THE HEFFERNAN GROUP

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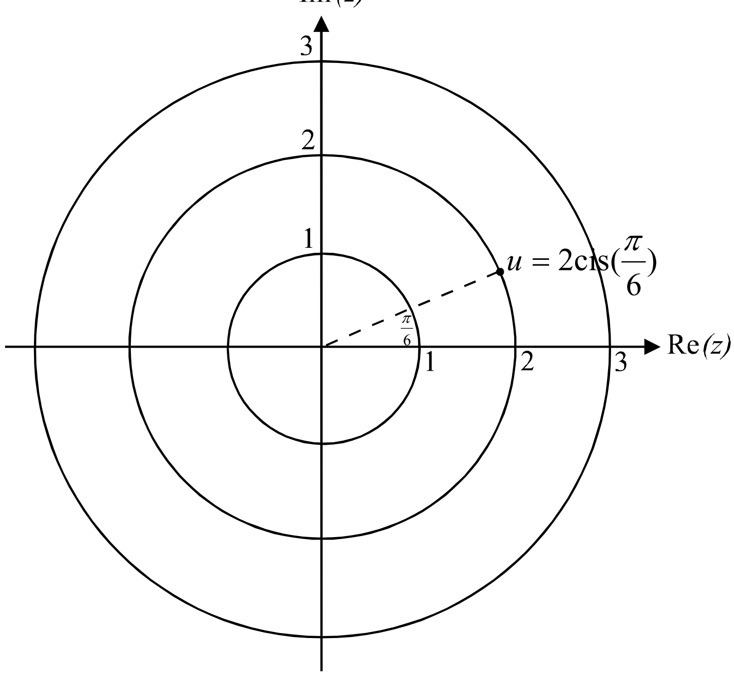
SPECIALIST MATHS TRIAL EXAMINATION 2 SOLUTIONS 2005

Question 1

a. **i.**



(1 mark)

ii. The roots of the equation $z^3 = a, a \in C$, are evenly spaced around a circle. Since we know that one of the roots is *u*, the circle has a radius of 2 and the roots are spaced $\frac{2\pi}{3}$ apart. So since $u = 2\text{cis}\left(\frac{\pi}{6}\right)$, the second root is

$2\text{cis}\left(\frac{\pi}{6}+\frac{2\pi}{3}\right) = 2\text{cis}\left(\frac{5\pi}{6}\right)$ and the third root is $2\text{cis}\left(\frac{5\pi}{6}+\frac{2\pi}{3}\right) = 2\text{cis}\left(\frac{3\pi}{2}\right)$

or $2\text{cis}\left(-\frac{\pi}{2}\right)$.

(2 marks)

iii. Method 1

$a = u^3$ **(1 mark)**

$= 2^3 \text{cis}\left(\frac{3\pi}{6}\right)$

$= 8\text{cis}\left(\frac{\pi}{2}\right)$

$= 8\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

$= 8i$ **(1 mark)**

Method 2

Now, $u = 2\text{cis}\left(\frac{\pi}{6}\right)$

$= 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

$= 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$= \sqrt{3} + i$

From part ii. we know that the second root; $2\text{cis}\left(\frac{5\pi}{6}\right)$, is a reflection of $2\text{cis}\left(\frac{\pi}{6}\right)$ in the *y*- axis. Therefore the second root must be a reflection of $\sqrt{3} + i$ in the *y*-axis.

The second root is therefore $-\sqrt{3} + i$ and the third root is $-2i$. **(1 mark)**

So,

$$(z - \sqrt{3} - i)(z + \sqrt{3} - i)(z + 2i) = 0$$

$$(z^2 + \sqrt{3}z - iz - \sqrt{3}z - 3 + \sqrt{3}i - iz - i\sqrt{3} - 1)(z + 2i) = 0$$

$$(z^2 - 2iz - 4)(z + 2i) = 0$$

$$z^3 + 2iz^2 - 2iz^2 + 4z - 4z - 8i = 0$$

$$z^3 - 8i = 0$$

$$z^3 = 8i$$

So $a = 8i$.

