SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1

2008 Trial Examination

SOLUTIONS

Question 1

a. $\hat{\mathbf{a}} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k}) \qquad \hat{\mathbf{b}} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} - \mathbf{k})$

A2

b. The angle bisector can be found by adding $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

$\hat{\mathbf{a}} + \hat{\mathbf{b}} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k}) + \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} - \mathbf{k})$ M1

$= \frac{2}{\sqrt{3}}(\mathbf{i} - \mathbf{j})$ A1

The magnitude of $\hat{\mathbf{a}} + \hat{\mathbf{b}}$ is $\frac{2\sqrt{2}}{\sqrt{3}}$. The unit vector of the angle bisector is

$\frac{\sqrt{3}}{2\sqrt{2}} \times \frac{2}{\sqrt{3}}(\mathbf{i} - \mathbf{j}) = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$ A1

Question 2

a. $z = 1 \qquad |z| = 1, Arg z = 0$, thus $z = cis0$ A1

b. $\sqrt[6]{1} = cis\frac{0 + 2k\pi}{6} = cis\frac{k\pi}{3}$, $k = 0,1,2,3,4,5$ or any other 6 integers A1

c. $\sqrt[12]{1} = cis\frac{2k\pi}{12} = cis\frac{k\pi}{6}$ A1

$\sqrt[6]{1} \times \sqrt[12]{1} = cis\frac{k\pi}{3} \times cis\frac{k\pi}{6} = cis\left(\frac{k\pi}{3} + \frac{k\pi}{6}\right)$ M1

$= cis\frac{k\pi}{2} = cis\frac{2k\pi}{4}$ which is $\sqrt[4]{1}$. A1

Question 3

a.

$\frac{1}{4}\cos 3\theta = \frac{1}{4}\cos(2\theta + \theta)$

$= \frac{1}{4}(\cos 2\theta \cos\theta - \sin 2\theta \sin\theta)$ M1

$= \frac{1}{4}(2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta)$ M1

$= \frac{1}{4}(2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta)$

$= \cos^3\theta - \frac{3}{4}\cos\theta$ A1

b. For $x = \frac{2}{3}\cos\theta$ we have $27 \times \frac{8}{27}\cos^3\theta - 9 \times \frac{2}{3}\cos\theta = 1$

$8\cos^3\theta - 6\cos\theta = 1$

$\cos^3\theta - \frac{3}{4}\cos\theta = \frac{1}{8}$ M1

By substituting the result from **a.** $\frac{1}{4}\cos 3\theta = \frac{1}{8}$

$\cos 3\theta = \frac{1}{2}, \quad \theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$ M1

The solutions are $x = \frac{2}{3}\cos\frac{\pi}{9}, \quad \frac{2}{3}\cos\frac{5\pi}{9}, \quad \frac{2}{3}\cos\frac{7\pi}{9}$ A1

Question 4

From $x = y\ln(xy)$, $\ln(xy) = \frac{x}{y}$.

By differentiating implicitly

$1 = \frac{dy}{dx}\ln(xy) + \frac{y}{xy}\left(y + x\frac{dy}{dx}\right)$. Substitute $\ln(xy) = \frac{x}{y}$ M1

$$1 = \frac{dy}{dx}\frac{x}{y} + \frac{y}{x} + \frac{dy}{dx}$$

$\frac{dy}{dx}\left(\frac{x}{y} + 1\right) = 1 - \frac{y}{x}$ M1

$$\frac{dy}{dx}\left(\frac{x+y}{y}\right) = \frac{x-y}{x}$$

$\frac{dy}{dx} = \frac{y(x-y)}{x(x+y)}$ A1

Question 5

$v\frac{dv}{dx} = -g - 0.1v^2$, $\frac{dv}{dx} = -\left(\frac{g + 0.1v^2}{v}\right)$

M1

$x = -\int \frac{v}{g + 0.1v^2}dv$ By substituting $u = g + 0.1v^2$, $\frac{du}{dv} = 0.2v$

$$x = -\frac{1}{0.2}\int \frac{1}{u}dt = -5\ln|u| + c$$
$$= -5\ln\left(g + 0.1v^2\right) + c$$

M1

When $x = 0$, $v = 80$, $c = 5\ln(g + 640)$

$$x = 5\ln\left(\frac{g + 640}{g + 0.1v^2}\right)$$

A1

For the maximum height $v = 0$

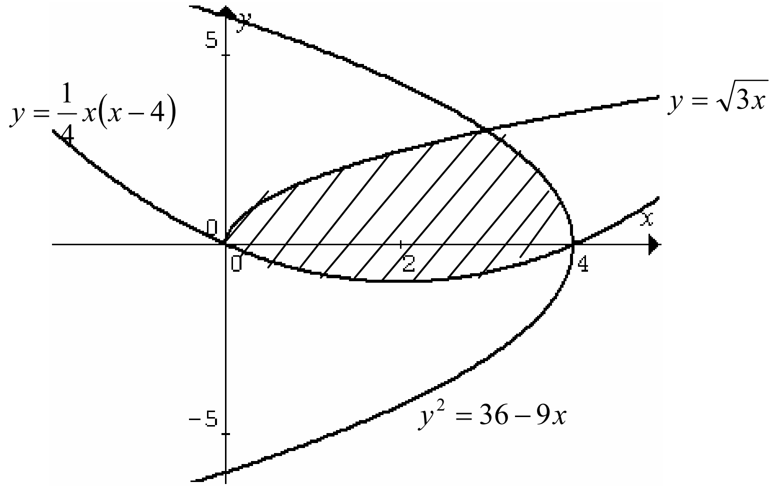
$$x = 5\ln\left(\frac{g + 640}{g}\right) = 5\ln\left(1 + \frac{640}{g}\right)$$

$a = 5$, $b = 640$

A1

Question 6

a.



