THE
HEFFERNAN
GROUP

P.O. Box 1180
Surrey Hills North VIC 3127
ABN 20 607 374 020
Phone 9836 5021
Fax 9836 5025

SPECIALIST MATHS
TRIAL EXAMINATION 1
2003
SOLUTIONS

Part I - Multiple-choice answers

1.	**D**	**7.**	**E**	**13.**	**C**	**19.**	**A**	**25.**	**D**
2.	**B**	**8.**	**E**	**14.**	**A**	**20.**	**B**	**26.**	**E**
3.	**C**	**9.**	**B**	**15.**	**B**	**21.**	**A**	**27.**	**A**
4.	**D**	**10.**	**C**	**16.**	**E**	**22.**	**C**	**28.**	**E**
5.	**B**	**11.**	**D**	**17.**	**C**	**23.**	**A**	**29.**	**D**
6.	**B**	**12.**	**B**	**18.**	**E**	**24.**	**B**	**30.**	**E**

Part I- Multiple-choice solutions

Question 1

The ellipse in the diagram has its centre at (0, 1). Its semi-major axis length (parallel to the *x*-axis) is 3 units and its semi-minor axis length (parallel to the *y*-axis) is 1 unit. The required equation is $\frac{x^2}{9}+\frac{(y-1)^2}{1}=1$.
The answer is D.

Question 2

Options C and E give the equation of an ellipse and hence there would be no asymptote. The equations of the asymptotes of the hyperbola with equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=\pm\frac{b}{a}x$. So the required equation would be $\frac{x^2}{9}-\frac{y^2}{25}=1$.
The answer is B.

Question 3

The graph is that of the function $y = \text{Sin}^{-1}\left(\frac{x}{a}\right)$.

Method 1

The graph of $y = \text{Sin}^{-1}\left(\frac{x}{a}\right)$ is produced when the graph of $y = \text{Sin}^{-1}(x)$ is dilated parallel to the x-axis by a factor of $\frac{1}{1/a} = a$.

Method 2

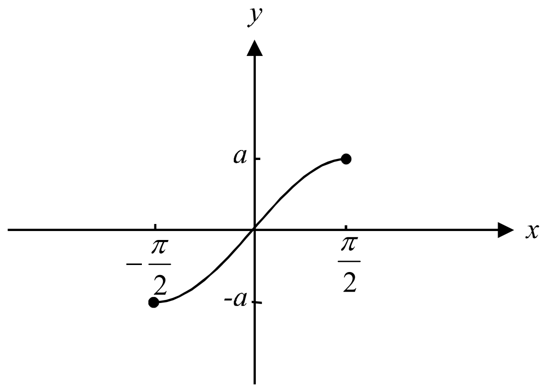
Consider the inverse function of this function which we find by swapping x and y.

$$x = \text{Sin}^{-1}\left(\frac{y}{a}\right)$$

Now rearrange,

So $$\frac{y}{a} = \text{Sin}(x)$$
$$y = a\text{Sin}(x)$$

The graph of this function involving the restricted sin (x) function is shown below.



The reflection of this graph in the line $y = x$ is what is given in the question.

The answer is C.

Question 4

$$\begin{aligned}
y &= \text{cosec}(2x-1) \\
&= \frac{1}{\sin(2x-1)} \\
&= (\sin(2x-1))^{-1} \\
\frac{dy}{dx} &= -1(\sin(2x-1))^{-2} \times \cos(2x-1) \times 2 \\
&= \frac{-2\cos(2x-1)}{\sin^2(2x-1)} \\
&= -2\cot(2x-1)\text{cosec}(2x-1)
\end{aligned}$$

The answer is D.

Question 5

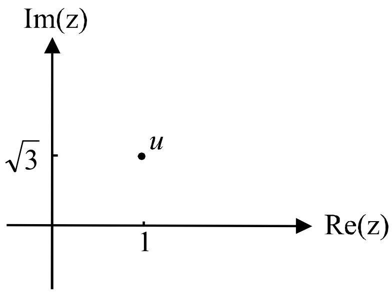
$$\begin{aligned}
z &= 4 - 3i \\
zi &= 4i + 3 \\
\bar{z} &= 4 + 3i \\
|z| &= \sqrt{16 + 9} \\
&= 5
\end{aligned}$$

So $z + zi - \bar{z} + |z|$

$$\begin{aligned}
&= 4 - 3i + 4i + 3 - 4 - 3i + 5 \\
&= 8 - 2i
\end{aligned}$$

The answer is B.

