Neap:

Trial Examination 2007

VCE Specialist Mathematics Units 3 & 4

Written Examination 2

Suggested Solutions

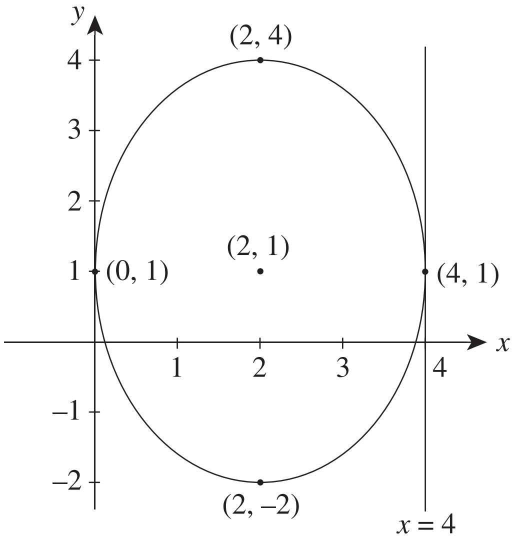
SECTION 1

1	A	B	C	**D**	E	12	A	B	**C**	D	E
2	A	B	C	D	**E**	13	A	B	C	**D**	E
3	**A**	B	C	D	E	14	**A**	B	C	D	E
4	A	**B**	C	D	E	15	**A**	B	C	D	E
5	A	**B**	C	D	E	16	A	B	**C**	D	E
6	A	B	C	**D**	E	17	A	**B**	C	D	E
7	A	B	**C**	D	E	18	A	**B**	C	D	E
8	A	B	**C**	D	E	19	A	B	C	D	**E**
9	A	B	**C**	D	E	20	**A**	B	C	D	E
10	A	B	C	D	**E**	21	A	**B**	C	D	E
11	A	B	C	**D**	E	22	A	B	C	**D**	E

SECTION 1

Question 1

The ellipse can be sketched as follows:



With reference to the general equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, the semi-minor axis length a is 2, the semi-major axis length b is 3 and the centre (h, k) is $(2, 1)$. Hence the equation is $\frac{(x-2)^2}{4} + \frac{(y-1)^2}{9} = 1$.

Answer D

Question 2

For $f(x) = \frac{a + bx^2}{x^2}$, $f(-x) = f(x)$ ('even' function), hence the graph of $y = f(x)$ is symmetrical about the y-axis. By division, $f(x) = \frac{a}{x^2} + b$. Since $f(x)$ is undefined when $x = 0$, the line $x = 0$ is a vertical asymptote for the graph. As the magnitude of x approaches infinity, $\frac{a}{x^2}$ approaches zero and $f(x)$ approaches b. Hence the line $y = b$ is a horizontal asymptote for the graph.

Answer E

Question 3

$z = \frac{4}{\sqrt{3}i - 1}$

$= \frac{4}{-1 + \sqrt{3}i} \times \frac{-1 - \sqrt{3}i}{-1 - \sqrt{3}i}$

$= \frac{4(-1 - \sqrt{3}i)}{1 + 3}$

$= -1 - \sqrt{3}i$

The complex conjugate of z, $\bar{z} = -1 + \sqrt{3}i$.

The modulus of $\bar{z} = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2 = a$.

The argument of $\bar{z} = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$ in the second quadrant $= \frac{2\pi}{3} = b$.

Answer A

Question 4

To find the three cube roots of $27i$, let $z^3 = 27i = 0 + 27i$.

Thus $z^3 = 27\text{cis}\left(\frac{\pi}{2} + 2k\pi\right)$ for k, an integer.

By de Moivre's theorem, $z = 3\text{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$ for k, an integer.

The first cube root ($k = 0$) is $3\text{cis}\left(\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$.

The other values of z may be obtained by progressively adding $\frac{2\pi}{3}$ or letting $k = 1, 2$.

$3\text{cis}\left(\frac{5\pi}{6}\right) = 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$ and $3\text{cis}\left(\frac{3\pi}{2}\right) = 3(-i)$

The truth of each statement can now be determined.

A. On an Argand diagram, z_1, z_2 and z_3 are separated by an angle of $\frac{2\pi}{3}$ radians. True: the three cube roots must be equally spaced on a circle.

