The Mathematical Association of Victoria

2001

MATHEMATICS: SPECIALIST

Trial Examination 2

Reading time: 15 minutes
Writing time: 1 hour 30 minutes

Student's Name: ____

Directions to students

This examination consists of five questions.

Answer all questions.

All working and answers should be written in the spaces provided.

The marks allotted to each part of each question appear at the end of each part.

There are **66 marks** available for this task.

A formula sheet is attached.

These questions have been produced to assist students in their preparation for the 2001 Specialist Mathematics Examination 2. The questions and associated answers and solutions do not necessarily reflect the views of the Victorian Curriculum and Assessment Authority (VCAA) Assessing Panels. The Association gratefully acknowledges the permission of the VCAA to reproduce the formula sheet.

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Specialist Mathematics Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
cosine rule:	$c^2 = a^2 + b^2 - 2ab\cos C$

Coordinate geometry

ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

hyperbola: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Circular (trigometric) functions

$\cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\cot^2 x + 1 = \operatorname{cosec}^2 x$

$\sin(x+y) = \sin x\cos y + \cos x\sin y$

$\sin(x-y) = \sin x\cos y - \cos x\sin y$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$

$\cos(x-y) = \cos x\cos y + \sin x\sin y$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$

$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\sin 2x = 2\sin x\cos x$

$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

function	Sin^{-1}	Cos^{-1}	Tan^{-1}
domain	$[-1, 1]$	$[-1,1]$	R
range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (Complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$

$|z| = \sqrt{x^2 + y^2} = r$

$-\pi < \operatorname{Arg} z \le \pi$

$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$

$z^n = r^n \operatorname{cis} n\theta$ (de Moivre's theorem)

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ $\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$ $\int e^{ax} dx = \frac{1}{a} e^{ax} + c$

$\frac{d}{dx}(\log_e x) = \frac{1}{x}$ $\int \frac{1}{x} dx = \log_e x + c$, for $x > 0$

$\frac{d}{dx}(\sin ax) = a\cos ax$ $\int \sin ax\, dx = -\frac{1}{a}\cos ax + c$

$\frac{d}{dx}(\cos ax) = -a\sin ax$ $\int \cos ax\, dx = \frac{1}{a}\sin ax + c$

$\frac{d}{dx}(\tan ax) = a\sec^2 ax$ $\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + c$

$\frac{d}{dx}\left(\text{Sin}^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$ $\int \frac{1}{\sqrt{a^2-x^2}} dx = \text{Sin}^{-1}\frac{x}{a} + c, a > 0$

$\frac{d}{dx}\left(\text{Cos}^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$ $\int \frac{-1}{\sqrt{a^2-x^2}} dx = \text{Cos}^{-1}\frac{x}{a} + c, a > 0$

$\frac{d}{dx}\left(\text{Tan}^{-1} x\right) = \frac{1}{1+x^2}$ $\int \frac{a}{a^2+x^2} dx = \text{Tan}^{-1}\frac{x}{a} + c$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

mid-point rule: $\int_a^b f(x)\,dx \approx (b-a) f\left(\frac{a+b}{2}\right)$

trapezoidal rule: $\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\big(f(a) + f(b)\big)$

Euler's method: If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h\,f(x_n)$

acceleration: $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at$ $s = ut + \frac{1}{2}at^2$ $v^2 = u^2 + 2as$ $s = \frac{1}{2}(u+v)t$

TURN OVER

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$|\underset{\sim}{r}| = \sqrt{x^2 + y^2 + z^2} = r$

$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2$

$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$

Mechanics

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

friction: $F \le \mu N$

END OF FORMULA SHEET

Mathematical Association of Victoria
2001

SPECIALIST MATHEMATICS

Written examination 2 (Analysis Task)

Question 1

The displacement of a particle is specified by the equation

$$\underset{\sim}{r}(t) = 8\cos 2t\,\underset{\sim}{i} + 6\sin 2t\,\underset{\sim}{j} \quad \text{where} \quad t: 0 \le t \le \frac{\pi}{2}$$

a. i. Show that at $t = 0$ the displacement and velocity of the particle are perpendicular.

