5. $\underset{\sim}{u} = \underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$

$\underset{\sim}{v} = \underset{\sim}{i} + a\,\underset{\sim}{j} + \underset{\sim}{k}$

The angle between $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is known to be 60°.

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}| \cdot |\underset{\sim}{v}| \cos 60°$$

$$1 - a - 1 = \sqrt{3}.\sqrt{2 + a^2}.\frac{1}{2}$$

$$-2a = \sqrt{3(2 + a^2)}$$

$$4a^2 = 6 + 3a^2$$

$$a^2 = 6$$

$$a = \pm\sqrt{6}$$

Consideration of the earlier equation,

$$-2a = \sqrt{3(2 + a^2)}$$

shows that the value of $a = -\sqrt{6}$ must be chosen since the right hand side is positive, $a = -\sqrt{6}$

6. $a = -2 + \sqrt{v^2 + 5}$

$$\frac{dv}{dt} = -2 + \sqrt{v^2 + 5}$$

$$\frac{dv}{dt} = \frac{1}{-2 + \sqrt{v^2 + 5}}$$

The time taken for the velocity to go from $v = 3$ to $v = 10$ is $t = \int_3^{10} \frac{dv}{-2 + \sqrt{v^2 + 5}}$

Using graphics calculator, $t = 1.7$ seconds.

2002 Specialist Mathematics
Written Examination 2 (Analysis task)
Suggested answers and solutions

1. a. $\overrightarrow{OC} = 2\underset{\sim}{i} - 6\underset{\sim}{j}$

$\overrightarrow{OS} = 8\underset{\sim}{i} - 4\underset{\sim}{j}$

$$\overrightarrow{CS} = \overrightarrow{CO} + \overrightarrow{OS}$$
$$= -\overrightarrow{OC} + \overrightarrow{OS}$$
$$= -2\underset{\sim}{i} + 6\underset{\sim}{j} + 8\underset{\sim}{i} - 4\underset{\sim}{j}$$
$$= 6\underset{\sim}{i} + 2\underset{\sim}{j}$$

$$|\overrightarrow{CS}| = \sqrt{36 + 4}$$
$$= \sqrt{40}$$
$$= 2\sqrt{10}$$
$$= 6.32\ldots$$

Distance between the cargo ship and the sailing ship at 12:00 midday is 6.3 kilometres to the nearest tenth of a kilometre.

b. i. $\overrightarrow{OP} = 6m\underset{\sim}{i} - 2m\underset{\sim}{j}$

$$\overrightarrow{PS} = \overrightarrow{PO} + \overrightarrow{OS}$$
$$= -\overrightarrow{OP} + \overrightarrow{OS}$$
$$= -6m\underset{\sim}{i} + 2m\underset{\sim}{j} + 8\underset{\sim}{i} - 4\underset{\sim}{j}$$
$$= (8 - 6m)\underset{\sim}{i} + (2m - 4)\underset{\sim}{j}$$

$$\overrightarrow{OP}.\overrightarrow{PS} = 6m(8 - 6m)\underset{\sim}{i} - 2m(2m - 4)\underset{\sim}{j}$$
$$= 48m - 36m^2 - 4m^2 + 8m$$
$$= 56m - 40m^2$$

b. ii. $\overrightarrow{OP}.\overrightarrow{PS} = 56m - 40m^2$

When $\overrightarrow{OP}.\overrightarrow{PS} = 0$

$\overrightarrow{OP}$ is perpendicular to $\overrightarrow{PS}$ if $OP \neq 0$ and $PS \neq 0$

If $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{PS}$ then this will be the sailing ship's closest point to the shore line.

