2004 Specialist Mathematics
Written Examination 1 (facts, skills and applications)
Suggested answers and solutions

Part I (Multiple-choice) Answers

1. **C**	2. **A**	3. **B**	4. **C**	5. **E**
6. **D**	7. **A**	8. **D**	9. **D**	10. **B**
11. **A**	12. **E**	13. **E**	14. **A**	15. **C**
16. **E**	17. **B**	18. **C**	19. **A**	20. **C**
21. **E**	22. **E**	23. **B**	24. **D**	25. **C**
26. **B**	27. **A**	28. **D**	29. **D**	30. **B**

1 General equation for an ellipse: **[C]**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$a=4$ and $b=9$

$$\therefore \frac{x^2}{16}+\frac{y^2}{81}=1$$

2 $\text{Cos}\left(\frac{7\pi}{6}\right)=\frac{^-\sqrt{3}}{2}$ **[A]**

$$\text{Sin}^{-1}\left(\frac{^-\sqrt{3}}{2}\right)=\frac{^-\pi}{3}$$

Note: $\frac{^-\pi}{2}\le \text{Sin}^{-1}(x)\le\frac{\pi}{2}$

3 $\sec^2(x)-1=\tan^2(x)$ **[B]**

$\cos(x)=-\frac{2}{3}$

$\Rightarrow \sec^2(x)=\frac{9}{4}$

$\therefore \tan^2(x)=\frac{9}{4}-1$

$=\frac{5}{4}$

$\tan(x)=\pm\frac{\sqrt{5}}{2}$

x is in the 2nd quadrant $\therefore \tan(x)=-\frac{\sqrt{5}}{2}$

4 $z^4-81=0$ **[C]**

$(z^2-9)(z^2+9)=0$

$(z-3)(z+3)(z-3i)(z+3i)=0$

$z=\pm3,\pm3i$

5 $z=2\text{cis}\left(\frac{5\pi}{6}\right)$ and $w=5\text{cis}\left(\frac{\pi}{3}\right)$ **[E]**

