2004 Specialist Mathematics Examination 1

Suggested Solutions

Question 1 [E]

$$y = \frac{-x^2 + 1}{2x}$$

$$= \frac{-x^2}{2x} + \frac{1}{2x}$$

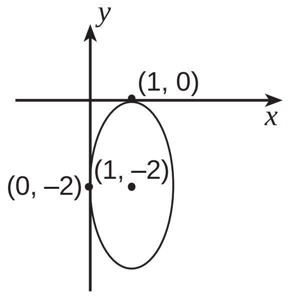
$$= \frac{-x}{2} + \frac{1}{2x}$$

Vertical asymptote at $x = 0$

As $x \to \pm\infty$, $\frac{1}{2x} \to 0$, $\therefore y \to \frac{-x}{2}$

So asymptote at $y = \frac{-x}{2}$

Question 2 [C]



Centre at (1, –2)

$\therefore a = 1, b = 2$

$$\frac{(x-1)}{1^2} + \frac{(y+2)^2}{2^2} = 1$$

$$(x-1)^2 + \frac{(y+2)^2}{4} = 1$$

Question 3 [E]

(A) $\dfrac{\sin(\frac{\pi}{5})}{\cos(\frac{\pi}{5})} = \tan(\frac{\pi}{5})$

(B) $\dfrac{1}{\cot(\frac{\pi}{5})} = \dfrac{1}{\frac{\cos(\frac{\pi}{5})}{\sin(\frac{\pi}{5})}}$

$$= \frac{\sin(\frac{\pi}{5})}{\cos(\frac{\pi}{5})}$$

$$= \tan(\frac{\pi}{5})$$

(C) $\cot(\frac{3\pi}{10}) = \cot(\frac{\pi}{2} - \frac{\pi}{5})$

$$= \frac{\cos(\frac{\pi}{2} - \frac{\pi}{5})}{\sin(\frac{\pi}{2} - \frac{\pi}{5})}$$

$$= \frac{\cos\frac{\pi}{2}\cos\frac{\pi}{5} + \sin\frac{\pi}{2}\sin\frac{\pi}{5}}{\sin\frac{\pi}{2}\sin\frac{\pi}{5} - \cos\frac{\pi}{2}\sin\frac{\pi}{5}}$$

$$= \frac{\sin\frac{\pi}{5}}{\cos\frac{\pi}{5}}$$

$$= \tan\frac{\pi}{5}$$

(D) $\dfrac{2\tan(\frac{\pi}{10})}{1-\tan^2(\frac{\pi}{10})} = \tan(2 \times \frac{\pi}{10})$

$= \tan\frac{\pi}{5}$

$= \tan(2(\frac{\pi}{5})$

(E) $2\tan(\frac{2\pi}{5})$

$= \tan(\frac{4\pi}{5}) = -\tan(\frac{\pi}{5})$ which is incorrect.

Question 4 [D]

Period of graph $= \pi \quad \therefore a = 2$

Horizontal shift of $\frac{\pi}{4}$ parallelto x axis $\quad \therefore b = \frac{\pi}{4}$

Question 5 [E]

(A) $\frac{1}{z} = \frac{1}{x-iy}$

$= \frac{x-iy}{x^2-y^2}$ which is not real.

(B) $\frac{1}{\bar{z}} = \frac{1}{x-iy}$

$= \frac{x+iy}{x^2+y^2}$ which is not real.

(C) $\frac{1}{z-\bar{z}} = \frac{1}{(x+iy)-(x-iy)}$

$= \frac{1}{2iy}$

$= \frac{-1}{2y}\, i$ which is not real.

(D) $\frac{1}{z} - \frac{1}{\bar{z}} = \frac{1}{(x+iy)} - \frac{1}{(x-iy)}$

$= \frac{x-iy}{x^2+y^2} - \frac{x+iy}{x^2+y^2}$

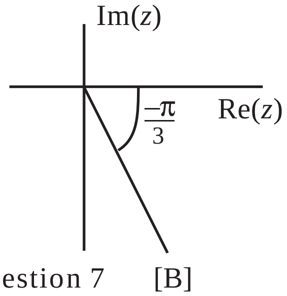
$= \frac{-2yi}{x^2+y^2}$ which is not real.

(E) $\frac{1}{z} + \frac{1}{\bar{z}} = \frac{1}{x+iy} + \frac{1}{x-iy}$

$= \frac{x-iy}{x^2+y^2} + \frac{x+iy}{x^2+y^2}$

$= \frac{2x}{x^2+y^2}$ which is real.