Let $\overrightarrow{OP}.\overrightarrow{PS} = 0$

$$\Rightarrow 56m - 40m^2 = 0$$
$$m(56 - 40m) = 0$$
$$\therefore m = 0 \text{ or } m = \frac{56}{40} = 1.4$$

Disregard $m = 0$ because for $m = 0$, $\overrightarrow{OP} = 0$

$P(6 \times 1.4, -2 \times 1.4)$

$= (8.4, -2.8)$

b. iii. $P(8.4, -2.8)$

$S(8, -4)$

$$d\overrightarrow{PS} = \sqrt{(8.4 - 8)^2 + (-2.8 + 4)^2}$$
$$= \sqrt{(0.4)^2 + (1.2)^2}$$
$$= \sqrt{0.16 + 1.44}$$
$$= \sqrt{1.60}$$
$$= 1.26$$

$= 1.3$ (to the nearest tenth of a kilometre)

c. i. $\underset{\sim}{v}_c = 15\underset{\sim}{i} - 5\underset{\sim}{j}$

$\underset{\sim}{s}_c = 15t\underset{\sim}{i} - 5t\underset{\sim}{j} + c$

at $t = 0$ $\underset{\sim}{s}_c = 2\underset{\sim}{i} - 6\underset{\sim}{j}$

$\therefore c = 2\underset{\sim}{i} - 6\underset{\sim}{j}$

$\Rightarrow \underset{\sim}{s}_c = 15t\underset{\sim}{i} - 5t\underset{\sim}{j} + 2\underset{\sim}{i} - 6\underset{\sim}{j}$

$= (15t + 2)\underset{\sim}{i} - (5t + 6)\underset{\sim}{j}$

c. ii. $\underset{\sim}{v}_s = 12\underset{\sim}{i} + (3\sin(t) - 8)\underset{\sim}{j}$

$\underset{\sim}{s}_s = 12t\underset{\sim}{i} + (-3\cos(t) - 8t)\underset{\sim}{j} + c$

$= 12t\underset{\sim}{i} - (3\cos(t) + 8t)\underset{\sim}{j} + c$

at $t = 0$ $\underset{\sim}{s}_s = 8\underset{\sim}{i} - 4\underset{\sim}{j}$

$\Rightarrow 8\underset{\sim}{i} - 4\underset{\sim}{j} = -3\underset{\sim}{j} + c$

$c = 8\underset{\sim}{i} - \underset{\sim}{j}$

$\therefore \underset{\sim}{s}_s = 12t\underset{\sim}{i} - (3\cos(t) + 8t)\underset{\sim}{j} + 8\underset{\sim}{i} - \underset{\sim}{j}$

$= (12t + 8)\underset{\sim}{i} - (3\cos(t) + 8t + 1)\underset{\sim}{j}$

d. When $t = 2$

$\underset{\sim}{s}_c(2) = (15 \times 2 + 2)\underset{\sim}{i} - (5 \times 2 + 6)\underset{\sim}{j}$

$= 32\underset{\sim}{i} - 16\underset{\sim}{j}$

$\underset{\sim}{s}_s(2) = (12 \times 2 + 8)\underset{\sim}{i} - (3\cos(2) + 8 \times 2 + 1)\underset{\sim}{j}$

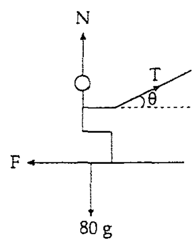
$= 32\underset{\sim}{i} - (3\cos(2) + 16 + 1)\underset{\sim}{j}$

$= 32\underset{\sim}{i} - 15.75\underset{\sim}{j}$

$\underset{\sim}{s}_c$ and $\underset{\sim}{s}_s$ have the same position in east west line because both have the same $\underset{\sim}{i}$ value.

On the north south line $\underset{\sim}{s}_c < \underset{\sim}{s}_s$ therefore the cargo ship is directly south of the sailing ship.