$zw=10\text{cis}\left(\frac{5\pi}{6}+\frac{\pi}{3}\right)$

$=10\text{cis}\left(\frac{5\pi}{6}+\frac{2\pi}{6}\right)$

$=10\text{cis}\left(\frac{7\pi}{6}\right)$

$\arg(zw)=\frac{7\pi}{6}$

$-\pi<\text{Arg}(zw)\le\pi$

$\therefore \text{Arg}(zw)=-\frac{5\pi}{6}$

6 $z = 2 + 3i$ **[D]**

$\bar{z} = 2 - 3i$

$\bar{z}w = (2 - 3i)(2 - i)$

$= 4 - 2i - 6i + 3i^2$

$= 1 - 8i$

7 $z = -\sqrt{3} + i$ **[A]**

$|z| = \sqrt{3+1} = 2$

$\theta = \text{Tan}^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

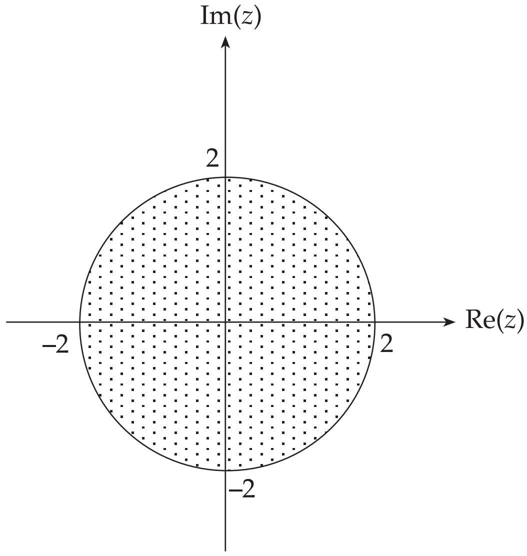
$z = 2\text{cis}\frac{5\pi}{6}$

$z^4 = 2^4\text{cis}\frac{20\pi}{6}$

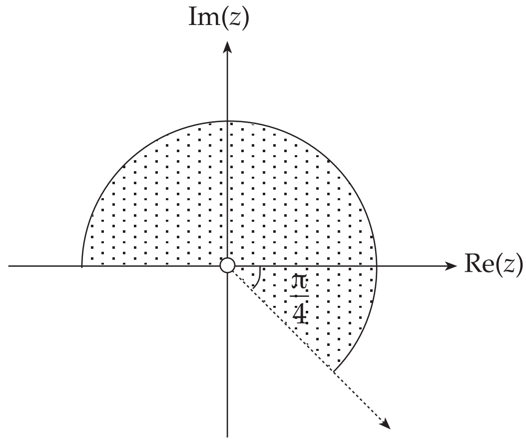
$= 16\text{cis}\left(\frac{10\pi}{3}\right)$

$= 16\text{cis}\left(\frac{-2\pi}{3}\right)$

8 **[D]**



$\{z : z\bar{z} \le 2\}$ or $\{z : |z| \le 2\}$



$\left\{z : \text{Arg}(z) \ge -\frac{\pi}{4}\right\}$

$\therefore \{z : |z| \le 2\} \cap \left\{z : \text{Arg}(z) \ge -\frac{\pi}{4}\right\}$

9 $\frac{3x-1}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$ **[D]**

$= \frac{A(x-2) + B(x+3)}{(x+3)(x-2)}$

$\Rightarrow 3x - 1 \equiv A(x-2) + B(x+3)$

Let $x = 2$

$5 = 5B$

$\Rightarrow B = 1$

Let $x = -3$

$-10 = -5A$

$\Rightarrow A = 2$

$\therefore A = 2$ and $B = 1$

10 $\frac{d\left(x\text{Tan}^{-1}x\right)}{dx} = \frac{x}{x^2+1} + \text{Tan}^{-1}(x)$ **[B]**

$\Rightarrow \text{Tan}^{-1}(x) = \frac{d\left(x\text{Tan}^{-1}x\right)}{dx} - \frac{x}{x^2+1}$

$\Rightarrow \int \text{Tan}^{-1}(x)dx = x\text{Tan}^{-1}(x) - \int \frac{x}{x^2+1}dx$

$= x\text{Tan}^{-1}(x) - \frac{1}{2}\ln\left(x^2+1\right)$

11 $\int_1^3 (2x+1)\sqrt{2x-1}\,dx$ **[A]**

Let $u = 2x - 1$

$\Rightarrow 2x = u + 1$

and $\frac{du}{dx} = 2$

$\Rightarrow dx = \frac{du}{2}$

For terminals: $x = 3$ $\quad u = 6 - 1$

$u = 5$

$x = 1$ $\quad u = 2 - 1$

$u = 1$

$\frac{1}{2}\int_1^5 (u+2)\sqrt{u}\,du$

12 $y = x$ $\qquad y = 2 - x^2$ **[E]**

$x = 2 - x^2$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = 1, -2$

$y = 1, -2$

$V = \pi\int y_1^2 dx - \pi\int y_1^2 dx$

where $y_1 = 2 - x^2$ and $y_2 = x$

$V = \pi\int_0^1 \left(2-x^2\right)^2 dx - \pi\int_0^1 \left(x^2\right) dx$

$= \pi\int_0^1 \left(2-x^2\right)^2 - x^2 dx$

13 $\int_0^1 x - \frac{-2}{\sqrt{4-x^2}}\,dx$ **[E]**

$= \int_0^1 x + \frac{-2}{\sqrt{4-x^2}}\,dx$

$= \left[\frac{x^2}{2} + 2\mathrm{Sin}^{-1}\left(\frac{x}{2}\right)\right]_0^1$

$= \left(\frac{1}{2} + 2\mathrm{Sin}^{-1}\left(\frac{1}{2}\right)\right) - (0+0)$

