Year 2004

VCE

Specialist Mathematics

Trial Examination 2

KILBAHA MULTIMEDIA PUBLISHING PO BOX 2227 KEW VIC 3101 AUSTRALIA	TEL: (03) 9817 5374 FAX: (03) 9817 4334 chemas@chemas.com www.chemas.com

STUDENT NUMBER

Figures

Words

VICTORIAN CERTIFICATE OF EDUCATION
2004

SPECIALIST MATHEMATICS

Trial Written Examination 2
(Analysis Task)

Reading time: 15 minutes
Total writing time: 1 hour 30 minutes

QUESTION AND ANSWER BOOK

Number of questions	*Number of questions to be answered*	*Number of marks*
5	5	60

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, up to four pages (two A4 sheets) of pre-written notes (typed or handwritten) and an approved scientific and/or graphics calculator (memory may be retained).
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or whiteout liquid/tape.

Materials supplied

- Question and answer book of 21 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Working space is provided throughout the book.

Instructions

- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other electronic communication devices into the examination room.

SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time. This formula sheet is provided for your reference.

Specialist Mathematics Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$
cosine rule:	$c^2=a^2+b^2-2ab\cos C$

Coordinate geometry

ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

hyperbola: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Circular (trigonometric) functions

$\cos^2 x+\sin^2 x=1$

$1+\tan^2 x=\sec^2 x$ $\qquad$ $\cot^2 x+1=\text{cosec}^2 x$

$\sin(x+y)=\sin x\cos y+\cos x\sin y$ $\qquad$ $\sin(x-y)=\sin x\cos y-\cos x\sin y$

$\cos(x+y)=\cos x\cos y-\sin x\sin y$ $\qquad$ $\cos(x-y)=\cos x\cos y+\sin x\sin y$

$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$ $\qquad$ $\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$

$\cos 2x=\cos^2 x-\sin^2 x=2\cos^2 x-1=1-2\sin^2 x$

$\sin 2x=2\sin x\cos x$ $\qquad$ $\tan 2x=\frac{2\tan x}{1-\tan^2 x}$

Function	Sin^{-1}	Cos^{-1}	Tan^{-1}
Domain	$[-1,1]$	$[-1,1]$	R
range	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0,\pi]$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Algebra (Complex Numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$

$|z| = \sqrt{x^2 + y^2} = r$ $\qquad -\pi < Arg\,z \le \pi$

$z_1 z_2 = r_1 r_2 \,\text{cis}(\theta_1 + \theta_2)$ $\qquad \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\,\text{cis}(\theta_1 - \theta_2)$

$z^n = r^n \text{cis}(n\theta)$ (de Moivre's theorem)

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$|\underset{\sim}{r}| = \sqrt{x^2 + y^2 + z^2} = r$ $\qquad \underset{\sim}{r_1} \cdot \underset{\sim}{r_2} = r_1 r_2 \cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2$

$\dot{\underset{\sim}{r}} = \dfrac{d\underset{\sim}{r}}{dt} = \dfrac{dx}{dt}\underset{\sim}{i} + \dfrac{dy}{dt}\underset{\sim}{j} + \dfrac{dz}{dt}\underset{\sim}{k}$

<u>Mechanics</u>

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

sliding friction: $F \le \mu N$

constant (uniform) acceleration:

$v = u + at$ $\qquad s = ut + \dfrac{1}{2}at^2$ $\qquad v^2 = u^2 + 2as$ $\qquad s = \dfrac{1}{2}(u + v)t$

acceleration: $a = \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

Calculus

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin ax) = a\cos ax$$

$$\frac{d}{dx}(\cos ax) = -a\sin ax$$

$$\frac{d}{dx}(\tan ax) = a\sec^2 ax$$

$$\frac{d}{dx}(\text{Sin}^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\text{Cos}^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\text{Tan}^{-1}x) = \frac{1}{1+x^2}$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c\text{ , } n \neq -1$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$$

$$\int \frac{1}{x}dx = \log_e x + c, \textit{ for } x > 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax + c$$

