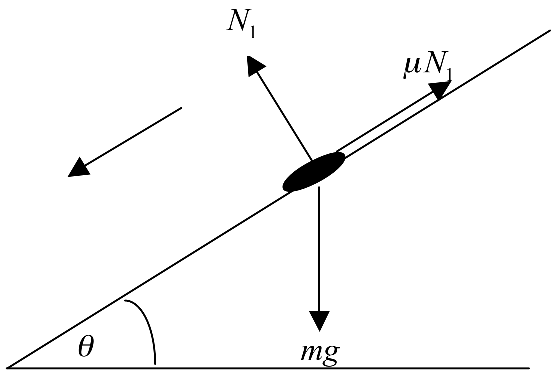
SPECIALIST MATHEMATICS EXAM 1 SOLUTIONS

Question 1

a. i.



A1

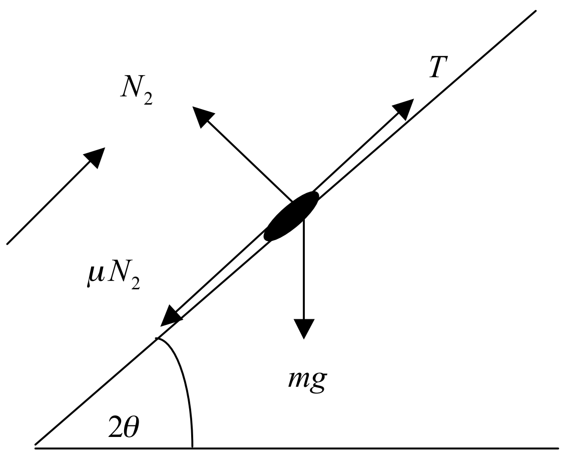
ii. Resolving down and parallel to the plane $mg\sin(\theta) - \mu N_1 = 0 \quad (1)$

Resolving perpendicular to the plane $N_1 - mg\cos(\theta) = 0 \quad (2)$

from (2) $N_1 = mg\cos(\theta)$ into (1) $mg\sin(\theta) = \mu mg\cos(\theta)$

so that $\mu = \tan(\theta)$ shown A1

b. i.



A1

ii. Resolving up and parallel to the plane $T - mg\sin(2\theta) - \mu N_2 = 0 \quad (3)$

Resolving perpendicular to the plane $N_2 - mg\cos(2\theta) = 0 \quad (4)$

from (4) $N_2 = mg\cos(2\theta)$ into (3) $T = \mu mg\cos(2\theta) + mg\sin(2\theta)$

but $\mu = \tan(\theta)$

$T = mg\left(\sin(2\theta) + \tan(\theta)\cos(2\theta)\right)$ M1

$T = mg\left(\sin(2\theta) + \frac{\sin(\theta)}{\cos(\theta)}\cos(2\theta)\right)$ A1

$T = mg\left(\frac{\sin(2\theta)\cos(\theta) + \sin(\theta)\cos(2\theta)}{\cos(\theta)}\right)$ by addition theorems

$T = \frac{mg\sin(3\theta)}{\cos(\theta)}$ shown A1

Question 2

$y = \cos\left(x^2\right)$ using the chain rule

$\frac{dy}{dx} = -2x\sin\left(x^2\right)$ now differentiating using the product rule A1

$\frac{d^2y}{dx^2} = -2\sin\left(x^2\right) - 4x^2\cos\left(x^2\right)$ A1

substituting into $x\frac{d^2y}{dx^2} + a\frac{dy}{dx} + bx^3y = 0$

$x\left(-2\sin\left(x^2\right) - 4x^2\cos\left(x^2\right)\right) - 2ax\sin\left(x^2\right) + bx^3\cos\left(x^2\right) = 0$ M1

$(b-4)x^3\cos\left(x^2\right) - 2x(a+1)\sin\left(x^2\right) = 0$

so that $b = 4$ $a = -1$ A1

Question 3

$2x^2 + 12x + y^2 - 8y + 22 = 0$ using implicit differentiation

$\frac{d}{dx}\left(2x^2\right) + \frac{d}{dx}(12x) + \frac{d}{dx}\left(y^2\right) - \frac{d}{dx}(8y) + \frac{d}{dx}(22) = 0$

