INSIGHT

Trial Exam Paper

2006

SPECIALIST MATHEMATICS

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations.

SECTION 1

Question 1

If $2i$ is a solution of the equation $z^3 - 5z^2 + 4z - mi = 0$, then the value of m will be

A. $-2i$

B. **−20i**

C. -20

D. 20

E. $20i$

Answer is B

Worked solution

Let $z = 2i$

$(2i)^3 - 5(2i)^2 + 4(2i) - mi = 0$

$-8i + 20 + 8i - mi = 0$

$20 - mi = 0$

$m = \dfrac{20}{i}$

$m = \dfrac{20}{i} \times \dfrac{i}{i}$

$m = -20i$

Question 2

If $z = -1 + \sqrt{3}$, then Arg(z^2) equals

A. $-\frac{2\pi}{3}$

B. $-\frac{\pi}{3}$

C. $\frac{\pi}{3}$

D. $\frac{2\pi}{3}$

E. $\frac{4\pi}{3}$

Answer is A

Worked solution

$z = -1 + \sqrt{3} = r\text{cis}\theta$

$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$z = 2\text{cis}\left(\frac{2\pi}{3}\right)$

$z^2 = 2^2\,\text{cis}\left(2 \times \frac{2\pi}{3}\right)$ by De Moivre's Theorem

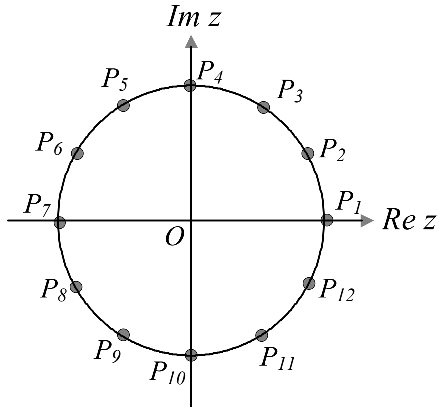
$z^2 = 4\text{cis}\left(\frac{4\pi}{3}\right)$

$z^2 = 4\text{cis}\left(-\frac{2\pi}{3}\right)$

$\therefore \text{Arg}(z^2) = -\frac{2\pi}{3}$

Question 3

Points P_1 to P_{12} are twelve equally spaced points around the circumference of a circle.



Point P_3 represents the complex number $z = a + ib$.

The complex number $i^{11}\bar{z}$ is represented by point

A. P_2

B. P_5

C. $\boldsymbol{P_8}$

D. P_9

E. P_{11}

Answer is C

Worked solution

Find expressions for complex numbers P_2, P_5, P_8, P_9, P_{11} in terms of a and b

$\angle P_1OP_2 = \frac{360°}{12} = 30° \quad \Rightarrow \quad \angle P_1OP_3 = 60°$

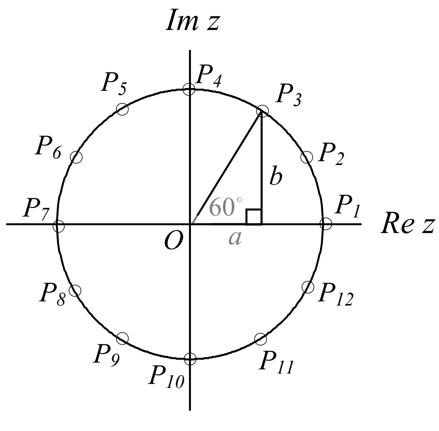
$z = a + bi$

$\tan 60° = \frac{b}{a}$

$\Rightarrow \sqrt{3} = \frac{b}{a}$

Taking reciprocals: $\frac{1}{\sqrt{3}} = \frac{a}{b}$

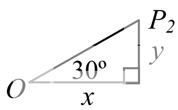
$\tan 30° = \frac{a}{b}$ K (1)



Finding P_2

Let $P_2 = x + yi$

$\tan 30° = \frac{y}{x}$



$\frac{a}{b} = \frac{y}{x}$ from (1)

$\therefore \quad a = y, \quad b = x$

Therefore $P_2 = b + ai$

By symmetry :
$P_5 = -a + bi, \quad P_6 = -b + ai$
$P_8 = -b - ai, \quad P_9 = -a - bi$
$P_{11} = a - bi, \quad P_{12} = b - ai$

Simplify $i^{11}\bar{z}$:
$i^{11}\bar{z} = i^{11}(a - bi)$
$= -i(a - bi)$
$= -ai + bi^2$
$= -ai - b$
$= -b - ai$
Point P_8

Question 4

The range of the function $f(x) = \cos^{-1}(x - \pi) - 1$ is

A. $[\pi - 1, \ \pi + 1]$

B. $[-\mathbf{1}, \ \boldsymbol{\pi} - \mathbf{1}]$

C. $[0, \ \pi]$

D. $[-2, \ 0]$

E. $[-1, \ 1]$

Answer is B

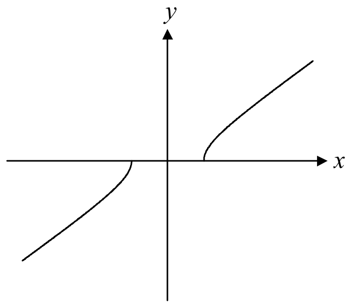
Worked solution

$g(x) = \cos^{-1}(x)$ has range $[0, \ \pi]$. This graph is translated horizontally by π units and vertically by -1 unit to give $f(x) = \cos^{-1}(x - \pi) - 1$
The range of this function is $[-1, \ \pi - 1]$

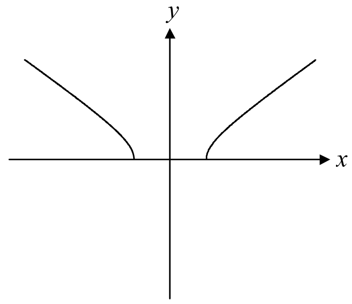
Question 5

A graph of the curve specified by the parametric equations $x = \sec(t)$, $y = \tan(t)$ where $t \in [0, \pi]$ could be

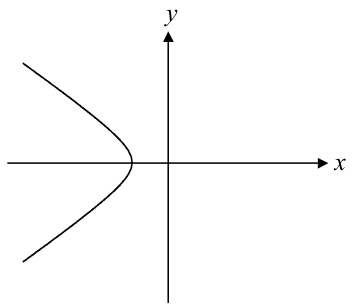
A.



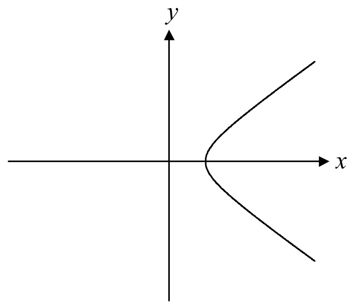
B.



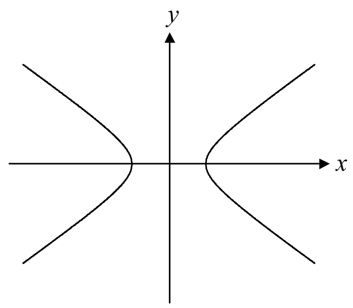
C.



D.



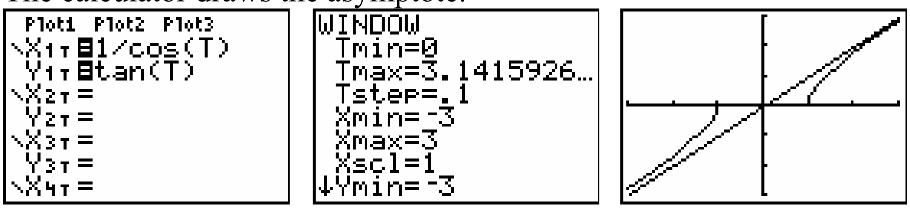
E.



Answer is A

Worked solution

When $t \in \left[0, \frac{\pi}{2}\right)$, $x = \sec(t) = \frac{1}{\cos(t)}$ is positive and $y = \tan(t)$ is positive.
Since both x and y are positive, a branch of the graph will be in the first quadrant.
When $t \in \left(\frac{\pi}{2}, \pi\right]$, $x = \sec(t)$ is negative and $y = \tan(t)$ is negative.
Since both x and y are negative, a branch of the graph will be in the third quadrant.
A and E satisfy these conditions, but E shows four quadrants, i.e. $t \in [0, 2\pi]$
The solution can found by sketching the graph on a calculator in parametric mode.
The calculator draws the asymptote.