(1 mark)

b. **i.** Method 1

$$u = 2\text{cis}\left(\frac{\pi}{6}\right)$$

S	A
T	C

$$\bar{u} = 2\text{cis}\left(\frac{-\pi}{6}\right)$$

$$= 2\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right)$$

$$= 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

$$= \sqrt{3} - i$$

Method 2

$$u = 2\text{cis}\left(\frac{\pi}{6}\right)$$

$$= 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

$$= 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= \sqrt{3} + i$$

So $\bar{u} = \sqrt{3} - i$

(1 mark)

ii. $z = x + iy, \quad x, y \in R$

$$|z - u| = |z - \bar{u}|$$

$$\left|x + iy - \sqrt{3} - i\right| = \left|x + iy - \sqrt{3} + i\right|$$

$$\sqrt{\left(x - \sqrt{3}\right)^2 + (y - 1)^2} = \sqrt{\left(x - \sqrt{3}\right)^2 + (y + 1)^2}$$ **(1 mark)**

$$y^2 - 2y + 1 = y^2 + 2y + 1$$

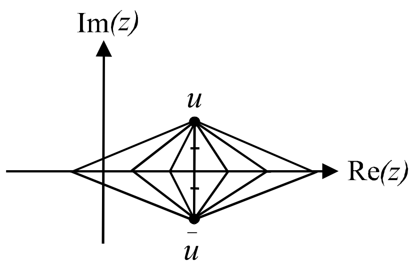
$$-4y = 0$$

$$y = 0$$

So $z = x + iy$ becomes $z = x$ as required **(1 mark)**

iii.

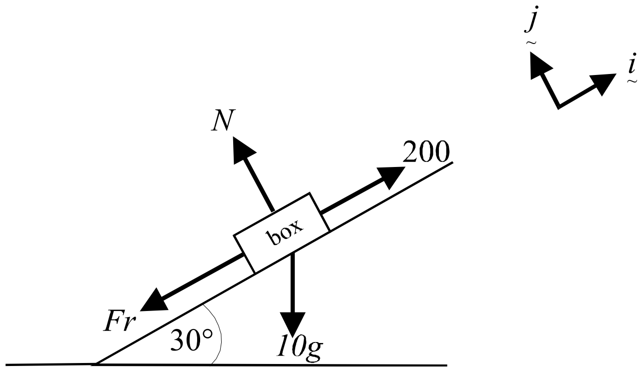
If $|z - u| = |z - \bar{u}|$ then the distance from the complex number z to the complex number u is the same as the distance from z to the complex number $\bar{u}$.
From the diagram, those complex numbers z for which this applies, lie along the real axis of the Argand diagram.
So $z = x$, that is, $y = 0$.



(1 mark)
Total 9 marks

Question 2

a. Draw a diagram showing all the forces.



$$\underset{\sim}{R} = \left(200 - 10g\sin\left(30^\circ\right) - Fr\right)\underset{\sim}{i} + \left(N - 10g\cos\left(30^\circ\right)\right)\underset{\sim}{j}$$ **(1 mark)**

$$= \left(200 - 5g - \mu N\right)\underset{\sim}{i} + \left(N - 5\sqrt{3}g\right)\underset{\sim}{j}$$

Also $\underset{\sim}{R} = m\underset{\sim}{a}$

$= 10a\underset{\sim}{i}$ **(1 mark)**

So $N - 5\sqrt{3}g = 0$

$N = 5\sqrt{3}g$

and $200 - 5g - \mu N = 10a$

$200 - 5g - 15g = 10a$ **(1 mark)**

$$a = \frac{200 - 20g}{10}$$

$= 20 - 2g$

$= 0 \cdot 4\text{ms}^{-2}$

(1 mark)

b. Since the acceleration is constant, we can use the formula

$$s = ut + \frac{1}{2}at^2$$ **(1 mark)**

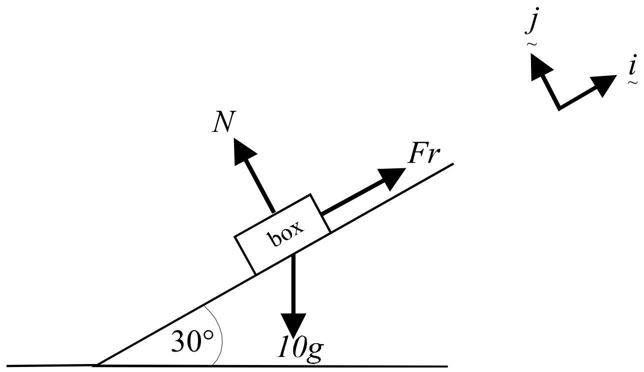
So, $2 \cdot 5 = 0 \times t + \frac{1}{2} \times 0 \cdot 4t^2$

$$t^2 = \frac{25}{2}$$

$$t = \frac{5\sqrt{2}}{2} \text{ secs} \qquad (t > 0)$$

(1 mark)

c. Draw a diagram showing all the forces.