2 marks

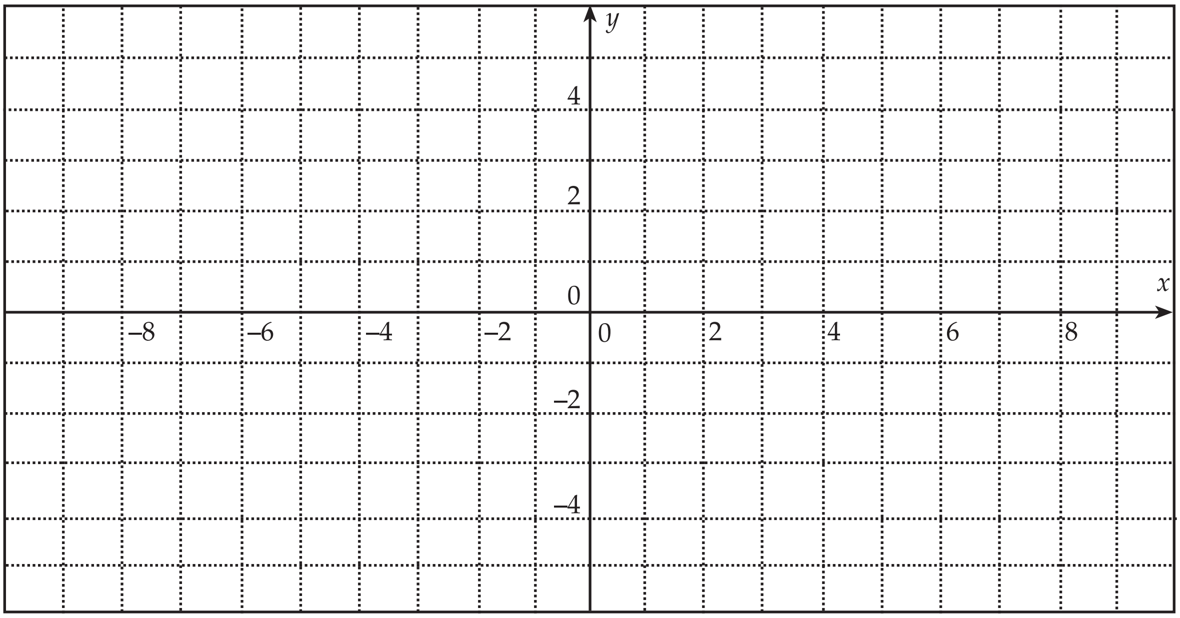
ii. Find the remaining times where the displacement and velocity of the particle are perpendicular.

3 marks

b. i. Find the cartesian equation of the path of the particle.

2 marks

ii. Sketch the graph of the path on the axes below.



2 marks

c. Find the maximum and minimum magnitude of the acceleration of the particle.

