The Mathematical Association of Victoria

Specialist Mathematics

2001 Written Examinations

Solutions

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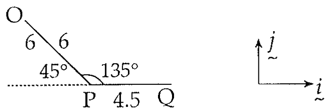
2001 Specialist Mathematics
Written Examination 2 (Analysis task)
Suggested answers and solutions

1.



a. $\overline{OQ}^2 = \overline{OP}^2 + \overline{PQ}^2 - 2\overline{OP}.\overline{PQ}\cos\theta$

$\overline{OP} = 6 \qquad \overline{PQ} = 4.5 \qquad \theta = 135°$

$\overline{OQ}^2 = 6^2 + 4.5^2 - 2 \times 6 \times 4.5 \times \cos 135°$

$= 94.434$

$\overline{OQ} = 9.7177$

$\simeq 9.72$ km

b. i. $\overrightarrow{OP} = 6\sin 45°\, \underset{\sim}{i} - 6\cos 45°\, \underset{\sim}{j}$

$= \frac{6}{\sqrt{2}}\underset{\sim}{i} - \frac{6}{\sqrt{2}}\underset{\sim}{j}$

$= 3\sqrt{2}\,\underset{\sim}{i} - 3\sqrt{2}\,\underset{\sim}{j}$

$\overrightarrow{PQ} = 4.5\,\underset{\sim}{i}$

$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$= \left(4.5 + 3\sqrt{2}\right)\underset{\sim}{i} - 3\sqrt{2}\,\underset{\sim}{j}$

ii. Let $\alpha = \angle POQ$

$\left|\overrightarrow{OP}\right| = 6 \qquad \left|\overrightarrow{OQ}\right| = 9.72$ (from Q1a)

$\overrightarrow{OP} = 3\sqrt{2}\,\underset{\sim}{i} - 3\sqrt{2}\,\underset{\sim}{j}$

$\overrightarrow{OQ} = \left(4.5 + 3\sqrt{2}\right)\underset{\sim}{i} - 3\sqrt{2}\,\underset{\sim}{j}$

$\overrightarrow{OP}.\overrightarrow{OQ} = 13.5\sqrt{2} + 18 + 18$

$= 36 + 13.5\sqrt{2}$

$\cos\alpha = \frac{\overrightarrow{OP} - \overrightarrow{OQ}}{\left|\overrightarrow{OP}\right|\left|\overrightarrow{OQ}\right|}$

$= \frac{36 + 13.5\sqrt{2}}{6 \times 9.7177}$

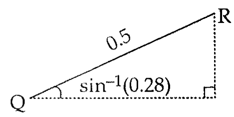
$= 0.9449\ldots$

$\alpha = \cos^{-1}(0.9449\ldots)$

$= 19.1°$

(correct to the nearest 10^{th} of a degree)
$\angle POQ = 19.1°$. Bearing of Q from O can be expressed as S(45° + 19.1°)E, i.e. S64.1°E.

c. i.



$\overrightarrow{RQ} = 0.5\left(\cos\left(\sin^{-1} 0.28\right)\underset{\sim}{i} + \sin\left(\sin^{-1} 0.28\right)\underset{\sim}{k}\right)$

$= 0.5 \times 0.96\,\underset{\sim}{i} + 0.5 \times 0.28\,\underset{\sim}{k}$

$= 0.48\,\underset{\sim}{i} + 0.14\,\underset{\sim}{k}$

$\overrightarrow{OR} = \overrightarrow{OQ} + \overrightarrow{QR} = 8.74\,\underset{\sim}{i} - 4.24\,\underset{\sim}{j} + 0.48\,\underset{\sim}{i} + 0.14\,\underset{\sim}{k}$

$= 9.22\,\underset{\sim}{i} - 4.24\,\underset{\sim}{j} + 0.14\,\underset{\sim}{k}$

c. ii. $|OR| = \sqrt{9.22^2 + 4.24^2 + 0.14^2}$

$= \sqrt{103.01}$

$= 10.15$

Her transmitter has insufficient range.