2. a.



b. $a = 2\text{m/s}^2$ $\theta = 60°$

$\underset{\sim}{i}: T\cos\theta - \mu N = F$

$T\cos 60° - 0.3N = 80 \times 2$ ①

$\underset{\sim}{j}: N + T\sin\theta - 80g = 0$

$N = 80g - T\sin 60°$ ②

Substitute ② into ①

$T\cos 60° - 0.3(80g - T\sin 60°) = 160$

$T\cos 60° - 24g + 0.3T\sin 60° = 160$

$T(\cos 60° + 0.3\sin 60°) = 160 + 24g$

$T = \dfrac{160 + 24g}{\cos 60° + 0.3\sin 60°}$

$= 520N$ (to the nearest integer)

c. i. $T = \dfrac{160 + 24g}{\cos\theta + 0.3\sin\theta}$

$T = (160 + 24g)(\cos\theta + 0.3\sin\theta)^{-1}$

$\dfrac{dT}{d\theta} = (160 + 24g) \times -1(\cos\theta + 0.3\sin\theta)^{-2}$ $(-\sin\theta + 0.3\cos\theta)$

$= \dfrac{-(160 + 24g)(-\sin\theta + 0.3\cos\theta)}{(\cos\theta + 0.3\sin\theta)^2}$

$\cos\theta \neq -0.3\sin\theta$
because $0 < \theta < 90°$

Let $\dfrac{dT}{d\theta} = 0$

$\sin\theta = 0.3\cos\theta$

$\text{Tan}\,\theta = 0.3$

$\theta = 16.6999...$

$\theta = 16.7°$ (to nearest tenth of a degree)

c. ii. $T = \dfrac{160 + 24g}{\cos\theta + 0.3\sin\theta}$ (from previous work)

$\theta = 16.6999...$

$T_{\min} = \dfrac{160 + 24g}{\cos(16.999) + 0.3\sin(16.999)}$

$T_{\min} = 378.53...$

$T_{\min} = 379$ N (to the nearest Newton)

3. a. i. $u = 0$ $t = 8$ $s = 400$

constant acceleration, so we can use

$s = ut + \frac{1}{2}at^2$

$400 = 0 + \frac{1}{2}(8)^2$

$a = \dfrac{2(400)}{64}$

$a = 12.5 \text{ m/s}^2$

a. ii. $u = 0$ $t = 8$ $s = 400$ $a = 12.5$

constant acceleration, so we can use

$v = u + at$

$v = 0 + 12.5 \times 8$

$v = 100$ (value as required)

b. i. $F = ma = -(5000 + 0.5v^2)$

$m = 400$

$400a = -(5000 + 0.5v^2)$

$a = -\left(\dfrac{5000}{400} + \dfrac{0.5v^2}{400}\right)$

$= -\left(\dfrac{10\,000 + v^2}{800}\right)$

b. ii. $a = -\left(\dfrac{10\,000 + v^2}{800}\right)$

$a = v\dfrac{dv}{dx} = -\dfrac{(10^4 + v^2)}{800}$

$\dfrac{dv}{dx} = -\dfrac{(10^4 + v^2)}{800v}$

b. iii. $\dfrac{dv}{dx} = -\dfrac{(10^4 + v^2)}{800v}$

$\Rightarrow \dfrac{dx}{dv} = -\dfrac{800v}{10^4 + v^2}$

$dx = -\dfrac{800v}{10^4 + v^2}.dv$

$x = \int -\dfrac{800v}{10^4 + v^2}.dv$

Let $u = 10^4 + v^2$

$\dfrac{du}{dv} = 2v$

$dv = \dfrac{du}{2v}$

$x = -\int \dfrac{800v}{u} \times \dfrac{du}{2v}$

$= -\int \dfrac{400}{u}du$

$x = -400\log_e u + c$

$= -400\log_e(10^4 + v^2) + c$

at $x = 0$ $v = 100$ (from **a. ii.**)