$= \frac{1}{2} + 2\times\frac{\pi}{6}$

$= \frac{1}{2} + \frac{\pi}{3}$

14 Using graphics calculator. **[A]**

$\int_0^1 \left(e^{x^2} + e^{-x^2}\right)^2 dx = 4.9626$

15 Midpoint at $x = \frac{1}{2}$ **[C]**

Approximate area $= \cos\left(2\times\frac{1}{2}\right)$

$= \cos(1)$

16 Inflow: $\frac{dx_1}{dt} = 0.1\times 2$ **[E]**

$= 0.2$ kg/min

Outflow: $\frac{dx_2}{dt} = \frac{x}{100}\times 2$

$= \frac{x}{50}$

$\frac{dx}{dt} =$ inflow – outflow

$\frac{dx}{dt} = 0.2 - \frac{x}{50}$

17 $\frac{d^2y}{dx^2} = \cos(2x)$ **[B]**

$\frac{dy}{dx} = \frac{1}{2}\sin(2x) + c$

$\frac{dy}{dx} = 0$ at $x = 0$

$\therefore c = 0$

$\frac{dy}{dx} = \frac{1}{2}\sin(2x)$

$y = -\frac{1}{4}\cos(2x) + d$

at $y = 0$ and $x = 0$

$d = \frac{1}{4}$

$y = -\frac{1}{4}\cos(2x) + \frac{1}{4}$

18 $\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$ **[C]**

$\Rightarrow \cos\theta = \frac{\underset{\sim}{a}.\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$\underset{\sim}{a}.\underset{\sim}{b} = 2 - 3 - 6 = -7$

$|\underset{\sim}{a}| = \sqrt{4+1+9} = \sqrt{14}$

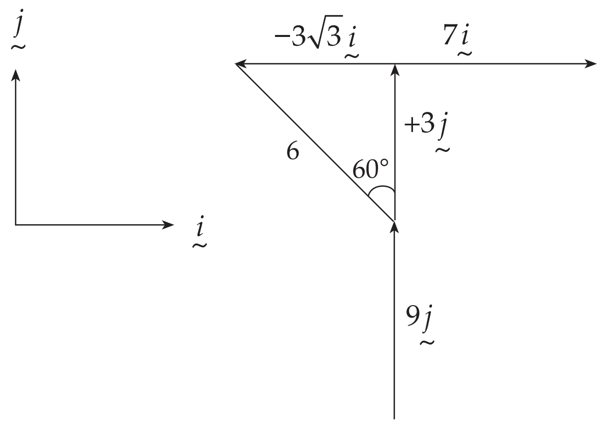
$|\underset{\sim}{b}| = \sqrt{1+9+4} = \sqrt{14}$

$\cos\theta = \frac{-7}{\sqrt{14}\times\sqrt{14}} = -\frac{1}{2}$

$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$

$= \frac{2\pi}{3}$

19 **[A]**



$(9+3)\underset{\sim}{j} + \left(7 - 3\sqrt{3}\right)\underset{\sim}{i}$

Position vector: $\left(7 - 3\sqrt{3}\right)\underset{\sim}{i} + 12\underset{\sim}{j}$

20 $\overrightarrow{OT}.\overrightarrow{TB} = |\overrightarrow{OT}||\overrightarrow{TB}|\cos\angle OTB$ **[C]**

$\angle OTB = 90°$

$\cos 90° = 0$

$\therefore \overrightarrow{OT}.\overrightarrow{TB} = 0$

21 $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{k}$ **[E]**

$4\underset{\sim}{a} = 8\underset{\sim}{i} + 4\underset{\sim}{j} + 20\underset{\sim}{k}$

$\underset{\sim}{b} = -4\underset{\sim}{i} + \underset{\sim}{j}$

$4\underset{\sim}{a} + \underset{\sim}{b} = 4\underset{\sim}{i} + 5\underset{\sim}{j} + 20\underset{\sim}{k}$