(1 mark)

There is now no pulling force up the ramp and so the tendency of the box would be to slip down the ramp and so the friction forces are directed up the ramp.

$$\underset{\sim}{R} = \left(Fr - 10g\sin\left(30^\circ\right)\right)\underset{\sim}{i} + \left(N - 10g\cos\left(30^\circ\right)\right)\underset{\sim}{j}$$

$$= \left(Fr - 5g\right)\underset{\sim}{i} + \left(N - 5\sqrt{3}g\right)\underset{\sim}{j}$$

(1 mark)

Once stationary, $a = 0$

so $\underset{\sim}{R} = m\underset{\sim}{a}$

$= 0\underset{\sim}{i}$

So $Fr - 5g = 0$

$Fr = 5g$

Also $N - 5\sqrt{3}g = 0$

$N = 5\sqrt{3}g$

So $\mu N = 15g$

So $Fr < \mu N$

So the box is not on the point of slipping down the ramp.

(1 mark)

d. From part c., we have

$\underset{\sim}{R} = \left(Fr - 5g\right)\underset{\sim}{i} + \left(N - 5\sqrt{3}g\right)\underset{\sim}{j}$

Also $\underset{\sim}{R} = -ma\underset{\sim}{i}$ where a represents acceleration down the ramp. **(1 mark)**

So,

$Fr - 5g = -10a$

$\mu N - 5g = -10a$

$2 \cdot 5\sqrt{3}g - 5g = -10a$

$a = 0 \cdot 65647...$ **(1 mark)**

Since the acceleration is constant,

$s = ut + \frac{1}{2}at^2$

$s = 0 \times 1 + \frac{1}{2} \times 0.65647...$

$= 0 \cdot 328...$

The box travels 33cm (to the nearest centimetre) in the first second.

(1 mark)
Total 12 marks

Question 3

a. $\frac{dT}{dt} = \sqrt{400 - T^2}, \quad t \geq 0$

$\frac{dt}{dT} = \frac{1}{\sqrt{400 - T^2}}$

$t = \int \frac{1}{\sqrt{400 - T^2}}\, dT$ **(1 mark)**

$t = \text{Sin}^{-1}\left(\frac{T}{20}\right) + c$ **(1 mark)**

When $t = 0, \quad T = 10\sqrt{3}$

$0 = \text{Sin}^{-1}\left(\frac{10\sqrt{3}}{20}\right) + c$

$c = -\text{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$c = -\frac{\pi}{3}$

So $\quad t = \text{Sin}^{-1}\left(\frac{T}{20}\right) - \frac{\pi}{3}$ **(1 mark)**

$t + \frac{\pi}{3} = \text{Sin}^{-1}\left(\frac{T}{20}\right)$

$\frac{T}{20} = \text{Sin}\left(t + \frac{\pi}{3}\right)$

$T = 20\text{Sin}\left(t + \frac{\pi}{3}\right)$ **(1 mark)**

b. Since the function $T(t)$ contains the principal valued sin function, then

$-\frac{\pi}{2} \leq t + \frac{\pi}{3} \leq \frac{\pi}{2}$

$-\frac{\pi}{2} - \frac{\pi}{3} \leq t \leq \frac{\pi}{2} - \frac{\pi}{3}$

$-\frac{5\pi}{6} \leq t \leq \frac{\pi}{6}$

but $\quad t \geq 0 \qquad$ so

we require $\left\{t : 0 \leq t \leq \frac{\pi}{6}\right\}$ or $t \in \left[0, \frac{\pi}{6}\right]$.

(1 mark) – upper limit
(1 mark) – lower limit

c. $\dfrac{dT}{dt}=\sqrt{400-T^2}$

The sign of $\dfrac{dT}{dt}$ can only be positive. Hence the gradient of the function $T(t)$ is always positive and hence the temperature increases as time increases and hence the maximum temperature occurs at the upper limit of the domain of the function $T(t)$; that is, at $t=\dfrac{\pi}{6}$.