Question 6

Consider the function $f : R \to R$ where $f(x) = 4x^3 - 3x^4$

Which one of the following statements is not true?

A. f has two stationary points

B. f has two points of inflexion

C. $\boldsymbol{f'}$ **is maximum when** $\boldsymbol{x = \frac{2}{3}}$

D. $\frac{1}{f}$ has three asymptotes

E. $f = \frac{1}{f}$ has three solutions

Answer is C

Worked solution

Graphing $f(x) = 4x^3 - 3x^4$ shows that stationary points occur at $x = 0$ and $x = 1$.
Therefore A is true.

$f'(x) = 12x^2 - 12x^3$

$f''(x) = 24x - 36x^2$

Points of inflexion occur where $f''(x) = 0$.

$24x - 36x^2 = 0$

$12x(2 - 3x) = 0$

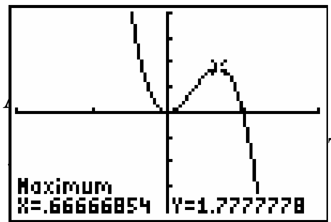
$x = 0, \quad x = \frac{2}{3}$

$f(x)$ has two points of inflexion. Therefore B is true.

$x = 0$ is a stationary point of inflexion.

$x = \frac{2}{3}$ is the point of maximum gradient over the interval $\left(-\frac{1}{3}, \quad \infty\right)$.

Sketching a graph of $f'(x)$ shows a local maximum at $x = \frac{2}{3}$; however this is not the maximum value of $f'(x)$ over $x \in R$.

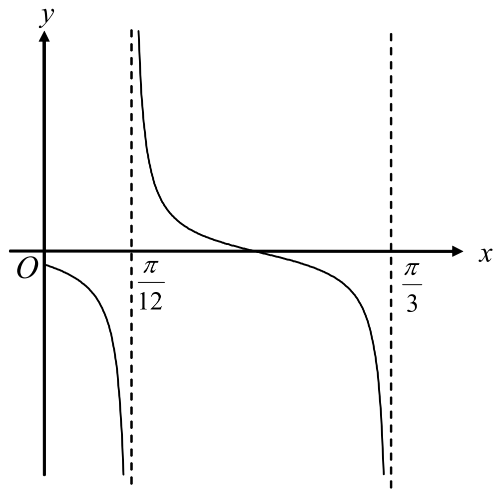


Therefore C is not true.

Graphing $f(x)$ and $\frac{1}{f(x)}$ will show that both D and E are true.

Question 7

A graph of $f:\left[0, \ \dfrac{\pi}{3}\right)$ where $f(x) = \cot\left(nx - \dfrac{\pi}{3}\right)$ is sketched below.



The value of n could be

A. $\dfrac{1}{4}$

B. $\dfrac{1}{3}$

C. 3

D. **4**

E. 8

Answer is D

Worked solution

Period of $y = \cot(nx - \dfrac{\pi}{3})$ is $\dfrac{\pi}{n}$

Period of this graph is $\dfrac{\pi}{3} - \dfrac{\pi}{12} = \dfrac{\pi}{4}$

$\Rightarrow \dfrac{\pi}{n} = \dfrac{\pi}{4}$

$\therefore \quad n = 4$

Question 8

The gradient of the curve $y^2 = 4x + 6y - 5$ is $-\frac{2}{3}$ at the point where y equals

A. **0**

B. 0.15

C. 1.25

D. 5

E. 6

Answer is A

Worked solution

$y^2 = 4x + 6y - 5$

Using implicit differentiation:

$2y\frac{dy}{dx} = 4 + 6\frac{dy}{dx} + 0$

$(2y - 6)\frac{dy}{dx} = 4$

$\frac{dy}{dx} = \frac{4}{2y - 6}$

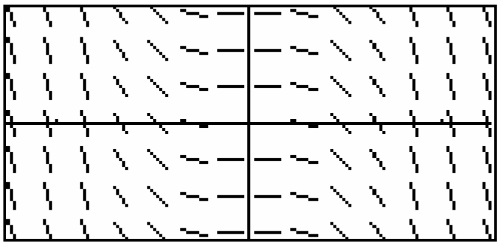
$-\frac{2}{3} = \frac{4}{2y - 6}$

$-4y + 12 = 12$

$y = 0$

Question 9

The slope field from a first order differential equation is shown below.



If $a \in R$, a solution of this differential equation could be

A. $y = a\log_e(x)$

B. $y = a\cos(x)$

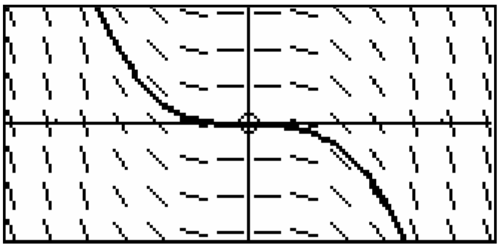
C. $y = a\tan^{-1}(x)$

D. $y = \frac{a}{x^2}$

E. $\boldsymbol{y = ax^3}$

Answer is E

Worked solution



$y = ax^3$ where $a \in (-\infty, 0)$

Question 10

Given $\frac{dy}{dx} = \sqrt{\sin(2x)}$ and $y = \sqrt{2}$ when $x = \frac{\pi}{12}$.

The value of y when $x = \frac{\pi}{3}$ is

A. 0.2500

B. 0.7298

C. 0.9306

D. **1.4369**

E. 2.1440

Answer is D

Worked solution

$$y = \int \sqrt{\sin(2x)}\, dx$$
$$y = f(x) + c$$

When $\quad x = \frac{\pi}{12}, \quad y = \frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} = f\left(\frac{\pi}{12}\right) + c$$

$$c = -f\left(\frac{\pi}{12}\right) + \frac{1}{\sqrt{2}} \qquad \ldots(1)$$

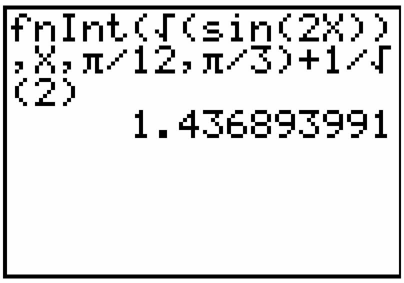
When $\quad x = \frac{\pi}{3}, \quad y = f\left(\frac{\pi}{3}\right) + c \qquad \ldots(2)$

Substitute (1) into (2):

$$y = f\left(\frac{\pi}{3}\right) - f\left(\frac{\pi}{12}\right) + \frac{1}{\sqrt{2}}$$

$$y = \int_{\frac{\pi}{12}}^{\frac{\pi}{3}} \sqrt{\sin(2x)}\, dx + \frac{1}{\sqrt{2}}$$

Using fnInt on calculator:



Question 11

Using a suitable substitution, $\int_5^{10} \frac{1}{x^2} e^{\frac{10}{x}}\, dx$ can be expressed as

A. $\int_1^2 \frac{100}{u^2} e^u\, du$

B. $100 \int_1^2 u^2 e^u\, du$

C. $-10 \int_2^1 e^u\, du$

D. $\mathbf{\frac{1}{10} \int_1^2 e^u\, du}$

E. $-\frac{1}{10} \int_5^{10} e^u\, du$

Answer is D

Worked solution

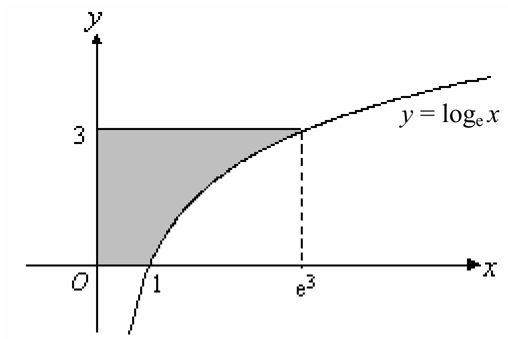
Let $u = \frac{10}{x}$, $\quad \frac{du}{dx} = -\frac{10}{x^2} \qquad \Rightarrow du = -\frac{10}{x^2}\, dx$