B. $z_1 = 3i$. False: one of the roots is $-3i$ but none is $3i$. Also, $(3i)^3 = -27i$. Note that the complex conjugate root theorem applies to polynomial equations with real coefficients.

C. $|z_2| = 3$. True: all roots have modulus 3 (they are spaced on a circle of radius 3).

D. $(z_3)^{-1} = \frac{1}{3}\text{cis}\left(-\frac{5\pi}{6}\right)$. Could be true if $z_3 = 3\text{cis}\left(\frac{5\pi}{6}\right)$, then by de Moivre's theorem

$(z_3)^{-1} = \frac{1}{3}\text{cis}\left(\frac{-5\pi}{6}\right)$.

E. $z_1 + z_2 + z_3 = 0$. True: $z_1 + z_2 + z_3 = 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) + 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) + (-3i) = 0$.

Answer B

Question 5

If $\text{Im}(z^2) \geq 4$, we can write $\text{Im}\{(x + iy)^2\} \geq 4$.

$\text{Im}\{x^2 + 2xyi + (iy)^2\} \geq 4$

$\text{Im}\{x^2 - y^2 + 2xyi\} \geq 4$

$2xy \geq 4$

$y \geq \dfrac{2}{x}$

This region is 'above' each branch of the hyperbola $y = \dfrac{2}{x}$.

The region defined by $-\dfrac{3\pi}{4} \leq \text{Arg}(z) \leq \dfrac{\pi}{4}$ is one half of the complex plane (with the origin excluded) bounded by the line $\text{Re}(z) = \text{Im}(z)$.

The intersection of the two regions is illustrated by alternative **B**.

Answer B

Question 6

The alternatives show a dilation from the y-axis, a translation parallel to the x-axis and possibly a reflection in the x-axis. Alternative **E**. can be eliminated first because its period of $\dfrac{2\pi}{b}$ does not match the graph.

Consider the stationary points and vertical asymptotes of $y = \pm\sec(a(x - b))$ and $y = \pm\text{cosec}(a(x - b))$.

Curve equation	**Stationary points**	**Vertical asymptotes (x values)**
$y = \sec(a(x - b))$	$(b, 1), \left(\dfrac{\pi}{a} + b, -1\right), \left(\dfrac{2\pi}{a} + b, 1\right), \ldots$	$\dfrac{\pi}{2a} + b, \dfrac{3\pi}{2a} + b, \ldots$
$y = -\sec(a(x - b))$	$(b, -1), \left(\dfrac{\pi}{a} + b, 1\right), \left(\dfrac{2\pi}{a} + b, -1\right), \ldots$	$\dfrac{\pi}{2a} + b, \dfrac{3\pi}{2a} + b, \ldots$
$y = \text{cosec}(a(x - b))$	$\left(\dfrac{\pi}{2a} + b, 1\right), \left(\dfrac{3\pi}{2a} + b, -1\right), \left(\dfrac{5\pi}{2a} + b, 1\right), \ldots$	$b, \dfrac{\pi}{a} + b, \dfrac{2\pi}{a} + b, \ldots$
$y = -\text{cosec}(a(x - b))$	$\left(\dfrac{\pi}{2a} + b, -1\right), \left(\dfrac{3\pi}{2a} + b, 1\right), \left(\dfrac{5\pi}{2a} + b, -1\right), \ldots$	$b, \dfrac{\pi}{a} + b, \dfrac{2\pi}{a} + b, \ldots$

Comparing these stationary points and asymptotes with those of the graph shows that the appropriate equation is $y = -\text{cosec}(a(x - b)) = \text{cosec}(a(b - x))$ since $\sin(-A) = -\sin(A)$.

Answer D

Question 7

The usual domain of $y = \sin(x)$ so that it is a one-to-one function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The range of $\sin(x)$ is $[-1, 1]$.

With $f(x) = \sin(2x)$ the period is halved, so the domain of $f(x)$ could be $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, with the same range as for $y = \sin(x)$. The domain and range of $f^{-1}(x)$ would then be $[-1, 1]$ and $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. Although no alternative has $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ as its range, the range of alternative **C**, $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$, is suitable because it is one period advanced from $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. Hence the original domain restriction of $f(x)$ corresponding to this will result in a one-to-one function, as required.

