2001 Specialist Mathematics Exam 1

Suggested Answers and Solutions

Part I (Multiple-choice) Answers

1. **A**	2. **D**	3. **C**	4. **D**	5. **C**
6. **C**	7. **D**	8. **A**	9. **D**	10. **A**
11. **B**	12. **D**	13. **D**	14. **B**	15. **C**
16. **A**	17. **C**	18. **C**	19. **E**	20. **C**
21. **D**	22. **D**	23. **B**	24. **B**	25. **A**
26. **A**	27. **E**	28. **A**	29. **D**	30. **E**

Solutions — Part I (multiple choice)

Question 1 [A]

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$
$$= \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$$

Question 2 [D]

$$|z-1| = |z+3|$$
$$|x+yi-1| = |x+yi+3|$$
$$(x-1)^2 + y^2 = (x+3)^2 + y$$
$$x^2 - 2x + 1 + y^2 = x^2 + 6x + 9 + y^2$$
$$8x + 8 = 0$$
$$x = -1$$

Question 3 [C]

$$2 + Sin^{-1}\left(\frac{x}{2}+1\right) = 2 + Sin^{-1}\left[\frac{1}{2}(x+2)\right]$$

$f(x) = Sin^{-1}(x)$, dom $f = [-1, 1]$

$g(x) = Sin^{-1}\left(\frac{1}{2}x\right)$ Dilated by a factor of 2.

dom $g = [-2, 2]$

$h(x) = Sin^{-1}\left[\frac{1}{2}(x+2)\right]$ is g(x) translated 2 units left.

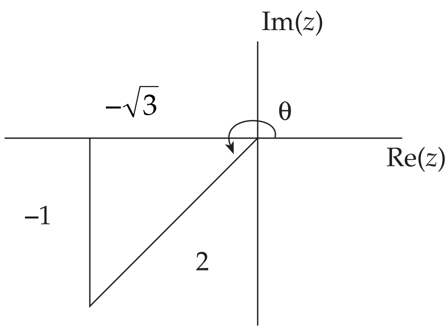
dom $h = [-4, 0]$

Question 4 [D]

$$\sin^2 x = 1 - \cos^2 x$$
$$= 1 - \frac{4}{5}$$
$$= \frac{1}{5}$$

$\sin x = \frac{1}{\sqrt{5}}$, since $\frac{\pi}{2} < x < \pi$

Question 5 [C]



$$\sin\theta = -\frac{1}{2}$$
$$\theta = \frac{-5\pi}{6}$$
$$-\sqrt{3} - i = 2cis\left(-\frac{5\pi}{6}\right)$$

Question 6 **[C]**

$$\int \frac{2}{1+9x^2}\,dx = 2\int \frac{1}{9\left(\frac{1}{9}+x^2\right)}\,dx$$

$$= \frac{2}{9} \times \frac{3}{1} \int \frac{1 \times \frac{1}{3}}{\left(\frac{1}{3}\right)^2 + x^2}\,dx$$

$$= \frac{2}{3} Tan^{-1}(3x)$$

Since question requests 'an antiderivative' '+c' is not required.

Question 7 **[D]**

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(3x) e^{\cos(3x)}\,dx$$

Let $u = \cos 3x$

$\frac{du}{dx} = -3\sin 3x$

Terminals:

$x = \frac{\pi}{2},\ u = 0$

$x = \frac{\pi}{3},\ u = -1$

$-\frac{1}{3}\int_{-1}^{0} e^u\,du$

Question 8 **[A]**

$$\underset{\sim}{a} = \frac{\underset{\sim}{F}}{m} = \frac{6\underset{\sim}{i} - 2\underset{\sim}{j}}{2} = 3\underset{\sim}{i} - \underset{\sim}{j}$$

$$a = \left|\underset{\sim}{a}\right| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$$