Question 6



$$\begin{aligned}
u &= 1 + \sqrt{3}i \\
|u| &= \sqrt{1 + 3} \\
&= 2 \\
\operatorname{Arg} u &= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) \\
&= \frac{\pi}{3} \quad \text{(1st quadrant from diagram)} \\
u &= 2\text{cis}\left(\frac{\pi}{3}\right) \\
\frac{u}{v} &= \frac{2\text{cis}\left(\frac{\pi}{3}\right)}{2\text{cis}\left(\frac{\pi}{6}\right)} \\
&= \text{cis}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) \\
&= \text{cis}\left(\frac{\pi}{6}\right)
\end{aligned}$$

The answer is B.

Question 7

Since the coefficients of the terms in the polynomial equation are real, the complex roots must occur in conjugate pairs. Since $(z - 3i)$ is a factor, $z = 3i$ and $z = -3i$ are roots. So options A and B are correct. We have 2 complex roots which are a conjugate pair and the third root must be real since the coefficients of the polynomial are real. So we have two complex roots and 1 real root. Option E is not correct.
The answer is E.

Question 8

$$z^2 = 2 + 2i$$

$$(r\text{cis}\theta)^2 = \sqrt{8}\,\text{cis}\left(\frac{\pi}{4}\right) \text{ since } r = \sqrt{2^2 + 2^2} = \sqrt{8} \text{ and } \theta = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4} \quad \text{(first quadrant)}$$

$$r^2\text{cis}(2\theta) = \sqrt{8}\,\text{cis}\left(\frac{\pi}{4}\right) \quad \text{(De Moivre)}$$

$$r^2 = \sqrt{8} \text{ and } 2\theta = \frac{\pi}{4} + 2k\pi \qquad k = \ldots -1, 0, 1, 2$$

$$r = \sqrt[4]{8} \qquad \theta = \frac{\pi}{8} + k\pi$$

Let $k = 0$, $\theta = \dfrac{\pi}{8}$

Let $k = 1$, $\theta = \dfrac{9\pi}{8}$

Let $k = -1$, $\theta = \dfrac{-7\pi}{8}$

The values of θ repeat themselves.

So the two solutions are $\sqrt[4]{8}\,cis\left(\dfrac{\pi}{8}\right)$ and $\sqrt[4]{8}\,cis\left(\dfrac{9\pi}{8}\right)$.

The answer is E.

Question 9

$$\begin{aligned}
\int \frac{3x}{\sqrt{x^2 - 2}}\,dx &= \int \frac{3}{2}\frac{du}{dx} u^{\frac{-1}{2}}\,dx \qquad \text{where } u = x^2 - 2, \quad \frac{du}{dx} = 2x \\
&= \frac{3}{2}\int u^{\frac{-1}{2}}\,du \\
&= \frac{3}{2}u^{\frac{1}{2}} \times 2 + c \\
&= 3\sqrt{x^2 - 2} + c
\end{aligned}$$

If $c = 0$, then an antiderivative is $3\sqrt{x^2 - 2}$.
The answer is B.

Question 10

$$\int x\sqrt{2x-1}\,dx = \int \frac{u+1}{2}u^{\frac{1}{2}} \times \frac{1}{2}\frac{du}{dx}\,dx$$ where $u = 2x-1$, $\frac{du}{dx} = 2$, $x = \frac{u+1}{2}$

$$= \frac{1}{4}\int\left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{4}\left(\frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3}\right) + c$$

$$= \frac{(2x-1)^{\frac{5}{2}}}{10} + \frac{(2x-1)^{\frac{3}{2}}}{6} + c$$

The answer is C.