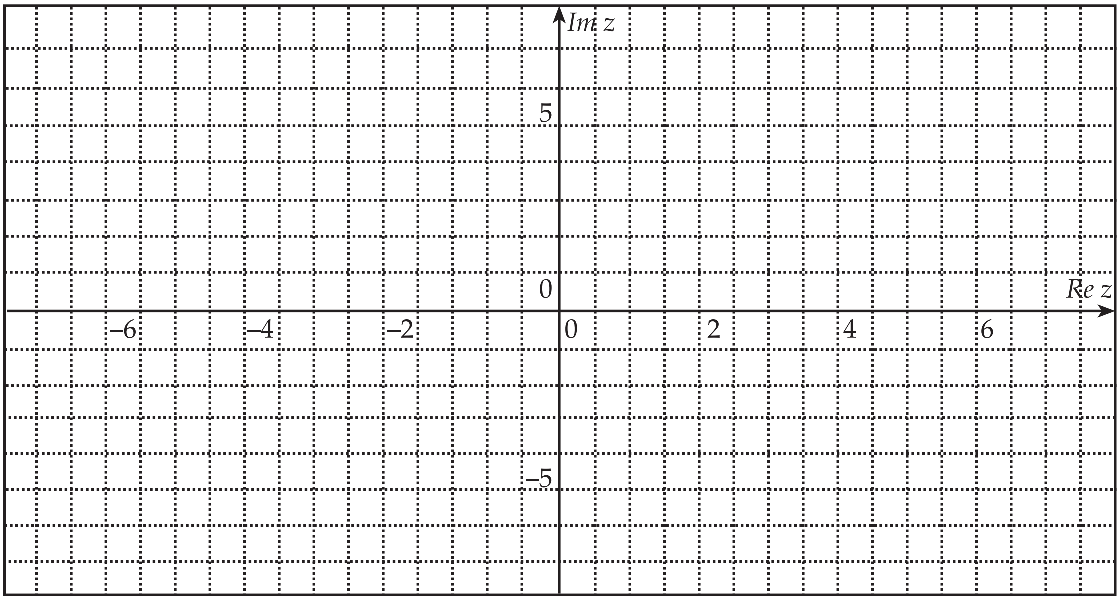
3 marks

Total marks: 12

Question 2

Let $w = 2 + 2i$

a. Plot, and clearly label, w and $\overline{w}$ on the Argand diagram below.



1 mark

b. Define the set T which is the disc whose boundary includes the circle that passes through the points w and $\overline{w}$, where the line joining w and $\overline{w}$ is a diameter of the disc.

1 mark

c. Show that $v = 3 + \sqrt{3}i$ is an element of T.

1 mark

d. Express v in polar form.

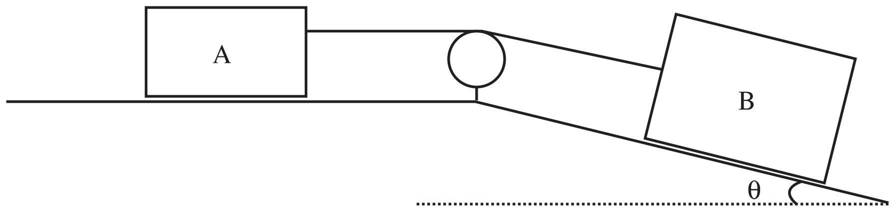
2 marks

e. Sketch, on the Argand diagram provided above, $U = T \cap \{z : z < \text{Arg}\, v\}$.

2 marks

Total marks: 7

Question 3



This diagram shows two particles A and B attached by a light inextensible string which passes over a smooth pulley. The mass of A is 4 kg and B has mass 8 kg. The coefficient of friction for both planes is 0.1.

a. Draw all the forces acting on the particles A and B on the diagram above. 2 marks

b. Find the angle θ such that the two particles would be just on the point of moving. Give your answer to the nearest degree.