$|4\underset{\sim}{a} + \underset{\sim}{b}| = \sqrt{16+25+400}$

$= \sqrt{441}$

$= 21$

22 $\underset{\sim}{a} = 6\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}$ **[E]**

$\underset{\sim}{b} = -6\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

$\hat{\underset{\sim}{a}} = \frac{1}{2\sqrt{19}}\left(6\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}\right)$

$\underset{\sim}{b} \cdot \hat{\underset{\sim}{a}} = \frac{1}{2\sqrt{19}}\left(-36 + 4 + 6\right)$

$= \frac{-26}{2\sqrt{19}}$

$= \frac{-13}{\sqrt{19}}$

$\left(\underset{\sim}{b} \cdot \hat{\underset{\sim}{a}}\right)\hat{\underset{\sim}{a}} = \frac{-13}{\sqrt{19}} \times \frac{1}{2\sqrt{19}}\left(6\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}\right)$

$= \frac{-13}{19}\left(3\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right)$

23 $x = t - 1$ **[B]**

$y = 5\left(t - 1\right)^2$

$y = 5(x)^2$

we know that $t \geq 0$

$\Rightarrow x \geq -1$

$\therefore y = 5x^2$ where $x \geq -1$

24 $\dot{\underset{\sim}{r}}(t) = 4\cos 2t\underset{\sim}{i} + 3e^t\underset{\sim}{j}$ **[D]**

$\underset{\sim}{r}(0) = 4\underset{\sim}{i} + 3\underset{\sim}{j}$

$\left|\underset{\sim}{r}(0)\right| = \sqrt{16 + 9}$

$= \sqrt{25}$

$= 5$

25 $\underset{\sim}{F}_1 = 2\underset{\sim}{i} + \underset{\sim}{j}$ $\quad\quad$ $\underset{\sim}{F}_2 = 8\underset{\sim}{i} - 5\underset{\sim}{k}$ **[C]**

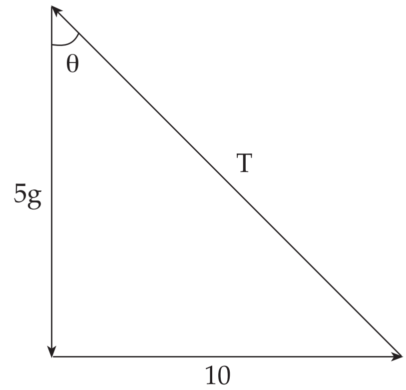
$\underset{\sim}{F}_T = 10\underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k}$

$\underset{\sim}{F}_T = m\underset{\sim}{a}$

$= 4\underset{\sim}{a} = 10\underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k}$

$\underset{\sim}{a} = 2.5\underset{\sim}{i} + 0.25\underset{\sim}{j} - 1.25\underset{\sim}{k}$

26 **[B]**



$T\sin\theta = 10$

$T\cos\theta = 5g$

$\tan\theta = \frac{10}{5g}$

$\theta = \tan^{-1}\left(\frac{2}{g}\right)$

$\theta \simeq 0.2013$

27 $a = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = x + 6$ **[A]**

$\frac{1}{2}v^2 = \frac{1}{2}x^2 + 6x + c$

$v^2 = x^2 + 12x + d$

$2c = d$

at $x = 1$, $v = 7$

$49 = 1 + 12 + d$

$d = 36$

$\therefore v^2 = x^2 + 12x + 36$

$v = \pm\left(x + 6\right)$

28 Distance travelled is area enclosed by graph over the interval: **[D]**

$A = \frac{1}{2} \times 6 \times 4 + \frac{1}{2} \times 6 \times 4 + 2 \times 6$

$= 12 + 12 + 12$

$= 36$

29 We know: $T - \mu m_1 g = m_1 a$ —— 1 **[D]**

$T = m_2 g$ —— 2

If $m_1 a > 0$ then the block will move.