$$\int \cos ax dx = \frac{1}{a}\sin ax + c$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \text{Sin}^{-1}\frac{x}{a} + c,\ a > 0$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}}dx = \text{Cos}^{-1}\frac{x}{a} + c,\ a > 0$$

$$\int \frac{a}{a^2 + x^2}dx = \text{Tan}^{-1}\frac{x}{a} + c$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

mid-point rule: $\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right)$

trapezoidal rule: $\int_a^b f(x)dx \approx \frac{1}{2}(b-a)(f(a) + f(b))$

Euler's method

If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x)$

Take the acceleration due to gravity to have magnitude g m/s^2, where $g = 9.8$

Question 1

i. Differentiate $x\,\text{Cos}^{-1}(x)$ with respect to x and hence show that
$\int \text{Cos}^{-1}(x)\,dx = x\,\text{Cos}^{-1}(x) - \sqrt{1-x^2}$

2 marks

ii. A cross-section of a hillside is of the form of $y = 4x\,\mathrm{Cos}^{-1}\left(\frac{4x^2}{9}\right)$ where x and y are measured in metres. If $y \geq 0$ for $0 \leq x \leq b$ show that $b = \frac{3}{2}$ and sketch the graph of $y = 4x\,\mathrm{Cos}^{-1}\left(\frac{4x^2}{9}\right)$ on the axes below.

1 mark

iii. Find the coordinates of the turning point on the graph of $y = 4x\,\mathrm{Cos}^{-1}\left(\frac{4x^2}{9}\right)$

Hence, find the maximum height of the hillside giving your answers correct to three decimal places.

1 mark

iv. A ball rolls down the hillside on the curve $y = 4x\text{Cos}^{-1}\left(\frac{4x^2}{9}\right)$ and at $x = 1$ the vertical component of its speed is 2 m/s. Find the horizontal component of the speed of the ball giving your answer correct to three decimal places.

3 marks

v. Find using the mid-point rule with three subintervals an approximation to the cross-sectional area of the hillside, give your answer correct to three decimal places. Re-draw the graph with rectangles on the diagram.

2 marks

vi. Find using calculus the exact area bounded by the curve $y = 4\,x\,\text{Cos}^{-1}\left(\frac{4x^2}{9}\right)$ and the x axis.

3 marks
Total 12 marks

Question 2

a. Sets of points in the complex plane are defined by

$T = \{z : 3\mathrm{Re}(z) - 4\,\mathrm{Im}(z) = 25\}$ and $U = \{z : |z| = |z - 6 + 8i|\}$

i. Find the Cartesian equation of T and U and show that $T = U$

2 marks

ii. Sets of points in the complex plane are defined by

$S = \{z : |z - 3 + 4i| = 5\}$ and $R = \{z : (z - 3 + 4i)(\overline{z} - 3 - 4i) = 25\}$

