Insight Publications

INSIGHT

Trial Exam Paper

2007

SPECIALIST MATHEMATICS

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations.

SECTION 1

Question 1

If $z = i\left(2i + i^3 - 3\right)$, then $\text{Re}(z)$ is equal to

A. -3
B. -2
C. -1
D. 1
E. 3

Answer is C.

Worked solution

$$z = i\left(2i + i^3 - 3\right)$$
$$z = 2i^2 + i^4 - 3i$$
$$z = -2 + 1 - 3i$$
$$z = -1 - 3i$$
$$\text{Re}(z) = -1$$

Question 2

In polar form $-\frac{1}{\sqrt{2}}(1+i)$ is equivalent to

A. $\text{cis}\left(\frac{\pi}{4}\right)$

B. $\text{cis}\left(-\frac{3\pi}{4}\right)$

C. $\frac{1}{\sqrt{2}}\text{cis}\left(\frac{\pi}{4}\right)$

D. $\frac{1}{\sqrt{2}}\text{cis}\left(\frac{5\pi}{4}\right)$

E. $\frac{1}{\sqrt{2}}\text{cis}\left(-\frac{3\pi}{4}\right)$

Answer is B.

Worked solution

Let $z = -\frac{1}{\sqrt{2}}(1+i) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

The complex number is in the third quadrant, so the argument is negative.

$$\theta = \text{Arg}(z) = \tan^{-1}\left(\frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}\right) = \tan^{-1}(1) = -\frac{3\pi}{4}$$

$$r = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

$$z = r\text{cis}(\theta)$$

$$z = \text{cis}\left(-\frac{3\pi}{4}\right)$$

Question 3

If $u = 3 - 4i$ and $v = 1 + 2i$ then $\left|\frac{u^2}{v^2}\right|$ is equal to

A. $\bar{u}$

B. $-u$

C. $|v|$

D. $|v|^2$

E. $\left(\frac{u}{v}\right)^2$

Answer is D.

Worked solution

$$\frac{u^2}{v^2} = \frac{(3-4i)^2}{(1+2i)^2} = -3 + 4i$$

$$\left|\frac{u^2}{v^2}\right| = |-3+4i| = \sqrt{(-3)^2 + (4)^2} = 5$$

Only two alternatives include a modulus sign, so eliminate A, B and E.

$|v| = \sqrt{1^2 + 2^2} = \sqrt{5}$ Hence, solution is not C.

$|v|^2 = \left(\sqrt{5}\right)^2 = 5$

Tip

- *Use your calculator to determine* $\frac{u^2}{v^2}$.

Question 4

The asymptotes of the curve $x^2 - 4y^2 + 6x + 16y = 11$ intersect at the point

A. $(-3, 2)$

B. $(-3, -8)$

C. $(-6, -4)$

D. $(3, 2)$

E. $(3, 8)$

Answer is A.

Worked solution

$x^2 - 4y^2 + 6x + 16y = 11$
$x^2 + 6x - 4(y^2 - 4y) = 11$
$(x^2 + 6x + 9) - 4(y^2 - 4y + 4) = 11 + 9 - 16$
$(x+3)^2 - 4(y-2)^2 = 4$
$\frac{(x+3)^2}{4} - (y-2)^2 = 1$
$(h, k) = (–3, 2)$
The asymptotes of the hyperbola intersect at the point (–3, 2).

Tip

- *Put hyperbola equation in the form* $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.
- *The asymptotes will intersect at the point (h, k).*

Question 5

The range of the function $f : [0, \frac{7\pi}{12}] \to R$, $f(x) = 1 - 3\operatorname{cosec}\left(x + \frac{\pi}{4}\right)$ is

A. R

B. $(-\infty, -2] \cup [2, \infty)$

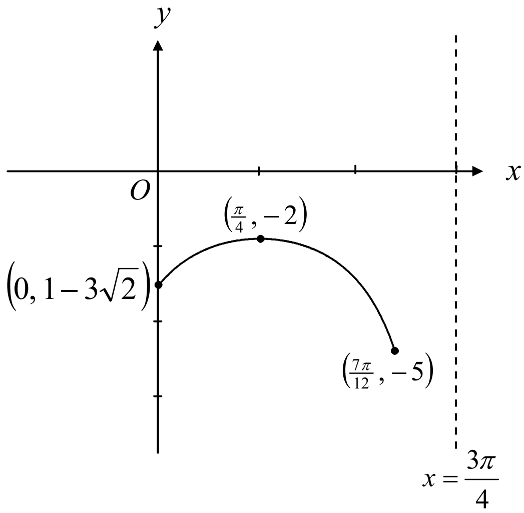
C. $(-\infty, -2]$

D. $[-5, -2]$

E. $[-5, 1-3\sqrt{2}]$

Answer is D.

Worked solution

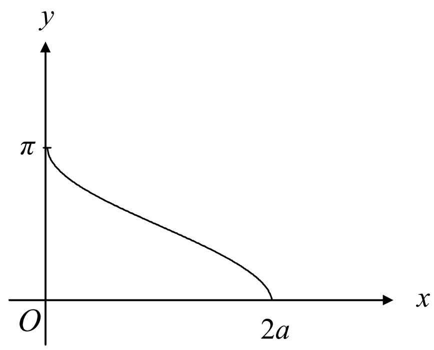


Minimum value is –5.
Maximum value is –2.
Range $[-5, -2]$.

Tip

- *Use the calculator to graph the function.*
- *Determine the coordinates of the maximum point and the end points over the domain* $\left[0, \frac{7\pi}{12}\right]$.

Question 6



The equation of the graph shown above could be

A. $y = \cos^{-1}(ax - 1)$

B. $y = \cos^{-1}(x - 2a)$

C. $y = \cos^{-1}\left(\frac{x}{2} - a\right)$

D. $y = \cos^{-1}\left(\frac{x-1}{a}\right)$

E. $y = \cos^{-1}\left(\frac{x}{a} - 1\right)$

Answer is E.

Worked solution

Graph of $y = \cos^{-1}(x)$ has been dilated by a factor of a units from the y-axis and then translated right by a units. The equation of the curve is

$$y = \cos^{-1}\left(\frac{1}{a}(x-a)\right)$$

$$y = \cos^{-1}\left(\frac{x}{a}-1\right)$$

Verifying this is the solution by finding the x-intercept.

$$0 = \cos^{-1}\left(\frac{x}{a}-1\right)$$

$$\cos(0) = \frac{x}{a}-1$$

$$1 = \frac{x}{a}-1$$

$$\frac{x}{a} = 2$$

$$x = 2a$$

Question 7

Given the function $f(x) = \dfrac{|\log_e(x)|}{\log_e|x|-1}$, which one of the following statements is false?

A. f has an asymptote at $x = e$.

B. The point $(1, 0) \in f(x)$.

C. The inverse, f^{-1}, exists for $x \in (0, 1]$.

D. The range of f is $R \setminus [0, 1)$.

E. f has more than two asymptotes.

Answer is D.

Worked solution

A is true. Vertical asymptote occurs where $\log_e|x|-1 = 0 \Rightarrow \quad x = e$

B is true. When $x = 1$, $f(1) = \dfrac{|\log_e(1)|}{\log_e|1|-1} = \dfrac{0}{-1} = 0$.

C is true. f is one-to-one over $x \in (0, 1]$, therefore the inverse f^{-1} exists.

D is false. The range of f is $(-\infty, 0] \cup (1, \infty)$. This is written as $R \setminus (0, 1]$.

E is true. The function has three asymptotes: $x = 0$, $x = e$, $y = 1$.

Tip

- *Eliminate the true alternatives.*

Question 8

At the point where $y = 2$, the gradient of the curve $y^2 + x^3 = 5$ is

A. -3

B. $-\frac{3}{4}$

C. $-\frac{3}{2}$

D. $\frac{1}{2}$

E. 2

Answer is B.

Worked solution

Use implicit differentiation.

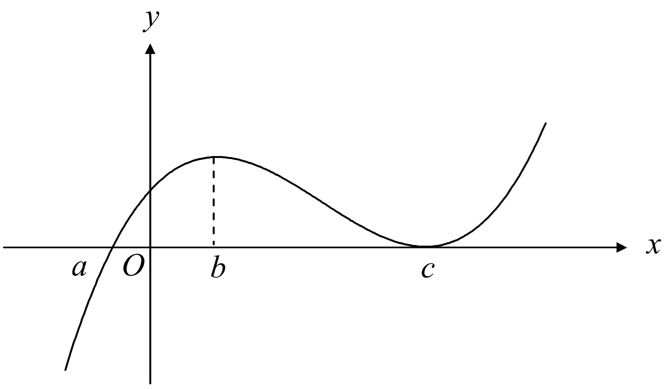
$$2y\frac{dy}{dx} + 3x^2 = 0$$

$$\frac{dy}{dx} = -\frac{3x^2}{2y}$$ When $y = 2$, $x = \sqrt[3]{5-2^2} = 1$.

At the point $x = 1$, $y = 2$:

$$\frac{dy}{dx} = -\frac{3\times 1^2}{2\times 2} = -\frac{3}{4}$$

Question 9



The graph of $y = f(x)$ is shown above.
Let $F(x)$ be an antiderivative of $f(x)$.

The graph of $y = F(x)$ has a

A. local maximum at $x = a$

B. stationary point at $x = b$

C. point of inflexion at $x = c$

D negative gradient for $b < x < c$

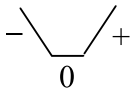
E. zero gradient at $x = 0$

Answer is C.

Worked solution

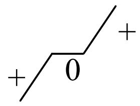
$f(x)$ is a graph of the gradient of $F(x)$.

A is incorrect. $x = a$ is a local minimum point on graph of $F(x)$.
When $x < a$, $f(x) < 0$.
When $x = a$, $f(x) = 0$.
When $x > a$, $f(x) > 0$.



B is incorrect. The point of greatest slope of $F(x)$ will occur at $x = b$ since $f(x)$ has a stationary point there.