Answer C

Question 8

The vector resolute of $2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$ in the direction of $3\underset{\sim}{i} - 2\underset{\sim}{k}$ is given by

$$\frac{2 \times 3 - 3 \times 0 - 1 \times (-2)}{3^2 + (-2)^2}$$

$$= \frac{8}{13}(3\underset{\sim}{i} - 2\underset{\sim}{k})$$

Answer C

Question 9

The gradient of the tangent to the curve $y = \cos^{-1}\left(\frac{x}{2}\right)$ is given by $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}} \times \frac{1}{2} \text{ applying the chain rule}$$

$$= -\frac{1}{\sqrt{4 - x^2}}$$

When $x = 1$, the gradient is $-\frac{1}{\sqrt{3}}$ and $y = \cos^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{3}$.

The tangent equation is $y - \frac{\pi}{3} = -\frac{1}{\sqrt{3}}(x - 1)$

Answer C

Question 10

In $\int_0^{\frac{\pi}{4}} \sqrt{1-\sin(2x)}(1-2\cos^2(x))\ dx$, replace $1-\sin(2x)$ with u.

$$\begin{aligned}\frac{du}{dx} &= -2\cos(2x)\\ &= -2(2\cos^2(x)-1)\\ &= 2(1-2\cos^2(x))\end{aligned}$$

When $x = \frac{\pi}{4}$, $u = 1-1 = 0$.

When $x = 0$, $u = 1-0 = 1$.

The integral becomes $\frac{1}{2}\int_1^0 u^{\frac{1}{2}}\ du$.

Answer E

Question 11

$$\int \sin^3(6x)dx = \int \sin^2(6x)\sin(6x)dx = \int (1-\cos^2(6x))\sin(6x)dx$$

Using the substitution $u = \cos(6x)$ so that $\frac{du}{dx} = -6\sin(6x)$, $x = \frac{\pi}{6}$ gives $u = -1$ and $x = 0$ gives $u = 1$.

$$\int_0^{\frac{\pi}{6}} (1-\cos^2(6x))\sin(6x)dx$$

$$= -\frac{1}{6}\int_0^{\frac{\pi}{6}} (1-\cos^2(6x))(-6\sin(6x))dx$$

$$= -\frac{1}{6}\int_1^{-1} (1-u^2)du$$

$$= \frac{1}{6}\int_1^{-1} (u^2-1)du$$

Answer D

Question 12

The initial point is (0, 1), i.e. $a = 0$ and $b = 1$.
Euler's method using a step size of 0.1 gives

$a = 0 \qquad f(a) = f(0) = 1$

$x_1 = 0.1 \qquad f(x_1) = f(0.1) = \cos^3(0.1)$

Using $y_{n+1} = y_n + hf(x_n)$

$$\begin{aligned} y_1 &= b + hf(a) \\ &= 1 + 0.1\cos^3(0) \\ &= 1.1 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + hf(x_1) \\ &= 1.1 + 0.1\cos^3(0.1) \end{aligned}$$

Answer C

Question 13

$$10a = -10g - \frac{v^2}{10}$$

$$\frac{dv}{dt} = -g - \frac{v^2}{100}$$

$$\frac{dv}{dt} = -\frac{100g + v^2}{100}$$

Answer D

Question 14

We require $\overrightarrow{AB} = \overrightarrow{DC}$ so that one pair of opposite sides are equal and parallel.

Also, we require that $|\overrightarrow{AB}| = |\overrightarrow{AD}|$. Hence we have adjacent sides of equal length.

Answer A

Question 15

$$\begin{aligned} \underset{\sim}{r}(t) &= \int (e^{-t}\underset{\sim}{i} - 3e^{-3t}\underset{\sim}{j})\,dt \\ &= -e^{-t}\underset{\sim}{i} + e^{-3t}\underset{\sim}{j} + \underset{\sim}{d} \end{aligned}$$

When $t = 0$, $\underset{\sim}{r} = \underset{\sim}{0}$.

$\underset{\sim}{d} = \underset{\sim}{i} - \underset{\sim}{j}$

$\underset{\sim}{r}(t) = (1 - e^{-t})\underset{\sim}{i} + (e^{-3t} - 1)\underset{\sim}{j}$

Answer A

Question 16

$$\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$$

$$\begin{aligned}\cos(\theta) &= \frac{\underset{\sim}{a}.\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\\ &= \frac{(\underset{\sim}{i} - 2\underset{\sim}{k}).(2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})}{\sqrt{5} \times 3}\\ &= \frac{2-4}{3\sqrt{5}}\\ &= \frac{-2}{3\sqrt{5}}\end{aligned}$$

Answer C

Question 17

Distance travelled is the area under the graph.