Question 9 **[D]**

$h'(x) = 2x\sqrt{1-x^2}$

Let $u = 1 - x^2$

$\frac{du}{dx} = -2x$

$$h(x) = \int -u^{\frac{1}{2}}\,du$$

$$= -\frac{2}{3}(1-x^2)^{\frac{3}{2}} + c$$

Since $h(0) = 1$, $1 = -\frac{2}{3} + c$, $c = \frac{5}{3}$

$$\therefore h(x) = \frac{5}{3} - \frac{2}{3}(1-x^2)^{\frac{3}{2}}$$

Question 10 **[A]**

$$\underset{\sim}{r}(t) = 2e^{-3t}\underset{\sim}{i} + \frac{5}{2}\sin(3t)\underset{\sim}{j}$$

$$\dot{\underset{\sim}{r}}(t) = -6e^{-3t}\underset{\sim}{i} + \frac{5}{2} \times 3\cos t\,\underset{\sim}{j}$$

$$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{36e^{-6t} + \frac{225}{4}\cos^2 3t}$$

$$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{36 + \frac{225}{4}}$$

$$= \frac{3\sqrt{41}}{2}$$

Question 11 **[B]**

$$f(x) = 2x^2 + x - 15$$

$$= (2x-5)(x+3)$$

Asymptotes of $\frac{1}{f(x)}$ occur where $f(x) = 0$

Hence asymptotes occur at $x = -3$, and $x = \frac{5}{2}$

Question 12 **[D}**

$$\int \sin^3(2x)dx$$

$$= \int \sin(2x)\sin^2(2x)\ dx$$

$$= \int \sin(2x)\times(1-\cos^2(2x))\ dx$$

Let $u = \cos(2x)$ $\qquad \frac{du}{dx} = -2\sin(2x)$

$$\int -\frac{1}{2}(1-u^2)du$$

Question 13 **[D]**

$$A = \frac{1}{2}(2+\sqrt{2})2 + \frac{1}{2}(\sqrt{2}+0)2$$

$$= 2 + 2\sqrt{2}$$

$$= 4.8284$$

Question 14 **[B]**

$$\int \frac{3x}{2x^2+3}dx$$

Let $u = 2x^2 + 3$

$$= 3\int \frac{1}{4}\frac{1}{u}\frac{du}{dx}dx \qquad \frac{du}{dx} = 4x$$

$$= \frac{3}{4}\log_e(2x^2+3) + c$$

Since question requests ‘an antiderivative’ ‘+c’ is not required.

Question 15 **[C]**

Inflow: $\frac{dQ}{dV} = 3\ \text{gL}^{-1},\ \frac{dV}{dt} = 5\ \text{Lmin}^{-1}$

$$\frac{dQ}{dt_{IN}} = \frac{dQ}{dV}\frac{dV}{dt} = 15\ \text{gmin}^{-1}$$

Given inflow of 5 Lmin^{-1} and outflow of 2 Lmin^{-1}, then the volume at any time t is $(40 + 3t)$ L.

Outflow: $\frac{dQ}{dV} = \frac{Q}{40+3t}\ \text{gL}^{-1},\ \frac{dV}{dt} = 2\ \text{Lmin}^{-1}$

$$\frac{dQ}{dt_{OUT}} = \frac{dQ}{dV}\frac{dV}{dt} = \frac{2Q}{40+3t}$$

$$\frac{dQ}{dt} = \frac{dQ}{dt_{IN}} - \frac{dQ}{dt_{OUT}}$$

$$= 15 - \frac{2Q}{40+3t}$$

Question 16 **[A]**

Volume of revolution about y-axis given by

$$V = \int_{y=a}^{y=b} \pi x^2 dy$$

$$V = \int_0^2 \pi(2y)^2 dy - \int_0^2 \pi(y^2)^2 dy$$

$$= \pi\int_0^2 (4y^2 - y^4)dy$$

Question 17 **[C]**

$$y = \log_e(\sin x)$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} \qquad \frac{d^2y}{dx^2} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x}$$

$$\sin 2x = 2\sin x\cos x$$

$$\frac{1}{\sin x\cos x}\frac{\cos x}{\sin x} - \frac{1}{\sin^2 x} = 0$$

Hence $\frac{2}{\sin 2x}\frac{dy}{dx} + \frac{d^2y}{dx^2} = 0$

Question 18 **[C]**

$$wz = 2\times 3cis\left(\frac{3\pi}{4} + \frac{\pi}{3}\right)$$

$$= 6cis\left(\frac{13\pi}{12}\right)$$

$$= 6cis\left(-\frac{11\pi}{12}\right)$$

Question 19 **[E]**

$$x = 3\cos(2t)$$
$$y = 4\sin(2t)$$

$$\frac{x}{3} = \cos(2t) \qquad \frac{y}{4} = \sin(2t)$$

Since $\sin^2(2t) + \cos^2(2t) = 1$

then $\frac{x^2}{9} + \frac{y^2}{16} = 1$

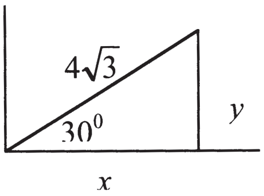
$$16x^2 + 9y^2 = 144 \qquad -3 \le x \le 3$$