C is correct. $x = c$ is a point of inflexion on graph of $F(x)$.
When $x < c$, $f(x) > 0$.
When $x = c$, $f(x) = 0$.
When $x > c$, $f(x) > 0$.



D is incorrect. $F(x)$ will have a positive gradient between $b < x < c$ because $f(x) > 0$.

E is incorrect. $f(x) > 0$ at $x = 0$, so the gradient of $F(x)$ is positive.

Question 10

Using the substitution $u = 2 - x$, the integral $\int_2^4 x(2-x)^6\,dx$ is equal to

A. $\int_2^4 \left(2u^6 - u^7\right)du$

B. $\int_2^4 \left(u^7 - 2u^6\right)du$

C. $\int_{-2}^0 \left(u^7 - 2u^6\right)du$

D. $\int_0^{-2} \left(u^7 - 2u^6\right)du$

E. $\int_0^{-2} \left(2u^6 - u^7\right)du$

Answer is D.

Worked solution

Let $u = 2 - x$, $\quad \frac{du}{dx} = -1 \quad \Rightarrow du = -dx$

Finding the terminals of integration: When $x = 4$, $u = 2 - 4 = -2$.
When $x = 2$, $u = 2 - 2 = 0$.

$$\int_2^4 x(2-x)^6\, dx$$

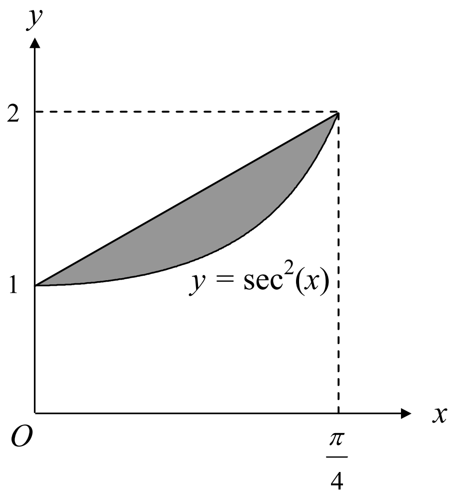
Substituting $u = 2 - x$

$$= \int_0^{-2} (2-u)u^6(-du)$$

$$= -\int_0^{-2} \left(2u^6 - u^7\right) du$$

$$= \int_0^{-2} \left(u^7 - 2u^6\right) du$$

Question 11



The shaded region shown above is formed between the graph of $y = \sec^2(x)$ and the line joining the points $(0, 1)$ and $\left(\frac{\pi}{4}, 2\right)$.

The area of the shaded region would be

A. $\frac{3\pi - 8}{8}$

B. $\frac{3\pi - 4}{8}$

C. $\frac{3\pi - 1}{8}$

D. $\frac{3\pi}{8}$

E. $\frac{\pi}{4}$

Answer is A.

Worked solution

Shaded region = area of trapezium – $\int_0^{\frac{\pi}{4}} \sec^2(x)\,dx$

$= \frac{1}{2}(1+2)\frac{\pi}{4} \quad - \quad \int_0^{\frac{\pi}{4}} \sec^2(x)\,dx$

$= \frac{3\pi}{8} - [\tan(x)]_0^{\frac{\pi}{4}}$

$= \frac{3\pi}{8} - 1$

$= \frac{3\pi - 8}{8}$

Question 12

Given $f'(x) = \frac{e^x}{e^{-x} + e^x}$ and $f(0) = 2$.

Using Euler's method with increments of 0.1, an approximate value of $f(0.2)$ is

A. 2.0500
B. 2.0550
C. 2.1050
D. 2.1099
E. 2.1484

Answer is C.

Worked solution

Using $f(x+h) \approx hf'(x) + f(x)$
Let $x = 0$, $h = 0.1$

$f(0.1) = f(0+0.1) \approx 0.1f'(0) + f(0)$ $\qquad f'(0) = \frac{e^0}{e^{-0} + e^0} = 0.5$

$f(0.1) \approx 0.1 \times 0.5 + 2$
≈ 2.05

$f(0.2) = f(0.1+0.1) \approx 0.1f'(0.1) + f(0.1)$ $\qquad f'(0.1) = \frac{e^{0.1}}{e^{-0.1} + e^{0.1}} = 0.5498$

$f(0.2) \approx 0.1 \times 0.5498 + 2.05$
≈ 2.1050

Question 13

A bowl of soup is heated to a temperature of 80°C. It is then left to cool in a room in which the air temperature is 20°C. The rate at which the temperature of the soup decreases is proportional to the difference between its temperature and the temperature of the room.

Let S °C be the temperature of the soup at any time t minutes after it is removed from the heat. Given k is a positive constant, the relationship between S and t may be modelled by the differential equation

A. $\frac{dS}{dt} = -k(S-20)$; $t = 0,\ S = 60$

B. $\frac{dS}{dt} = -k(S-20)$; $t = 0,\ S = 80$

C. $\frac{dS}{dt} = -k(S-60)$; $t = 0,\ S = 80$

D. $\frac{dS}{dt} = -k(S-60)$; $t = 0,\ S = 20$

E. $\frac{dS}{dt} = -k(S-80)$; $t = 0,\ S = 20$

Answer is B.

Worked solution

The difference between the temperature of the soup and the temperature of the room is $(S-20)$. Initially, the temperature of the soup is 80°C.

Therefore, $\frac{dS}{dt} = -k(S-20)$ when $t = 0,\ S = 80$.

Question 14

The solution of the differential equation $\frac{dy}{dx} = e^{\sin(x)}$, given $y = 3$ when $x = 2$, is

A. $y = \int_2^x e^{\sin(u)}du + 3$

B. $y = \int_3^x e^{\sin(u)}du + 2$

C. $y = \int_2^3 e^{\sin(x)}dx$

D. $y = \int_2^3 e^{\sin(x)}dx + 3$

E. $y = \int_2^3 e^{\sin(x)}dx + 2$

Answer is A.

Worked solution

$$\frac{dy}{dx} = e^{\sin(x)}$$
$$y = \int e^{\sin(x)}\ dx$$
$\Rightarrow y = f(x) + c$ ….(1) where $f'(x) = e^{\sin(x)}$

When $x = 2,\quad y = 3$

$$3 = f(2) + c$$
$$c = -f(2) + 3 \quad \text{….(2)}$$

$$y = f(x) - f(2) + 3$$
$$y = \left[f(u)\right]_2^x + 3$$
$$y = \int_2^x e^{\sin(u)} du + 3$$

Question 15

A particle moves in a straight line such that its velocity is given by

$v = \log_e\left|\cos\left(\frac{x}{4}\right)\right|$, $x \in [0, 2\pi)$, where x is its displacement from the origin O.

The particle's acceleration $\frac{4\pi}{3}$ units from O is

A. $-\frac{\sqrt{3}}{4}\log_e(2)$

B. $-\log_e(2)$

C. $-\frac{\sqrt{3}}{4}$

D. $\log_e(2)$

E. $\frac{\sqrt{3}}{4}\log_e(2)$

Answer is E.

Worked solution

$$v = \log_e\left|\cos\left(\frac{x}{4}\right)\right|$$
$$\frac{dv}{dx} = \frac{-\frac{1}{4}\sin\left(\frac{x}{4}\right)}{\cos\left(\frac{x}{4}\right)} = -\frac{1}{4}\tan\left(\frac{x}{4}\right)$$

$$a = v\,\frac{dv}{dx}$$

$$a = -\frac{1}{4}\tan\left(\frac{x}{4}\right) \times \log_e \left|\cos\left(\frac{x}{4}\right)\right|$$

When $x = \frac{4\pi}{3}$, $\quad a = -\frac{1}{4}\tan\left(\frac{\frac{4\pi}{3}}{4}\right) \times \log_e \left|\cos\left(\frac{\frac{4\pi}{3}}{4}\right)\right|$

$$a = -\frac{1}{4}\tan\left(\frac{\pi}{3}\right) \times \log_e \left|\cos\left(\frac{\pi}{3}\right)\right|$$

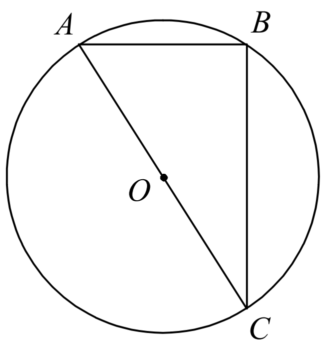
$$a = -\frac{1}{4} \times \sqrt{3} \times \log_e\left(\frac{1}{2}\right)$$

$$a = \frac{\sqrt{3}}{4}\log_e(2)$$

Question 16

A, *B* and *C* are three points on the circumference of a circle with centre *O*.

AC passes through *O*.



Which one of the following statements is **not** true?

A. $\vec{AB}.\vec{BC} = 0$

B. $\vec{AB} + \vec{BC} + \vec{CA} = \underset{\sim}{0}$

C. $\left(\vec{AB} + \vec{BC}\right).\vec{AC} = \left|\vec{AC}\right|^2$

D. $\vec{AB}.\vec{AC} = \left|\vec{AB}\right|\left|\vec{AC}\right|\cos(A)$

E. $\left|\vec{AC}\right| = \left|\vec{AB}\right| + \left|\vec{BC}\right|$

Answer is E.

Worked solution

A is true. $\angle B$ is a right angle, therefore $\overrightarrow{AB}.\overrightarrow{BC} = 0$.
B is true. Vectors are added head to tail.

C is true. $\left(\overrightarrow{AB} + \overrightarrow{BC}\right).\overrightarrow{AC} = \overrightarrow{AC}.\overrightarrow{AC} = \left|\overrightarrow{AC}\right|^2$

D is true. It is the dot product of two vectors $\overrightarrow{AB}.\overrightarrow{AC} = \left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\cos(A)$

E is not true. For a right-angled triangle $\left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2$

However, $\left|\overrightarrow{AC}\right| \neq \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$,

$\left|\overrightarrow{AC}\right| < \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$

Question 17

Let $\underset{\sim}{m} = 4\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{n} = \underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$.