A1

b. $A = \int_0^3 \sqrt{3x}\,dx + \int_3^4 \sqrt{36-9x}\,dx - \int_0^4 \frac{1}{4}x(x-4)\,dx$

correct limits
A1

or $A = \int_0^3 \left(\sqrt{3x} - \frac{1}{4}x(x-4)\right)dx + \int_3^4 \left(\sqrt{36-9x} - \frac{1}{4}x(x-4)\right)dx$ correct integrals

A1

c.

$$A = \left[\frac{2}{9}(3x)^{3/2}\right]_0^3 - \left[\frac{2}{27}(36-9x)^{3/2}\right]_3^4 - \left[\frac{x^3}{12} - \frac{x^2}{2}\right]_0^4$$ M1

$$= \frac{2}{9} \times 9^{3/2} + \frac{2}{27} \times 9^{3/2} - \frac{64}{12} + \frac{16}{2}$$

$$= \frac{32}{3}$$ sq units A1

Question 7

$\mathbf{r}(t) = (2 + 3\cos 2t)\mathbf{i} + (-3 + \sin 2t)\mathbf{j}$, $t \geq 0$.

a.

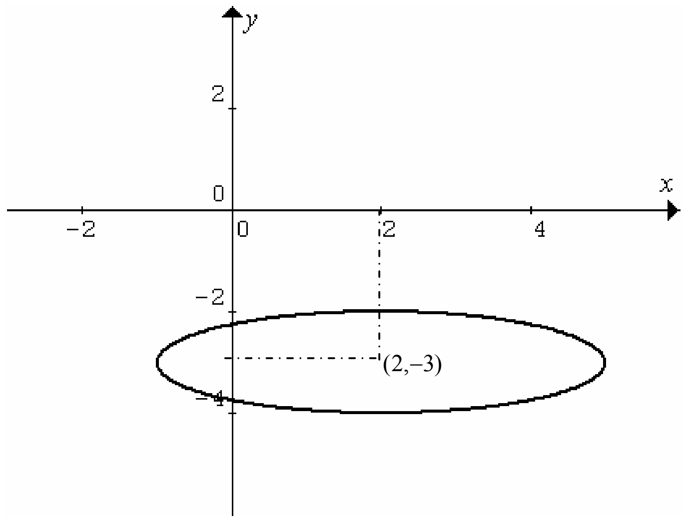
$x = 2 + 3\cos 2t$ $y = -3 + \sin 2t$

$\cos 2t = \frac{x-2}{3}$ $\sin 2t = y + 3$

M1

Substituting into $\sin^2 2t + \cos^2 2t = 1$ gives the ellipse $\frac{(x-2)^2}{9} + (y+3)^2 = 1$ A1

b.



A1

c. When $t = 0$ $x = 2 + 3\cos 0 = 5$ $y = -3 + \sin 0 = -3$. Initial position is $(5, -3)$.

The particle is moving anticlockwise as for $t = \frac{\pi}{4}$ its position is $(2, -2)$.

A1

It returns to its initial position after $t = \pi$.

A1

Question 8

$$\int_{-2}^{a} \frac{3}{(x+2)^2 + 16} dx = \frac{3\pi}{16}$$

$$\frac{3}{4}\left[\tan^{-1}\left(\frac{x+2}{4}\right)\right]_{-2}^{a} = \frac{3\pi}{16}$$

M1

$$\frac{3}{4}\tan^{-1}\left(\frac{a+2}{4}\right) = \frac{3\pi}{16}$$

M1

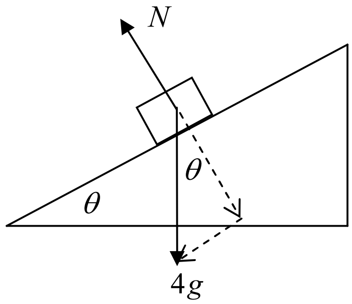
$$\frac{a+2}{4} = 1$$

$$a = 2$$

A1

Question 9

a.

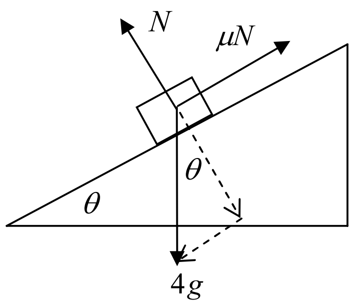


$4g\sin\theta = 4a$

$a = g\sin\theta$

A1

b.



$N = 4g\cos\theta, \qquad a = \frac{1}{4}g\sin\theta$

$4g\sin\theta - \mu \times 4g\cos\theta = 4 \times \frac{1}{4}g\sin\theta$

$3\sin\theta = 4\mu\cos\theta$

$\mu = \frac{3\sin\theta}{4\cos\theta} = \frac{3}{4}\tan\theta$

M2

A1