$c = 400\log_e(10^4 + 10^4)$

$= 400\log_e(2 \times 10^4)$

$\therefore x = -400\log_e(10^4 + v^2) + 400(2 \times 10^4)$

$= 400\log_e\left(\dfrac{2 \times 10^4}{10^4 + v^2}\right)$

at $v = 0$

$x = 400\log_e\left(\dfrac{2 \times 10^4}{10^4}\right)$

$= 400\log_e 2$

$= 277$ metres

c. From 3 **b. i.** we know that:

$$a = -\left(\frac{10^4 + v^2}{800}\right)$$

$$\Rightarrow \frac{dv}{dt} = -\left(\frac{10^4 + v^2}{800}\right)$$

$$\Rightarrow \frac{dt}{dv} = \frac{-800}{10^4 + v^2}$$

$$\frac{dt}{dv} = \frac{-800}{10^2} \times \frac{10^2}{(10^2)^2 + v^2}$$

$$t = -8\text{Tan}^{-1}\frac{v}{100} + c$$

at $t = 0$ $v = 100$

$$c = 8\text{Tan}^{-1}(1)$$

$$= 8 \times \frac{\pi}{4}$$

$$= 2\pi$$

$$t = -8\text{Tan}^{-1}\frac{v}{100} + 2\pi$$

at $v = 0$

$t = 0 + 2\pi$

$= 6.283$

$= 6$ seconds (to the nearest second)

4. a. $(2 - x)(2 + x) > 0$

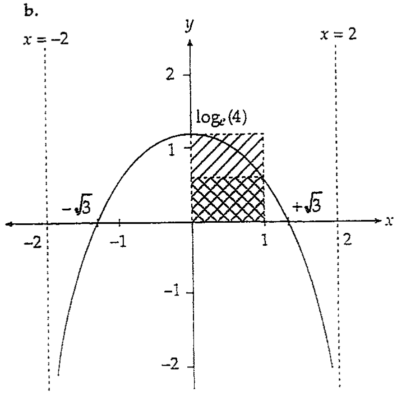
$\Rightarrow 2 - x > 0$ and $2 + x > 0$

$\Rightarrow -x > -2$ and $x > -2$

$x < 2$

$\therefore D_{DM} f = D = (-2, 2)$ or $\{x: -2 < x < 2\}$

b.



c. Rectangle (hatched) $l \times w = \log_e 4 \times 1 = \log_e 4$

Rectangle (cross-hatched) $l = \log_e(4 - 1) = \log_e 3$

$w = 1$

$l \times w = \log_e 3 \times 1 = \log_e 3$

$\therefore \log_e 3 < A < \log_e 4$

d. i. $y = x\log_e(4 - x^2)$

$$\frac{dy}{dx} = \log_e(4 - x^2) + x \times \frac{-2x}{4 - x^2}$$

$$= \log_e(4 - x^2) - \frac{2x^2}{4 - x^2}$$

d. ii. $\frac{x^2}{4 - x^2} = -1 + \frac{4}{4 - x^2}$

$$\frac{4}{4 - x^2} = \frac{4}{(2 - x)(2 + x)} = \frac{A}{2 - x} + \frac{B}{2 + x}$$

$4 \equiv A(2 + x) + B(2 - x)$

at $x = -2$

$4 = 4B$ $\quad B = 1$

at $x = 2$

$4 = 4A$ $\quad A = 1$

$$\therefore \frac{x^2}{4 - x^2} = -1 + \frac{1}{2 - x} + \frac{1}{2 + x}$$

$$\int \frac{x^2}{4 - x^2}dx = \int -1 + \frac{1}{2 - x} + \frac{1}{2 + x}dx$$

$$= -x - \log_e(2 - x) + \log_e(2 + x) + c$$

$$= \log_e\frac{(2 + x)}{(2 - x)} - x$$

d. iii. $\int_0^1 \log_e(4 - x^2)dx$

From part **d. i.**

$$\frac{dx\log_e(4 - x^2)}{dx} = \log_e(4 - x^2) - \frac{2x^2}{4 - x^2}$$

$$\Rightarrow \int \log_e(4 - x^2)dx = x\log_e(4 - x^2) + \int \frac{2x^2}{4 - x^2}dx$$

