VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

**Victorian Certificate of Education 2003**

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

**STUDENT NUMBER**

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| Figures | | | | | | | | | | |
| Words | | | | | | | | | | |

# SPECIALIST MATHEMATICS

# Written examination 2
# (Analysis task)

**Wednesday 5 November 2003**

**Reading time: 11.45 am to 12.00 noon (15 minutes)**
**Writing time: 12.00 noon to 1.30 pm (1 hour 30 minutes)**

## QUESTION AND ANSWER BOOK

**Structure of book**

| *Number of questions* | *Number of questions to be answered* | *Number of marks* |
|---|---|---|
| 5 | 5 | 60 |

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, up to four pages (two A4 sheets) of pre-written notes (typed or handwritten) and an approved scientific and/or graphics calculator (memory may be retained).
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

**Materials supplied**
- Question and answer book of 15 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Working space is provided throughout the book.

**Instructions**
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.

**Students are NOT permitted to bring mobile phones and/or any other electronic communication devices into the examination room.**

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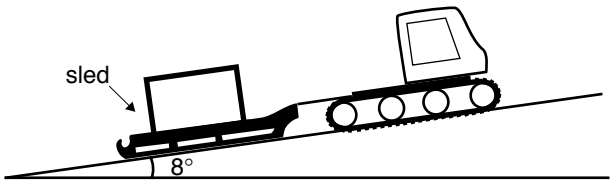
## Instructions

- Answer **all** questions in the spaces provided.
- A decimal approximation will not be accepted if an **exact** answer is required to a question.
- Where an **exact** answer is required to a question, appropriate working must be shown.
- In questions where more than one mark is available, appropriate working must be shown.
- Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- Take the **acceleration due to gravity** to have magnitude $g$ m/s$^2$, where $g = 9.8$.

Working space

**Question 1**

A loaded sled of total mass 1200 kg is connected by a tow bar to a snow tractor. The sled and snow tractor are on a ski slope inclined at an angle of 8° to the horizontal and the tow bar is parallel to the ski slope.




**a.** The sled and snow tractor are moving up the slope with constant acceleration of 0.25 m/s$^2$.

If the coefficient of friction between the sled and the ski slope is 0.09, find the magnitude of the tension in the tow bar to the nearest newton.

4 marks

Later, the sled and snow tractor are parked on the slope with the snow tractor's brakes applied.

Let $T$ newtons be the magnitude of the tension in the tow bar and $F$ newtons be the magnitude of the frictional force between the sled and the ski slope.

**b.** Express $T$ in terms of $F$.

2 marks

**c.** Find $T$, correct to the nearest integer, if the coefficient of friction between the sled and the ski slope is

**i.** 0.09

1 mark

**ii.** 0.15

1 mark

Total 8 marks

**Question 2**

**a.** Using the fact that $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$, show that

$$\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4} \text{ and } \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

3 marks

Let $u = \text{cis}\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4}i$, and let $v$ be the complex conjugate of $u$.

**b.** Find the **exact** value of $\text{Arg}(v - u)$.

3 marks

**c.** Find the **exact** value of $\text{Arg}\left(\frac{v}{u}\right)$.

2 marks

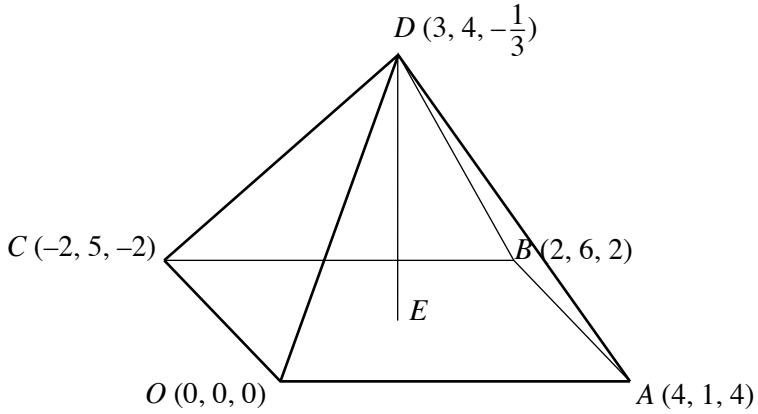
**d.** Let $u$ and $v$ satisfy the quadratic equation $z^2 + az + b = 0$.
Show that $b = 1$ and find the **exact** value of $a$ in surd form.

3 marks

Total 11 marks

**Question 3**

*OABCD* is a pyramid. The height of the pyramid is the length of *DE* where *E* is the point on the base *OABC* such that *DE* is perpendicular to the base.