$600 = \frac{1}{2}(36 + t) \times 24$

$50 = 36 + t$

$t = 14$

The particle travels at 24 m/s for 14 seconds. Hence the particle travels at a constant velocity for $24 \times 14 = 336$ metres.

Answer B

Question 18

$$\begin{aligned}\text{initial momentum} &= 8 \times 7\\ &= 56 \text{ (kg m/s)}\\ \text{final momentum} &= 8 \times 1\\ &= 8 \text{ (kg m/s)}\\ \text{change in momentum} &= \text{final momentum} - \text{initial momentum}\\ &= 8 - 56\\ &= -48 \text{ (kg m/s)}\end{aligned}$$

Answer B

Question 19

$\sin(\theta) = \frac{5}{13}$ and so $\cos(\theta) = \frac{12}{13}$

$$\begin{aligned}mg\sin(\theta) - \mu N &= ma \text{ (parallel to the plane)}\\ N &= mg\cos(\theta) \text{ (perpendicular to the plane)}\\ \frac{5mg}{13} - \frac{12\mu mg}{13} &= ma\\ a &= \frac{g}{13}(5 - 12\mu)\end{aligned}$$

Answer E

Question 20

$\underset{\sim}{r} = -\sqrt{3}\cos(t)\underset{\sim}{i} + \sin(t)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}} = \sqrt{3}\sin(t)\underset{\sim}{i} + \cos(t)\underset{\sim}{j}$

$\ddot{\underset{\sim}{r}} = \sqrt{3}\cos(t)\underset{\sim}{i} - \sin(t)\underset{\sim}{j}$

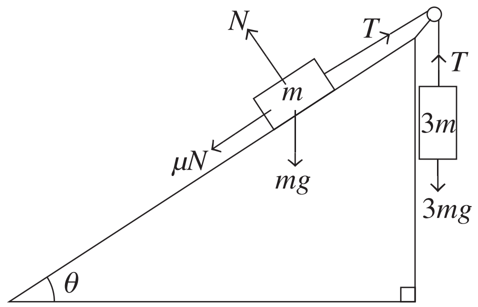
From Newton's 2nd law of motion, $\underset{\sim}{F} = m\ddot{\underset{\sim}{r}}$.

$\underset{\sim}{F} = 2\sqrt{3}\cos(t)\underset{\sim}{i} - 2\sin(t)\underset{\sim}{j}$

$|\underset{\sim}{F}| = \sqrt{12\cos^2(t) + 4\sin^2(t)}$

Answer A

Question 21



$3m$ particle: $T = 3mg$ (1)

m particle: $T - \mu N - mg\sin(\theta) = 0$ (2)

$N = mg\cos(\theta)$ (3)

Substituting (1) and (3) into (2) gives

$3mg - \mu mg\cos(\theta) - mg\sin(\theta) = 0$

$3 - \mu\cos(\theta) - \sin(\theta) = 0$

$\mu\cos(\theta) = 3 - \sin(\theta)$

$\mu = \dfrac{3 - \sin(\theta)}{\cos(\theta)}$

Answer B

Question 22

Given $v^2 = 12 - 4x^2$

$\frac{1}{2}v^2 = 6 - 2x^2$

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Hence $a = -4x$.

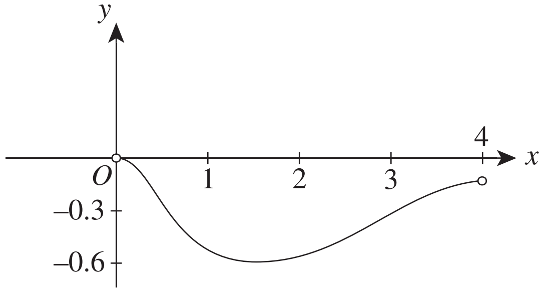
We are looking for a linear graph that passes through the origin with a negative gradient. Graph **D** is correct as the motion could also be along the negative x-axis.

Answer D

SECTION 2

Question 1

a.