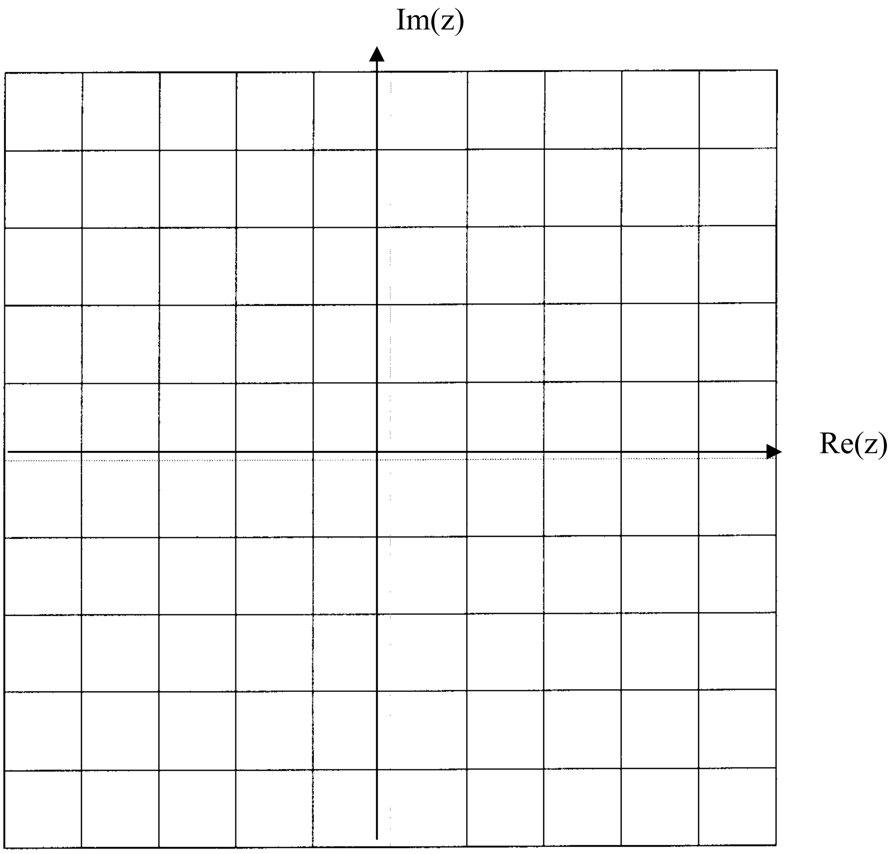
Find the Cartesian equation of S and R and show that $S = R$

4 marks

iii. Let z_A be the point $7-i$ and z_B be the point $-1-7i$, show that both z_A and z_B **both** belong to **both** S and T

2 marks

iv. Given the points $O = (0,0)$, $A = (7,-1)$, $B = (-1,-7)$, $C = (3,-4)$ $D = (6,-8)$
Sketch the set of points S and T, along with the points O, A, B, C and D on the one Argand diagram below.



2 marks

v. Use vectors to show that $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{OC}$ and that C lies on $\overrightarrow{AB}$.

Write down the values of $\left|\overrightarrow{AC}\right|$ and $\left|\overrightarrow{AB}\right|$

3 marks

vi. Explain geometrically the set T in relationship to the points O and D

1 mark

b. A particle P of mass 2 kg moves so that its position vector at a time t seconds is given by $\underset{\sim}{r}(t) = (3 + 5\cos(2t))\underset{\sim}{i} + (-4 + 5\sin(2t))\underset{\sim}{j}$ metres where $t \geq 0$

i. Find the Cartesian equation of the path and show that P moves on S

1 mark

ii. Find the magnitude of the momentum of the particle.

1 marks
Total 16 marks

Question 3

A hammer of mass 5 kg is dropped from a building site from a height of 100 metres

a. Assuming no air resistance, find correct to two decimal places

i. the time in seconds it takes to hit the ground.

1 mark

ii. the speed in m/s upon hitting the ground.

1 mark

b. If on its downward path the hammer is retarded by a variable force of $0.01v^2$ Newtons, where v m/s is its velocity at a time t seconds, and x m is its distance down from the point of release.

i. Show that the differential equation which describes the downward motion of the hammer is $v\frac{dv}{dx} = \frac{49 - 0.01v^2}{5}$

1 mark

ii. Express x in terms of the velocity v and show that $x = 250 \log_e\left(\frac{49}{49 - 0.01v^2}\right)$

2 marks

iii. Find correct to two decimal places the speed in m/s at which the hammer now strikes the ground.

1 mark

iv. Show that $t = \frac{25}{7}\log_e\left(\frac{70+v}{70-v}\right)$

4 marks

v. Find the time T correct to two decimal places, that is how long before the hammer hits the ground.

1 mark

vi. Show that $v(t)=\dfrac{70\left(1-e^{-0.28t}\right)}{1+e^{-0.28t}}$ for $0 \le t \le T$, where T is the time when the hammer hits the ground.

2 marks

vii. Sketch the velocity time graph, on the axes below, clearly marking the scale.