Finding the terminals of integration: When $x = 10, \quad u = \frac{10}{10} = 1$

When $x = 5, \quad u = \frac{10}{5} = 2$

$$\begin{aligned}
\int_5^{10} \frac{1}{x^2} e^{\frac{10}{x}}\, dx &= -\frac{1}{10} \int_5^{10} -\frac{10}{x^2} e^{\frac{10}{x}}\, dx \\
&= -\frac{1}{10} \int_5^{10} e^{\frac{10}{x}} \left(-\frac{10}{x^2}\, dx \right) \\
&= -\frac{1}{10} \int_2^1 e^u\, du \\
&= \frac{1}{10} \int_1^2 e^u\, du
\end{aligned}$$

Question 12



The graph of $y = \log_e x$ is shown above. The volume of the solid of revolution formed when the shaded region is rotated around the y-axis is given by

A. $\pi \int_0^3 (3 - \log_e x)^2 \, dx$

B. $\pi \int_1^{e^3} (\log_e x)^2 \, dx$

C. $\pi \int_1^{e^3} (3 - e^y)^2 \, dy$

D. $\pi \int_0^3 e^y \, dy$

E. $\boldsymbol{\pi \int_0^3 e^{2y} \, dy}$

Answer is E

Worked solution

Rotating around the y-axis: Volume $= \pi \int_0^3 x^2 dy$

Finding expression for x^2: $y = \log_e x$

$x = e^y$

$x^2 = (e^y)^2$

$V = \pi \int_0^3 (e^y)^2 \, dy$

$V = \pi \int_0^3 e^{2y} \, dy$

Question 13

A spherical ice ball initially has radius 0.9 cm. It is placed in a drink and melts at a constant rate of 1.5 cm^3/minute. When the radius is 0.6 cm, the rate, in cm/minute, at which the radius is decreasing is

A. $\frac{5}{24\pi}$

B. $\frac{25}{72\pi}$

C. $\mathbf{\frac{25}{24\pi}}$

D. $\frac{54\pi}{25}$

E. $\frac{36\pi}{25}$

Answer is C

Worked solution

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$1.5 = 4\pi r^2 \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1.5}{4\pi r^2}$$

When $r = 0.6$ cm

$$\frac{dr}{dt} = \frac{1.5}{4\pi \times 0.6^2} = \frac{25}{24\pi}$$

Question 14

A tank initially contains 200 litres of pure water. A salt solution with a concentration of 0.2 kg/litre is poured into the tank at a rate of 5 litres/minute. The mixture is kept uniform by stirring and flows out of the tank at a rate of 3 litres/minute.

Let Q be the amount of salt in the tank after t minutes.

$\frac{dQ}{dt}$ is equal to

A. $5-\frac{3Q}{200+2t}$

B. $5-\frac{3Q}{200}$

C. $(5-3t)\frac{Q}{200}$

D. $1-\frac{3Q}{200-2t}$

E. $\mathbf{1-\frac{3Q}{200+2t}}$

Answer is E

Worked solution

The volume of mixture in the tank after t minutes is $200+2t$ litres

The concentration of salt in the tank after t minutes is $\frac{Q}{200+2t}$ kg/litre

Rate of inflow of salt is $5\times 0.2=1$ kg/minute

Rate of outflow of salt is $3\times\frac{Q}{200+2t}$ kg/minute

$\frac{dQ}{dt}=$ rate of inflow – rate of outflow

$\frac{dQ}{dt}=1-\frac{3Q}{200+2t}$ kg/minute

Question 15

Let $\underset{\sim}{u} = 6\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$.

The vector resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$ is

A. $\frac{1}{49}(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$

B. $\frac{1}{7}(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$

C. $\mathbf{\frac{1}{14}(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})}$

D. $\frac{1}{\sqrt{14}}(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$

E. $\frac{1}{7\sqrt{14}}(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$

Answer is C

Worked solution

Vector resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$ is

$$(\underset{\sim}{u}.\hat{\underset{\sim}{v}})\hat{\underset{\sim}{v}} = \left((6\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}).\frac{(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})}{\sqrt{4+1+9}}\right).\frac{(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})}{\sqrt{4+1+9}}$$

$$= \left(\frac{12-2-9}{\sqrt{14}}\right)\frac{(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})}{\sqrt{14}}$$

$$= \frac{1}{14}(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$$

Question 16

Points A, B and C are collinear such that $AB : BC = 1:3$

If $\vec{OA} = \underset{\sim}{a}$ and $\vec{OC} = \underset{\sim}{c}$ then $\vec{OB}$ equals

A. $\mathbf{\frac{1}{4}(3\underset{\sim}{a} + \underset{\sim}{c})}$

B. $\frac{1}{4}(\underset{\sim}{a} + 3\underset{\sim}{c})$

C. $\frac{1}{4}(5\underset{\sim}{a} - \underset{\sim}{c})$

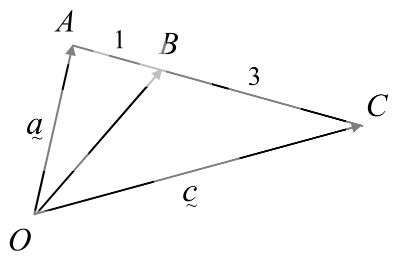
D. $\frac{1}{3}(2\underset{\sim}{a} + \underset{\sim}{c})$

E. $\frac{1}{3}(\underset{\sim}{a} - 3\underset{\sim}{c})$

Answer is A

Worked Solution

Draw a vector diagram



$\vec{AC} = -\underset{\sim}{a} + \underset{\sim}{c}$

$\vec{AB} = \frac{1}{4}\vec{AC} = \frac{1}{4}(-\underset{\sim}{a} + \underset{\sim}{c})$

$\vec{OB} = \vec{OA} + \vec{AB}$

$\vec{OB} = \underset{\sim}{a} + \frac{1}{4}(-\underset{\sim}{a} + \underset{\sim}{c})$

$\vec{OB} = \frac{3}{4}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{c}$

$\vec{OB} = \frac{1}{4}(3\underset{\sim}{a} + \underset{\sim}{c})$

Question 17

The position of a particle at time t is given by $\underset{\sim}{r}(t) = (t^3 + 2t)\underset{\sim}{i} + 5t\underset{\sim}{j} - t^2\underset{\sim}{k}$.

The magnitude of its acceleration when $t = 1$ is

A. $3\underset{\sim}{i} + 5\underset{\sim}{j} - \underset{\sim}{k}$

B. $6\underset{\sim}{i} - 2\underset{\sim}{k}$

C. $\mathbf{2\sqrt{10}}$

D. $3\sqrt{6}$

E. $\sqrt{35}$

Answer is C

Worked solution

$\underset{\sim}{r}(t) = (t^3 + 2t)\underset{\sim}{i} + 5t\underset{\sim}{j} - t^2\underset{\sim}{k}$

$\dot{\underset{\sim}{r}}(t) = (3t^2 + 2)\underset{\sim}{i} + 5\underset{\sim}{j} - 2t\underset{\sim}{k}$

$\ddot{\underset{\sim}{r}}(t) = 6t\underset{\sim}{i} - 2\underset{\sim}{k}$

$\ddot{\underset{\sim}{r}}(1) = 6\underset{\sim}{i} - 2\underset{\sim}{k}$

$\left|\ddot{\underset{\sim}{r}}(1)\right| = \sqrt{6^2 + (-2)^2} = \sqrt{40} = 2\sqrt{10}$

Question 18

A particle is moving in a straight line with an acceleration of $-20x+20$ m/s^2, where x is its displacement, in metres, from a fixed point O. If the particle is travelling with a velocity of 6 m/s when it is 3 metres to the right of O, its maximum speed, in m/s, is

A. 6.0

B. 9.8

C. 10.0

D. **10.8**

E. 12.0

Answer is D

Worked solution

$a=-20x+20$

$\frac{d\left(\frac{1}{2}v^2\right)}{dx}=-20x+20$

$\frac{1}{2}v^2=\frac{-20}{2}x^2+20x+c$

When $x=3,\quad v=6:$

$\frac{1}{2}\times 6^2=-10\times 3^2+20\times 3+c$

$18=-90+60+c$

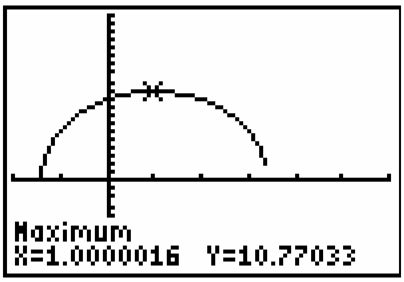
$c=48$

$\Rightarrow\quad \frac{1}{2}v^2=-10x^2+20x+48$

$v^2=-20x^2+40x+96$

$v=\sqrt{-20x^2+40x+96}$

Draw a graph of velocity on calculator:



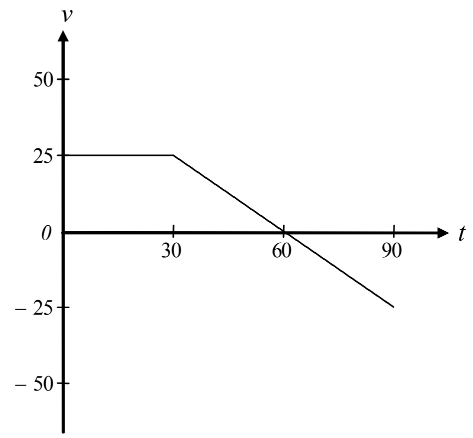
The maximum speed is 10.8 m/s. This occurs when the particle is 1 m from O.