(1 mark) – stating $\dfrac{dT}{dt}$ is always positive
(1 mark) – rest of explanation

Total 8 marks

Question 4

a. We require that $\sqrt{1+e^x} > 0$

$$1+e^x > 0$$
$$e^x > -1$$

Now $e^x > 0$ for $x \in R$

So $e^x > -1$ for $x \in R$

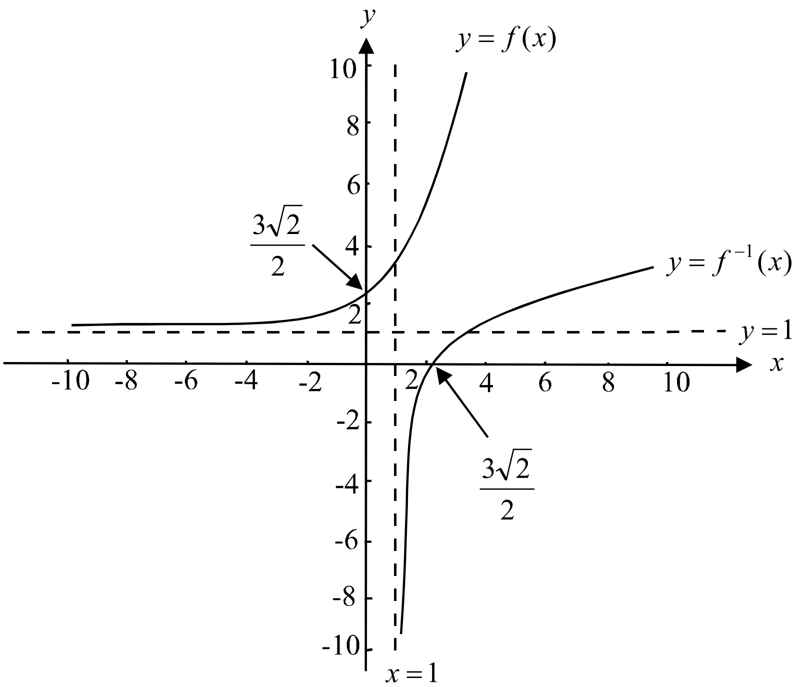
So $D = R$

(1 mark)

b. and **c.**

The y-intercept occurs when $x = 0$

$$f(0) = \frac{1}{\sqrt{1+1}} + \sqrt{1+1}$$
$$= \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{1}$$
$$= \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}}$$
$$= \frac{3\sqrt{2}}{2}$$



There is an asymptote of $y = 1$ since as $x \to -\infty$, $e^x \to 0$ and so $f(x) \to 1^+$.

(1 mark) correct shape of $y = f(x)$

(1 mark) – correct y-intercept

(1 mark) - showing correct asymptote on graph

(1 mark) - showing graph of $y = f^{-1}(x)$ as a reflection of whatever has been drawn for graph of $y = f(x)$.

d. Stationary points occur when $f'(x)=0$.

Now, $f(x)=\dfrac{e^x}{\sqrt{1+e^x}}+\sqrt{1+e^x}$

$$\text{so } f'(x)=\frac{\left(1+e^x\right)^{\frac{1}{2}}e^x-e^x\times\frac{1}{2}\left(1+e^x\right)^{-\frac{1}{2}}\times e^x}{\left(\sqrt{1+e^x}\right)^2}+\frac{1}{2}\left(1+e^x\right)^{-\frac{1}{2}}\times e^x \qquad \textbf{(1 mark)}$$

$$=\frac{e^x\sqrt{1+e^x}}{1+e^x}-\frac{e^{2x}}{2\left(1+e^x\right)\sqrt{1+e^x}}+\frac{e^x}{2\sqrt{1+e^x}}$$

$$=\frac{2e^x\left(1+e^x\right)-e^{2x}+e^x\left(1+e^x\right)}{2\left(1+e^x\right)\sqrt{1+e^x}}$$

$$=\frac{2e^x+2e^{2x}-e^{2x}+e^x+e^{2x}}{2\left(1+e^x\right)^{\frac{3}{2}}}$$

$$=\frac{3e^x+2e^{2x}}{2\left(1+e^x\right)^{\frac{3}{2}}} \qquad \textbf{(1 mark)}$$

If $\quad f'(x)=0,$

then $3e^x+2e^{2x}=0$

$$3e^x=-2e^{2x}$$

$$-\frac{3}{2}=\frac{e^{2x}}{e^x}$$

$$e^x=-\frac{3}{2} \qquad \textbf{(1 mark)}$$

Since $e^x>0$ for $x\in R$, there is no solution to $e^x=-\dfrac{3}{2}$ and therefore there is no stationary point on the graph of $y=f(x)$.