Suitable y-axis scale. A1
Correct graph shape and location of endpoints. A1

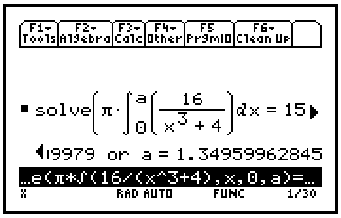
b. The maximum rate of decrease is at the local minimum of the graph of $y = f'(x)$.
The coordinates of the local minimum are (1.47, –0.67). A1
Hence the corresponding point on the graph of f has coordinates (1.47, 1.49). A1

c. The maximum rate of decrease is 0.67. A1

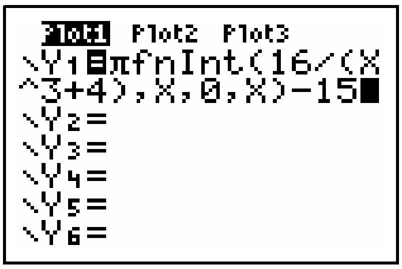
d. $$\pi\int_0^a\left(\frac{4}{\sqrt{x^3+4}}\right)^2 dx = 15$$ A1

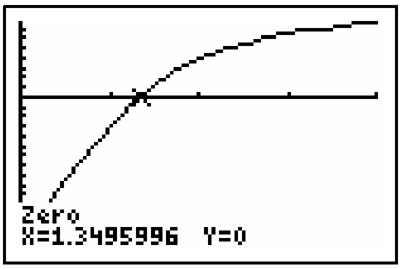
Solving this equation for a gives $a = 1.35$. A1

Using CAS:



Using a graphics calculator:





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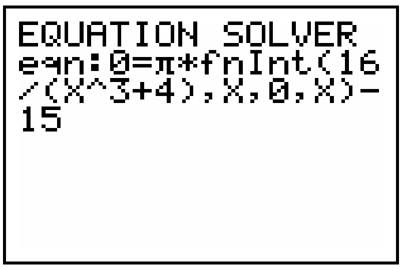
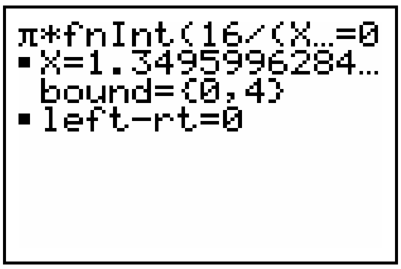




Question 2

a. $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \dfrac{1}{\sqrt{3}}(\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k})$ A1

b. **i.** $\underset{\sim}{b}.\underset{\sim}{c} = (2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}).(4\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k})$

$= 8 - 3 - 5$ A1

$= 0$

As $\underset{\sim}{b}.\underset{\sim}{c} = 0$ and $|\underset{\sim}{b}|,\ |\underset{\sim}{c}| \neq 0,$ then $\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{c}$.

ii. Given $\hat{\underset{\sim}{n}} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$.

$\underset{\sim}{b}.\hat{\underset{\sim}{n}} = 0,\ 2x + 3y - z = 0$ (1)

$\underset{\sim}{c}.\hat{\underset{\sim}{n}} = 0,\ 4x - y + 5z = 0$ (2)

$|\hat{\underset{\sim}{n}}| = 1,\ x^2 + y^2 + z^2 = 1$ (3) A1 for (1), (2) and (3)

For example, $2 \times (1) - (2)$ gives $y = z$.

Substituting $y = z$ into (1) for example gives $x = -y$. M1

Substituting $y = z$ and $x = -y$ into (3) gives $x = -\dfrac{1}{\sqrt{3}},\ y = \dfrac{1}{\sqrt{3}}$ and $z = \dfrac{1}{\sqrt{3}},$ since $x < 0$.

Hence $\hat{\underset{\sim}{n}} = -\dfrac{1}{\sqrt{3}}(\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k})$. A1

$\underset{\sim}{a} = -\sqrt{3}\hat{\underset{\sim}{n}},$ i.e. $\underset{\sim}{a}$ is perpendicular to both $\underset{\sim}{b}$ and $\underset{\sim}{c}$, so $\underset{\sim}{a},\ \underset{\sim}{b}$ and $\underset{\sim}{c}$ are mutually perpendicular. A1

iii. $V = |\overrightarrow{OA}| \times |\overrightarrow{OB}| \times |\overrightarrow{OC}|$

$= \sqrt{3} \times \sqrt{14} \times \sqrt{42}$ M1 A1

$= 42$

c. $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$ where $m,\ n \neq 0$

$4\underset{\sim}{i} + p\underset{\sim}{j} + 5\underset{\sim}{k} = m(\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}) + n(2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k})$

By equating components we obtain:

$4 = m + 2n$ (1)

$p = -m + 3n$ (2)

$5 = -m - n$ (3) A1 for (1), (2) and (3)

$(1) + (3)$ gives $n = 9$ A1

Substituting $n = 9$ into (1) or (3) gives $m = -14$. A1

Hence $p = 41$. A1

Question 3

a. Resolving perpendicular to the slope, $N = 60g\cos(\theta)$.