Question 19

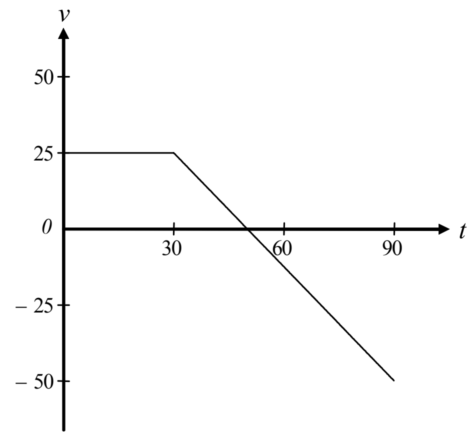
A particle travels in a straight line with a constant velocity of 25 m/s for 30 seconds. It then decelerates for 60 seconds and returns to its original position.

The velocity-time graph that best represents the motion of the particle is

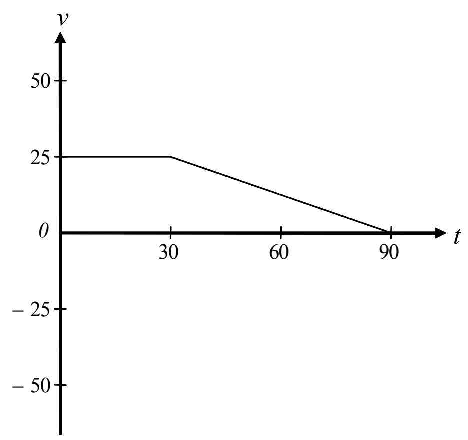
A.



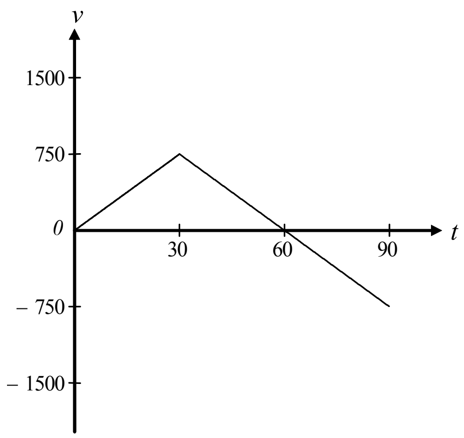
B.



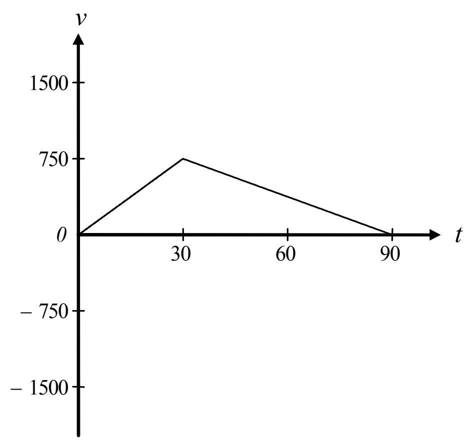
C.



D.



E.



Answer is B

Worked solution

Velocity is constant for the first 30 seconds, therefore discard D and E, as these graphs show constant acceleration.

The total signed area under velocity-time graph must be zero after 90 seconds since the particle returns to its original position.

Discard A and C since the total signed area is not zero.

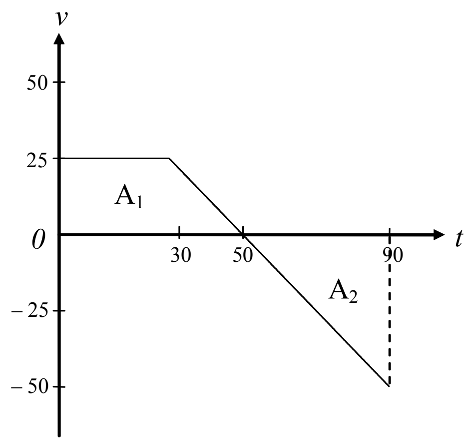
For B, calculate the *t* intercept of the line segment joining (30, 25) and (90, –50):

$$\frac{25-0}{30-t}=\frac{25+50}{30-90}$$

$$\frac{25}{30-t}=\frac{75}{-60}$$

$$-1500=2250-75t$$

$$t=50$$



Calculate the signed area of trapezium A_1.

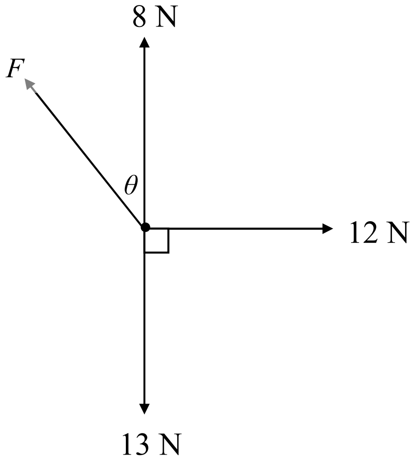
$$A_1=\frac{1}{2}(50+30)\times 25=1000$$

Calculate the signed area of triangle A_2

$$A_2=\frac{1}{2}\times 40\times -50=-1000$$

Total area $A_1+A_2=0$

Question 20



Four forces are acting on a particle as shown in the diagram above.

The particle will be in equilibrium when F, measured in newtons, is equal to

A. $5\cos\theta$

B. $12\sin\theta$

C. $\dfrac{\cos\theta}{12}$

D. 5

E. **13**

Answer is E

Worked solution

Resolving forces in a horizontal direction:

$$F\sin\theta = 12 \quad \text{K (1)}$$

Resolving forces in a vertical direction:

$$F\cos\theta + 8 = 13$$

$$F\cos\theta = 5 \quad \text{K (2)}$$

From (1) and (2):

$$(F\sin\theta)^2 + (F\cos\theta)^2 = 12^2 + 5^5$$

$$F^2\left(\sin^2\theta + \cos^2\theta\right) = 144 + 25$$

$$F^2 = 169$$

$$F = 13$$

Question 21

A motorbike is travelling at a speed of 60 km/hr on a straight road. A school zone is observed in the distance and over the next 10 seconds it reduces speed to 40 km/hr.

If the mass of the motorbike is 900 kg, the change in momentum, measured in kg m/s, in the direction of motion is

A. −6480

B. **−5000**

C. −1800

D. −500

E. −180

Answer is B

Worked solution

Speed must be converted to m/s.

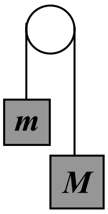
$$\Delta \underset{\sim}{p} = m\underset{\sim}{v}_2 - m\underset{\sim}{v}_1$$

$$\Delta \underset{\sim}{p} = 900 \times 40 \times \frac{1}{3.6} - 900 \times 60 \times \frac{1}{3.6}$$

$$\Delta \underset{\sim}{p} = -5000 \text{ kg m/s}$$

Question 22

A mass of m kg is attached to a second mass of M kg, $m < M$, by a light string passing over a smooth pulley as shown below. The tension in the string is T newtons.