2. a. $\frac{ds}{dt} = \frac{kt}{12 + t^4}$

at $t = 4$

$\frac{ds}{dt} = \frac{4 \times 10^6}{12 \times 256}$

$= 1.49 \times 10^4$

b. i. $\dfrac{ds}{dt} = \dfrac{kt}{12 + t^4}$

$$\frac{d^2s}{dt^2} = \frac{k(12 + t^4) - kt \times 4t^3}{(12 + t^4)^2}$$

max value occurs when $\dfrac{d^2s}{dt^2} = 0$

$$\Rightarrow k(12 + t^4) - 4kt^4 = 0$$
$$12 + t^4 - 4t^4 = 0$$
$$3t^4 = 12$$
$$t^4 = 4$$
$$t = \sqrt{2}$$

$\therefore a = \sqrt{2}$

b. ii. Maximum rate occurs at $t = \sqrt{2}$

$$\frac{ds}{dt} = \frac{kt}{12 + t^2}$$

at $t = \sqrt{2}$

$$\frac{ds}{dt} = \frac{\sqrt{2} \times 10^6}{16} = 8.84 \times 10^4 \text{ litres per day}$$

c. i. $v = \displaystyle\int \frac{t}{12 + t^4}\,dt$

Let $u = t^2 \quad t = \sqrt{u}$

$$\frac{du}{dt} = 2t$$

$$\frac{dt}{du}.du = \frac{du}{2t}$$

Substituting $\sqrt{u}$ for t and $\dfrac{du}{2t}$ for dt

$$v = \int \frac{t}{12 + u^2}\,\frac{du}{2t}$$

$$v = \frac{1}{2\sqrt{12}} \int \frac{\sqrt{12}}{12 + u^2}\,du$$

$$v = \frac{1}{2\sqrt{12}} \text{Tan}^{-1}\left(\frac{u}{\sqrt{12}}\right)$$

substituting t^2 for u

$$v = \frac{1}{2\sqrt{12}} \text{Tan}^{-1}\left(\frac{t^2}{\sqrt{12}}\right)$$

OR $= \dfrac{1}{4\sqrt{3}} \text{Tan}^{-1}\left(\dfrac{t^2}{2\sqrt{3}}\right)$

c. ii. $v = \displaystyle\int \frac{kt}{12 + t^4}\,dt$

$$v = k\left(\frac{1}{2\sqrt{12}} \text{Tan}^{-1}\left(\frac{t^2}{\sqrt{12}}\right)\right) + c$$

at $t = 0 \qquad v = 0$

$$0 = k\left(\frac{1}{\sqrt{12}} \text{Tan}^{-1}(0)\right) + c$$

$$0 = c$$

$$\therefore v = \frac{k}{2\sqrt{12}} \text{Tan}^{-1}\left(\frac{t^2}{\sqrt{12}}\right)$$

c. iii. For $v = \dfrac{k}{2\sqrt{12}} \text{Tan}^{-1}\left(\dfrac{t^2}{\sqrt{12}}\right)$

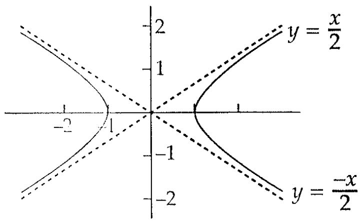
Since $\text{Tan}^{-1}(t)$ is less than $\dfrac{\pi}{2}$ for all t

$$v < \frac{10^6}{4\sqrt{3}} \times \frac{\pi}{2}$$

i.e. $v < 226725$

This suggests that less than 300,000 litres will spill into the sea in the long term.