$4x + 12 + 2y\frac{dy}{dx} - 8\frac{dy}{dx} = 0$ M1

$4x + 12 = (8 - 2y)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{4x+12}{8-2y} = \frac{2(x+3)}{4-y}$ A1

when $\frac{dy}{dx} = 0$ $x = -3$ A1

Question 4

a. $P(z) = z^4 + pz^2 - 8$

$P(2i) = (2i)^4 + p(2i)^2 - 8 = 0$

$P(2i) = 16 - 4p - 8 = 0$

$4p = 8$

so that $p = 2$ A1

b. $P(z) = z^4 + 2z^2 - 8$

$P(z) = \left(z^2 + 4\right)\left(z^2 - 2\right) = 0$ M1

$z = \pm 2i \quad \pm\sqrt{2}$ A1

Question 5

a. $y = \tan^{-1}\left(\frac{3x^2}{4}\right) = \tan^{-1}\left(\frac{u}{4}\right)$ where $u = 3x^2$

$\frac{dy}{du} = \frac{4}{16+u^2}$ $\frac{du}{dx} = 6x$ M1

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = \frac{24x}{16+u^2}$

$\frac{dy}{dx} = \frac{24x}{9x^4+16}$ A1

b. $\int \frac{24x}{9x^4+16}dx = \tan^{-1}\left(\frac{3x^2}{4}\right)$

$\int_0^{\frac{2\sqrt{3}}{3}} \frac{x}{9x^4+16}dx$

$= \left[\frac{1}{24}\tan^{-1}\left(\frac{3x^2}{4}\right)\right]_0^{\frac{2\sqrt{3}}{3}}$ M1

$= \frac{1}{24}\left(\tan^{-1}\left(\frac{3}{4} \times \frac{12}{9}\right) - \tan^{-1}(0)\right)$

$= \frac{1}{24}\left(\tan^{-1}(1) - \tan^{-1}(0)\right)$

$= \frac{\pi}{96}$ A1

Question 6

a. $y = \frac{x}{\sqrt{2x-3}} = \frac{u}{v}$ quotient rule

$u = x$ $v = \sqrt{2x-3} = (2x-3)^{\frac{1}{2}}$

$\frac{du}{dx} = 1$ $\frac{dv}{dx} = \frac{1}{2} \times 2(2x-3)^{-\frac{1}{2}} = \frac{1}{\sqrt{2x-3}}$ M1

$\frac{dy}{dx} = \frac{\sqrt{2x-3} - \frac{x}{\sqrt{2x-3}}}{2x-3}$ M1

$\frac{dy}{dx} = \frac{\left[\frac{(2x-3)-(x)}{\sqrt{2x-3}}\right]}{2x-3}$

$\frac{dy}{dx} = \frac{\left[\frac{x-3}{\sqrt{2x-3}}\right]}{2x-3} = \frac{x-3}{\sqrt{(2x-3)^3}}$

Therefore $a = 1$ $b = -3$ A1

b. $A = \int_2^6 \frac{x}{\sqrt{2x-3}}\,dx$

let $u = 2x-3$ $\frac{du}{dx} = 2$ $dx = \frac{du}{2}$ $2x = u+3$ $x = \frac{1}{2}(u+3)$

terminals when $x = 6$ $u = 9$ and when $x = 2$ $u = 1$ M1

$$A = \frac{1}{4}\int_1^9 \frac{u+3}{\sqrt{u}}\,du$$

$$A = \frac{1}{4}\int_1^9 \left(u^{\frac{1}{2}} + 3u^{-\frac{1}{2}}\right)du$$

$$A = \frac{1}{4}\left[\frac{2}{3}u^{\frac{3}{2}} + 6u^{\frac{1}{2}}\right]_1^9$$ A1

$$A = \frac{1}{4}\left(\left(\frac{2}{3} \text{ x } 9^{\frac{3}{2}} + 6\sqrt{9}\right) - \left(\frac{2}{3} \text{ x } 1^{\frac{3}{2}} + 6\sqrt{1}\right)\right)$$

$$A = \frac{1}{4}\left(18 + 18 - \frac{2}{3} - 6\right)$$

$A = 7\frac{1}{3}$ square units A1

Question 7

a. $\underset{\sim}{r}(t) = (3 + 4\cos(2t))\underset{\sim}{i} + (-2 + 3\sin(2t))\underset{\sim}{j}$ for $t \geq 0$