Question 6 [C]



i.e. $\theta = \frac{-\pi}{3}$

$\therefore \tan\theta = \frac{a}{1} = -\sqrt{3}$

so $a = -\sqrt{3}$

Question 7 [B]

If $p(z)$ is of degree 4 and has real coefficients, then the Complex Conjugate Rule applies. So $p(z)$ must either have one pair of complex conjugate roots and two real roots

or two pairs of complex conjugate roots or four real roots.

So $p(z)$ cannot have an *odd* number of non-real roots.

Question 8 [A]

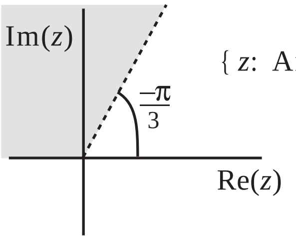
If $|W| = 1.5$ then $w = 1.5 \text{ cis } \theta$

so $W^{-1} = (1.5)^{-1} \text{ cis } (-\theta)$

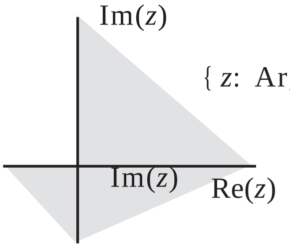
$= \frac{2}{3} \text{ cis } (-\theta)$

So P is best representation.

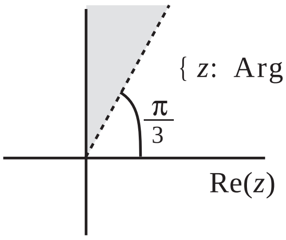
Question 9 [C]



$\{ z: \text{ Arg } (z) > \frac{\pi}{3} \}$



$\{ z: \text{ Arg } (z) < \frac{\pi}{2} \}$



$\{ z: \text{ Arg } z > \frac{\pi}{3} \} \cap \{ z: \text{ Arg } z < \frac{\pi}{2} \}$

Question 10 [D]

$$\int \frac{1}{x^2 + 16} \, dx = \frac{1}{4} \int \frac{4}{x^2 + 16} \, dx$$

$$= \frac{1}{4} \text{ Tan}^{-1}\left(\frac{x}{4}\right)$$

Question 11 [B]

$$\int_o^a \left(\sin^2\left(\frac{3x}{2}\right) - \cos^2\left(\frac{3x}{2}\right)\right) = -\int_o^a \cos\left(2\left(\frac{3x}{2}\right)\right) dx$$

$$= -\int_o^a \cos(3x) \, dx$$

$$= \left[-\frac{1}{3} \sin(3x)\right]_o^a$$

$$= -\frac{1}{3} \sin(3a)$$

Question 12 [A]

$$\int_{o}^{\frac{\pi}{3}} \cos^2(x) - \sin^3(x)\, dx = \int_{0}^{\frac{\pi}{3}} \cos^2(x) \, . \, \sin^2(x) \, . \, \sin(x)\, dx$$

$$= \int_{0}^{\frac{\pi}{3}} \cos^2(x)\,(1 - \cos^2(x)) \sin(x)\, dx$$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$

$-du = \sin(x)\, dx$

When $x = 0$, $u = \cos(0) = 1$

When $x = \frac{\pi}{3}$, $u = \cos(\frac{\pi}{3}) = \frac{1}{2}$

$$= \int_{1}^{\frac{1}{2}} u^2 (1 - u^2) \, . -1\, du$$

$$= \int_{\frac{1}{2}}^{1} u^2 (1 - u^2)\, du$$

Question 13 [D]

$$\int \frac{2}{(3-x)^2} - \frac{1}{3-x}\, dx = \int 2(3-x)^{-2} - \frac{1}{3-x}\, dx$$

$$= 2(3-x)^{-1} + \log_e(3-x)$$

$$= \frac{2}{(3-x)} + \log_e(3-x)$$

Question 14 [D]

For $x = \frac{\pi}{2}$, $f(x) > 0$

$x = \frac{\pi}{2}$, $f(x) = 0$

$\frac{\pi}{2} < x$, $f(x) < 0$ So local maximum at $x = \frac{\pi}{2}$

For $\frac{\pi}{2} < x < \pi$, $f(x) < 0$

$x = \pi$, $f(x) = 0$

$\pi < x < \frac{3\pi}{2}$, $f(x) < 0$ So stationary point of inflection at $x = \pi$

For $\pi < x < \frac{3\pi}{2}$, $f(x) < 0$

$x = \frac{3\pi}{2}$, $f(x) = 0$

$x < \frac{3\pi}{2}$, $f(x) > 0$ So local minimum at $x = \frac{3\pi}{2}$

Question 15 [C]