A unit vector in the direction of $\underset{\sim}{m} - 2\underset{\sim}{n}$ would be

A. $\frac{1}{7}\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}\right)$

B. $\frac{1}{3}\left(2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\right)$

C. $\frac{1}{\sqrt{17}}\left(2\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}\right)$

D. $\frac{1}{\sqrt{41}}\left(6\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}\right)$

E. $\frac{1}{\sqrt{29}}\left(3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}\right)$

Answer is A.

Worked solution

$$\begin{aligned}\underset{\sim}{m} - 2\underset{\sim}{n} &= \left(4\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}\right) - 2\left(\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)\\ &= 4\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} - 2\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}\\ &= 2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}\end{aligned}$$

Unit vector is $\frac{1}{\sqrt{2^2 + (-3)^2 + 6^2}}\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}\right)$

$$= \frac{1}{\sqrt{49}}\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}\right)$$

$$= \frac{1}{7}\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}\right)$$

Question 18

The velocity of a particle at time *t*, $t \ge 0$ is given by $\underset{\sim}{v} = 4\sin(2t)\underset{\sim}{i} + 6\cos(3t)\underset{\sim}{j}$.

If the particle was initially at $\underset{\sim}{i} + 2\underset{\sim}{j}$, its position after $\frac{\pi}{2}$ seconds will be

A. $-3\underset{\sim}{i} + 4\underset{\sim}{j}$

B. $3\underset{\sim}{i} - 2\underset{\sim}{j}$

C. $3\underset{\sim}{i} + 3\underset{\sim}{j}$

D. $5\underset{\sim}{i}$

E. $5\underset{\sim}{i} - \underset{\sim}{j}$

Answer is D.

Worked solution

Finding the position of the particle at any time *t*:

$\underset{\sim}{r} = \int \underset{\sim}{v}\, dt$

$\underset{\sim}{r} = \int \left(4\sin(2t)\underset{\sim}{i} + 6\cos(3t)\underset{\sim}{j} \right) dt$

$\underset{\sim}{r} = -2\cos(2t)\underset{\sim}{i} + 2\sin(3t)\underset{\sim}{j} + \underset{\sim}{c}$

At $t = 0$, $\underset{\sim}{r} = \underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{i} + 2\underset{\sim}{j} = -2\cos(0)\underset{\sim}{i} + 2\sin(0)\underset{\sim}{j} + \underset{\sim}{c}$

$\underset{\sim}{c} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{r} = -2\cos(2t)\underset{\sim}{i} + 2\sin(3t)\underset{\sim}{j} + 3\underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{r} = (3 - 2\cos(2t))\underset{\sim}{i} + (2\sin(3t) + 2)\underset{\sim}{j}$

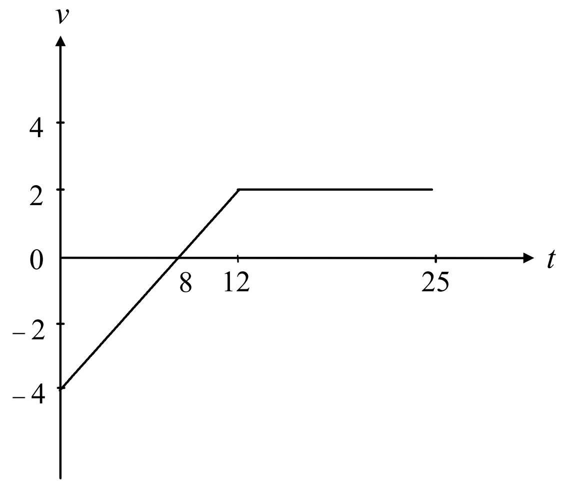
When $t = \frac{\pi}{2}$, $\underset{\sim}{r} = (3 - 2\cos(\pi))\underset{\sim}{i} + \left(2\sin\left(\frac{3\pi}{2}\right) + 2 \right)\underset{\sim}{j}$

$\underset{\sim}{r} = 5\underset{\sim}{i} + 0\underset{\sim}{j}$

$\underset{\sim}{r} = 5\underset{\sim}{i}$

Question 19

The graph below shows the velocity, v m/s, of a particle moving in a straight line for 25 s.



How many metres is the particle from its starting point after 25 s?

A. 6
B. 14
C. 30
D. 46
E. 50

Answer is B.

Worked solution

The particle travels with a negative velocity for the first 8 s of motion.

Distance travelled: $d_1 = \frac{1}{2} \times (8-0) \times -4 = -16$ m

At $t = 8$ s, it stops moving momentarily, turns around and travels back over the same path.

Distance travelled: $d_2 = \frac{1}{2} \times (12-8) \times 2 = 4$ m

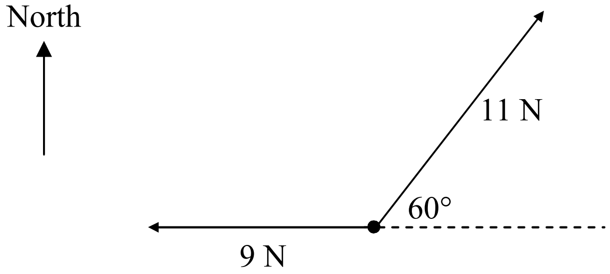
At $t = 12$ s, it continues moving in the same direction with a constant velocity of 2 m/s.

Distance travelled: $d_3 = (25-12) \times 2 = 26$ m

$\therefore$ Displacement $= -16 + 4 + 26 = 14$ m

Question 20

Two forces act simultaneously on a particle, as shown in the diagram below. One force of 9 N acts due west and another force of 11 N acts at an angle of $60°$ in an anticlockwise direction from due east.



Correct to the nearest degree, the resultant force acting in an anticlockwise direction from due east will be

A. 50°

B. 70°

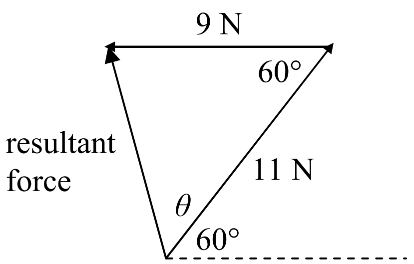
C. 110°

D. 120°

E. 130°

Answer is C.

Worked solution



Add forces head to tail.

The resultant force, R, acts at an angle $(\theta + 60)°$ from due east.

Use the cosine rule to find the magnitude of the resultant force.

$$|R| = \sqrt{9^2 + 11^2 - 2 \times 9 \times 11 \cos(60°)}$$

$$|R| = \sqrt{103} \text{ N}$$

Finding θ, using the sine rule:

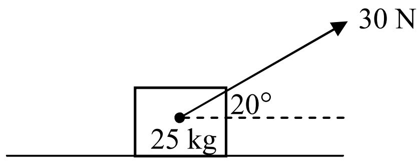
$$\frac{\sin(\theta)}{9} = \frac{\sin(60°)}{\sqrt{103}}$$

$\theta = 50°$

The resultant force acts at an angle of 110°.

Question 21

A mass of 25 kg is pulled across a smooth horizontal surface by a force of 30 newtons acting at and angle of 20° to the horizontal level.



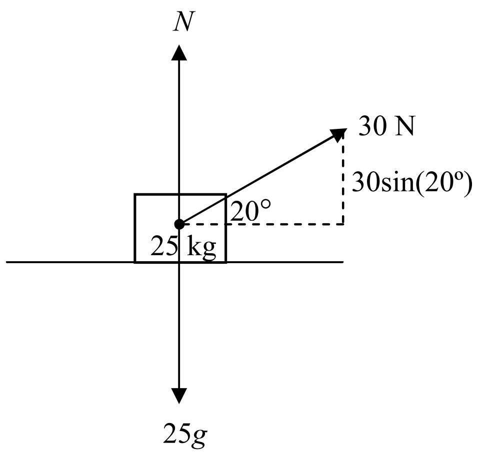
The magnitude of the normal reaction of the surface on the mass, in newtons, is closest to

A. 215
B. 217
C. 235
D. 245
E. 255

Answer is C.

Worked solution

Let N be the normal reaction of the surface on the mass.



Resolving forces in a vertical direction

$N + 30\sin(20°) = 25g$

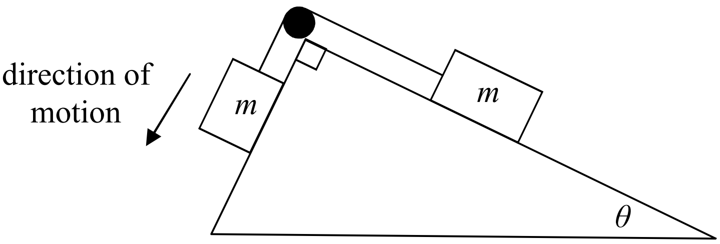
$N = 25g - 30\sin(20°)$

$N = 245 - 10.3$

$N = 234.7$

$N = 235$ newtons

Question 22



Two bodies each of mass, *m* kg, are connected on a back-to-back plane by a light string passing over a smooth pulley, as shown in the diagram. The coefficient of friction of the surface of each plane is μ.

If the system is on the point of moving in the direction shown, the value of μ will be

A. $\tan(\theta)$

B. $\cot(\theta)$

C. $\sin(\theta)-\cos(\theta)$

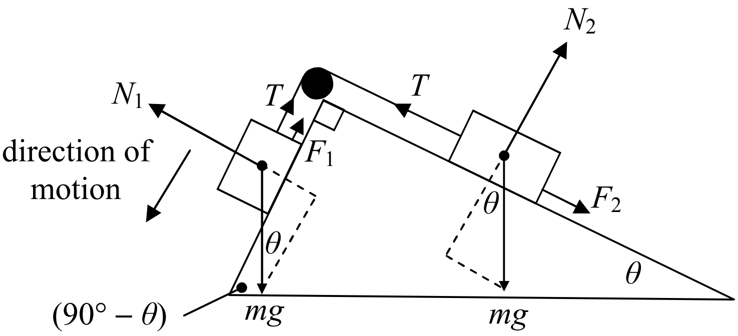
D. $\dfrac{\sin(\theta)-\cos(\theta)}{\sin(\theta)+\cos(\theta)}$

E. $\dfrac{\cos(\theta)-\sin(\theta)}{\cos(\theta)+\sin(\theta)}$

Answer is E.