From 1 and 2:

$m_2 g - \mu m_1 g = m_1 a$

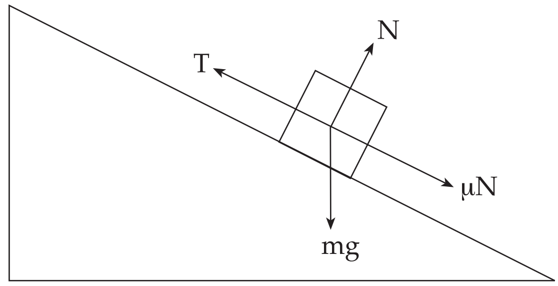
If $m_1 a > 0$

then $m_2 g - \mu m_1 g > 0$

$m_2 g > \mu m_1 g$

$m_2 > \mu m_1$

30 **[B]**



Short answer solutions

Question 1

a **i** $f(x) = \dfrac{-1}{1+x^2}$

Let $u = 1 + x^2$

$\dfrac{du}{dx} = 2x$

$\Rightarrow y = -u^{-1}$

$\dfrac{dy}{du} = u^{-2}$ **(M1)**

$f'(x) = \dfrac{dy}{dx} = 2x\left(1+x^2\right)^{-2}$

$= \dfrac{2x}{\left(1+x^2\right)^2}$ **(A1)**

a **ii** $f'(1) = \dfrac{2}{(1+1)^2}$

$= \dfrac{1}{2}$ **(A1)**

b $y_2 = y_1 + hf'(x_1)$

$y_1 = f(1) = -\dfrac{1}{2}$

$h = 0.01$

$f'(x_1) = f'(1) = \dfrac{1}{2}$

$y_2 = -\dfrac{1}{2} + 0.01 \times \dfrac{1}{2}$

$= -0.495$ **(A1)***

* must include working

Question 2

$\displaystyle\int_1^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}}\,dx$

$= \left[\text{Sin}^{-1}\left(\dfrac{x}{2}\right)\right]_1^{\sqrt{2}}$ **(M1)**

$= \text{Sin}^{-1}\left(\dfrac{\sqrt{2}}{2}\right) - \text{Sin}^{-1}\left(\dfrac{1}{2}\right)$

$= \dfrac{\pi}{4} - \dfrac{\pi}{6} = \dfrac{\pi}{12}$ **(A1)**

Question 3

a $\underset{\sim}{r}(t) = 4\cos t\,\underset{\sim}{i} + 4\sin t\,\underset{\sim}{j} + 3t\underset{\sim}{k}$

$\dot{\underset{\sim}{r}}(t) = -4\sin t\,\underset{\sim}{i} + 4\cos t\,\underset{\sim}{j} + 3\,\underset{\sim}{k}$ **(M1)**

$\dot{\underset{\sim}{r}}(0) = 4\underset{\sim}{j} + 3\,\underset{\sim}{k}$

$\left|\dot{\underset{\sim}{r}}(0)\right| = \sqrt{16+9}$ **(M1)**

$= 5$ **(A1)**

b $\dot{\underset{\sim}{r}}(t) = -4\sin t\,\underset{\sim}{i} + 4\cos t\,\underset{\sim}{j} + 3\,\underset{\sim}{k}$