2 marks
Total 15 marks

Question 4

A golf ball is hit from the second level of a golfing range, so that its position vector at a time t seconds above ground level is given by
$\underset{\sim}{r}(t) = 10t\,\underset{\sim}{i} + 90t\,\underset{\sim}{j} + \left(4 + 4\sin(\pi t)\right)\underset{\sim}{k}$ where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors of magnitude one metre in the directions of east north and vertically upwards from ground level respectively,

i. Show that the golf ball takes 1.5 seconds to hit ground level.

1 mark

ii. Find the initial speed of projection correct to two decimal places and the initial angle correct to the nearest degree at which the golf ball is hit.

3 marks

iii. Find how far from the initial point of projection the golf ball strikes the ground, giving your answer correct to the nearest metre.

2 mark

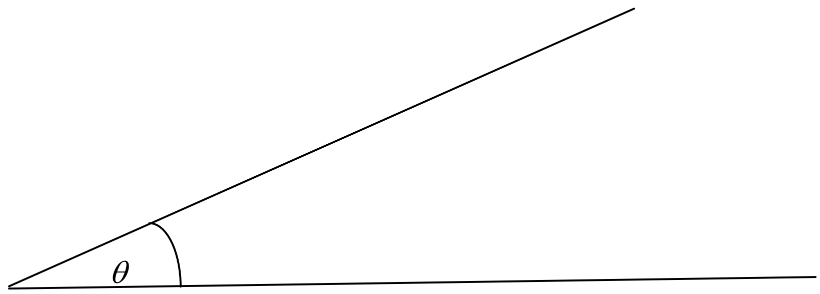
iv. Find the when the golf ball reaches its maximum height and give its position vector at this time.

2 mark
Total 8 marks

Question 5

A boy of mass m kg is on a slope inclined at an angle of θ to the horizontal.
He is trying to climb his way to the top of the slope.
He is just supported from sliding down the slope by holding on to a rope.
The rope acts up and parallel to the slope and has a tension of P newtons.
The coefficient of friction between the boy and the plane is 0.5

i. On the diagram below mark in all the forces acting on the boy.



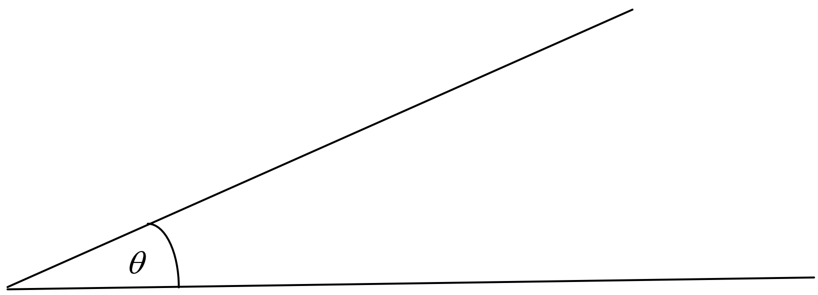
1 mark

ii. Show that $P = mg(\sin\theta - 0.5\cos\theta)$

2 marks

When some friends come along they all pull on the rope, such that the rope makes an angle of 15^0 with the slope and then the tension in the rope is doubled, (that is has a magnitude of $2P$)
The boy then moves up the rough slope with a constant acceleration of 1.0 m/s^2
The coefficient of friction between the boy and the plane is still 0.5

iii. On the diagram below mark in all the forces now acting on the boy.



1 mark

iv. Show that $11.669\sin\theta - 15.634\cos\theta = 1$

4 marks

v. Hence find the inclination of the slope θ to the horizontal, correct to the nearest degree.

1 mark

Total 9 marks

End of 2004 Specialist Mathematics Trial Examination 2
Question and Answer Book

KILBAHA MULTIMEDIA PUBLISHING	**TEL: (03) 9817 5374**
PO BOX 2227	**FAX: (03) 9817 4334**
KEW VIC 3101	**chemas@chemas.com**
AUSTRALIA	www.chemas.com