Worked solution

Draw all forces acting.



N is the normal reaction
F is the friction.
T is the tension in the string.
The remaining angle in the triangle is (90° – θ).

Resolving forces perpendicular to the planes to find F_1 and F_2:

$N_1 = mg\sin(\theta)$ $\qquad$ $N_2 = mg\cos(\theta)$

$F_1 = \mu N_1 = \mu mg\sin(\theta)$ $\qquad$ $F_2 = \mu N_2 = \mu mg\cos(\theta)$

The system is on the point of moving. The mass on the left will move down the plane and the mass on the right will move up.

$\underset{\sim}{a} = 0$

Resolving forces (parallel to the plane) for the mass on the left:

$$mg\cos(\theta) - T - F_1 = 0$$

$$T = mg\cos(\theta) - F_1$$

$$T = mg\cos(\theta) - \mu mg\sin(\theta) \quad ....(1)$$

Resolving forces (parallel to the plane) for the mass on the right:

$$T - mg\sin(\theta) - F_2 = 0$$

$$T = mg\sin(\theta) + F_2$$

$$T = mg\sin(\theta) + \mu mg\cos(\theta) \quad ....(2)$$

Equate (1) and (2):

$$mg\sin(\theta) + \mu mg\cos(\theta) = mg\cos(\theta) - \mu mg\sin(\theta)$$

$$\mu\left(\cos(\theta) + \sin(\theta)\right) = \cos(\theta) - \sin(\theta)$$

$$\mu = \frac{\cos(\theta) - \sin(\theta)}{\cos(\theta) + \sin(\theta)}$$

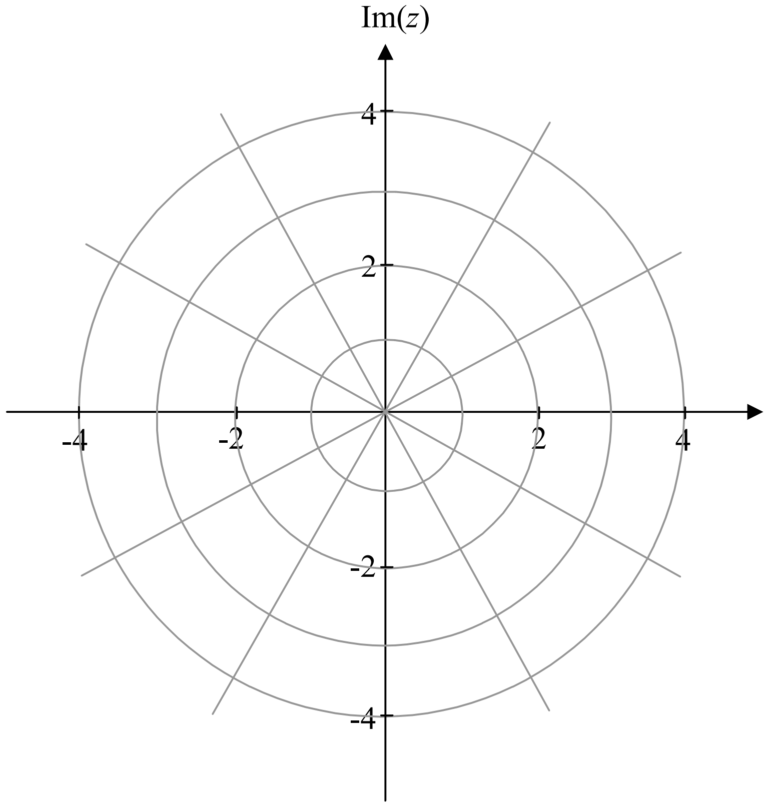
END OF SECTION 1

SECTION 2

Question 1

Let $u = 2\text{cis}\left(\frac{\pi}{3}\right)$.

1a. **i.** u, $\bar{u}$ and v are solutions of the equation $\{z : z^3 = k,\ z \in C\}$.
Plot u, $\bar{u}$ and v on the Argand plane below.

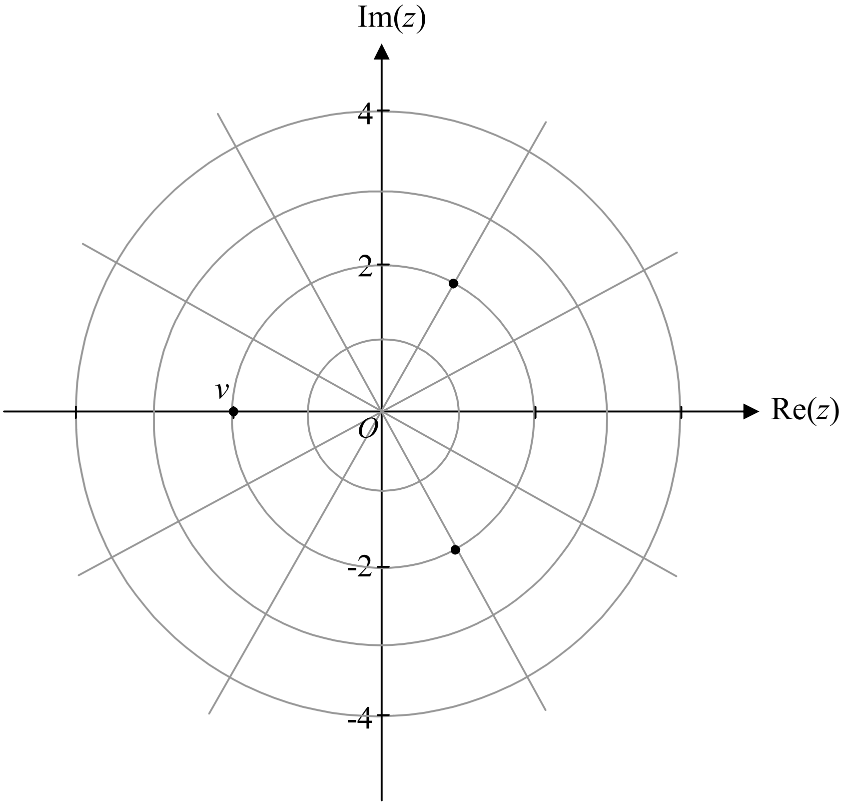


Worked solution

The solutions of the cubic equation are spaced equally around the circumference of a circle.
The angle between the three solutions will be $\frac{2\pi}{3}$ (or 120°).

$|u| = 2$; therefore, the circle has radius 2 units.

$$u = 2\text{cis}\left(\frac{\pi}{3}\right)$$
$$\bar{u} = 2\text{cis}\left(-\frac{\pi}{3}\right)$$
$$v = 2\text{cis}(\pi)$$



1A
1A
1A

3 marks

Mark allocation

- 1 mark for each point plotted correctly.

1a. **ii.** Determine the value of *k*.

Worked solution

u, $\bar{u}$ and v are solutions of the equation $z^3 = 8$. Hence, $k = 8$. 1A

1 mark

Mark allocation

- 1 mark for correct answer.

1b. Show that $u \in \left\{z : \mathrm{Re}(z) + \sqrt{3}\,\mathrm{Im}(z) = 4\right\}$.

Worked solution

Find the real and imaginary components of u.

$u = 2\mathrm{cis}\left(\frac{\pi}{3}\right) = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

$u = 1 + \sqrt{3}i$

$\therefore\ \mathrm{Re}(u) = 1,\ \mathrm{Im}(u) = \sqrt{3}$ 1A

If $u \in \left\{z : \mathrm{Re}(z) + \sqrt{3}\,\mathrm{Im}(z) = 4\right\}$, then $\mathrm{Re}(u) + \sqrt{3}\,\mathrm{Im}(u) = 4$.

$$\begin{aligned}\text{LHS} &= \text{Re}(u) + \sqrt{3}\,\text{Im}(u)\\ &= 1 + \sqrt{3} \times \sqrt{3}\\ &= 4\\ &= \text{RHS}\end{aligned}$$

1A

2 marks

Mark allocation

- 1 mark for finding the real and imaginary components of *u*.
- 1 mark for showing these values satisfy the equation.

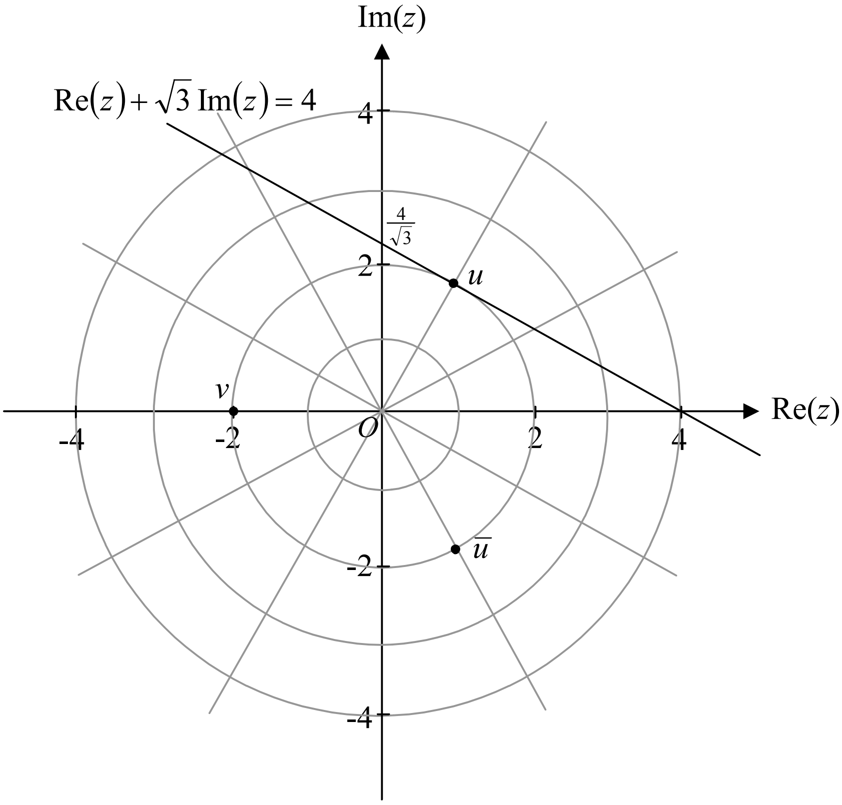
1c. i. Sketch $\{z : \text{Re}(z) + \sqrt{3}\,\text{Im}(z) = 4\}$ on the Argand plane given in part **a**, showing the exact intercepts with the axes.