4 marks

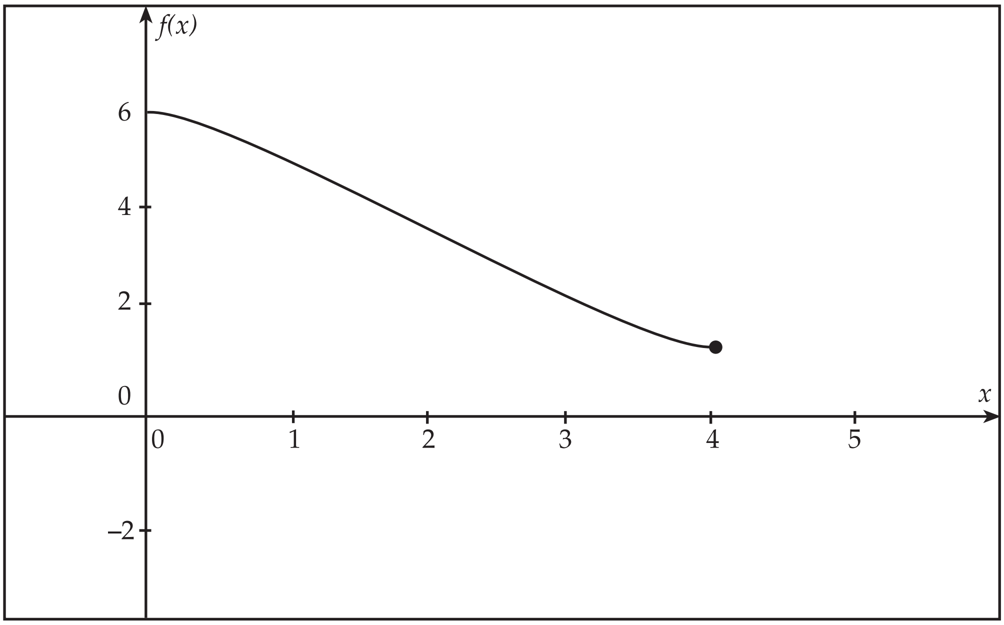
Total marks: 6

Question 4

The graph of the function

$$f:[0,4] \to R \text{ where } f(x) = \frac{12}{\sqrt{x^2+4}} - \frac{x}{3}$$

is shown below.



a. i. Find $f''(x)$.

3 marks

ii. For what exact value of x is the magnitude of the gradient of the graph a maximum?

1 mark

b. Find an anti-derivative of $\dfrac{x}{\sqrt{x^2+4}}$

2 marks

A space capsule is composed of two parts, an inner compartment and a re-entry shield.

The inner compartment takes the shape of a volume of revolution of the graph of f about the x-axis. All measurements are in metres.

c. Use calculus to find the volume of the inner compartment to the nearest cubic metre.

4 marks

The external wall of the re-entry shield is the shape of the parabola $g:[0,5] \to R$ where $g(x) = 3\sqrt{5-x}$ rotated about the x-axis.

The thickness of the re-entry shield is the cross-sectional distance between its external wall and the wall of the inner compartment measured parallel to the y-axis for $\{x: 0 \le x \le 4\}$

d. i. Write down an expression as a function of x which gives the thickness of the re-entry shield.

1 mark

ii. Evaluate the maximum thickness of the re-entry shield correct to 3 decimal places.

1 mark

Total marks: 12

Question 5

a. i. Find $\frac{d}{dx}\left[\log_e(\cos x)\right]$

2 mark

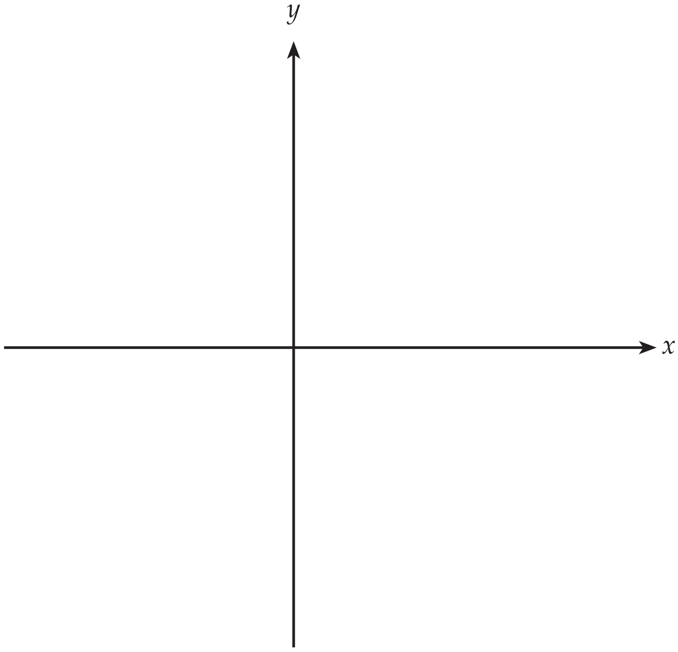
ii. Hence find an antiderivative of $\tan x$

1 mark

b. Find $\int \tan^3 x \, dx$

4 marks

c. Sketch $f(x) = \tan^3 x$, for $0 \le x \le \pi$



2 marks

d. Find the exact value of the area enclosed by the curve $f(x) = \tan^3 x$, the y-axis and the straight line $y = 1$.

