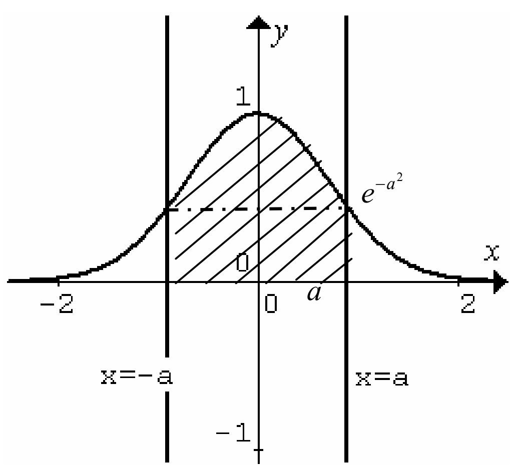
A1

$$= -\frac{1}{2}e^{-4} + \frac{1}{2} = \frac{1-e^{-4}}{2}$$

A1

**c.**

**i.**




Correct shading

A1

**ii.** The volume of this solid of revolution consists of a cylinder of radius $a$ and height $e^{-a^2}$ and a solid obtained by revolving part of the curve from $e^{-a^2}$ to 1 about the $y$-axis.

$$V = a^2\pi\, e^{-a^2} + \pi \int_{e^{-a^2}}^{1} x^2 dy .$$

M1

From $y = e^{-x^2}$, $x^2 = -\log_e y$. Therefore $V = a^2\pi\, e^{-a^2} - \pi \int_{e^{-a^2}}^{1} \log_e y dy$

A1

**d.** $V = a^2\pi\, e^{-a^2} - \pi\left[y\log_e y - y\right]_{e^{-a^2}}^{1}$

M1

$$= a^2\pi\, e^{-a^2} - \pi\left[-1 - e^{-a^2}\log_e e^{-a^2} + e^{-a^2}\right]$$

M1

$$= a^2\pi\, e^{-a^2} + \pi - a^2\pi e^{-a^2} - \pi e^{-a^2}$$
$$= \pi\left(1 - e^{-a^2}\right)$$

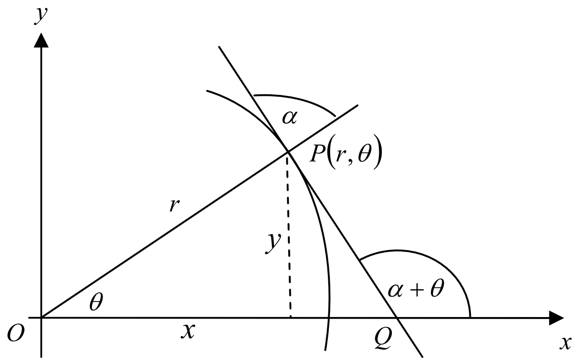
A1

**e.** When $a \to \infty$, $e^{-a^2} = \dfrac{1}{e^{a^2}} \to 0$ and so the volume approaches $\pi$.

A1

Total 12 marks

**Question 3**




**a.** From $\cos\theta = \frac{x}{r}, \sin\theta = \frac{y}{r}$, $x = r\cos\theta, y = r\sin\theta$ . Given that $r = Ae^{k\theta}$, it follows
$x = Ae^{k\theta}\cos\theta, \quad y = Ae^{k\theta}\sin\theta$.

M1 + A1

**b.** $\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx}$

$\frac{dy}{d\theta} = kAe^{k\theta}\sin\theta + Ae^{k\theta}\cos\theta$
$= Ae^{k\theta}(k\sin\theta + \cos\theta)$

$\frac{dx}{d\theta} = kAe^{k\theta}\cos\theta - Ae^{k\theta}\sin\theta$
$= Ae^{k\theta}(k\cos\theta - \sin\theta)$

M2

$$\frac{dy}{dx} = \frac{Ae^{k\theta}(k\sin\theta + \cos\theta)}{Ae^{k\theta}(k\cos\theta - \sin\theta)} = \frac{(k\sin\theta + \cos\theta)}{(k\cos\theta - \sin\theta)}.$$

Dividing each term by $\cos\theta$:

M1

$$\frac{dy}{dx} = \frac{\frac{k\sin\theta}{\cos\theta} + \frac{\cos\theta}{\cos\theta}}{\frac{k\cos\theta}{\cos\theta} - \frac{\sin\theta}{\cos\theta}} = \frac{k\tan\theta + 1}{k - \tan\theta}$$

A1

**c.** The angle that the tangent makes with the *x*- axis is the exterior angle of the triangle OPQ, which is equal to the sum of two interior angles $(\alpha + \theta)$. The gradient of the line is $\tan(\alpha + \theta)$.

A1

**d.** From **b.** and **c.**

$$\frac{k\tan\theta+1}{k-\tan\theta}=\tan(\alpha+\theta).$$

Using the addition formula for tan: $\dfrac{k\tan\theta+1}{k-\tan\theta}=\dfrac{\tan\alpha+\tan\theta}{1-\tan\alpha\tan\theta}$

M1

Cross multiplying gives:

$$k\tan\theta-k\tan\alpha\tan^2\theta+1-\tan\alpha\tan\theta=k\tan\alpha+k\tan\theta-\tan^2\theta-\tan\alpha\tan\theta$$
$$k\tan\alpha+k\tan\alpha\tan^2\theta=1+\tan^2\theta$$
$$k\tan\alpha(1+\tan^2\theta)=1+\tan^2\theta$$