3. a.



b. $v = \pi \displaystyle\int_0^h x^2\,dy$

$$x^2 = 1 + 4y^2$$

$$v = \pi \int_0^h 1 + 4y^2\,dy$$

$$= \pi\left[y + \frac{4y^3}{3}\right]_0^h$$

$$= \pi\left(\left(h + \frac{4h^3}{3}\right) - 0\right)$$

$$= \pi\left(\frac{4h^3}{3} + h\right)$$

c. IN : $\frac{dv}{dt} = 0.003 \text{ m}^3/s$

OUT : $\frac{dv}{dt} = 0.004\sqrt{h}$

$\frac{dv}{dt} = 0.003 - 0.004\sqrt{h}$

$\frac{dv}{dh} = \pi\left(4h^2 + 1\right)$

$\frac{dh}{dt} = \frac{dv}{dt} \times \frac{dh}{dv}$

$= \frac{0.003 - 0.004\sqrt{h}}{\pi\left(4h^2 + 1\right)}$

at $h = 0.25$

$\frac{dh}{dt} = \frac{0.001}{3.927}$

$\approx 2.5 \times 10^{-4} \text{ m/s}$

d. $\frac{dt}{dh} = \frac{\pi\left(4h^2 + 1\right)}{0.003 - 0.004\sqrt{h}}$

$t = \int_0^{0.5} \frac{\pi\left(4h^2 + 1\right)}{0.003 - 0.004\sqrt{h}}$

$= 3525$ sec

e. Depth will stabilise when inflow and outflow are the same.

i.e. when

$0.003 - 0.004\sqrt{h} = 0$

$\Rightarrow \sqrt{h} = \frac{3}{4}$

$h = \frac{9}{16}$ metres

$\simeq 0.5625$ metres

4. a. $z^2 - 6z + 25 = 0$

$z^2 - 6z + 9 - 9 + 25 = 0$

$(z - 3)^2 + 16 = 0$

$(z - 3 - 4i)(z - 3 + 4i) = 0$

$z_1 = 3 + 4i \qquad z_2 = 3 - 4i$

$z_1 + z_2 = 3 + 4i + 3 - 4i$

$= 6$

$z_1 \times z_2 = (3 + 4i)(3 - 4i)$

$= 9 + 16 = 25$

b. i. If u and v are roots

then $(z - u)(z - v) = 0$

$z^2 - uz - vz + uv = 0$

$z^2 - (u + v)z + uv = 0$

given $z^2 + bz + c = 0$

equating co-efficients

$b = -(u + v)$

$\Rightarrow u + v = -b \qquad uv = c$

b. ii. $u = p + qi$

$v = \overline{u} = p - qi$

$u + v = p + qi + p - qi$

$= 2p$

From 4. b. i $\quad u + v = -b$

Given $p \in R$

then $v \in R$

$uv = (p + qi)(p - qi)$

$= p^2 + q^2$

From 4. b. i. $uv = c$

Given $p + q \in R$

then $p^2 + q^2 \in R$

$\therefore c \in R$

c. Let $a = 2 + \sqrt{5}i$ and $b = -2 + \sqrt{5}i$

$(z - a)(z - b) = 0$

$z - (a + b)z + ab = 0$

$a + b = 2\sqrt{5}i$

$ab = \left(+2 + \sqrt{5}i\right)\left(-2 + \sqrt{5}i\right)$

$= -4 + 2\sqrt{5}i + 5i^2 - 2\sqrt{5}i$

$= -4 - 5 = -9$

Quadratic equation with roots

$2 + \sqrt{5}i$ and $-2 + \sqrt{5}i$

is $z^2 - 2\sqrt{5}iz - 9 = 0$

d. $u + v = -3 \qquad uv = 4$

From **4. b. ii.** we know

$u = p + qi \qquad v = p - qi$

$\& \;\; 2p = -3 \qquad p^2 + q^2 = 4$

$p = \frac{-3}{2} \qquad \frac{9}{4} + q^2 = 4$

$q^2 = \frac{16-9}{4} = \frac{7}{4}$

$q = \frac{\pm\sqrt{7}}{2}$

$u = \frac{-3}{2} + \frac{\sqrt{7}i}{2} \qquad v = \frac{-3}{2} - \frac{\sqrt{7}i}{2}$