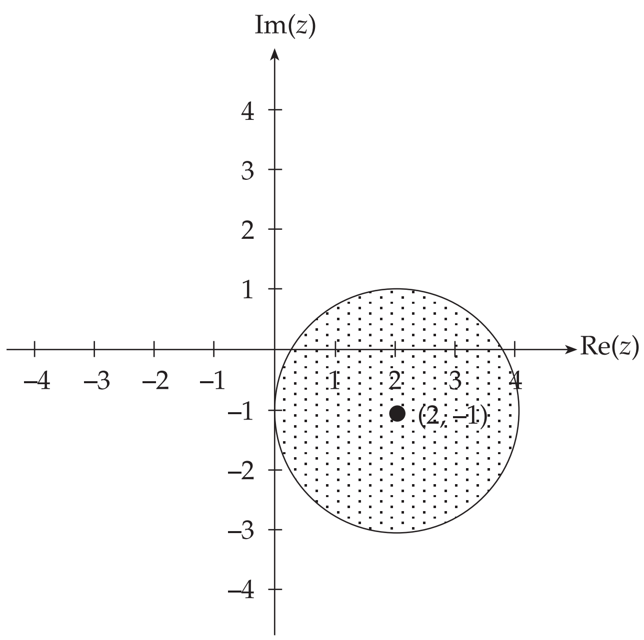
$\ddot{\underset{\sim}{r}}(t) = -4\cos(t)\underset{\sim}{i} - 4\sin(t)\underset{\sim}{j} + 0\underset{\sim}{k}$ **(M1)**

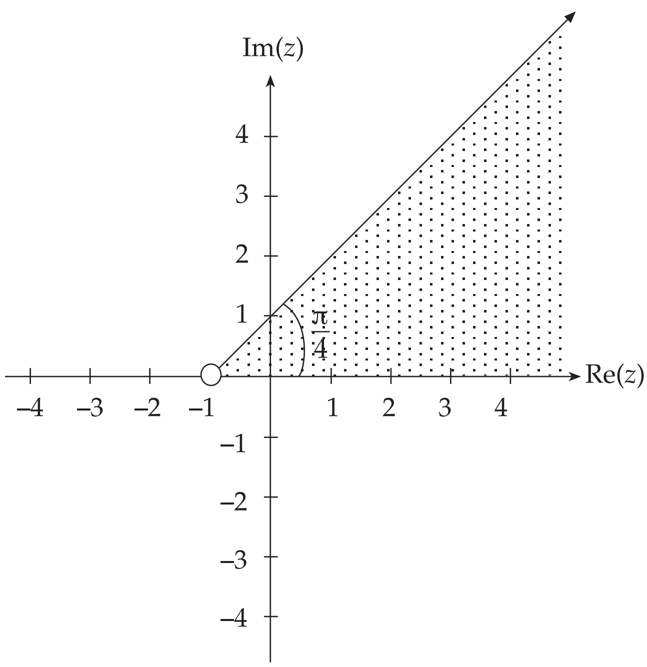
$\dot{\underset{\sim}{r}}(t).\ddot{\underset{\sim}{r}}(t) = 16\sin(t)\cos(t) - 16\sin(t)\cos(t) + 0$ **(M1)**

$\dot{\underset{\sim}{r}}(t).\ddot{\underset{\sim}{r}}(t) = 0$ for all t, and $\dot{\underset{\sim}{r}}(t) \neq 0$ and $\ddot{\underset{\sim}{r}}(t) \neq \underset{\sim}{0}$

$\therefore \dot{\underset{\sim}{r}}(t)$ is perpendicular to $\ddot{\underset{\sim}{r}}(t)$ **(A1)**

Question 4





Circle of radius 2, centre at (2, –1) **(A1)**

Correct shading **(A1)**

Both rays starting at (–1, 0) **(A1)**

Correct shading **(A1)**

Question 5

a $\left(\cos\theta + i\sin\theta\right)^3$

Using De Moivre's Theorem

$\left(\cos\theta + i\sin\theta\right) = \cos 3\theta + i\sin 3\theta$ **(A1)**

Using Binomial expansion

$\left(\cos\theta + i\sin\theta\right)^3 = \cos^3\theta + 3\cos^2\theta i\sin\theta + 3\cos\theta i^2\sin^2\theta + i^3\sin^3\theta$ **(M1)**

$= \cos^3\theta - 3\cos\theta\sin^2\theta + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)$

Equating real parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ **(A1)**

b Equating imaginary parts

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ **(A1)**