Question 11

$$\int \cos^3(3x)\,dx$$

$$= \int \cos^2(3x)\cos(3x)\,dx$$

$$= \int \left(1-\sin^2(3x)\right)\cos(3x)\,dx$$

$$= \int (1-u^2).\frac{1}{3}\frac{du}{dx}\,dx$$ where $u = \sin(3x)$ and $\frac{du}{dx} = 3\cos(3x)$

$$= \frac{1}{3}\int (1-u^2)\,du$$

$$= \frac{1}{3}\left(u - \frac{u^3}{3}\right) + c$$

$$= \frac{\sin(3x)}{3} - \frac{1}{3} \times \frac{\sin^3(3x)}{3} + c$$

$$= \frac{\sin(3x)}{3} - \frac{\sin^3(3x)}{9} + c$$

The answer is D.

Question 12

$$\int_1^{\sqrt{3}} \frac{1}{x^2+3}\,dx = \frac{1}{\sqrt{3}}\int_1^{\sqrt{3}} \frac{\sqrt{3}}{3+x^2}\,dx$$

$$= \frac{1}{\sqrt{3}}\left[\operatorname{Tan}^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_1^{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}\left\{\operatorname{Tan}^{-1}(1) - \operatorname{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right)\right\}$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{12\sqrt{3}}$$

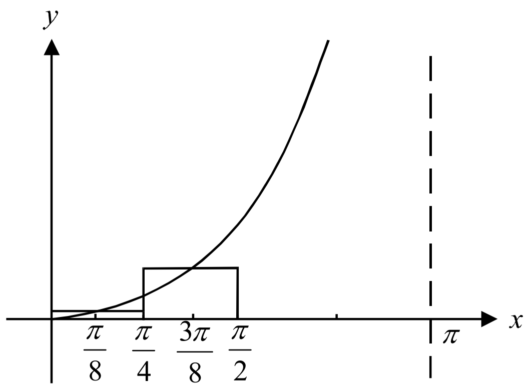
$$= \frac{\sqrt{3}\pi}{36}$$

The answer is B.

Question 13

Find the value of this definite integral using your graphics calculator. Note that the decimal approximations given as the alternatives give a clue that a graphics calculator should be involved. Note also that radian mode is required.
The answer is C.

Question 14



The area of the two rectangles shown is $f\left(\frac{\pi}{8}\right) \times \frac{\pi}{4} + f\left(\frac{3\pi}{8}\right) \times \frac{\pi}{4}$

where $f(x) = \tan\left(\frac{x}{2}\right)$

So area $= \tan\left(\frac{\pi}{16}\right) \times \frac{\pi}{4} + \tan\left(\frac{3\pi}{16}\right) \times \frac{\pi}{4}$

$= 0.68$ square units

The answer is A.

Question 15

$$\text{Area of shaded region} = \int_0^a \left(e^{-2x} - \sin(x)\right)dx$$
$$= \left[\frac{-e^{-2x}}{2} + \cos(x)\right]_0^a$$
$$= \left\{\left(\frac{-e^{-2a}}{2} + \cos(a)\right) - \left(\frac{-e^{0}}{2} + \cos(0)\right)\right\}$$
$$= \frac{-1}{2e^{2a}} + \cos(a) + \frac{1}{2} - 1$$
$$= \frac{-1}{2e^{2a}} + \cos(a) - \frac{1}{2}$$

The answer is B.

Question 16

If $f''(m) > 0$ then the graph of $y = f(x)$ is "concave up" i.e.  at $x = m$.

From the graph of $y = f(x)$, we see that the graph is concave up for $0 < x < a$, and for $c < x < e$. So for these values $f''(x) > 0$.
The answer is E.

Question 17

From the formulae sheet we have, if $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$,
then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$
Now, $x_0 = 0$ and $y_0 = 1$ and $\frac{dy}{dx} = f(x) = \frac{e^{-3x}}{x-1}$. Also, $h = 0.2$

So, $x_1 = 0 + 0.2$ and $y_1 = 1 + 0.2 \times f(0)$
$= 0.2$ $\quad = 1 + 0.2 \times -1$
$\quad = 0.8$

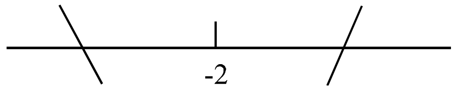
Therefore,

$x_2 = 0.2 + 0.2$ and $y_2 = 0.8 + 0.2 \times f(0.2)$
$= 0.4$ $\quad = 0.6628$ to 4 decimal places

So, the value obtained for y when $x = 0.4$ is 0.6628 (to 4 places).
The answer is C.