Worked solution

$\text{Re}(z) + \sqrt{3}\,\text{Im}(z) = 4$ is a line.

Vertical intercept at $\text{Im}(z) = \dfrac{4}{\sqrt{3}}$.

Horizontal intercept at $\text{Re}(z) = 4$.



1A

1 mark

Mark allocation

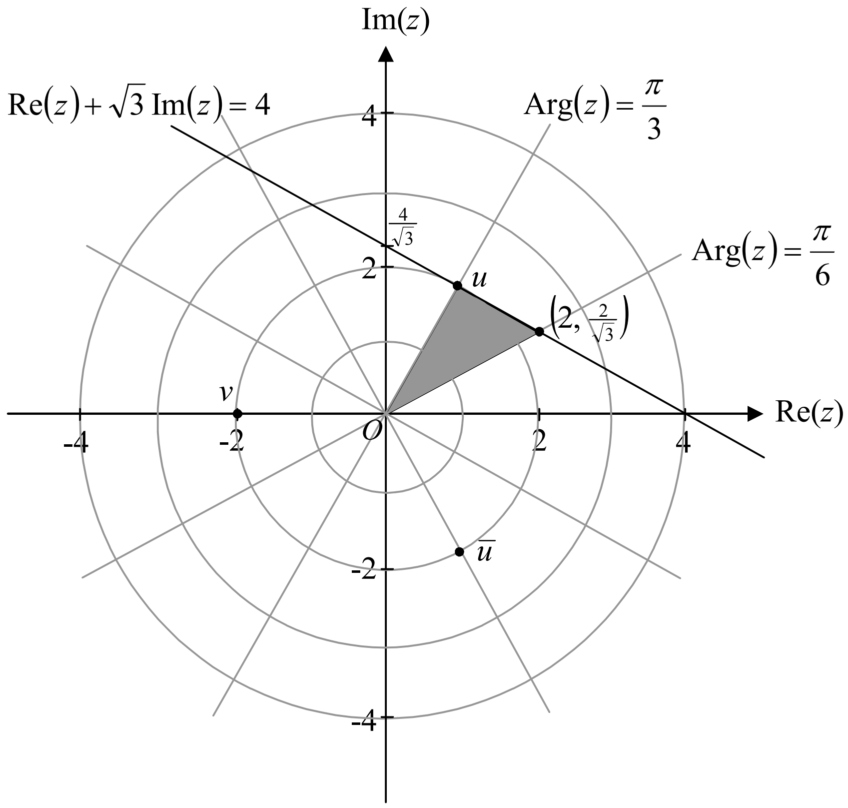
- 1 mark for line drawn correctly with intercepts shown in exact form.

1c. ii. Hence, shade the region represented by

$$\left\{ z : \frac{\pi}{6} \le \operatorname{Arg}(z) \le \frac{\pi}{3} \right\} \cap \left\{ z : \operatorname{Re}(z) + \sqrt{3}\operatorname{Im}(z) \le 4 \right\}.$$

Worked solution

Region is shaded below.



1A

1 mark

Mark allocation

- 1 mark for correct region shaded.

1c. iii. Calculate the exact area of this region.

Worked solution

$$\operatorname{Arg}(z) \le \frac{\pi}{6} \qquad \operatorname{Re}(z) + \sqrt{3}\operatorname{Im}(z) = 4$$

$$\frac{y}{x} = \tan^{-1}\left(\frac{\pi}{6}\right) \qquad x + \sqrt{3}y = 4 \quad \ldots(2)$$

$$y = \frac{1}{\sqrt{3}}x \quad \ldots(1)$$

Substitute (1) into (2): $x + \sqrt{3} \times \frac{1}{\sqrt{3}}x = 4$

$$2x = 4$$

$$x = 2 \qquad y = \frac{2}{\sqrt{3}}$$

The lines intersect at $\left(2, \frac{2}{\sqrt{3}}\right)$ 1A

Distance of this point from origin is $\sqrt{2^2 + \left(\frac{2}{\sqrt{3}}\right)^2} = \sqrt{4 + \frac{4}{3}} = \frac{4}{\sqrt{3}}$.

Distance of point u from origin is $|u|$.

$|u| = 2$

Angle $\theta = \left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{6}$

Now, use formula for area of a triangle: $A = \frac{1}{2}ab\sin\theta$.

$A = \frac{1}{2} \times 2 \times \frac{4}{\sqrt{3}} \sin\left(\frac{\pi}{6}\right)$ 1M

$A = \frac{2}{\sqrt{3}}$

Area of region is $\frac{2\sqrt{3}}{3}$ square units. 1A

3 marks

Mark allocation

- 1 mark for finding the point of intersection.
- 1 mark for using a correct method to find the area of the triangle.
- 1 mark for correct answer.

Tip

- *The point of intersection of the lines* $\operatorname{Arg}(z) \le \frac{\pi}{6}$ *and* $\operatorname{Re}(z) + \sqrt{3}\operatorname{Im}(z) = 4$ *is needed to find the required area. It is easiest to find this point in Cartesian form.*

Total 3 + 1 + 2 + 1 + 1 + 3 = 11 marks

Question 2

Given $f: R \to R,\ f(x) = \dfrac{x}{x^2+2} + 1$

2a. i. Show that $f'(x) = \dfrac{2-x^2}{\left(x^2+2\right)^2}$.

Worked solution

Applying the quotient rule:

$$\frac{dy}{dx} = \frac{1\left(x^2+2\right) - 2x.x}{\left(x^2+2\right)^2}$$

$$\frac{dy}{dx} = \frac{2-x^2}{\left(x^2+2\right)^2} \qquad \text{1A}$$

1 mark

Mark allocation

- 1 mark for showing how to find the correct derivative.

2a. ii. Hence, determine the exact coordinates of any stationary points of *f*.

Worked solution

Stationary points occur where $\dfrac{dy}{dx} = 0$

$$0 = \frac{2-x^2}{\left(x^2+2\right)^2} \qquad \text{1M}$$

$$2 - x^2 = 0$$

$$x = \pm\sqrt{2}$$

Stationary points occur at $\left(\sqrt{2},\ \dfrac{\sqrt{2}}{4}+1\right)$ and $\left(-\sqrt{2},\ 1-\dfrac{\sqrt{2}}{4}\right)$. 1A

2 marks

Mark allocation

- 1 mark for applying a correct method to find *x* coordinate of stationary point.
- 1 mark for two correct stationary points, given in exact form.

2b. i. Use calculus to show that f has points of inflexion.

Worked solution

$$f'(x) = \frac{2-x^2}{\left(x^2+2\right)^2}$$

Apply quotient rule to find second derivative.

$$f''(x) = \frac{-2x\left(x^2+2\right)^2 - 4x\left(2-x^2\right)\left(x^2+2\right)}{\left(x^2+2\right)^4} \qquad \text{1M}$$

$$f''(x) = \frac{\left(x^2+2\right)\left(-2x\left(x^2+2\right) - 4x\left(2-x^2\right)\right)}{\left(x^2+2\right)^4}$$

$$f''(x) = \frac{\left(-2x^3 - 4x - 8x + 4x^3\right)}{\left(x^2+2\right)^3}$$

$$f''(x) = \frac{2x^3 - 12x}{\left(x^2+2\right)^3}$$

Points of inflexion occur where $f''(x) = 0$.

$$\frac{2x^3 - 12x}{\left(x^2+2\right)^3} = 0$$

$$2x^3 - 12x = 0$$

$$2x\left(x^2 - 6\right) = 0$$

$x = 0$, $x = \pm\sqrt{6}$ 1A

Hence, *f* has three non-stationary points of inflexion.

2 marks

Mark allocation

- 1 mark for applying the quotient rule
- 1 mark for the correct answer

2b. ii. Find the exact coordinates of all points of inflexion and explain what each point represents.

Worked solution

The coordinates of the non-stationary points of inflexion are:

$(0,\ 1)$, $\left(\sqrt{6},\ \frac{\sqrt{6}}{8}+1\right)$, $\left(\sqrt{6},\ 1-\frac{\sqrt{6}}{8}\right)$ 1A

Points of inflexion are points where the rate of change of the gradient function is zero.

These are the points of maximum or minimum gradient.

(0, 1) is the point at which the gradient is a maximum.