The acceleration, in m/s^2, of the M kg mass is

A. g

B. Mg

C. $\dfrac{Mg - T}{m}$

D. $\boldsymbol{\dfrac{g(M - m)}{(M + m)}}$

E. $\dfrac{g(M + m)}{(M - m)}$

Answer is D

Worked solution

Let a m/s^2 be the acceleration of the system.
Since $m < M$, mass M is accelerating downwards.

Resolving forces:

m kg mass: $T - mg = ma$ .....(1)

M kg mass: $Mg - T = Ma$ .....(2)

Solving (1) and (2) for a:

$$Mg - mg = Ma + ma$$

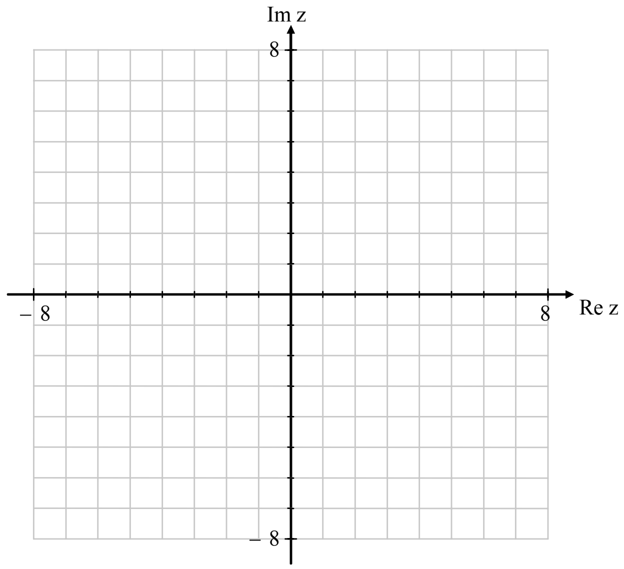
$$g(M - m) = a(M + m)$$

$$a = \frac{g(M - m)}{(M + m)}$$

END OF SECTION 1

SECTION 2

Question 1



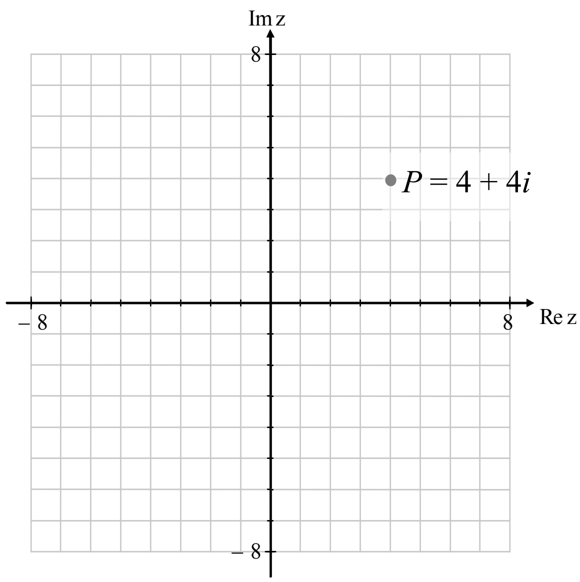
1a. Let $P = 4\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$.

Express P in Cartesian form and plot and label this point in the Argand plane above.

Worked solution

$P = 4\sqrt{2}\cos\left(\frac{\pi}{4}\right) + 4\sqrt{2}\sin\left(\frac{\pi}{4}\right)i$

$P = 4 + 4i$ Point must be plotted correctly in Argand plane. 1A



1 mark

1b. **i.** Find an equivalent Cartesian equation for
$\{z : |z + 2 - 4i| = |z - 2|, z \in C\}$

Worked solution

$|z + 2 - 4i| = |z - 2|$

Let $z = x + yi$

$|x + yi + 2 - 4i| = |x + yi - 2|$

$\sqrt{(x+2)^2 + (y-4)^2} = \sqrt{(x-2)^2 + y^2}$ 1M

$x^2 + 4x + 4 + y^2 - 8y + 16 = x^2 - 4x + 4 + y^2$

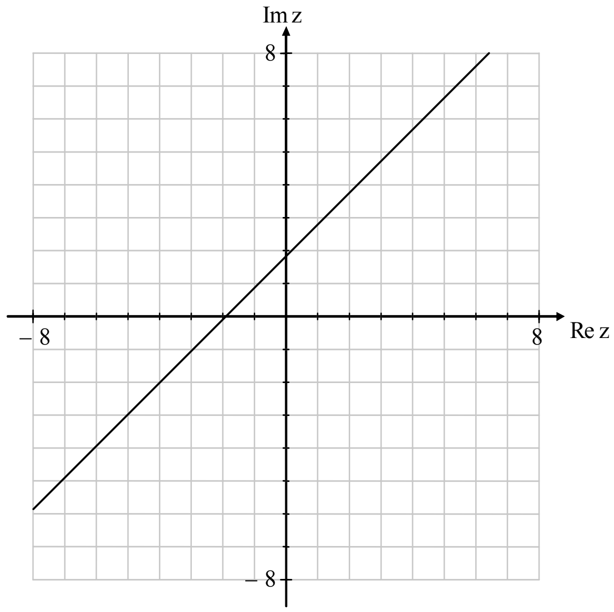
$8x - 8y + 16 = 0$

$y = x + 2$ 1A

2 marks

1b. **ii.** Hence sketch $\{z : |z + 2 - 4i| = |z - 2|, z \in C\}$ on the Argand plane below.

Worked solution



Graph of $y = x + 2$ sketched above. 1H

1 mark

1c. Describe the key features of the relation defined by $\{z : |z - i| = 5\}$

Worked solution

$|x + yi - i| = 5$

$|x + (y-1)i| = 5$

$\sqrt{x^2 + (y-1)^2} = 5$

$x^2 + (y-1)^2 = 25$

The relation represents a circle with centre $0 + i$. 1A
The circle has radius 5 units. 1A

2 marks

SECTION 2 – continued

1d. *M* and *N* are the points of intersection of the relations $\{z : |z - i| = 5\}$ and $\{z : |z + 2 - 4i| = |z - 2|\}$. Determine points *M* and *N* in Cartesian form using your graphics calculator.

Answer

The points are $M = -4 - 2i$, $N = 3 + 5i$ (or vice versa). 2A

2 marks

1e. Use vectors to prove that points *M*, *N* and *P* are the vertices of a right-angled triangle.

Worked solution

The vectors are:

$\overrightarrow{OP} = 4\underset{\sim}{i} + 4\underset{\sim}{j}$, $\quad \overrightarrow{OM} = -4\underset{\sim}{i} - 2\underset{\sim}{j}$, $\quad \overrightarrow{ON} = 3\underset{\sim}{i} + 5\underset{\sim}{j}$

Find vectors $\overrightarrow{MN}$ and $\overrightarrow{NP}$:

$\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON}$ $\qquad$ $\overrightarrow{NP} = \overrightarrow{NO} + \overrightarrow{OP}$

$\overrightarrow{MN} = -(-4\underset{\sim}{i} - 2\underset{\sim}{j}) + 3\underset{\sim}{i} + 5\underset{\sim}{j}$ $\qquad$ $\overrightarrow{NP} = -(3\underset{\sim}{i} + 5\underset{\sim}{j}) + 4\underset{\sim}{i} + 4\underset{\sim}{j}$

$\overrightarrow{MN} = 7\underset{\sim}{i} + 7\underset{\sim}{j}$ $\qquad$ $\overrightarrow{NP} = \underset{\sim}{i} - \underset{\sim}{j}$ 1A

Find the dot product of the vectors:

$\overrightarrow{MN} . \overrightarrow{NP} = (7\underset{\sim}{i} + 7\underset{\sim}{j}).(\underset{\sim}{i} - \underset{\sim}{j})$ 1M

$\overrightarrow{MN} . \overrightarrow{NP} = 7 - 7$

$\overrightarrow{MN} . \overrightarrow{NP} = 0$

Since the dot product is zero, the angle between $\overrightarrow{MN}$ and $\overrightarrow{NP}$ is 90°. 1A

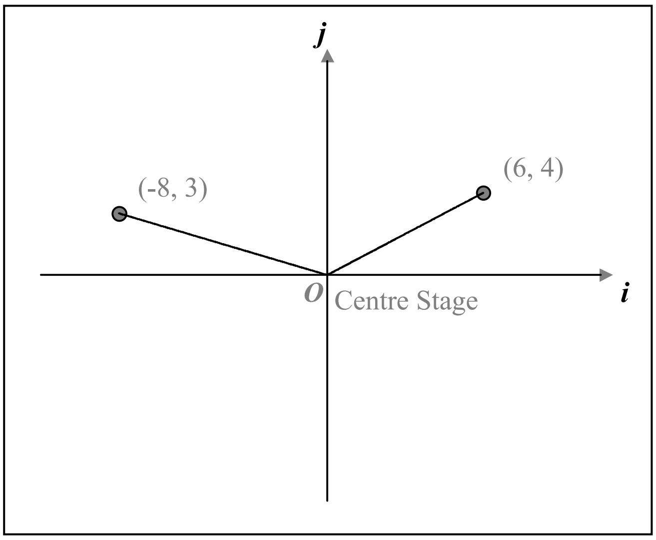
$\therefore$ Points *M*, *N* and *P* are the vertices of a right-angled triangle.