Question 18

The graph of $y = f(x)$ gives the gradient function of the graph of $y = F(x)$.
The gradient of $y = F(x)$ at $x = -2$ is zero.
For $x < -2$ the gradient is negative and for $x > -2$ the gradient is positive.
So at $x = -2$, on the graph of $y = F(x)$ there is a local minimum.



Similarly the gradient of $y = F(x)$ at $x = 1$ is zero.
For $x < 1$ the gradient is positive and for $x > 1$ the gradient is positive. So at $x = 1$ on the graph of $y = F(x)$, there is a stationary point of inflection.
The answer is E.

Question 19

The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. (from formulae sheet)

Since $r = 2h$, we have

$$V = \frac{1}{3}\pi(2h)^2 h$$

So, $$V = \frac{4}{3}\pi h^3$$

and therefore $$\frac{dV}{dh} = 4\pi h^2$$

Also, $$\frac{dV}{dt} = 2$$

Now, $$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} \quad \text{(Chain rule)}$$

So, $$2 = 4\pi h^2 \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{2\pi h^2}$$

So when $h = 2$, $\frac{dh}{dt} = \frac{1}{8\pi}$

The answer is A.

Question 20

$$a = \frac{1}{x+1}$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{1}{x+1}$$

$$\frac{1}{2}v^2 = \int \frac{1}{x+1}\,dx$$

$$= \log_e(x+1) + c$$

$$v^2 = 2\log_e(x+1) + 2c$$

When $v = 0,\ x = 0,$

so, $0 = 2\log_e 1 + 2c$

$$c = 0$$

$$v^2 = 2\log_e(x+1)$$

$$\frac{v^2}{2} = \log_e(x+1)$$

$$e^{\frac{v^2}{2}} = x+1$$

$$x = e^{\frac{v^2}{2}} - 1$$

When $v = \sqrt{2}$

$$x = e^1 - 1$$

The answer is B.

Question 21

Now, $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$$= (-3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}) + (2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k})$$

So, $\overrightarrow{OQ} = -\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

The answer is A.

Question 22

Let $\underset{\sim}{u} = 2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$

$$\hat{\underset{\sim}{u}} = \frac{1}{\sqrt{4+9+1}}\left(2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}\right)$$
$$= \frac{1}{\sqrt{14}}\left(2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}\right)$$

The vector we require is $\frac{5}{\sqrt{14}}\left(2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}\right)$

The answer is C.

Question 23

The vector resolute of $2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ perpendicular to $-\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ is given by

$$\left(2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) - \left(\left(2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) \bullet \frac{1}{\sqrt{1+4+9}}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)\right)\frac{1}{\sqrt{14}}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)$$
$$= \left(2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) - \frac{1}{\sqrt{14}}(-2-2-3)\frac{1}{\sqrt{14}}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)$$
$$= \left(2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) + \frac{7}{14}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)$$
$$= \frac{3}{2}\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{k}$$

The answer is A.

Question 24

$$\underset{\sim}{r}(t) = \mathrm{Sin}^{-1}\left(\frac{t}{3}\right)\underset{\sim}{i} + \mathrm{Cos}^{-1}\left(\frac{t}{3}\right)\underset{\sim}{j}, \quad 0 \le t \le 3$$
$$\dot{\underset{\sim}{r}}(t) = \frac{1}{\sqrt{9-t^2}}\underset{\sim}{i} + \frac{-1}{\sqrt{9-t^2}}\underset{\sim}{j}$$
$$\text{speed} = \left|\dot{\underset{\sim}{r}}(t)\right|$$
$$= \sqrt{\frac{1}{9-t^2} + \frac{1}{9-t^2}}$$
$$= \sqrt{\frac{2}{9-t^2}}$$

When $t = 1$, $\quad \text{speed} = \sqrt{\frac{2}{8}}$
$$= \frac{1}{2}$$

The answer is B.