$\left(\sqrt{6},\ \frac{\sqrt{6}}{8}+1\right)$ and $\left(\sqrt{6},\ 1-\frac{\sqrt{6}}{8}\right)$ are the points at which the gradient

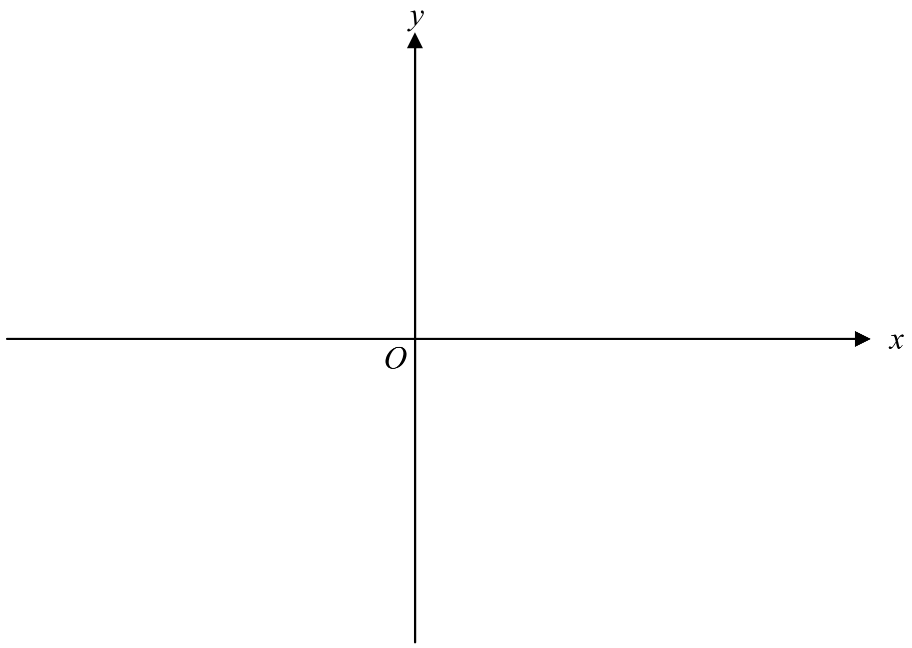
is a minimum. 1A

2 marks

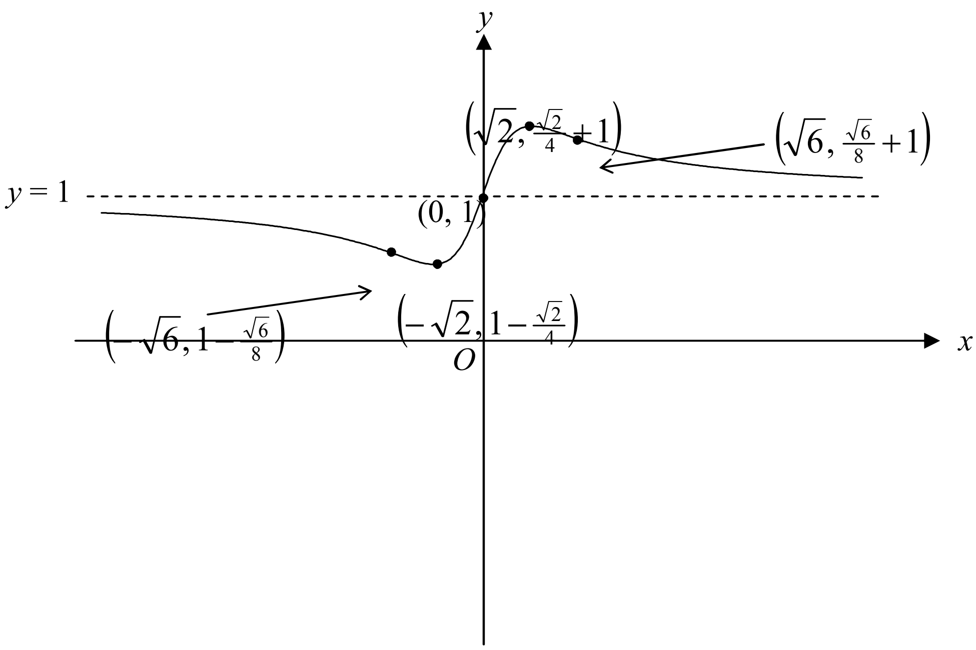
Mark allocation

- 1 mark for the correct coordinates, given in exact form.
- 1 mark for a correct explanation.

2c. Sketch a graph of f on the axes below, labelling its features clearly.



Worked solution



1A
1A

2 marks

Mark allocation

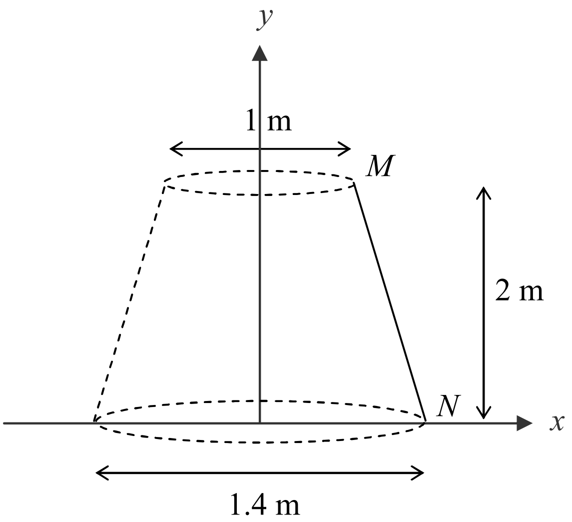
- 1 mark for correct shape and asymptote.
- 1 mark for all relevant points shown correctly, in exact form.

Total 1 + 2 + 2 + 2 + 2 = 9 marks

SECTION 2 –continued

Question 3

The tank shown below has the shape of a truncated cone with base diameter 1.4 m, top diameter 1 m and height 2 m. It may be modelled by rotating the line segment *MN* around the *y*-axis.



3a. The line segment *MN* has equation $ax + by + c = 0$. Show that $a = 10$, $b = 1$ and $c = -7$.

Worked solution

Point *M* (0.7, 0) and point *N* (0.5, 2).
Find the line in the form $y = mx + c_1$.

Find gradient: $m = \dfrac{2-0}{0.5-0.7} = -10$ 1M

$y = -10x + c_1$

Find *y*-intercept: $0 = -10 \times 0.7 + c_1$

$c_1 = 7$

Equation of line: $y = -10x + 7$ 1A

Place in the required form:
$\therefore\ 10x + y - 7 = 0$
$a = 10$, $b = 1$ and $c = -7$

2 marks

Mark allocation

- 1 mark for using a correct method.
- 1 mark for showing how to find correct values.

3b. i. Show that the volume of the tank is $\frac{10\pi}{3}\left((h-7)^3+343\right)$ L, where h is the vertical height to which the tank is filled.

Worked solution

Volume of revolution rotating around the y-axis

$V=\pi\int_0^h x^2 dy$ Given $10x+y-7=0$

$\Rightarrow \quad x=\frac{7-y}{10}$

$V=\pi\int_0^h\left(\frac{7-y}{10}\right)^2 dy$ 1M

$V=\frac{\pi}{100}\int_0^h\left(y-7\right)^2 dy$

$V=\frac{\pi}{100}\left[\frac{1}{3}\left(y-7\right)^3\right]_0^h$ 1A

$V=\frac{\pi}{300}\left[(h-7)^3-(0-7)^3\right]\text{ m}^3$

$V=\frac{\pi}{300}\left((h-7)^3+343\right)\times 1000\text{ L}$ 1A

$V=\frac{10\pi}{3}\left((h-7)^3+343\right)\text{ L}$

3 marks

Mark allocation

- 1 mark for correctly substituting values into formula.
- 1 mark for correct integration.
- 1 mark for correct working, leading to given formula.

3b. ii. **Hence,** find the capacity of the tank when full. Write your answer correct to the nearest litre.

Worked solution

The water tank is full when $h = 2$.

$V=\frac{10\pi}{3}\left((2-7)^3+343\right)$

$V=\frac{2180\pi}{3}$

$V=2283\text{ L}$ 1A

1 mark

Mark allocation

- 1 mark for correct answer.

3c. Suppose the tank is filled initially to a height of 1 m. Liquid then drains from a tap at the base of the tank at the rate of $2\sqrt{h}$ m^3/h. Use calculus to determine how long it takes before the tank is empty. Write your answer correct to the nearest minute.

Worked solution

Given $\dfrac{dV}{dt} = -2\sqrt{h}$ and $V = \dfrac{\pi}{300}\left((h-7)^3 + 343\right)$ m^3

$\Rightarrow \dfrac{dV}{dh} = \dfrac{\pi}{100}(h-7)^2$

Apply the chain rule to related rates problem.

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$-2\sqrt{h} = \frac{\pi}{100}(h-7)^2 \times \frac{dh}{dt} \qquad \text{1M}$$

$$\frac{dh}{dt} = \frac{-2\sqrt{h}}{\frac{\pi}{100}(h-7)^2}$$

$$\frac{dt}{dh} = -\frac{\pi(h-7)^2}{200\sqrt{h}} \qquad \text{1A}$$

$$t = -\frac{\pi}{200}\int \frac{(h-7)^2}{\sqrt{h}}$$

$$t = -\frac{\pi}{200}\int \left(h^{\frac{3}{2}} - 14h^{\frac{1}{2}} + 49h^{-\frac{1}{2}}\right)dh$$

$$t = -\frac{\pi}{200}\left(\frac{2}{5}h^{\frac{5}{2}} - \frac{28}{3}h^{\frac{3}{2}} + 98h^{\frac{1}{2}}\right) + c \qquad \text{1A}$$

Let $t = 0$ when $h = 1$: $\quad 0 = -\dfrac{\pi}{200}\left(\dfrac{2}{5}(1)^{\frac{5}{2}} - \dfrac{28}{3}(1)^{\frac{3}{2}} + 98(1)^{\frac{1}{2}}\right) + c$

$$\therefore c = \frac{167\pi}{375}$$

$$t = -\frac{\pi}{200}\left(\frac{2}{5}h^{\frac{5}{2}} - \frac{28}{3}h^{\frac{3}{2}} + 98h^{\frac{1}{2}}\right) + \frac{167\pi}{375} \qquad \text{1A}$$

When $h = 0$: $\quad t = -\dfrac{\pi}{200}\left(\dfrac{2}{5}(0)^{\frac{5}{2}} - \dfrac{28}{3}(0)^{\frac{3}{2}} + 98(0)^{\frac{1}{2}}\right) + \dfrac{167\pi}{375}$

$$t = \frac{167\pi}{375} = 1.40 \text{ h}$$

$t = 84$ min $\qquad$ 1A

5 marks

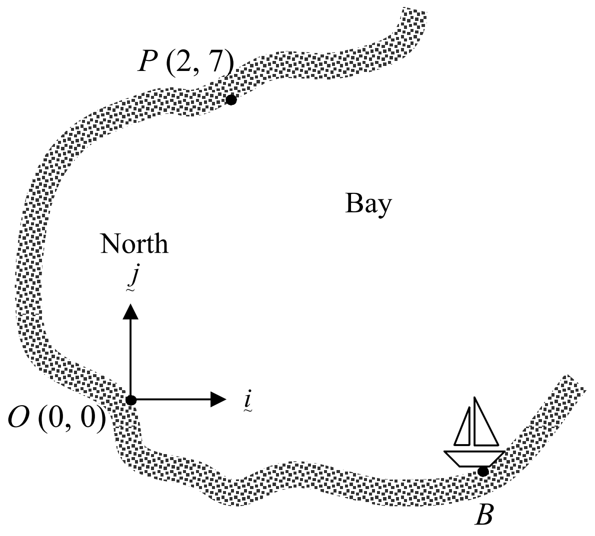
Mark allocation

- 1 mark for applying the chain rule.
- 1 mark for finding $\dfrac{dt}{dh}$.
- 1 mark for correct integration.
- 1 mark for correct expression for *t*.
- 1 mark for correct answer.