Resolving parallel to the slope, $60g\sin(\theta) = 60a$. A1

Hence $a = g\sin(\theta)$

$= 9.8 \times \dfrac{1}{20}$ A1

$= 0.49\ \text{m/s}^2$ A1

b. Michael travels down the slope with constant acceleration. If v is his final speed, then v can be determined from $v^2 = u^2 + 2as$.

$v^2 = 0^2 + 2 \times 0.49 \times 5$ A1

$v^2 = 4.9$

$v = 2.21$ m/s A1

c. The time taken to reach the bottom can be found from $v = u + at$.

$2.21 = 0 + 0.49t$ A1

$$t = \frac{2.21}{0.49}$$

To the nearest second, this is 5 seconds. A1

d. On the level, the equation of Michael's motion is $a = -\frac{k}{v^2}$, where k is a constant.

i. This question concerns distance and speed. Use $a = v\frac{dv}{dx} = -\frac{k}{v^2}$. A1

$$\frac{dv}{dx} = -\frac{k}{v^3}$$

$$\frac{dx}{dv} = -\frac{v^3}{k}$$

$$x = \int -\frac{v^3}{k} dv$$

*From here, technology can be used to find the solution – see alternative solution.

$$x = -\frac{v^4}{4k} + c$$ A1

When $x = 0$, $v = 2.21$

$$0 = \frac{-(2.21)^4}{4k} + c$$

$$c = \frac{(2.21)^4}{4k}$$ M1

When $x = 5$, $v = 1.5$

$$5 = \frac{-(1.5)^4}{4k} + \frac{(2.21)^4}{4k}$$

$$4k = \frac{23.8544 - 5.0625}{5}$$

$4k = 3.75839$

$$c = \frac{23.8544}{3.75839}$$

$$x = \frac{23.8544 - v^4}{3.75839}$$ A1

When $v = 1$, $x = \frac{22.8544}{3.75839} = 6.0809$.

After approximately 6 m of travel on the level, Michael reaches a speed of 1 m/s. A1

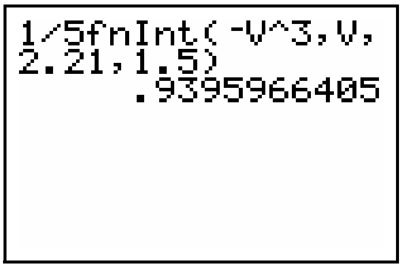
***Alternative Solution (using technology):**

To find k, $5 = \int_{2.21}^{1.5} -\frac{v^3}{k}\, dv$

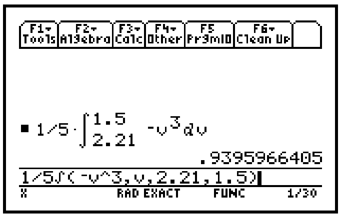
$$k = \frac{1}{5}\int_{2.21}^{1.5} -v^3 \, dv \qquad \text{M1}$$

$$\cong 0.9396 \qquad \text{M1}$$

Using a graphics calculator:



Using CAS:



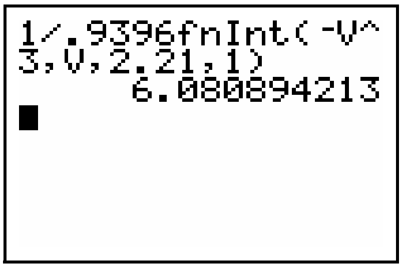
Then $x = \int_{2.21}^{1} \left(-\frac{v^3}{k}\right) dv$

$$= \frac{1}{0.9396}\int_{2.21}^{1} (-v^3)\,dv \qquad \text{M1}$$

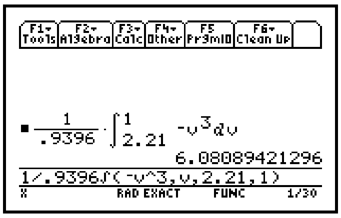
$$= 6.0809$$

After approximately 6 m of travel on the level, Michael reaches a speed of 1 m/s. A1