4 marks

Total marks: 13

Question 6

A tree fully laden with apricots has 20 apricots infected with a fungal infection called brown rot. After a thunderstorm, the fungal infection spreads rapidly through the remaining apricots. The rate of spread of the fungus is dependent on the number of infected apricots and the number of apricots yet to be infected. This infestation can be modelled by the equation:

$\frac{dA}{dt} = kA(2000 - A)$, where A is the total number of infected apricots, t days after the thunderstorm.

a. i. Express $\frac{1}{A(2000 - A)}$ in partial fraction form.

3 marks

ii. Show that the solution to $\frac{dA}{dt} = kA(2000 - A)$ can be written as:

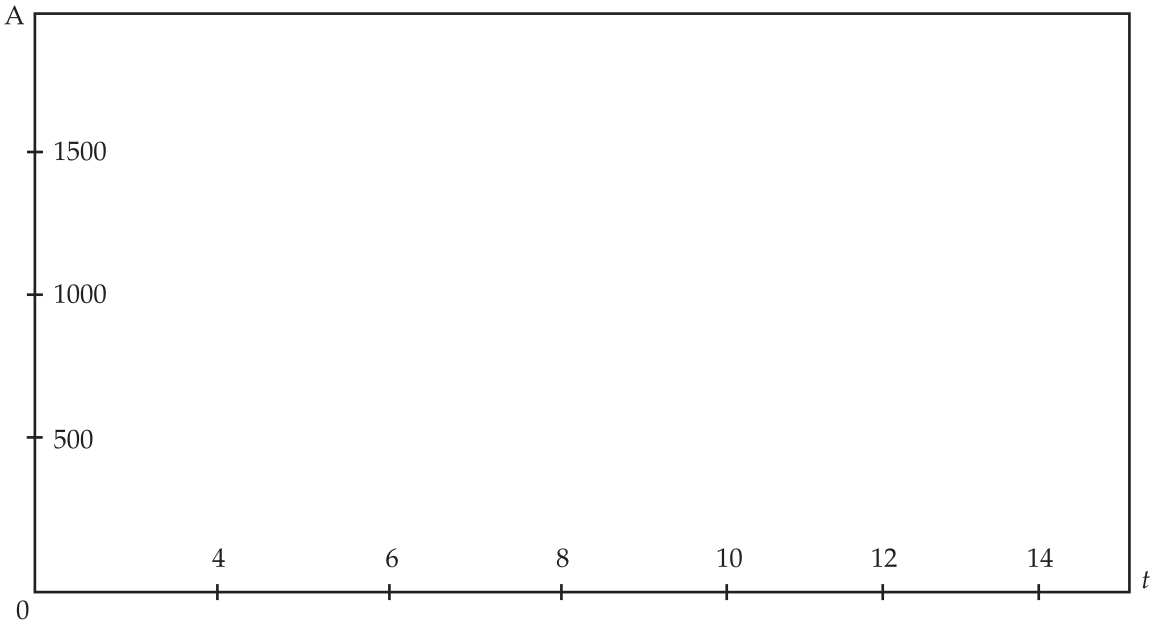
$$A = \frac{2000e^{2000kt}}{99 + e^{2000kt}}$$

6 marks

b. Find the *exact* value of k, given that 80 apricots are infected after 2 days.

2 marks

c. i. Sketch the graph of A against t, for $t \leq 15$.



2 marks

ii. Explain the relationship between the k-value and the rate of infection.

1 mark

d. After how many days will 1000 of the apricots be infected?

2 marks

Total 16 marks