Total 2 + 3 + 1 + 5 = 11 marks

Question 4

A yacht, initially at point B, will sail to point P (2, 7) on the other side of the bay. Distances are measured in kilometres in relation to the origin, O.



4a. Write a vector $\overrightarrow{OP}$ that gives the position of point P.

Worked solution

$\overrightarrow{OP} = 2\underset{\sim}{i} + 7\underset{\sim}{j}$ 1A

1 mark

Mark allocation

- 1 mark for correct answer.

The yacht leaves B, sailing with a velocity of $\underset{\sim}{v} = -2\underset{\sim}{i} + 2t\underset{\sim}{j}$ km/h. It reaches P after 3 h.

4b. i. Determine the speed of the yacht when it reaches P.
Write your answer correct to 1 decimal place.

Worked solution

When $t = 3$, $\underset{\sim}{v} = -2\underset{\sim}{i} + (2 \times 3)\underset{\sim}{j}$

$\underset{\sim}{v} = -2\underset{\sim}{i} + 6\underset{\sim}{j}$

Speed $|v| = \sqrt{(-2)^2 + 6^2}$

$|v| = \sqrt{40}$

$|v| = 6.3$ km/h 1A

1 mark

Mark allocation

- 1 mark for correct speed.

4b. ii. Show that the yacht's position at any time t hours is given by the vector $\underset{\sim}{r} = (8-2t)\underset{\sim}{i} + (t^2-2)\underset{\sim}{j}$.

Worked solution

$\underset{\sim}{r} = \int \underset{\sim}{v}\, dt$

$\underset{\sim}{r} = \int \left(-2\underset{\sim}{i} + 2t\underset{\sim}{j}\right) dt$

$\underset{\sim}{r} = -2t\underset{\sim}{i} + t^2\underset{\sim}{j} + \underset{\sim}{c}$ 1M

Now, find $\underset{\sim}{c}$.

At $t = 3$, $\underset{\sim}{r} = 2\underset{\sim}{i} + 7\underset{\sim}{j}$ (given)

$2\underset{\sim}{i} + 7\underset{\sim}{j} = -2\times 3\underset{\sim}{i} + 3^2\underset{\sim}{j} + \underset{\sim}{c}$

$\therefore \underset{\sim}{c} = 8\underset{\sim}{i} - 2\underset{\sim}{j}$

$\underset{\sim}{r} = -2t\underset{\sim}{i} + t^2\underset{\sim}{j} + 8\underset{\sim}{i} - 2\underset{\sim}{j}$ 1A

$\underset{\sim}{r} = (8-2t)\underset{\sim}{i} + (t^2-2)\underset{\sim}{j}$, as required.

2 marks

Mark allocation

- 1 mark for using integration to obtain a position vector.
- 1 mark for correct working, leading to the required expression.

4c. Write a vector $\vec{OB}$ that gives the position of point B.

Worked solution

When $t = 0$, $\underset{\sim}{r} = (8-2\times 0)\underset{\sim}{i} + (0^2-2)\underset{\sim}{j}$

$\therefore \vec{OB} = 8\underset{\sim}{i} - 2\underset{\sim}{j}$ 1A

1 mark

Mark allocation

- 1 mark for correct answer.

4d. Determine $\angle BOP$ in degrees, correct to 1 decimal place.

Worked solution

$$\cos\theta = \frac{\vec{OB}.\vec{OP}}{\left|\vec{OB}\right| \times \left|\vec{OP}\right|}$$

$$\cos\theta = \frac{\left(8\underset{\sim}{i} - 2\underset{\sim}{j}\right).\left(2\underset{\sim}{i} + 7\underset{\sim}{j}\right)}{\sqrt{8^2+(-2)^2} \times \sqrt{2^2+7^2}}$$ 1M

$$\cos\theta = \frac{8\times 2 - 2\times 7}{\sqrt{68}\times\sqrt{53}}$$

SECTION 2 –continued

$\theta = \cos^{-1}\left(\frac{2}{\sqrt{3604}}\right)$

$\angle BOP = 88.1°$ 1A

2 marks

Mark allocation

- 1 mark for using a correct method.
- 1 mark for correct answer.

4e. How long after the yacht commences sailing will it be on a bearing north-east of O? Write your answer in hours, correct to 2 decimal places.

Worked solution

Yacht is north-east of O when the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ component of $\underset{\sim}{r}$ are equal.

This occurs when $8-2t = t^2 - 2$ 1M

$t^2 + 2t - 10 = 0$

From calculator:

$t = -4.32, \quad 2.32 \qquad (t \geq 0)$

After 2.32 h the boat is on a bearing north-east of O. 1A

2 marks

Mark allocation

- 1 mark for using a correct method.
- 1 mark for correct answer.

4f. What is the closest distance the yacht comes to O when sailing from B to P? Give an exact answer.

Worked solution

Distance to origin is $|r| = \sqrt{(8-2t)^2 + (t^2-2)^2}$ 1A

$|r| = \left((8-2t)^2 + (t^2-2)^2\right)^{\frac{1}{2}}$

$\frac{d|r|}{dt} = \frac{1}{2}\left((8-2t)^2 + (t^2-2)^2\right)^{-\frac{1}{2}} \times \left(-4(8-2t) + 4t(t^2-2)\right)$

$0 = \frac{4t^3 - 32}{2\left((8-2t)^2 + (t^2-2)^2\right)^{\frac{1}{2}}}$

$4t^3 - 32 = 0$

$t = 2$ h 1A

The closest distance is $\sqrt{(8-2\times 2)^2 + (2^2-2)^2} = 2\sqrt{5}$ km 1A

3 marks

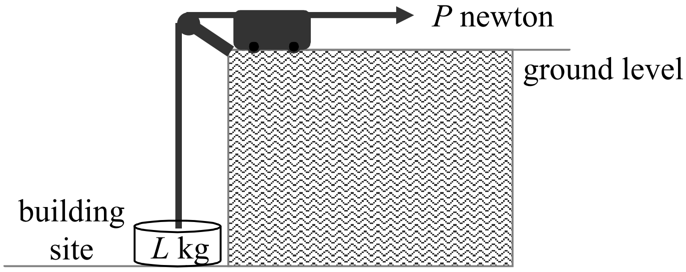
Mark allocation

- 1 mark for finding a correct expression for the distance.
- 1 mark for finding t correctly.
- 1 mark for finding the shortest exact distance.

Total 1 + 1 + 2 + 1 + 2 + 2 + 3 = 12 marks

Question 5

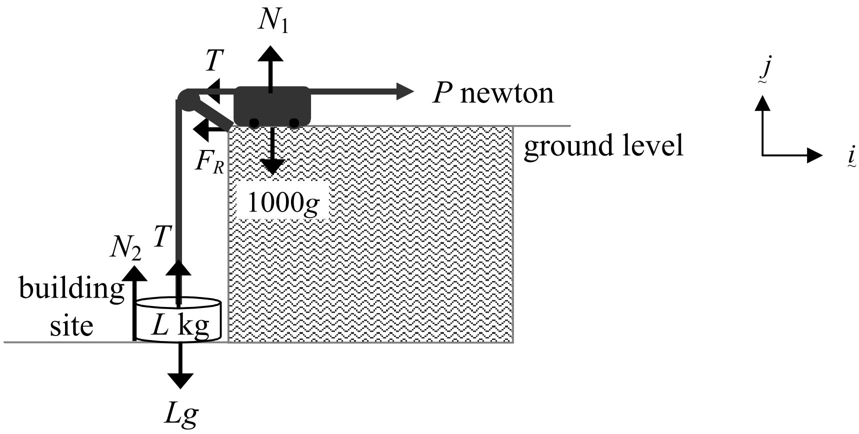
A wire rope passing over a smooth pulley is used to transport materials to and from a building site situated below ground level. A 1 tonne engine at ground level applies a horizontal force of P newton along a track to pull a load of L kg upwards. The coefficient friction between the engine and the track is 0.25.



5a. The load, L kg, is on the point of moving.

5a. i. On the diagram above, show all forces acting.

Worked solution



1A

N_1 is the normal reaction of the ground on the engine.

N_2 is the normal reaction of the building site on the load. (*Note:* If the load is on the point of moving then $N_2 = 0$, so it can be omitted.)

T is the tension in the wire rope.

F_R is the frictional force acting to oppose motion.

1 mark

Mark allocation

- 1 mark for showing all forces.

5a. ii. Find P in terms of L.

Worked solution

Resolving forces around the engine:

$m\underset{\sim}{a} = (P - T - F_R)\underset{\sim}{i} + (N_1 - 1000g)\underset{\sim}{j}$ 1M

Load is not yet moving, so $\underset{\sim}{a} = 0$.