Question 20 [C]

$\frac{dy}{dx} = 3x^2 - 1, \quad y_{n+1} = y_n + hf(x_n), \quad h = 0.1$

x	y
1	3
1.1	$3 + 0.1(3(1)^2 - 1) = 3.2$
1.2	$3.2 + 0.1(3(1.1)^2 - 1) = 3.463$

Question 21 [D]



$x = 4\sqrt{3}\cos(30°)$ $\quad y = 4\sqrt{3}\sin(30°)$

$= 6$ $\quad = 2\sqrt{3}$

$\therefore \underset{\sim}{v} = 6\underset{\sim}{i} + 2\sqrt{3}\underset{\sim}{j}$

Question 22 [D]

$\left|\underset{\sim}{b}\right| = \sqrt{2} \quad \underset{\sim}{a}.\hat{\underset{\sim}{b}} = \frac{1}{\sqrt{2}}(5-3) = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\left(\underset{\sim}{a}.\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} = \underset{\sim}{i} + \underset{\sim}{j}$

$\underset{\sim}{a} - \left(\underset{\sim}{a}.\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} = 4\underset{\sim}{i} - 4\underset{\sim}{j}$

Question 23 [B]

$\frac{dt}{dv} = \frac{v}{4}$

$t = \frac{v^2}{8} + c$

$t = 0, \ v = -2, \ 0 = \frac{1}{2} + c$

$t = \frac{v^2}{8} - \frac{4}{8}$

$v^2 = 8t + 4$

$v = \pm\sqrt{8t+4}$

$v = -2\sqrt{2t+1}$, since when $t = 0, \ v = -2$

Question 24 [B]

$\left|\underset{\sim}{a}\right| = 3, \ \left|\underset{\sim}{b}\right| = 5, \quad \cos\theta = \frac{\underset{\sim}{a}.\underset{\sim}{b}}{\left|\underset{\sim}{a}\right| \times \left|\underset{\sim}{b}\right|}$

$\cos\theta = \frac{11}{15}, \quad \theta = 42.83^0$

Question 25 [A]

$y = Cos^{-1}\left(\frac{3}{2x}\right) \qquad$ Let $u = \frac{3}{2x} = \frac{3}{2}x^{-1}$

$\frac{du}{dx} = -\frac{3}{2}x^{-2}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{-1}{\sqrt{1 - \frac{9}{4x^2}}} \times \frac{-3}{2x^2}$

$= \frac{3}{2\sqrt{x^4 - \frac{9}{4}x^2}}$

$= \frac{3}{2x\sqrt{\frac{4x^2}{4} - \frac{9}{4}}}$

$= \frac{3}{x\sqrt{4x^2 - 9}}$

Question 26 [A]

Since accelerating upwards, resultant force is upwards.

Hence (force up) - (force down) = ma

$T - 75g = 75(2.5)$

$T = 75(g + 2.5)$

Question 27 [E]