M1

$$\tan\alpha=\frac{1}{k},\quad \alpha=\tan^{-1}\frac{1}{k}$$

A1

Total 10 marks

**Question 4**

**a.** The acceleration of the missile is $\mathbf{a_m}=-9.8\mathbf{j}$.

A1

The initial velocity of the missile is $\mathbf{v}_m=-30\mathbf{i}+3\mathbf{j}$. The velocity at time $t$ will be

$$\mathbf{v}_m=-9.8t\mathbf{j}+(-30\mathbf{i}+3\mathbf{j})=-30\mathbf{i}+(3-9.8t)\mathbf{j}$$

A1

As the initial position of the missile is $\mathbf{r}_m=1000\mathbf{i}+500\mathbf{j}$, the position at time $t$ is

$$\mathbf{r}_m(t)=-30t\mathbf{i}+\left(-4.9t^2+3t\right)\mathbf{j}+1000\mathbf{i}+500\mathbf{j}$$
$$\mathbf{r}_m(t)=(-30t+1000)\mathbf{i}+\left(-4.9t^2+3t+500\right)\mathbf{j}$$

A1

**b.** The acceleration of the anti-missile is $\mathbf{a}_a=-9.8\mathbf{j}$

The initial velocity is

$\mathbf{v}_a=100\cos\theta\,\mathbf{i}+100\sin\theta\,\mathbf{j}$ and the velocity at time $t$ is

A1

$$\mathbf{v}_a=100\cos\theta\,\mathbf{i}+(-9.8t+100\sin\theta)\mathbf{j}\ .$$

M1

The anti-missile is initially at the origin, so its position at time $t$ is obtained by integrating $\mathbf{v}_a=100\cos\theta\,\mathbf{i}+(-9.8t+100\sin\theta)\mathbf{j}$ with respect to $t$.

Therefore $\mathbf{r}_a(t)=100t\cos\theta\,\mathbf{i}+\left(-4.9t^2+100t\sin\theta\right)\mathbf{j}$.

A1

**c.** For an interception, the position vectors must be equal.

M1

$$100t\cos\theta\,\mathbf{i}+\left(-4.9t^2+100t\sin\theta\right)\mathbf{j}=\left(-30t+1000\right)\mathbf{i}+\left(-4.9t^2+3t+500\right)\mathbf{j}$$

Equating **i** and **j** components:

$$100t\cos\theta=-30t+1000 \quad (1)$$
$$-4.9t^2+100t\sin\theta=-4.9t^2+3t+500 \quad (2)$$

M1

From (1) $t=\dfrac{1000}{100\cos\theta+30}$

Substituting into (2) $\dfrac{100000\sin\theta}{100\cos\theta+30}=\dfrac{3000}{100\cos\theta+30}+500$

M2

A calculator can be used to solve this equation without simplification, or when simplified, it becomes:

$$50\sin\theta-25\cos\theta-9=0 \text{ and } \theta=35.8^\circ$$

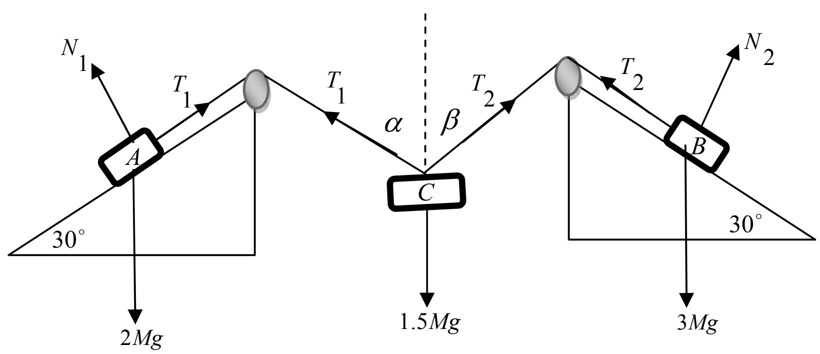
A1

**d.** Substituting $\theta=35.8^\circ$ into $t=\dfrac{1000}{100\cos\theta+30}$ gives 9 seconds.

A1

Total 12 marks

**Question 5**




**a.** Two different tensions and two different normal reaction forces shown

A1

Weights shown

A1

**b.**

$T_1-2Mg\sin 30^\circ=0$ $\qquad$ $T_2-3Mg\sin 30^\circ=0$

$T_1=Mg$ $\qquad$ $T_2=\dfrac{3Mg}{2}$

A2

**c.** Resolving forces acting on C horizontally:

$T_1 \sin\alpha = T_2 \sin\beta$

M1

$Mg\sin\alpha = \frac{3Mg}{2}\sin\beta$

$2\sin\alpha = 3\sin\beta$

A1

Vertically:

$T_1\cos\alpha + T_2\cos\beta - 1.5Mg = 0$

M1

$Mg\cos\alpha + \frac{3Mg}{2}\cos\beta - 1.5Mg = 0$

$2\cos\alpha + 3\cos\beta = 3$

A1

**d.** Equations $2\sin\alpha = 3\sin\beta$ and $2\cos\alpha + 3\cos\beta = 3$ need to be solved simultaneously.

From $\sin\alpha = \frac{3}{2}\sin\beta$, $\cos\alpha = \sqrt{1 - \frac{9}{4}\sin^2\beta} = \frac{1}{2}\sqrt{4 - 9\sin^2\beta}$

M1

Substituting $\cos\alpha$ into $2\cos\alpha + 3\cos\beta = 3$ gives $\sqrt{4 - 9\sin^2\beta} + 3\cos\beta - 3 = 0$

M1

This equation should be again solved on the calculator by plotting its graph. It gives a solution of $\beta = 38.94244^\circ = 39^\circ$ to the nearest degree.

A1

$\sin\alpha = \frac{3}{2}\sin 38.94244$, $\alpha = 70.52877^\circ = 71^\circ$ to the nearest degree.

A1

Total 12 marks