(1 mark) – explanation

e. **i.** Now $y=\dfrac{e^x}{\sqrt{1+e^x}}+\sqrt{1+e^x}$

$$y^2=\left(\frac{e^x}{\sqrt{1+e^x}}+\sqrt{1+e^x}\right)\left(\frac{e^x}{\sqrt{1+e^x}}+\sqrt{1+e^x}\right)$$

$$=\frac{e^{2x}}{1+e^x}+\frac{2e^x\sqrt{1+e^x}}{\sqrt{1+e^x}}+1+e^x$$

$$=\frac{e^{2x}}{1+e^x}+2e^x+1+e^x$$

$$=\frac{e^{2x}}{1+e^x}+3e^x+1$$

(1 mark)

ii.

$$1+e^x \overline{\big)\, e^{2x}}$$ quotient e^x

$$\frac{e^{2x}+e^x}{-e^x}$$

So $\dfrac{e^{2x}}{1+e^x}=e^x-\dfrac{e^x}{1+e^x}$

Check: $e^x-\dfrac{e^x}{1+e^x}$

$$=\frac{e^x\left(1+e^x\right)-e^x}{1+e^x}$$

$$=\frac{e^x+e^{2x}-e^x}{1+e^x}$$

$$=\frac{e^{2x}}{1+e^x}$$

So $a=e^x$ and $b=-e^x$.

(1 mark)

f. volume $=\pi\int_0^1 y^2\,dx$ **(1 mark)**

$=\pi\int_0^1\left(\dfrac{e^{2x}}{1+e^x}+3e^x+1\right)dx$ From part e.i.

$=\pi\int_0^1\left(e^x-\dfrac{e^x}{1+e^x}+3e^x+1\right)dx$ from part e. ii.

$=\pi\int_0^1\left(4\mathrm{e}^x+1\right)dx-\pi\int_0^1\dfrac{e^x}{1+e^x}\,dx$

$=\pi\int_0^1\left(4e^x+1\right)dx-\pi\int_2^{1+e}u^{-1}\dfrac{du}{dx}\,dx$ **(1 mark)**

where $u=1+e^x$

$\dfrac{du}{dx}=e^x$

$x=1, u=1+e$

$x=0,\ u=2$

$=\pi\left[4e^x+x\right]_0^1-\pi\left[\log_e(u)\right]_2^{1+e}$ **(1 mark)**

$=\pi\left\{(4e+1)-\left(4e^\circ+0\right)\right\}-\pi\left\{\log_e(1+e)-\log_e(2)\right\}$

$=\pi\left\{4e+1-4-\log_e(1+e)+\log_e(2)\right\}$

$=\pi\left(4e-3+\log_e\left(\dfrac{2}{(1+e)}\right)\right)$ **(1 mark)**

(1 mark) change of terminals

Total 16 marks

Question 5

a. The feather is released at $t = 0$. The vertical component above the ground is given by $-\log_e\left(\frac{t+1}{20}\right)$.

When $t = 0$, $\quad -\log_e\left(\frac{t+1}{20}\right) = -\log_e\left(\frac{1}{20}\right)$ **(1 mark)**

$$= 2 \cdot 9957...$$

The feather is dropped from a height of 3m (to the nearest metre).

(1 mark)

b. The feather reaches the ground when the component of the $\underset{\sim}{k}$ coordinate equals zero.

$$-\log_e\left(\frac{t+1}{20}\right) = 0$$

$$\log_e\left(\frac{t+1}{20}\right) = 0$$

$$e^0 = \frac{t+1}{20}$$

$$20 \times 1 = t + 1$$

$$t = 19$$

It takes 19 seconds.

(1 mark)

c. We need to find $\left|\underset{\sim}{r}(10)\right|$.

$$r(t) = \sin\left(\frac{t}{10}\right)\underset{\sim}{i} + \left(1 - \cos\left(\frac{t}{10}\right)\right)\underset{\sim}{j} - \log_e\left(\frac{t+1}{20}\right)\underset{\sim}{k}$$

$$\underset{\sim}{r}(10) = \sin(1)\underset{\sim}{i} + (1 - \cos(1))\underset{\sim}{j} - \log_e\left(\frac{11}{20}\right)\underset{\sim}{k}$$

$$\left|\underset{\sim}{r}(10)\right| = \sqrt{\sin^2(1) + (1 - \cos(1))^2 + \left(\log_e\left(\frac{11}{20}\right)\right)^2}$$

(1 mark)

The feather is 1.13m (correct to 2 decimal places) from the sister at $t = 10$ secs.