Using a graphics calculator:



Using CAS:



ii. Since this part concerns time and speed, use $a = \frac{dv}{dt} = -\frac{3.75839}{4v^2}$.

$$\frac{dt}{dv} = -\frac{4v^2}{3.75839}$$

$$t = \int -\frac{4v^2}{3.75839}\,dv$$

$$t = -\frac{4v^3}{(3.75839 \times 3)} + c$$ A1

When $t = 0$, $v = 2.21$.

$$0 = -\frac{4(2.21)^3}{(3.75839 \times 3)} + c$$

$$c = \frac{4(2.21)^3}{(3.75839 \times 3)}$$

$$t = \frac{4((2.21)^3 - v^3)}{3.75839 \times 3}$$ A1

When $v = 1$, t is given by

$$t = \frac{4((2.21)^3 - 1^3)}{3.75839 \times 3}$$

$t = 3.474$

Michael reaches a speed of 1 m/s on the level after 3 seconds (correct to the nearest second). A1

Alternative Solution (using technology):

$$\frac{dv}{dt} = -\frac{0.9396}{v^2}$$

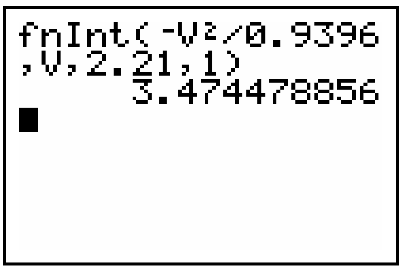
$$\frac{dt}{dv} = -\frac{v^2}{0.9396}$$ M1

$$t = \int_{2.21}^{1} \left(-\frac{v^2}{0.9396}\right) dv$$ M1

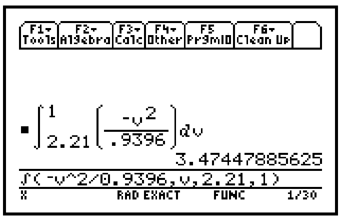
$= 3.474$

Michael reaches a speed of 1 m/s on the level after 3 seconds (correct to the nearest second). A1

Using a graphics calculator:



Using CAS:



Question 4

a. $\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{(9t - 3t^2)^2 + (\log_e(1 + (t-3)^4))^2}$

$\left|\dot{\underset{\sim}{r}}(1)\right| = \sqrt{(9-3)^2 + (\log_e(1 + (1-3)^4))^2}$ M1

Hence $\left|\dot{\underset{\sim}{r}}(1)\right| = 6.64$, i.e. the particle's speed is 6.64 m/s (correct to two decimal places). A1

b. Attempting to solve $9t - 3t^2 = 0$ **and** $\log_e(1 + (t-3)^4) = 0$ for t: M1

From $9t - 3t^2 = 0$ we obtain $t = 0$, 3 and from $\log_e(1 + (t-3)^4) = 0$ we obtain $t = 3$. A1

Hence the particle is a rest at $t = 3$. A1

c. The gradient of the curve is given by $\dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)}$. M1

At $t = 1$, $\dfrac{dy}{dx} = \dfrac{\log_e(17)}{6} = 0.47$ (correct to two decimal places). A1

d. $y(t) = y(0) + \displaystyle\int_0^T \log_e(1 + (t-3)^4)dt$ M1

$y(1) = 2 + \displaystyle\int_0^1 \log_e(1 + (t-3)^4)dt$ A1

Hence the y-coordinate at P is 5.67 (correct to two decimal places). A1

Question 5

a. Let $z = \text{cis}\left(\dfrac{\pi}{6}\right)$

i. $z^2 = \left(\text{cis}\left(\dfrac{\pi}{6}\right)\right)^2 = \text{cis}\left(\dfrac{2\pi}{6}\right)$ using de Moivre's theorem A1

$= \cos\left(\dfrac{\pi}{3}\right) + i\sin\left(\dfrac{\pi}{3}\right)$

$= \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$ A1

ii. $z^4 = \left(\text{cis}\left(\dfrac{\pi}{6}\right)\right)^4 = \text{cis}\left(\dfrac{4\pi}{6}\right)$ using de Moivre's theorem