When $x = 3.5$, $y = 1.803$

When $x = 4.5$, $y = 3.354$

Area $\approx 1.803 \times 1 + 3.354 \times 1$

$= 5.157$

$= 5.16$ (to two decimal places)

Question 16 [D]

$y = \sqrt{x^2 - 9}$

$\therefore y^2 = x^2 - 9$

$\therefore x^2 = y^2 + 9$

$V = \pi \int x^2 \, dy$

$= \pi \int_0^4 25 \, dy - \pi \int_0^4 y^2 + 9 \, dy$

$= \pi \int_0^4 (25 - y^2 - 9) \, dy$

$= \pi \int_0^4 \sqrt{(16 - y^2)} \, dy$

Question 17 [A]

$\underset{\sim}{P} = 2\,\underset{\sim}{i} + x\,\underset{\sim}{j} + 3\,\underset{\sim}{k}$

$|\underset{\sim}{P}| = 4 \quad \therefore \; 4 + x^2 + 9 = 4$

$x^2 + 13 = 16$

$x^2 = 3$

$x = \pm\sqrt{3}$

$\underset{\sim}{q}$ is parallel to $\underset{\sim}{p}$ so $\underset{\sim}{q} = -2\,\underset{\sim}{p}$

so $y = -2x$

If $x = \sqrt{3}$, $y = -2\sqrt{3}$

If $x = -\sqrt{3}$, $y = 2\sqrt{3}$ (not an alternative)

Question 18 [D]

$\underset{\sim}{u} \cdot (\underset{\sim}{u} - 2\underset{\sim}{v})$

$= (3\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}) \cdot ((3\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}) - 2(2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k})$

$= (3\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}) \cdot (-\underset{\sim}{i} - 10\underset{\sim}{j} + 7\underset{\sim}{k})$

$= -3 + 40 + 35$

$= 72$

Question 19 [C]

(A) $|\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2$ (Pythagoras Rule) $\therefore$ True

Now $\underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{a} \quad \therefore \; \underset{\sim}{a} - \underset{\sim}{c} = \underset{\sim}{b}$

(B) So $\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = \underset{\sim}{b} \cdot \underset{\sim}{b}$

$= |\underset{\sim}{b}|^2$ True

(C) $\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = \underset{\sim}{b} \cdot \underset{\sim}{c}$

$= |\underset{\sim}{b}|\,|\underset{\sim}{c}| \cos(90°)$

$= 0$

$\neq |\underset{\sim}{b}|\,|\underset{\sim}{c}|$ Not true

(D) $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos(\theta)$ (Dot Product Rule) True

(E) $\underset{\sim}{a} \cdot \underset{\sim}{c} = |\underset{\sim}{a}|\,|\underset{\sim}{c}| \cos(90° - \theta)$

$= |\underset{\sim}{a}|\,|\underset{\sim}{c}| \sin(\theta)$ True

Question 20 [E]

$\underset{\sim}{r}(t) = 2\sin(t)\, i + \cos(t)\, \underset{\sim}{j}$

so $x = 2\sin(t) + \sin(t) = \frac{x}{2}$

and $y = \cos(t)$

$\sin^2(t) + \cos^2(t) = 1$

$\therefore \quad (\frac{x}{2})^2 + y^2 = 1$

$\frac{x^2}{4} + y^2 = 1$

$0 \le t \le \pi$ so when $t = 0,\ x = 0$

$t = \frac{\pi}{2},\ x = 2$

$t = \pi,\ x = 0$

so $0 \le x \le 2$

Question 21 [B]

$\underset{\sim}{r}(t) = \int 3\sin(2t)\, \underset{\sim}{i} + 4\underset{\sim}{j}\ dt$

$= (\frac{-3}{2}\cos(2t) + c)\, \underset{\sim}{i} + (4t + d)\, \underset{\sim}{j}$

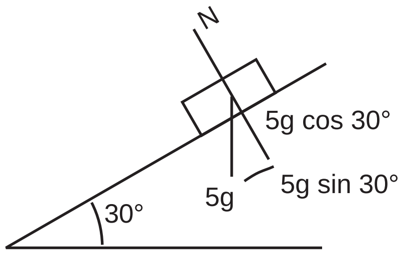
$\underset{\sim}{r}(0) = \frac{3}{2}\underset{\sim}{i}$ so $\frac{-3}{2}\cos(0) + c = \frac{3}{2}$