$x = 3 + 4\cos(2t)$ $\cos(2t) = \frac{x-3}{4}$ M1

$y = -2 + 3\sin(2t)$ $\sin(2t) = \frac{y+2}{3}$

$\sin^2(2t) + \cos^2(2t) = 1$

$$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{9} = 1$$ A1

b. $\dot{\underset{\sim}{r}}(t) = -8\sin(2t)\underset{\sim}{i} + 6\cos(2t)\underset{\sim}{j}$ M1

$$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{64\sin^2(2t) + 36\cos^2(2t)}$$

$$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{64\sin^2(2t) + 36(1 - \sin^2(2t))}$$

$$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{36 + 28\sin^2(2t)}$$

when $\sin(2t) = 1$ $\left|\dot{\underset{\sim}{r}}(t)\right|_{\max} = \sqrt{64} = 8$ A1

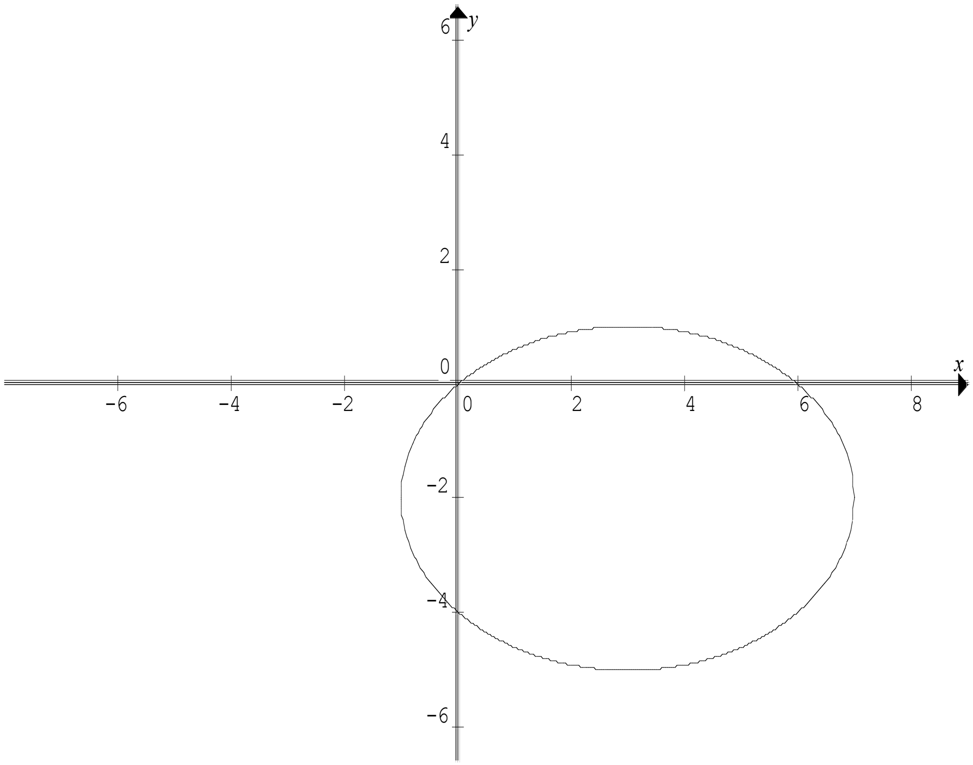
when $\sin(2t) = 0$ $\left|\dot{\underset{\sim}{r}}(t)\right|_{\min} = \sqrt{36} = 6$ A1