3 marks

Total 11 marks

Question 2

Two dancers, Ari, A, and Ben, B, are standing on stage at the start of a performance. Their position coordinates, in metres, in relation to point O at the centre of the stage are shown in the diagram below.



2a. Write vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$ to describe the positions of Ari and Ben at the start of the performance.

Worked solution

$\overrightarrow{OA} = 6\underset{\sim}{i} + 4\underset{\sim}{j}$

$\overrightarrow{OB} = -8\underset{\sim}{i} + 3\underset{\sim}{j}$ 1A

1 mark

2b. Find the obtuse angle AOB in degrees correct to one decimal place.

Worked solution

$$\cos\theta = \frac{\overrightarrow{OA}.\overrightarrow{OB}}{|OA|\,|OB|}$$

$$\cos\theta = \frac{(6\underset{\sim}{i} + 4\underset{\sim}{j}).(-8\underset{\sim}{i} + 3\underset{\sim}{j})}{\sqrt{6^2 + 4^2}.\sqrt{(-8)^2 + 3^2}}$$ 1M

$$\cos\theta = \frac{-48 + 12}{\sqrt{52}.\sqrt{73}}$$

$$\theta = \cos^{-1}\left(\frac{-36}{\sqrt{52}.\sqrt{73}}\right)$$

$\theta = 125.8°$ 1A

2 marks

As the performance starts spotlight, *r*, is beamed onto the stage. The path the spotlight follows around the stage is given by the equation $\underset{\sim}{r} = 10\cos(t)\underset{\sim}{i} + 5\sin(t)\underset{\sim}{j}, \quad t \geq 0$.

2c. Write a vector that describes the position of spotlight *r* initially.

Worked solution

When $t = 0$:

$$\underset{\sim}{r} = 10\cos(0)\underset{\sim}{i} + 5\sin(0)\underset{\sim}{j}$$

$$\underset{\sim}{r} = 10\underset{\sim}{i}$$ 1A

1 mark

2d. Show that both Ari and Ben are standing in the path traced out by spotlight *r*.

Worked solution

Method 1

If Ari is standing in the path of the spotlight then $10\cos(t)\underset{\sim}{i} + 5\sin(t)\underset{\sim}{j} = 6\underset{\sim}{i} + 4\underset{\sim}{j}$.

Equate the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components and solve for *t*: 1M

$10\cos(t) = 6$ and $5\sin(t) = 4$

$\cos(t) = 0.6$ $\quad\quad$ $\sin(t) = 0.8$

$t = 0.9273$ $\quad\quad$ $t = 0.9273$ 1A

The value of *t* is the same for $\underset{\sim}{i}$ and $\underset{\sim}{j}$

$\therefore$ Ari is standing in the path of the spotlight.

If Ben is standing in the path of the spotlight then $10\cos(t)\underset{\sim}{i} + 5\sin(t)\underset{\sim}{j} = -8\underset{\sim}{i} + 3\underset{\sim}{j}$.

Equate the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components and solve for *t*:

$10\cos(t) = -8$ and $5\sin(t) = 4$

$\cos(t) = -0.8$ $\quad\quad$ $\sin(t) = 0.6$

$\cos(t)$ is negative and $\sin(t)$ is positive, therefore *t* is in the second quadrant.

$t = 2.4981$ $\quad\quad$ $t = \pi - 0.6435 = 2.4981$ 1A

The value of *t* is the same for $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

$\therefore$ Ben is standing in the path of the spotlight.

Method 2 (alternative)

Change $\underset{\sim}{r} = 10\cos(t)\underset{\sim}{i} + 5\sin(t)\underset{\sim}{j}$ to Cartesian form:

$x = 10\cos(t)$ $\quad\quad$ $y = 5\sin(t)$

$\Rightarrow \cos(t) = \frac{x}{10}$ K (1) $\quad\quad$ $\Rightarrow \sin(t) = \frac{y}{5}$ K (2)

From (1) and (2):

$$\left(\frac{x}{10}\right)^2 + \left(\frac{y}{5}\right)^2 = \cos^2(t) + \sin^2(t)$$

$$\therefore \quad \frac{x^2}{100} + \frac{y^2}{25} = 1$$

Spotlight follows an elliptical path.

Ari's position coordinates are (6, 4). Substitute $x = 6$ and $y = 4$ into $\frac{x^2}{100} + \frac{y^2}{25}$:

$\frac{6^2}{100} + \frac{4^2}{25} = \frac{36}{100} + \frac{16}{25} = 1$ $\quad\quad$ $\Rightarrow$ Point (6, 4) lies on ellipse.

Ben's position coordinates are (–8, 3). Substitute these into the equation for the ellipse:

$$\frac{(-8)^2}{100} + \frac{3^2}{25} = \frac{64}{100} + \frac{9}{25} = 1$$

$\Rightarrow$ Point (–8, 3) lies on ellipse

3 marks

2e. How long after the spotlight passes Ari does it reach Ben? Write your answer in seconds correct to two decimal places.

Worked solution

From part d, method 1:

Spotlight passed Ari when $t = 0.9273$ and Ben when $t = 2.4981$ $\quad$ 1M

$\therefore$ Spotlight reaches Ben 1.57 seconds after it passed Ari. $\quad$ 1A

2 marks

A second spotlight, *s*, starts moving at the same time as spotlight *r*. It follows a path given by the equation $\underset{\sim}{s} = 5\sin(t)\underset{\sim}{i} + 10\cos(t)\underset{\sim}{j}, \quad t \geq 0$.

2f. Find the times and position coordinates of the points on stage where the spotlights meet. Write your answers correct to two decimal places.

Worked solution

The spotlights meet when $\underset{\sim}{r} = \underset{\sim}{s}$.

$10\cos(t)\underset{\sim}{i} + 5\sin(t)\underset{\sim}{j} = 5\sin(t)\underset{\sim}{i} + 10\cos(t)\underset{\sim}{j}$ $\quad$ 1M

Equating $\underset{\sim}{i}$ components:

$10\cos(t) = 5\sin(t)$

$\Rightarrow$

$$\frac{\sin(t)}{\cos(t)} = \frac{10}{5}$$

$\tan(t) = 2$

$t = 1.1071, \quad \pi + 1.1071$

t = 1.1071 and 4.2487 seconds 1A

Equating $\underset{\sim}{j}$ components:

$5\sin(t) = 10\cos(t)$ (same equation)

When $t = 1.1071$, $\underset{\sim}{r} = 5\sin(1.1071)\underset{\sim}{i} + 10\cos(1.1071)\underset{\sim}{j}$