$\therefore c = 3$

and $d = 0$

so $\underset{\sim}{r}(t) = (\frac{-3}{2}\cos(2t) + 3)\, \underset{\sim}{i} + 4t\, \underset{\sim}{j}$

Question 22 [B]

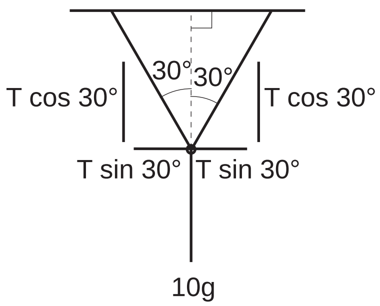


$5g\cos 30° = 5a$

$5g \cdot \frac{1}{2} = 5a$

$a = \frac{g}{2}$

Question 23 [D]



$2T\cos(30°) = 10g$

$2T \cdot \frac{\sqrt{3}}{2} = 10g$

$T = \frac{10g}{\sqrt{3}}$

$= \frac{10\sqrt{3}}{3}\, g$

Question 24 [A]

$\sum \underset{\sim}{F} = 5\underset{\sim}{a}$

$(2\underset{\sim}{i} + \underset{\sim}{j}) + (\underset{\sim}{i} + 10\underset{\sim}{j}) + (3\underset{\sim}{i} - 3\underset{\sim}{j}) = 5\underset{\sim}{a}$

$6\underset{\sim}{i} + 8\underset{\sim}{j} = 5\underset{\sim}{a}$

$|\underset{\sim}{a}| = \frac{\sqrt{6^2 + 8^2}}{5}$

$= \frac{10}{5}$

$= 2$

Question 25 [B]

$u = -21,\ t = 10$

$a = 9.8$

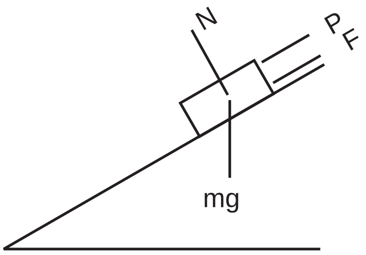
$s = ut + \frac{1}{2}at^2$

$\therefore\ h = -21 \times 10 + \frac{1}{2} \times 9.8 \times 10^2$

$= 280$ m

Question 26 [A]

Body is on a point of sliding down, so Frictionl Force (F) must be *up* the plane



Question 27 [B]

$v = \frac{dx}{dt} = 2.5 - \frac{9}{2}\sin\left(\frac{t}{2}\right)$

minimum when $\frac{dv}{dt} = 0$

$\frac{dv}{dt} = -\frac{9}{4}\cos\left(\frac{t}{2}\right) = 0$

$\therefore \cos\left(\frac{t}{2}\right) = 0$

$t = \pi$

so $v = 2.5 - \frac{9}{2}\sin\left(\frac{\pi}{2}\right)$

$= 2.5 - 4.5$

$= -2$

Question 28 [C]

$y = \sin(2x)$

$\frac{dy}{dx} = 2\cos(2x)$

$\frac{d^2y}{dx^2} = -4\sin(2x)$

(A) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 4\cos(2x)$

$-4\sin(2x) + 8\cos(2x) + \sin(2x) \neq 4\cos 2x$

(B) $\frac{d^2y}{dx} + 2\frac{dy}{dx} - 4y = 4\cos(2x)$

$-4\sin(2x) + 4\cos(2x) - \sin(2x) \neq 4\cos(2x)$

(C) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 4\cos(2x)$

$-4\sin(2x) + 4\cos(2x) + 4\sin(2x) = 4\cos 2x$

(D) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 4\cos(2x)$

$-4\sin(2x) - 4\cos(2x) - \sin(2x) \neq 4\cos(2x)$

(E) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 4\cos(2x)$

$-4\sin(2x) - 4\cos(2x) + 4\sin(2x) \neq 4\cos(2x)$

Question 29 [E]

$\frac{dy}{dt}\ \alpha\ y - T_s \qquad T_s = 4$

so $\frac{dy}{dt} = -k\,(y - 4) \qquad t = 0,\ y = 20$

Question 30 [C]

$a = 16x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 16x$

$\frac{1}{2}v^2 = \int 16x\ dx$

$\frac{1}{2}v^2 = 8x^2 + c$

$v = -5,\ x = 0 \ \therefore\ \frac{25}{2} = c$

so $\frac{1}{2}v^2 = 8x^2 + \frac{25}{2}$

$v^2 = 16x^2 + 25$

$v^2 = \pm\sqrt{16x^2 + 25}$

But $v = -5$ at $x = 0$ so $v = -\sqrt{16x^2 + 25}$

2004 Mathematical Methods, Specialist Examination 1, Part II

Question 1

(a) $\frac{d}{dt}(\sin^{-1}(\sqrt{2x})) = \frac{1}{\sqrt{1-(\sqrt{2x})^2}} \times \frac{1}{2}(2n)^{-0.5}.\ 2$