$0\underset{\sim}{i} + 0\underset{\sim}{j} = (P - T - F_R)\underset{\sim}{i} + (N_1 - 1000g)\underset{\sim}{j}$

Equating $\underset{\sim}{j}$ components:

$N_1 - 1000g = 0$
$N_1 = 1000g$
$N_1 = 9800$ newton

$F_R = \mu N_1$
$F_R = 0.25 \times 9800$
$F_R = 2450$ newton 1A

Equating $\underset{\sim}{i}$ components:

$P - T - F_R = 0$
$P = T + F_R$
$P = T + 2450 \quad \ldots(1)$ 1A

Resolving forces around the load:

$0\underset{\sim}{j} = (T + N_2 - Lg)\underset{\sim}{j}$

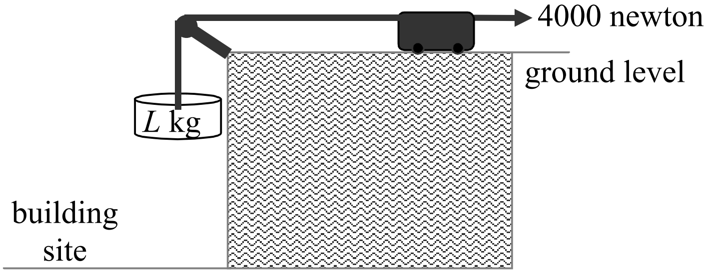
Since load is on the point of moving $N_2 = 0$
$T = Lg$
$T = 9.8L \quad \ldots(2)$
Substituting (2) into (1) gives:
$P = 9.8L + 2450$ newton 1A

4 marks

Mark allocation

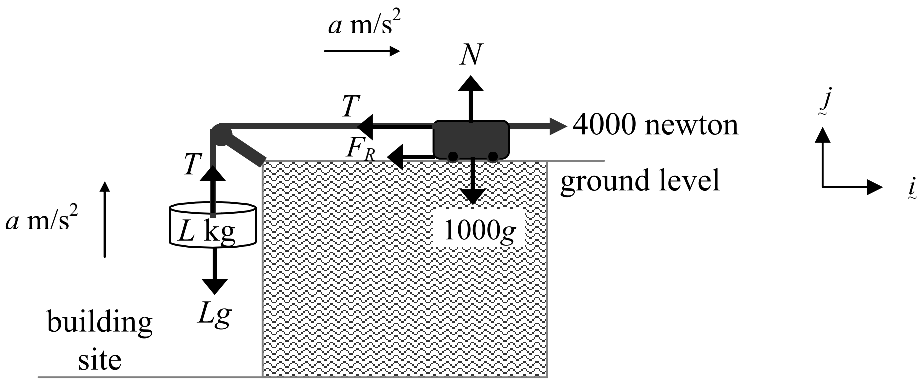
- 1 mark for method used to find the equation of motion for the engine.
- 1 mark for finding the friction (anywhere).
- 1 mark for finding an expression P in terms of T and F_R .
- 1 mark for correct answer.

5b. The engine exerts a horizontal force of 4000 newton. It pulls the load vertically upwards with an acceleration of 0.3 m/s^2.



5b. i. Find the mass of load, L, in kg, correct to 1 decimal place.

Worked solution



b. i.

Resolving forces around the engine:

$m\underset{\sim}{a} = (P - T - F_R)\underset{\sim}{i} + (N - 1000g)\underset{\sim}{j}$ (from part **a**)

$(0.3\times1000)\underset{\sim}{i} + 0\,\underset{\sim}{j} = (4000 - T - 2450)\underset{\sim}{i} + (N - 1000g)\underset{\sim}{j}$ 1A

Equating $\underset{\sim}{i}$ components:

$4000 - T - 2450 = 300$
$T = 4000 - 2450 - 300$
$T = 1250$ newton 1A

Resolving forces around the load:

$m\underset{\sim}{a} = (T - Lg)\underset{\sim}{j}$

$L\times0.3\,\underset{\sim}{j} = (1250 - L\times9.8)\underset{\sim}{j}$

$10.1L = 1250$
$L = 123.8$ kg 1A
The load has a mass of 123.8 kg.

3 marks

Mark allocation

- 1 mark for substituting values correctly into equation of motion for engine.
- 1 mark for correct tension.
- 1 mark for correct answer.

5b. ii. Determine the time taken, in seconds, to raise the load from rest to a point 15 m above the level of the building site.

Worked solution

The load is moving under constant acceleration.

$u = 0,\ s = 15,\ a = 0.3$

$s = ut + \frac{1}{2}at^2$

$15 = 0 \times t + \frac{1}{2} \times 0.3 \times t^2$

$t = \sqrt{100}$

$t = 10$ 1A

It takes 10 s to raise the load 15 m.

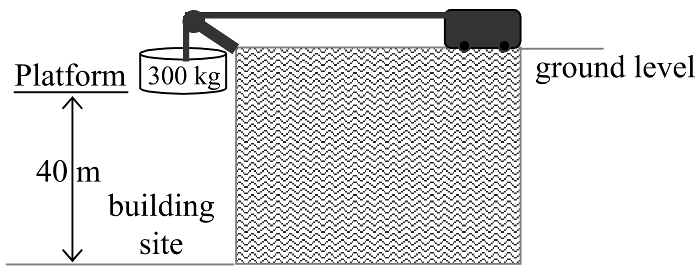
1 mark

Mark allocation

- 1 mark for correct answer.

When the engine reaches the end of the track it applies its breaks so that the load will remain stationary at the loading platform.

The diagram below shows a 300 kg load suspended at the loading platform 40 m above the building site. It is stationary.



SECTION 2 –continued

5c. Find the minimum force that the engine's breaks need to apply in order to keep the 300 kg load stationary in this position.

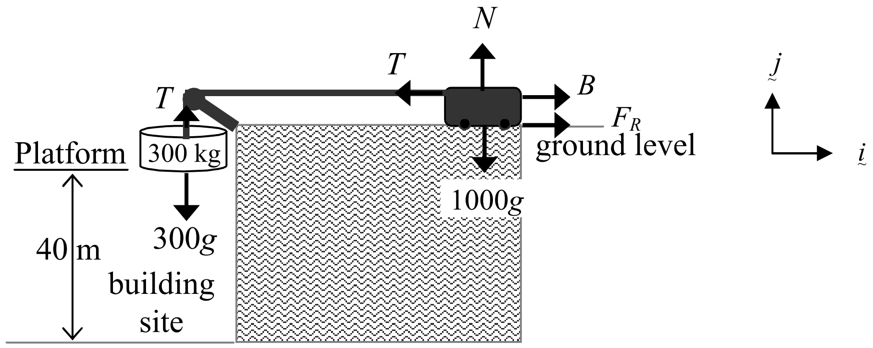
Worked solution

The pulley system is in equilibrium.

Friction is acting in the same direction as the engine's brakes in order to keep the engine stationary. (Otherwise it will be pulled backwards because the load will move down under the force of gravity.)

Let B be the minimum force applied by the engine's brakes.

The forces acting are shown in the diagram below.



Resolving forces around the stationary engine (i.e. $a = 0$):

$m\underset{\sim}{a} = (B + F_R - T)\underset{\sim}{i} + (N - 1000g)\underset{\sim}{j}$ ($F_R = 2450$ newton from part **a**)

$0\underset{\sim}{i} + 0\underset{\sim}{j} = (B + 2450 - T)\underset{\sim}{i} + (N - 1000g)\underset{\sim}{j}$ 1A

Equating $\underset{\sim}{i}$ components:

$B + 2450 - T = 0$ …(1)

Resolving forces around the stationary 300 kg load:
$\underset{\sim}{R} = (300g - T)\underset{\sim}{j}$

$0 = 300g - T$
$T = 300g = 2940$ newton 1A

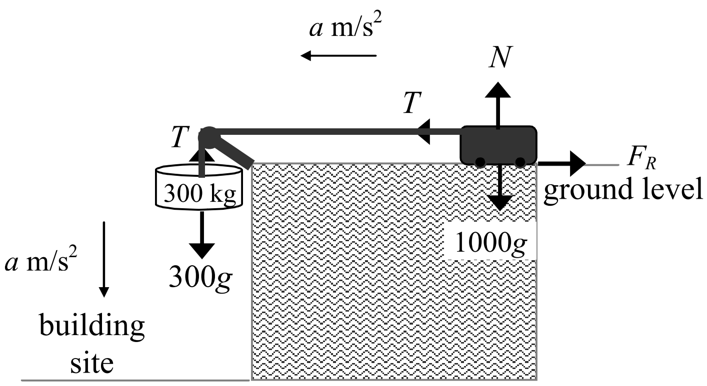
Substitute T into (1):
$B + 2450 - 2940 = 0$
$B = 490$ newton 1A
The minimum force the brakes apply to keep the load stationary is 490 newton.

3 marks

5d. The brakes are released when a load is to be lowered to the building site and the engine moves backwards from rest.

Unfortunately, a problem occurs when the 300 kg load is being lowered. At a point 30 m above the building site, the wire rope breaks and the load falls downwards. Find the speed of the load when it hits the building site. Write your answer in m/s, correct to 1 decimal place.

Worked solution



Resolving horizontal forces around the engine:

$m\underset{\sim}{a} = T - F_R$

$1000\underset{\sim}{a} = T - 2450$ ($F_R = 2450$ newton from part **a**)

$T = 1000a + 2450$ …(1)

Resolving vertical forces around the load:

$300g - T = 300a$ …(2)

Substituting (1) into (2) gives:

$300g - (1000a + 2450) = 300a$

$1300a = 300g - 2450$

$1300a = 490$

$a = \frac{49}{130}$ m/s² 1A

When the rope breaks the load is moving downwards with a constant acceleration of $\frac{49}{130}$ m/s². It has travelled 10 m.

Now, find the velocity of the load.

$a = \frac{49}{130}$, $u = 0$, $s = 10$

$v^2 = u^2 + 2as$

$v^2 = 0^2 + 2 \times \frac{49}{130} \times 10$

$v^2 = \frac{98}{13}$

$v = \sqrt{\frac{98}{13}}$ m/s 1A

After the rope breaks, the load falls 30 m to the building site under the force of gravity.

Need to find the velocity of the load when it hits the building site.

$a = g$, $u = \sqrt{\frac{98}{13}}$, $s = 30$

$v^2 = u^2 + 2as$

$v^2 = \left(\sqrt{\frac{98}{13}}\right)^2 + 2g \times 30$

$v^2 = 595.54$

$v = 24.4$ m/s 1A

3 marks

Mark allocation

- 1 mark for finding the acceleration of the system initially.
- 1 mark for finding the velocity of the mass when the rope breaks.
- 1 mark for finding the correct answer.

Total 1 + 4 + 3 + 1 + 3 + 3 = 15 marks

END OF WORKED SOLUTIONS