[OR $u = \frac{-3}{2} - \frac{\sqrt{7}i}{2} \qquad v = \frac{-3}{2} + \frac{\sqrt{7}i}{2}$]

taking the first set of values

$u + v = \frac{-3}{2} - \frac{3}{2} \qquad u - v = \frac{\sqrt{7}i}{2} + \frac{\sqrt{7}i}{2}$

$= -3 \qquad = \sqrt{7}i$

$(z+3)(z - \sqrt{7}i) = 0$

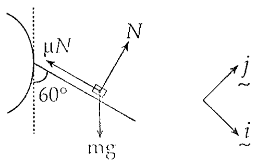
$z^2 - \sqrt{7}iz + 3z - 3\sqrt{7}i$

$z^2 + (3 - \sqrt{7}i)z - 3\sqrt{7}i$

[the second set of values gives:

$z^2 + (3 + \sqrt{7}i)z + 3\sqrt{7}i = 0$] only one equation is required

5. a.



b. $\underset{\sim}{i} \;:\; F = mg\cos 60° - \mu N$

$\underset{\sim}{b} \;:\; mg\sin 60° = N$

$F = mg\cos 60° - \mu\, mg\sin 60°$

$Ma = mg\cos 60° - \frac{1}{5} mg\sin 60°$

$a = g\left(\cos 60° - \frac{1}{5}\sin 60°\right)$

$= g\left(\frac{1}{2} - \frac{1}{5} \times \frac{\sqrt{3}}{2}\right)$

$= \frac{g}{2}\left(1 - \frac{\sqrt{3}}{5}\right)$ Note: acceleration is constant

c. i. $u = 0 \qquad a = \frac{g}{2}\left(1 - \frac{\sqrt{3}}{2}\right) \qquad s = 6$

$s = ut + \frac{1}{2}at^2$

Note: acceleration is constant

$6 = 0 + \frac{g}{2}\left(1 - \frac{\sqrt{3}}{5}\right)t^2$

$6 = 1.6013t^2$

$t^2 = 3.747$

$t = 1.94$ sec (correct to 2 decimal places)

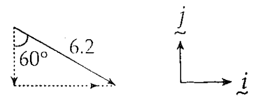
c. ii. $v = u + at$

$v = 0 + \frac{g}{2}\left(1 - \frac{\sqrt{3}}{5}\right) \times 1.94$

$= 6.199$

≈ 6.2 m/s

d.



$\underset{\sim}{v}(t) = 6.2\sin 60°\,\underset{\sim}{i} - 6.2\cos 60°\,\underset{\sim}{j} - gt\,\underset{\sim}{j}$

$= 3.1\sqrt{3}\,\underset{\sim}{i} - (3.1 + gt)\,\underset{\sim}{j}$

$\underset{\sim}{r}(t) = 3.1\sqrt{3}t\,\underset{\sim}{i} - \left(3.1t + \frac{1}{2}gt^2\right)\underset{\sim}{j} + c$

consider vertical distance

$3.1t + \frac{1}{2}gt^2 = 2$

$4.9t^2 + 3.1t - 2 = 0$

$t = 0.3966$ (positive value)

horizontal distance at $t = 0.3966$

given by $r = 3.1\sqrt{3} \times 0.3966$

$= 2.13$

≈ 2.1 metres

e. $\underset{\sim}{p} = m\underset{\sim}{v}$

$\underset{\sim}{v}(t) = 3.1\sqrt{3}\,\underset{\sim}{i} - (3.1 + gt)\,\underset{\sim}{j}$

$\underset{\sim}{v}(0.3966) = 3.1\sqrt{3}\,\underset{\sim}{i} - (3.1 + 9.8 \times 0.3966)\,\underset{\sim}{j}$

$= 5.369\,\underset{\sim}{i} - 6.987\,\underset{\sim}{j}$

$\left|\underset{\sim}{v}(0.3966)\right| = 8.812$

$p = mv \qquad m = 75$ kg

$p = 75 \times 8.812$

$= 661$ kg m/s