$= \frac{1}{\sqrt{1-2x}} \times \frac{1}{\sqrt{2x}}$

$= \frac{1}{\sqrt{2x\,(1-2x)}}$

(b) $\int_{.125}^{.25} \frac{1}{\sqrt{2x(1-2x)}}\, dx = [\ \text{Sin}^{-1}\sqrt{(2x)}]_{0.125}^{0.25}$

$= \text{Sin}^{-1}(\sqrt{0.5}) - \text{Sin}^{-1}(\sqrt{0.25})$

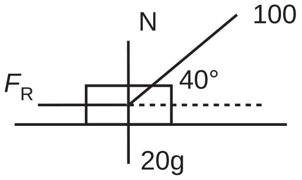
$= \text{Sin}^{-1}(\frac{1}{\sqrt{2}}) - \text{Sin}^{-1}(\frac{1}{2})$

$= \frac{\pi}{4} - \frac{\pi}{6}$

$= \frac{\pi}{12}$

Question 2

(a)



(b) $\sum \underset{\sim}{F} = m\underset{\sim}{a}$

$(100\cos(40°) - F_R)\underset{\sim}{i} + (\text{N} + 100\sin(40°) - 20g)\underset{\sim}{j} = 20\,(a\underset{\sim}{i} + 0\underset{\sim}{j})$

$F_R = \mu N = 0.34\,(20\text{g} - 100\sin(40°))$

$= 44.7852$

So $100\cos(40°) - 44.7852 = 20a$

$a = 1.59\ \ m/s^2$

Question 3

$f(x) = \int 15x\sqrt{2-x}\,dx$

let $u = 2 - x$

$\frac{du}{dx} = -1$

So $-du = dx$

and $x = 2 - u$

$\therefore\ f(x) = \int -15\,(2-u)\sqrt{u}\,du$

$= \int 15u^{1.5} - 30u^{0.5}\,du$

$= 0.4 \times 15u^{2.5} - \frac{60}{3}u^{0.5} + c$

$= 6u^{2.5} - 20u^{1.5} + c$

$= u^{1.5}(6u - 20) + \text{c}$

$= (2-x)^{1.5}(6(2-x) - 20) + c$

$= (2-x)^{1.5}(-6x - 8) + c$

$f(2) = 0$ so $c = 0$

$\therefore f(x) = (2 - x)^{1.5}(-6x - 8) = (ax + b)(2 - x)^{1.5}$
so $a = -6$ and $b = -8$

Question 4

(a) $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = (6\underset{\sim}{i} + 2\underset{\sim}{j}) \cdot \dfrac{(4\underset{\sim}{i} - 3\underset{\sim}{j})}{\sqrt{16 + 9}}$

$= \frac{1}{5}(24 - 6)$

$= \frac{18}{5}$

(b) $a \cdot \hat{b} = |\overrightarrow{OP}|$

$|\overrightarrow{PA}|^2 = |\overrightarrow{OP}|^2 - |\overrightarrow{OP}|^2$

$= 6^2 + 2^2 - (\frac{18}{5})^2$

$= 27.04$

$\therefore |\overrightarrow{PA}| = 5.2$

Question 5

$W = \frac{-\sqrt{3}}{2} - \frac{1}{2} i$

(a) $r = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

$\tan \theta = \dfrac{\frac{-1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$ (third quadrant)

so $\theta = -\frac{5\pi}{6}$

$W = \text{cis}\,(\frac{-5\pi}{6})$

(b) $W^k = \text{cis}\,(k \times \frac{-5\pi}{6}) = 1$

$\text{cis}\,(\frac{-5k\pi}{6}) = 1 = \cos(\frac{-5k\pi}{6}) + i \sin(\frac{-5k\pi}{6})$

$= 1 + 0i$

so $\cos(\frac{-5k\pi}{6}) = 1$

$\therefore \frac{-5k\pi}{6} = \ldots, -10\pi, -8\pi, -6\pi, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$

$k = \ldots \frac{-60\pi}{-5\pi}, \frac{-48\pi}{-5\pi}, \frac{-24\pi}{-5\pi}, \ldots$

But k is a positive integer, so $k = 12$ is the least positive integer.