Question 25

$$\ddot{\underset{\sim}{r}}(t) = e^{-2t}\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}, \qquad t \geq 0$$

$$\dot{\underset{\sim}{r}}(t) = \frac{e^{-2t}}{-2}\underset{\sim}{i} + \frac{\sin(2t)}{2}\underset{\sim}{j} + \underset{\sim}{c}$$

Now, $\dot{\underset{\sim}{r}}(0) = -\frac{1}{2}\underset{\sim}{i}$

So $\quad -\frac{1}{2}\underset{\sim}{i} = -\frac{1}{2}\underset{\sim}{i} + 0\,\underset{\sim}{j} + \underset{\sim}{c}$

$$\underset{\sim}{c} = \underset{\sim}{0}$$

So $\quad \dot{\underset{\sim}{r}}(t) = \frac{e^{-2t}}{-2}\underset{\sim}{i} + \frac{\sin(2t)}{2}\underset{\sim}{j}$

So $\quad \underset{\sim}{r}(t) = \frac{e^{-2t}}{4}\underset{\sim}{i} - \frac{\cos(2t)}{4}\underset{\sim}{j} + \underset{\sim}{d}$

Now $\quad \underset{\sim}{r}(0) = \frac{1}{4}\underset{\sim}{i}$

So $\quad \frac{1}{4}\underset{\sim}{i} = \frac{1}{4}\underset{\sim}{i} - \frac{1}{4}\underset{\sim}{j} + \underset{\sim}{d}$

$$\underset{\sim}{d} = \frac{1}{4}\underset{\sim}{j}$$

So $\quad \underset{\sim}{r}(t) = \frac{e^{-2t}}{4}\underset{\sim}{i} - \frac{\cos(2t)}{4}\underset{\sim}{j} + \frac{1}{4}\underset{\sim}{j}$

$$= \frac{e^{-2t}}{4}\underset{\sim}{i} + \frac{1}{4}\left(1 - \cos(2t)\right)\underset{\sim}{j}$$

The answer is D.

Question 26

$x^2 = 49 + 2 - 2 \times 7 \times \sqrt{2}\cos(135^\circ)$ (Cosine rule)

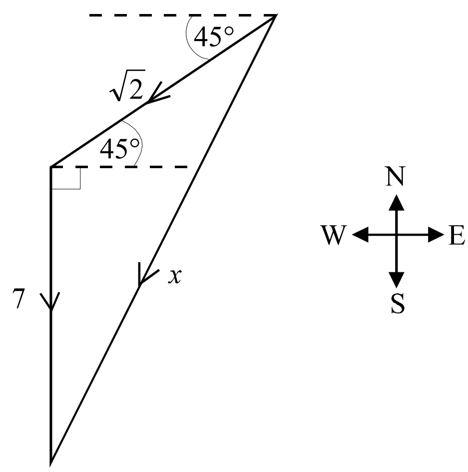
$= 51 + 14\sqrt{2} \times \frac{1}{\sqrt{2}}$

$= 65$

$x = \sqrt{65}$

The resulting force has a magnitude of $\sqrt{65}$ newtons.

The answer is E.



Question 27

$$\text{Initital momentum of particle} = (3 \times 20)\text{kg ms}^{-1}$$
$$= 60\,\text{kg ms}^{-1}$$

Momentum of particle after 4 seconds

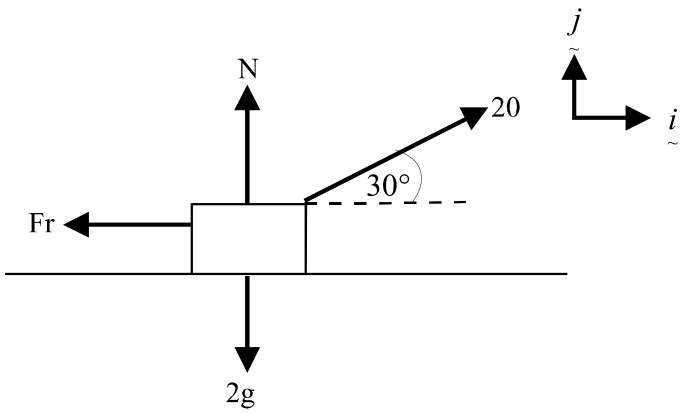
$$= (3 \times 15)\text{kg ms}^{-1}$$
$$= 45\,\text{kg ms}^{-1}$$

$$\text{The change in momentum} = 45 - 60$$
$$= -15\,\text{kg ms}^{-1}$$

The answer is A.