**a**. Show that the base *OABC* is a rhombus.

3 marks

**b.** Use a vector method to find $\angle AOC$ correct to the nearest tenth of a degree.

3 marks

**Question 3** – continued

The unit vector $p\underset{\sim}{\mathrm{i}} + q\underset{\sim}{\mathrm{j}} + r\underset{\sim}{\mathrm{k}},\ p > 0$, is perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

**c.** **i.** Show that $q = 0$ and find the **exact** values of $p$ and $r$.

4 marks

**ii.** Hence find the **exact** height of the pyramid.

3 marks

Total 13 marks

**TURN OVER**

**Question 4**

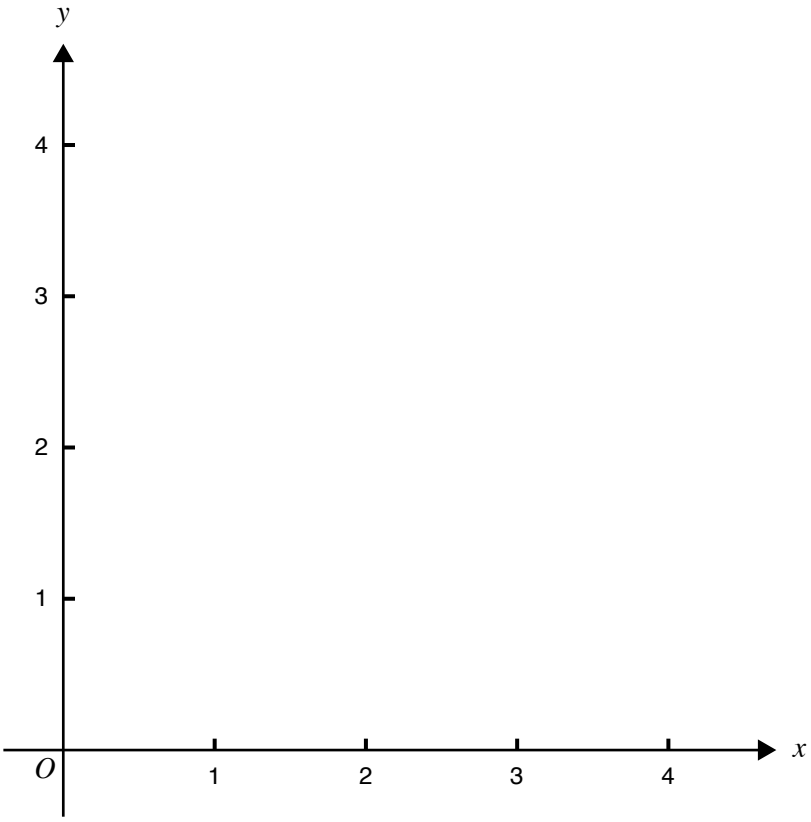
Consider the function $f$: $[0, 3) \rightarrow R$ where $f(x) = -2 + 2\sec\left(\frac{\pi x}{6}\right)$.

**a.** Evaluate $f(2)$.

1 mark

Let $f^{-1}$ be the inverse function of $f$.

**b.** On the axes below, sketch the graphs of $f$ and $f^{-1}$, showing their points of intersection.




2 marks

**c.** The rule for $f^{-1}$ can be written as $f^{-1}(x) = a\,\text{Cos}^{-1}\left(\frac{2}{x+2}\right)$.
Find the **exact** value of $a$.

2 marks

Let $A$ be the magnitude of the area enclosed by the graphs of $f$ and $f^{-1}$.

**d.** Write a definite integral expression for $A$ and evaluate it correct to three decimal places.

2 marks

**e.** **i.** Given that $0 < kx < \frac{\pi}{2}$, show that the derivative of $\log_e\left(\frac{1+\sin(kx)}{\cos(kx)}\right)$ is $k\sec(kx)$.

3 marks

**ii.** Hence find the **exact** value of $A$, the magnitude of the area enclosed by the graphs of $f$ and $f^{-1}$.

4 marks

Total 14 marks

Working space

**Question 5**

Nick takes a bottle of milk from the refrigerator for baby Alex. To heat the bottle, Nick puts it in a saucepan of continuously boiling water. Let $y$°C be the temperature of the milk at time $t$ minutes after the baby's bottle is placed in the boiling water.

A differential equation that models the increase in temperature of the milk while the bottle is in the boiling water is $\frac{dy}{dt} = a(100 - y)$ where $a > 0$.

**a.** The milk's temperature when the bottle is put into the boiling water is 5°C.

Solve the differential equation to show that $y = 100 - 95e^{-at}$ for $0 \leq t \leq T$, where $T$ is the time when Nick takes the bottle out of the boiling water.

4 marks

When Nick takes the bottle out of the boiling water at time $T$, the temperature of the milk is 48°C. He realises that this is too hot to give to baby Alex and so he puts the bottle into cold water. The temperature of the cold water is 10°C and the milk cools according to Newton's law of cooling: $\frac{dy}{dt} = -b(y - 10)$ where $b > 0$.

**b.** Verify, by differentiation, that for $t \geq T$, $y = 10 + Ae^{-b(t-T)}$, and evaluate $A$.

3 marks

Nick lets the milk cool to a temperature of 36°C to give to baby Alex. It takes three times as long for the milk to cool to this temperature from 48°C as it previously took to heat up from 5°C to 48°C.

**c.** Sketch a graph of $y$ in terms of $t$ from when the baby's bottle is put into the boiling water to when the milk is ready to give to baby Alex.


3 marks

**d.** Find the ratio $\frac{a}{b}$ correct to three significant figures.

4 marks

Total 14 marks

**END OF QUESTION AND ANSWER BOOK**