$= \cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right)$

$= -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$ A1

b. $z^4 - z^2 + 1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 1$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2} - \frac{\sqrt{3}}{2}i + 1$ A1

$= 0$, as required A1

c. **i.** Using the factor theorem, one linear factor is $z - \operatorname{cis}\left(\frac{\pi}{6}\right)$. The other will be $z - \operatorname{cis}\left(-\frac{\pi}{6}\right)$ (conjugate root theorem). M1

$\operatorname{cis}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{i}{2}$ and $\operatorname{cis}\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{i}{2}$

In Cartesian form, these factors are $\left(z - \frac{\sqrt{3}}{2} - \frac{i}{2}\right)$ and $\left(z - \frac{\sqrt{3}}{2} + \frac{i}{2}\right)$. A1

ii. The product of these factors is $\left(z - \frac{\sqrt{3}}{2} - \frac{i}{2}\right)\left(z - \frac{\sqrt{3}}{2} + \frac{i}{2}\right)$.

$= \left(z - \frac{\sqrt{3}}{2}\right)^2 - \left(\frac{i}{2}\right)^2$

$= z^2 - \sqrt{3}z + \frac{3}{4} + \frac{1}{4}$

$= z^2 - \sqrt{3}z + 1$ A1

d. $z^4 - z^2 + 1 = z^4 + 2z^2 + 1 - 3z^2$

$= (z^2 + 1)^2 - (\sqrt{3}z)^2$

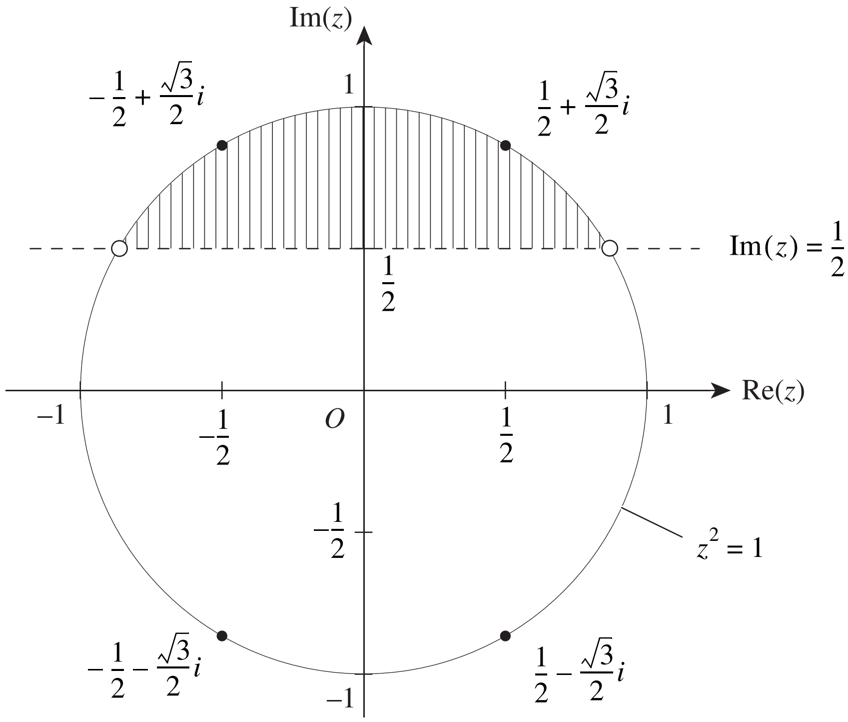
$= (z^2 + 1 - \sqrt{3}z)(z^2 + 1 + \sqrt{3}z)$ A1

Comparing the quadratic factors with the result of **c.ii.** enables us to obtain the other set of linear factors as $\left(z + \frac{\sqrt{3}}{2} - \frac{i}{2}\right)$ and $\left(z + \frac{\sqrt{3}}{2} + \frac{i}{2}\right)$. A2

Alternative solution:

A more elegant method is to recognise $z^4 - z^2 + 1$ as an even function of z.
Hence if $f(a) = 0$, then $f(-a) = 0$ also. M1
Thus if $(z - a)$ is a factor, so is $(z + a)$, A1
leading to the same result for the factors. A1

e.



i. All points correct and labelled. A1

ii. Circle $|z| = 1$ and line $\text{Im}(z) = \frac{1}{2}$ shown correctly and labelled. A1

Region of intersection, with boundary markings, correctly shown. A1