Question 28

Draw a diagram.



$$\underset{\sim}{R} = m\,\underset{\sim}{a}$$

$$(20\cos 30^\circ - Fr)\underset{\sim}{i} + (N + 20\sin 30^\circ - 2g)\underset{\sim}{j} = 2 \times 3\,\underset{\sim}{i}$$

So,

$10\sqrt{3} - Fr = 6$ and $N + 10 - 2g = 0$

$Fr = 10\sqrt{3} - 6$ $N = 2g - 10$

So $\mu N = 10\sqrt{3} - 6$

$$\mu = \frac{10\sqrt{3} - 6}{2g - 10}$$
$$= \frac{5\sqrt{3} - 3}{g - 5}$$

The answer is E.

Question 29

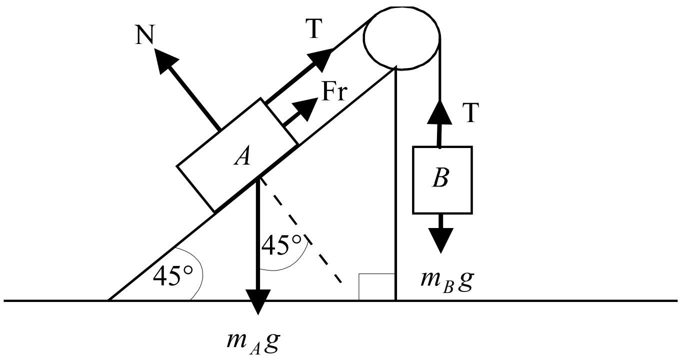
The pulling force P could be acting up or down the plane. The weight force acting vertically downwards has magnitude $10g$ newtons. Hence eliminate options A and B. The normal force acting perpendicularly from the plane is N not μN so eliminate option C. The frictional force is Fr and not μFr so eliminate option E.

Option D is correct. Note that the pulling force could be quite weak and hence the particle could be on the verge of sliding down the inclined plane which explains why the friction force is directed up the plane opposing the imminent movement.

The answer is D.

Question 30

Draw a diagram.



Because the system is on the point of sliding down the slope, the friction force opposes this and acts up the slope.

So, $\quad T + Fr = m_A g \sin 45^\circ$

$$T = \frac{m_A g}{\sqrt{2}} - \mu N$$

$$= \frac{m_A g}{\sqrt{2}} - \mu\left(m_A g \cos 45^\circ\right)$$

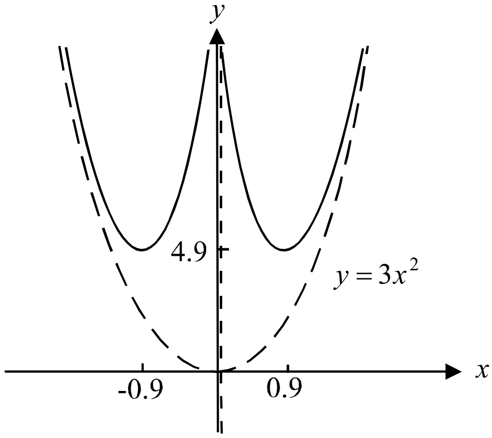
$$T = \frac{m_A g}{\sqrt{2}}(1 - \mu)$$

The answer is E.

PART II

Question 1

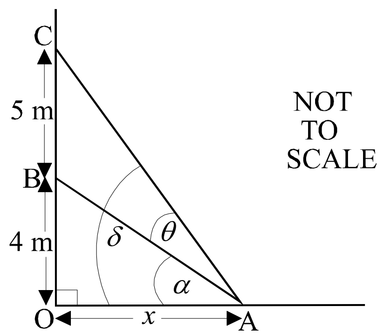
$y = 3x^2 + \frac{2}{x^2}$



The asymptotes are given by $x = 0$ and $y = 3x^2$.
Using a graphics calculator locate the minimum turning points. These are at $(-0.9, 4.9)$ and $(0.9, 4.9)$ where the coordinates are correct to 1 decimal place.