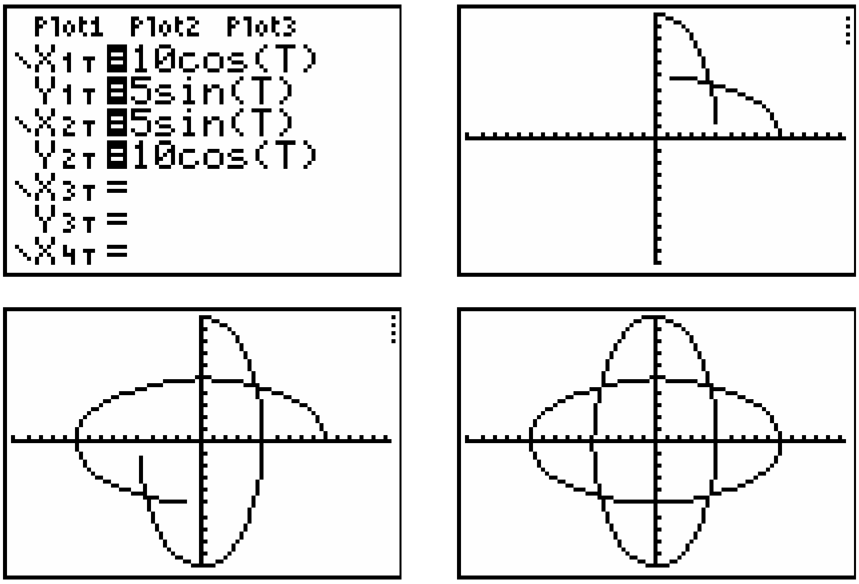
$$\underset{\sim}{r} = 4.4721\underset{\sim}{i} + 4.4721\underset{\sim}{j}$$

When $t = 4.2487$, $\underset{\sim}{r} = 5\sin(4.2487)\underset{\sim}{i} + 10\cos(4.2487)\underset{\sim}{j}$

$$\underset{\sim}{r} = -4.4721\underset{\sim}{i} - 4.4721\underset{\sim}{j}$$

Position coordinates are (4.47, 4.47) and (-4.47, -4.47) 1A

Note that the paths of the spotlights cross in four places, but the spotlights only *meet* (i.e., are in the same position at the same time) on two occasions. This can be seen by graphing the curves simultaneously on a calculator using parametric mode.



3 marks

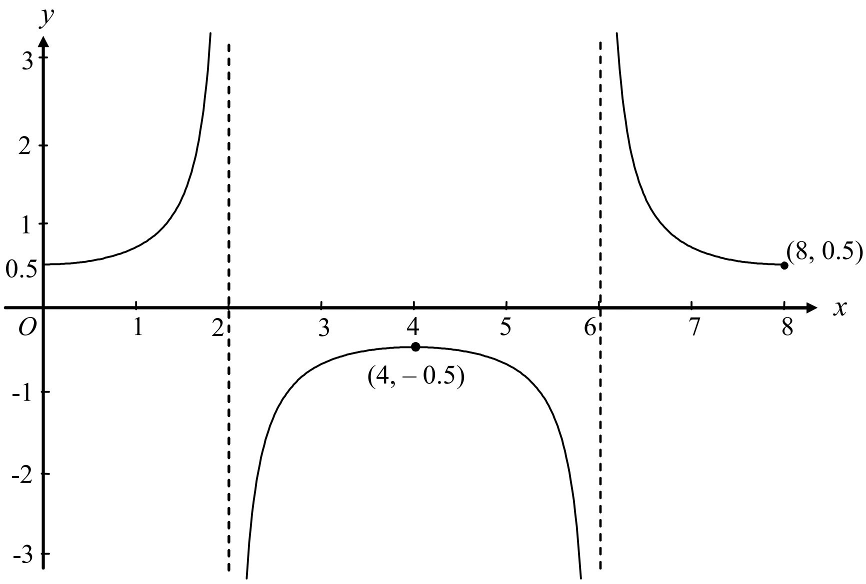
Total 12 marks

Question 3

Consider the function $f : D \to R$ where $f(x) = 0.5\text{cosec}\left(\frac{\pi}{4}(2-x)\right)$

3a. **i.** On the axes below, sketch a graph of *f* over the interval $[0, 8]$, labelling all features clearly.

Worked solution



Correct shape 1A

Local maximum of (4, – 0.5), Asymptotes and endpoints 1A

2 marks

3a. **ii.** Determine the domain and range of f over this interval.

Worked solution

Domain $[0, 8] \setminus \{2,6\}$ or $[0, 2) \cup (2, 6) \cup (6, 8]$ 1A

Range $R \setminus (-0.5, 0.5)$ or $(-\infty, -0.5] \cup [0.5, \infty)$ 1A

2 marks

3b. An equivalent rule for f is $f_1(x) = \dfrac{1}{a\cos(bx + c)}$ where $a, b, c \in R$

Give values for *a*, *b*, and *c*.

Worked solution

The simplest solution is

$a = 2$

$b = \frac{\pi}{4}$ Two values correct 1A

$c = 0$ All three values correct 1A

There are many other solutions – for example: $a = -2$, $b = \frac{\pi}{4}$, $c = -\pi$,

or $a = 2$, $b = \frac{\pi}{4}$, $c = 2\pi$.

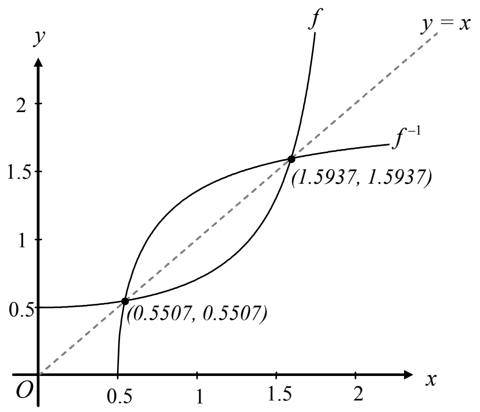
2 marks

3c. Let $D = [0, 2)$.

Sketch f and f^{-1} on the axes below, clearly showing the key features.

Worked solution

To graph f^{-1}, reflect graph of f in the line $y = x$.



Position and shape 1A

1 mark

3d. Write a definite integral that will give the area enclosed by f and f^{-1}. Using your graphics calculator, evaluate this integral correct to three decimal places.

Worked solution

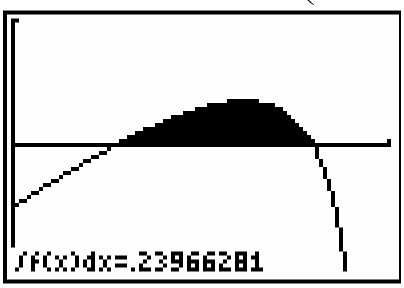
f and f^{-1} intersect at (0.5507, 0.5507) and (1.5937, 1.5937). 1H

These coordinates are found by graphing $y = x$ and $y = 0.5\operatorname{cosec}\left(\frac{\pi}{4}(2-x)\right)$ on a calculator.

The area enclosed by f and f^{-1} is twice the area between $y = x$ and $y = 0.5\operatorname{cosec}\left(\frac{\pi}{4}(2-x)\right)$.

Area $= 2\times\int_{0.5507}^{1.5937}\left[x - 0.5\operatorname{cosec}\left(\frac{\pi}{4}(2-x)\right)\right]dx$ 1A

Graph $y = x - 0.5\operatorname{cosec}\left(\frac{\pi}{4}(2-x)\right)$ on calculator and find area above the x-axis.



Area enclosed by f and f^{-1} is $2\times 0.2397 = 0.479$ square units. 1A

3 marks

Total 10 marks

SECTION 2 – continued

Question 4

A box of mass *m* kg is dropped from a hot air balloon. Its motion is retarded by a variable force of $\frac{mv}{5}$ newton, where *v* m/s is the velocity of the box *t* seconds after it is dropped.

4a. Taking vertically downwards as positive, show that the differential equation $\frac{dv}{dt} = \frac{5g - v}{5}$, where $g = 9.8$ m/sec^2 is the acceleration due to gravity, applies to this situation.

Worked solution

$ma = mg - \frac{mv}{5}$ 1A

$a = g - \frac{v}{5}$

$\frac{dv}{dt} = \frac{5g - v}{5}$ 1M

2 marks

4b. Hence, show that $t = 5\log_e\left(\frac{5g}{5g - v}\right)$

Worked solution

$\frac{dv}{dt} = \frac{5g - v}{5}$

$\frac{dt}{dv} = \frac{5}{5g - v}$

$t = \int \frac{5}{5g - v} dv$ 1M

$t = -5\int \frac{-1}{5g - v} dv$

$t = -5\log_e(5g - v) + c$

When $t = 0, \quad v = 0$:

$c = 5\log_e(5g)$

$t = -5\log_e(5g - v) + 5\log_e(5g)$ 1M

$t = 5\log_e\left(\frac{5g}{5g - v}\right)$

2 marks

SECTION 2 – continued

4c. Show that at time *t* the velocity of the box is $5g\left(1-e^{-0.2t}\right)$ m/s.