Question 8

a. The domain is $\left[-1,7\right]=\left[c-4,c+4\right]$ so that $c=3$ A1

b. $$\frac{\left(x-3\right)^2}{16}+\frac{\left(y+2\right)^2}{9}=1$$

ellipse centre $\left(3,-2\right)$ semi-major axes 4, semi-minor 3 A1



Correct graph A1

Question 9

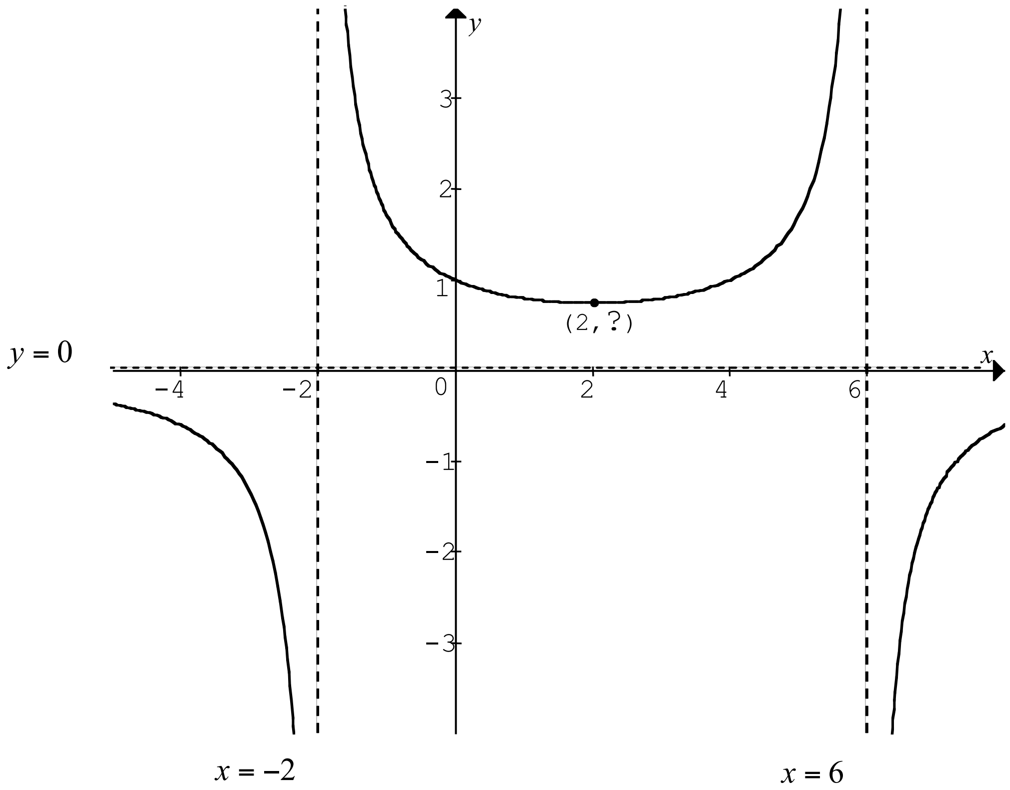
a. $$y=\frac{12}{12+4x-x^2}=\frac{12}{\left(x+2\right)\left(6-x\right)}$$

vertical asymptotes at $x=-2$ and $x=6$ A1

horizontal asymptote at $y=0$

turning point of $12+4x-x^2$ occurs when $4-2x=0$ $x=2$ $f\left(2\right)=\frac{12}{16}=\frac{3}{4}$

turning point at $\left(2,\frac{3}{4}\right)$ A1



Correct graph A1

b. the area $A = \int_0^3 \frac{12}{12+4x-x^2}\,dx = \int_0^3 \frac{12}{(x+2)(6-x)}\,dx$

by partial fractions $\frac{12}{12+4x-x^2} = \frac{A}{x+2} + \frac{B}{6-x} = \frac{12}{(x+2)(6-x)}$

$\frac{A(6-x)+B(x+2)}{(x+2)(6-x)} = \frac{x(B-A)+6A+2B}{12+4x-x^2}$

$(1)\ B-A=0$ and $(2)\ 6A+2B=12$

from $(1)\ A=B$ into (2)

$8B=12$ so that $A=B=\frac{3}{2}$ Note that alternative methods are possible M1

$A = \frac{3}{2}\int_0^3 \left(\frac{1}{x+2} + \frac{1}{6-x}\right)dx$

$A = \frac{3}{2}\left[\log_e|x+2| - \log_e|6-x|\right]_0^3$

$A = \frac{3}{2}\left[\log_e\left|\frac{x+2}{6-x}\right|\right]_0^3$ M1

$A = \frac{3}{2}\left(\log_e\left(\frac{5}{3}\right) - \log_e\left(\frac{1}{3}\right)\right)$

$A = \frac{3}{2}\log_e(5) = \log_e\left(\sqrt{5^3}\right) = \log_e\left(\sqrt{125}\right) = \log_e\left(\sqrt{p}\right)$

so $p = 125$ A1