(1 mark)

d. The feather is directly north-east when the component in the $\underset{\sim}{i}$ direction and the component in the $\underset{\sim}{j}$ direction are equal; that is when

$$\sin\left(\frac{t}{10}\right) = 1 - \cos\left(\frac{t}{10}\right)$$ **(1 mark)**

Using a graphics calculator we see that this happens when $t = 15 \cdot 707....$

So, the feather is directly north-east at $t = 15 \cdot 7$ secs (correct to 1 decimal place).

(1 mark)

e. $\underset{\sim}{r}(t) = \sin\left(\frac{t}{10}\right)\underset{\sim}{i} + \left(1 - \cos\left(\frac{t}{10}\right)\right)\underset{\sim}{j} - \log_e\left(\frac{t+1}{20}\right)\underset{\sim}{k}$

$$\underset{\sim}{v}(t) = \frac{1}{10}\cos\left(\frac{t}{10}\right)\underset{\sim}{i} + \frac{1}{10}\sin\left(\frac{t}{10}\right)\underset{\sim}{j} - \left(\frac{1}{20} \div \frac{t+1}{20}\right)\underset{\sim}{k}$$

$$= \frac{1}{10}\cos\left(\frac{t}{10}\right)\underset{\sim}{i} + \frac{1}{10}\sin\left(\frac{t}{10}\right)\underset{\sim}{j} - \left(\frac{1}{t+1}\right)\underset{\sim}{k}$$

(1 mark)

f. At $t = 9$ secs,

speed $= |\underset{\sim}{v}(9)|$ **(1 mark)**

$$= \left|\frac{1}{10}\cos(0\cdot 9)\underset{\sim}{i} + \frac{1}{10}\sin(0\cdot 9)\underset{\sim}{j} - \frac{1}{10}\underset{\sim}{k}\right|$$

$$= \sqrt{\frac{1}{100}\cos^2(0\cdot 9) + \frac{1}{100}\sin^2(0\cdot 9) + \frac{1}{100}}$$

$$= \sqrt{\frac{1}{100}\left(\cos^2(0\cdot 9) + \sin^2(0\cdot 9)\right) + \frac{1}{100}}$$

$$= \sqrt{\frac{2}{100}}$$

$$= \frac{\sqrt{2}}{10}\ \text{ms}^{-1}$$

(1 mark)

g. We need to find the angle between $\underset{\sim}{v}$ and $\underset{\sim}{j}$.

Now $\underset{\sim}{v} \bullet \underset{\sim}{j} = \left(\frac{1}{10}\cos\left(\frac{t}{10}\right)\underset{\sim}{i} + \frac{1}{10}\sin\left(\frac{t}{10}\right)\underset{\sim}{j} - \frac{1}{t+1}\underset{\sim}{k}\right) \bullet \underset{\sim}{j}$

$$= \frac{1}{10}\sin\left(\frac{t}{10}\right)$$ **(1 mark)**

Also, $\underset{\sim}{v} \bullet \underset{\sim}{j} = |\underset{\sim}{v}||\underset{\sim}{j}|\cos\theta$ (Scalar product)

So, $\frac{1}{10}\sin\left(\frac{t}{10}\right) = \sqrt{\frac{1}{100} + \frac{1}{(t+1)^2}} \times 1\cos\theta$ (using working from part f.)

The feather reaches the ground at $t = 19$ seconds.

So, $\frac{1}{10}\sin(1\cdot 9) = \sqrt{\frac{1}{100} + \frac{1}{400}}\cos\theta$ **(1 mark)**

$$= \frac{1}{\sqrt{80}}\cos\theta$$

$$\cos\theta = \frac{\sqrt{80}}{10}\sin(1\cdot 9)$$

$\theta = 32^\circ 11'$ (to the nearest minute)

(1 mark)

h. The component of $\underset{\sim}{r}$ in the $\underset{\sim}{i}$ direction gives the position of the feather in the east direction, that is, $\sin\left(\frac{t}{10}\right)$.

The maximum value for $\sin\left(\frac{t}{10}\right)$ is 1. **(1 mark)**

Check when this occurs to make sure that it is before the feather hits the ground.

Now, $\sin\left(\frac{t}{10}\right)=1$ when $\frac{t}{10}=\frac{\pi}{2}$; that is when $t=5\pi$ which is at approximately $t=15.7$ seconds. This is before the feather hits the ground.

So the furthest distance east reached by the feather is 1 m.

(1 mark)

e. **Total 15 marks**