using result of part **d. ii.**

$$\int \log_e(4 - x^2)dx = x\log_e(4 - x^2) + 2\int \frac{x^2}{4 - x^2}dx$$

$$= x\log_e(4 - x^2) - 2x - 2\log_e(2 - x) + 2\log_e(2 + x)$$

$$\int_0^1 \log_e(4 - x^2)dx$$

$$= \log_e 3 - 2 - 2\log_e 1 + 2\log_e 3 - 0 + 0 + 2\log_e 2 - 2\log_e 2$$

$$= -2 + 3\log_e 3$$

e. i. $y_{n+1} = y_n + hf(x_n)$

$n + 1 = 20$ $\quad n = 19$ $\quad h = 0.05$

$y_{20} = y_{19} + 0.05f(x_{19})$

$y_{19} = 19 \times 0.5$ $\quad f(x) = \log_e(4 - x^2)$

$y_{20} = y_{19} + 0.05\log_e(4 - (19 \times 0.5)^2)$

$= y_{19} + 0.05\log_e(3.0975)$

e. ii. $y_{20} = 1.2464 + 0.05\log_e(3.0975)$

$= 1.3029$ correct to 4 decimal places

e. iii. y_{20} is an approximation to the solution of the differential equation

$$\frac{dy}{dx} = \log_e(4 - x^2)$$

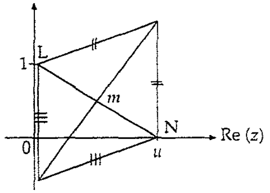
We know $A = \int_0^1 \log_e(4 - x^2)dx$ which is the solution of $\frac{dy}{dx}$ at $x = 1$ subtract the solution of $\frac{dy}{dx}$ at $x = 0$.

We know that $\frac{dy}{dx} = 0$ at x= 0

Therefore $y_{20} \approx A = \int_0^1 \log_e(4 - x^2)dx$

5. a. i. When z lies on $\overline{LN}$ it is equal distance from L and N that means it is on the midpoint, m.

When z lies on either side of the $\overline{LN}$ it is equidistant from L and N, and forms an isosceles triangle. The line drawn from the vertex of an isosceles triangle bisecting the base of the triangle is perpendicular to the base.



a. ii. $|z - i| = |z - u|$

$|x + iy - i| = |x + iy - u|$

$|x + i(y - 1)| = |x - u + iy|$

$\Rightarrow x^2 + (y-1)^2 = (x-u)^2 + y^2$

$x^2 + y^2 - 2y + 1 = x^2 - 2xu + u^2 + y^2$

$-2y + 1 = -2ux + u^2$

$2y - 1 = 2ux - u^2$

$2y = 2ux - u^2 + 1$

b. i. $y = ux - \frac{u^2}{2} + \frac{1}{2}$

at $x = u$

$y = u^2 - \frac{u^2}{2} + \frac{1}{2}$

$= \frac{1}{2}\left(u^2 + 1\right)$

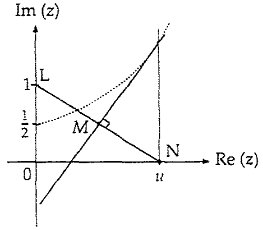
$w = x + yi$

$w = u + \frac{1}{2}\left(u^2 + 1\right)i$

b. ii. $w = u + \frac{1}{2}\left(u^2 + 1\right)i$

$x = u \qquad y = \frac{1}{2}(u^2 + 1)$

$y = \frac{1}{2}\left(x^2 + 1\right), \; x > 0$



c. $y = \frac{1}{2}(x^2 + 1)$

$\frac{dy}{dx} = x$

Tangent at $x = u \quad y = \frac{1}{2}(u^2 + 1)$

Given by $y - y_1 = \frac{dy}{dx}(x - x_1)$ at $x = u$

$y = \frac{1}{2}(u^2 + 1) = u(x - u)$

$2y - u^2 - 1 = 2ux - 2u^2$

$2y = 2ux - u^2 + 1$

From **a. ii.** we know that this is the equation of the perpendicular bisector of LN therefore the perpendicular bisector is tangent to the curve at w.