(1 mark) general shape of graph including asymptotes
(1 mark) for turning points

Question 2



Let $\angle BAO = \alpha$

Let $\angle CAO = \delta$

So, $\theta = \delta - \alpha$

So, $\tan(\theta) = \tan(\delta - \alpha)$

$$= \frac{\tan(\delta) - \tan(\alpha)}{1 + \tan(\delta)\tan(\alpha)} \quad \textbf{(1 mark)}$$

$$= \frac{\frac{9}{x} - \frac{4}{x}}{1 + \frac{9}{x} \times \frac{4}{x}}$$

$$= \frac{5}{x} \div \left(1 + \frac{36}{x^2}\right)$$

$$= \frac{5}{x} \div \left(\frac{x^2 + 36}{x^2}\right)$$

$$= \frac{5}{x} \times \frac{x^2}{x^2 + 36}$$

So, $\tan\theta = \frac{5x}{x^2 + 36}$ as required. **(1 mark)**

Question 3

To prove: $\overrightarrow{BD} \bullet \overrightarrow{DO} = 0$ **(1 mark)**

$$
\begin{aligned}
LS &= \overrightarrow{BD} \bullet \overrightarrow{DO} \\
&= \frac{1}{2}\overrightarrow{BA} \bullet \left(\overrightarrow{DB} + \overrightarrow{BO}\right) \\
&= \frac{1}{2}\left(\overrightarrow{BO} + \overrightarrow{OA}\right) \bullet \left(\frac{1}{2}\overrightarrow{AB} + \overrightarrow{BO}\right) \\
&= \frac{1}{2}\left(\overrightarrow{BO} + \overrightarrow{OA}\right) \bullet \left(\frac{1}{2}\left(\overrightarrow{AO} + \overrightarrow{OB}\right) + \overrightarrow{BO}\right) \\
&= \frac{1}{2}\left(\overrightarrow{BO} + \overrightarrow{OA}\right) \bullet \frac{1}{2}\left(\overrightarrow{AO} + \overrightarrow{BO}\right) \\
&= \frac{1}{4}\left(-\underset{\sim}{v} + \underset{\sim}{u}\right) \bullet \left(-\underset{\sim}{u} - \underset{\sim}{v}\right) \\
&= \frac{1}{4}\left(\underset{\sim}{v} - \underset{\sim}{u}\right) \bullet \left(\underset{\sim}{v} + \underset{\sim}{u}\right) \qquad \textbf{(1 mark)} \\
&= \frac{1}{4}\left(\underset{\sim}{v} \bullet \underset{\sim}{v} - \underset{\sim}{u} \bullet \underset{\sim}{u}\right) \\
&= \frac{1}{4}\left(\left|\underset{\sim}{v}\right|\left|\underset{\sim}{v}\right|\cos(0°) - \left|\underset{\sim}{u}\right|\left|\underset{\sim}{u}\right|\cos(0°)\right) \\
&= \frac{1}{4}\left(\left|\underset{\sim}{v}\right|^2 - \left|\underset{\sim}{v}\right|^2\right) \text{ since } \left|\underset{\sim}{v}\right| = \left|\underset{\sim}{u}\right| \\
&= 0 \\
&= RS \qquad \text{Have proved} \qquad \textbf{(1 mark)}
\end{aligned}
$$

Question 4

i.

$$
\begin{aligned}
v(t) &= \frac{5}{\sqrt{25 - t^2}} \\
&= 5\left(25 - t^2\right)^{\frac{-1}{2}} \\
a(t) &= \frac{-5}{2}\left(25 - t^2\right)^{\frac{-3}{2}} \times -2t \\
&= \frac{5t}{\left(25 - t^2\right)^{\frac{3}{2}}}
\end{aligned}
$$

$$
\text{At } t = 2,\ a(2) = \frac{10}{21^{\frac{3}{2}}} = \frac{10}{\left(\sqrt{21}\right)^3} \text{ ms}^{-2}
$$