Worked solution

Transpose $t = 5\log_e\left(\dfrac{5g}{5g-v}\right)$ to make *v* the subject

$e^{\frac{t}{5}} = \dfrac{5g}{5g-v}$ 1M

$e^{0.2t}(5g-v) = 5g$

$5g - v = 5g\,e^{-0.2t}$

$v = 5g - 5g\,e^{-0.2t}$

$v = 5g\left(1-e^{-0.2t}\right)$ 1M

2 marks

4d. Write an expression for the limiting velocity of the box. Show how you deduced your result.

Worked solution

At time *t* seconds, the velocity of the box is $v = 5g\left(1-e^{-0.2t}\right)$ m/s

As $t \to \infty$, $e^{-0.2t} \to 0$, therefore $v \to 5g(1+0) = 5g$ 1A

The limiting velocity is 5*g* m/s (49 m/s). 1A

2 marks

4e. Determine the time taken for the box to reach half its limiting velocity. Write your answer in seconds correct to two decimal places.

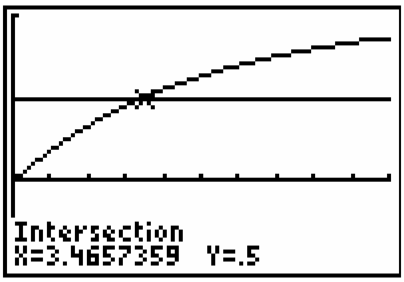
Worked solution

Finding *t* when $v = \dfrac{5g}{2}$

$\dfrac{5g}{2} = 5g\left(1-e^{-0.2t}\right)$ 1M

$\dfrac{1}{2} = 1 - e^{-0.2t}$

Equation can be solved graphically on calculator



$e^{-0.2t} = \dfrac{1}{2}$

$t = -\dfrac{1}{0.2}\log_e\left(\dfrac{1}{2}\right)$

$t = 5\log_e(2)$

$t = 3.47$ seconds

It takes 3.47 seconds for the box to reach half its limiting velocity. 1H

2 marks

4f. Find the distance travelled by the box in the first 10 seconds of motion. Write your answer correct to the nearest metre.

Worked solution

$\frac{dx}{dt} = 5g\left(1 - e^{-0.2t}\right)$

$x = 5g\int\left(1 - e^{-0.2t}\right)dt$

$x = 5g\left(t + 5e^{-0.2t}\right) + c$ 1M

When $t = 0$, $x = 0$:

$0 = 5g\left(0 + 5e^{0}\right) + c$

$c = -25g$

$x = 5g\left(t + 5e^{-0.2t}\right) - 25g$ 1A

$\therefore \quad x = 5g\left(t + 5e^{-0.2t} - 5\right)$

When $t = 10$:

$x = 5 \times 9.8\left(10 + 5e^{-0.2\times10} - 5\right)$

$x = 278$ metres 1A

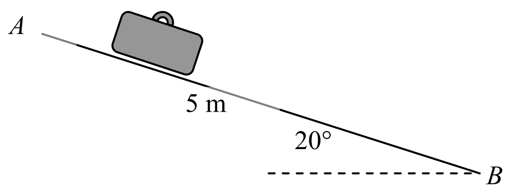
The box travels 278 metres in the first 10 seconds.

3 marks

Total 13 marks

Question 5

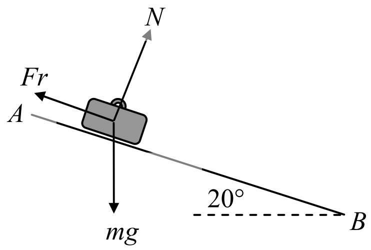
Baggage handlers use ramps to transport luggage. Ramp *AB* is 5 metres in length and inclined at an angle of 20° to the horizontal. A 20 kg suitcase, initially at rest at *A*, slides down ramp *AB* under the force of gravity. The coefficient of friction between the suitcase and the ramp is 0.2. Take $g = 9.8$ m/sec^2.



5a. On the diagram above, draw all forces acting on the suitcase as it slides down the ramp.

Worked solution

The forces are: normal reaction *N*, weight force *mg* (20*g*), friction *Fr*.

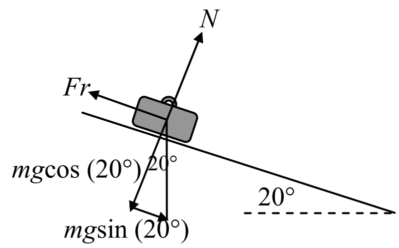


1A

1 mark

5b. Show that the suitcase slides down the ramp with an acceleration of 1.51 m/s^2.

Worked solution



Resolving forces acting perpendicular to the ramp:

$N = mg\cos(20°)$

Resolving forces acting parallel to the ramp:

$ma = mg\sin(20°) - Fr$

$ma = mg\sin(20°) - \mu N$ 1M

$ma = mg\sin(20°) - 0.2mg\cos(20°)$

$a = g(\sin(20°) - 0.2\cos(20°))$ 1A

$a = 1.51 \text{ m/s}^2$

2 marks

5c. Find the time taken for the suitcase to reach point *B*. Write your answer in seconds correct to two decimal places.

Worked solution

The suitcase is moving under constant acceleration.

$u = 0 \quad a = 1.51 \quad s = 5$

$s = ut + \frac{1}{2}at^2$

$5 = 0 + \frac{1}{2} \times 1.51t^2$ 1M

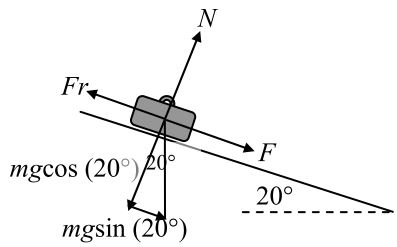
$t = \sqrt{\frac{5}{0.755}}$

$t = 2.57$ seconds 1A

2 marks

5d. Some time later, an identical 20 kg suitcase, initially at rest at *A*, is pushed down the ramp with a force of $100 - 200t$ newtons for the first 0.5 seconds of motion. Show that at time *t*, $0 < t < 0.5$, the acceleration of this suitcase is $6.51 - 10t$ m/s^2.

Worked solution



Let $F = 100 - 200t$

Resolving forces acting perpendicular to the ramp:

$N = mg\cos(20°)$

Resolving forces acting parallel to the ramp:

$ma = F + mg\sin(20°) - Fr$

$ma = 100 - 200t + mg\sin(20°) - \mu N$ 1M

$ma = 100 - 200t + mg\sin(20°) - 0.2mg\cos(20°)$

$20a = 100 - 200t + 20g\sin(20°) - 0.2 \times 20g\cos(20°)$

$20a = 130.2 - 200t$ 1A

$a = 6.51 - 10t$ m/s^2

2 marks

5e. Find the speed of the suitcase when $t = 0.5$. Write your answer in m/s, correct to two decimal places.

Worked solution

$v = \int a \, dt$

$v = \int (6.51 - 10t) dt$

$v = 6.51t - 5t^2 + c$

When $t = 0, v = 0 : c = 0$

$v = 6.51t - 5t^2$ 1A

When $t = 0.5$, $v = 2.01$.

After 0.5 seconds of motion the suitcase is moving at a speed of 2.01m/s. 1A

2 marks

5f. Determine the speed of this suitcase when it reaches point *B*. Write your answer in m/s, correct to two decimal places.

Worked solution

Find how far the suitcase travels whilst being pushed.

$x = \int v \, dt$

$x = \int (6.51t - 5t^2) dt$

$x = 3.255t^2 - 1.\dot{6}t^3 + c$ 1M

When $t = 0$, $x = 0$, so $c = 0$.

$x = 3.255t^2 - \frac{5}{3}t^3$

When $t = 0.5, x = 0.605$.

The suitcase travels 0.61 m while being pushed. 1A

For the remaining distance down the ramp, the suitcase travels with constant acceleration of 1.51 m/s^2.

$u = 2.01, \quad a = 1.51, \quad s = 5 - 0.605 = 4.395$

$v^2 = u^2 + 2as$

$v^2 = 2.01^2 + 2 \times 1.51 \times 4.395$

$v = \sqrt{17.31}$

$v = 4.16$ m/s 1A

The speed of this suitcase at point *B* is 4.16 m/s.

3 marks

12 marks

END OF SOLUTIONS BOOK