Since object moves with constant speed, forces up plane = forces down plane

$F + 5g\sin 25° = T\cos 40°$

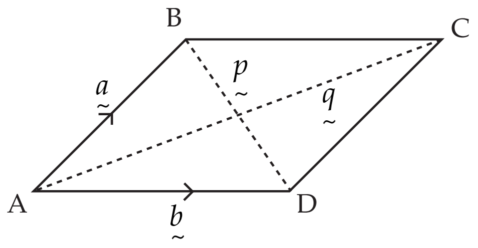
Frictional force $F = \mu N = 0.2N$

Hence $T\cos 40^o = 0.2N + 5g\sin 25°$

Question 28 **[A]**

$\tilde{p} = (\tilde{a} + \tilde{b})$

$\tilde{q} = (\tilde{a} - \tilde{b})$



need to prove $\tilde{p} \cdot \tilde{q} = 0$

hence $(\tilde{a} + \tilde{b}) \cdot (\tilde{a} - \tilde{b}) = 0$

Question 29 **[D]**

$2 \times T \sin\theta = 4g$

$T = \dfrac{2g}{\sin\theta}$

Question 30 **[E]**

2 kg mass:	$T - 2g = 2a$	(I)
3 kg mass:	$3g - T = 3a$	(II)
(I) + (II)	$g = 5a$	
	$a = \dfrac{g}{5}$	

Answers — Part II: Short answers

Question 1a

$\overrightarrow{AC} = \tilde{c} - \tilde{a}$ **[A1]**

$\overrightarrow{BC} = \tilde{c} + \tilde{a}$ **[A1]**

Question 1b

If $\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$ then ACB is a right angle

$\overrightarrow{AC} \cdot \overrightarrow{BC} = (\tilde{c} - \tilde{a}) \cdot (\tilde{c} + \tilde{a})$ **[M1]**

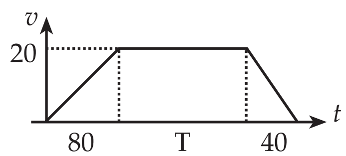
$= \tilde{c} \cdot \tilde{c} + \tilde{a} \cdot \tilde{c} - \tilde{a} \cdot \tilde{c} - \tilde{a} \cdot \tilde{a}$

$= |\tilde{c}|^2 - |\tilde{a}|^2$ **[M1]**

$|\tilde{c}| = |\tilde{a}|$ because both are radii of a circle

$\therefore \overrightarrow{AC} \cdot \overrightarrow{BC} = 0$ **[A1]**

Question 2a



[M1] shape
[A1] values

Note: Steepness in third section must be greater than in first interval for "shape" mark. 80 & 40 may be implicit in the working for (**2b**).

Question 2b

$1600 = 800 + 20T + 400$ **[M1]**

$20T = 400$

$T = 20$

Total time = 80 + 20 + 40 = 140 seconds **[A1]**

Question 3a

$(z-2)(z-\sqrt{3}i)$

$= z^2 - \sqrt{3}iz - 2z + 2\sqrt{3}i$

$= z^2 - (\sqrt{3}i + 2)z + 2\sqrt{3}i$ **[A1]**

Question 3b

$(z-2)(z-\sqrt{3}i)(z+\sqrt{3}i)$ **[M1]**

$= (z-2)(z^2+3)$

$= z^3 - 2z^2 + 3z - 6$ **[A1]**

Question 4

$$\int_{\frac{-1}{2}}^{\frac{3}{2}} x\sqrt{1+2x}\,dx$$

Let $u = 1 + 2x$

$$x = \frac{u-1}{2}$$

$$\frac{du}{dx} = 2$$

$$dx = \frac{du}{2}$$

$$x = \frac{3}{2}, u = 4$$

[M1]

$$u = \frac{-1}{2}, u = 0$$

$$\int_0^4 \frac{(u-1)\sqrt{u}}{2} \times \frac{du}{2}$$

[M1]

$$\int_0^4 \frac{u^{\frac{3}{2}} - u^{\frac{1}{2}}}{4}\,du$$

$$\frac{1}{2}\left[\frac{u^{\frac{5}{2}}}{5} - \frac{u^{\frac{3}{2}}}{3}\right]_0^4$$ **[A1]**

$$= \frac{56}{30} = \frac{28}{15}$$ **[A1]**

Question 5

$$\cos 2x = 1 - 2\sin^2 x$$

using $x = \frac{\pi}{8}$

$$\cos\frac{\pi}{4} = 1 - 2\sin^2\frac{\pi}{8}$$ **[M1]**

$$\frac{1}{\sqrt{2}} = 1 - 2\sin^2\frac{\pi}{8}$$

$$\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$$ **[A1]**

$$\cos\frac{15\pi}{8} = \cos\frac{\pi}{8}$$ **[A1]**

$$= \frac{\sqrt{2+\sqrt{2}}}{2}$$ **[A1]**