(1 mark)

ii. In this case the distance travelled by the particle is the same as its displacement.
So, distance travelled

$$= \int_0^3 \frac{5}{\sqrt{25-t^2}}\,dt + \int_3^{5.5} \left(-0.5t + 2.75\right) dt$$ **(1 mark)**

$$= 5\left[\text{Sin}^{-1}\left(\frac{t}{5}\right)\right]_0^3 + \left[\frac{-0.5t^2}{2} + 2.75t\right]_3^{5.5}$$ **(1 mark)**

$$= 5\left\{\text{Sin}^{-1}\left(\frac{3}{5}\right) - \text{Sin}^{-1}(0)\right\} + \{(-7.5625 + 15.125) - (-2.25 + 8.25)\}$$

$$= 5\,\text{Sin}^{-1}\left(\frac{3}{5}\right) + 1.5625$$

(1 mark)

Question 5

i. $p(z) = z^4 - 6z^3 + 16z^2 - 22z + 15$

When $p(z)$ is divided by $z - 3i$, the remainder is given by

$$p(3i) = 81 + 162i - 144 - 66i + 15$$
$$= -48 + 96i$$

(1 mark)

ii. Since $2 - i$ is a solution to the equation $p(z) = 0$ and since the coefficients of the terms in the equation are real, then, by the conjugate root theorem, $2 + i$ must also be a solution.

So $(z - 2 + i)(z - 2 - i)$

$$= z^2 - 2z - iz - 2z + 4 + 2i + iz - 2i + 1$$
$$= z^2 - 4z + 5 \text{ is a factor of } p(z)$$ **(1 mark)**

So

$$\begin{array}{r|l} & z^2 - 2z + 3 \\ \hline z^2 - 4z + 5 & z^4 - 6z^3 + 16z^2 - 22z + 15 \\ & \underline{z^4 - 4z^3 + 5z^2} \\ & -2z^3 + 11z^2 - 22z \\ & \underline{-2z^3 + 8z^2 - 10z} \\ & 3z^2 - 12z + 15 \\ & \underline{3z^2 - 12z + 15} \end{array}$$

(1 mark)

So $p(z) = \left(z^2 - 4z + 5\right)\left(z^2 - 2z + 3\right)$

$$= (z - 2 + i)(z - 2 - i)\left(\left(z^2 - 2z + 1\right) - 1 + 3\right)$$
$$= (z - 2 + i)(z - 2 - i)\left(z - 1 - \sqrt{2}i\right)\left(z - 1 + \sqrt{2}i\right)$$

The solutions to $p(z) = 0$ are $z = 2 \pm i$ and $1 \pm \sqrt{2}i$.

(1 mark)

Question 6

i. $\dfrac{dQ}{dt}$ = rate of inflow - rate of outflow

$$= \frac{dQ_{in}}{dl_{in}} \cdot \frac{dl_{in}}{dt_{in}} - \frac{dQ_{out}}{dl_{out}} \cdot \frac{dl_{out}}{dt_{out}}$$ (where l represents litres)

$$= 15 \times 20 - \frac{Q}{1000} \times 20$$

$$= 300 - \frac{Q}{50}$$

as required.

(1 mark)

ii.

$$\frac{dQ}{dt} = 300 - \frac{Q}{50}$$

$$= \frac{15000 - Q}{50}$$

$$\frac{dt}{dQ} = \frac{50}{15000 - Q}$$ **(1 mark)**

$$t = 50\int \frac{1}{15000 - Q}\, dQ$$

$$= -50\log_e(15000 - Q) + c$$ **(1 mark)**

$t = 0,\quad Q = 10 \times 1000$

$= 10\,000$

$$0 = -50\log_e(15000 - 10\,000) + c$$

$$c = 50\log_e(5000)$$

So $t = -50\log_e(15000 - Q) + 50\log_e(5000)$

$$= 50\log_e\left(\frac{5000}{15000 - Q}\right)$$ **(1 mark)**

$$\frac{t}{50} = \log_e\left(\frac{5000}{15000 - Q}\right)$$

$$e^{\frac{t}{50}} = \frac{5000}{15000 - Q}$$

$$e^{\frac{-t}{50}} = \frac{15000 - Q}{5000}$$

$$5000e^{\frac{-t}{50}} = 15000 - Q$$

$$Q = 15000 - 5000e^{\frac{-t}{50}}$$

$$Q = 5000\left(3 - e^{\frac{-t}{50}}\right)$$ **(1 mark)**