SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

										Letter
Figures										
Words										

Victorian Certificate of Education 2000

SPECIALIST MATHEMATICS

Examination 2 (Analysis Task)

Wednesday 1 November 2000: 11.45 am to 1.30 pm
Reading time: 11.45 am to 12 noon
Writing time: 12 noon to 1.30 pm
Total writing time: 1 hour 30 minutes

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*
6	6

Directions to students

Materials

Question and answer book of 17 pages. There is a detachable sheet of miscellaneous formulas in the centrefold. Working space is provided throughout the book.

You may bring to the examination up to four pages (two A4 sheets) of pre-written notes.

You may use an approved scientific and/or graphics calculator, ruler, protractor, set square and aids for curve sketching.

The task

Detach the formula sheet from the centre of this book during reading time.

Ensure that you write your **student number** in the space provided on the cover of this book.

Answer **all** questions.

The marks allotted to each part of each question are indicated at the end of each part.

There is a total of 60 marks available for the task.

You need not give numerical answers as decimals unless instructed to do so. Alternative forms may involve, for example, π, e, surds or fractions. A decimal approximation will not be accepted if an **exact** answer is required to a question.

Where an **exact** answer is required to a question, appropriate working must be shown and calculus must be used to evaluate derivatives and definite integrals.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the acceleration due to gravity to have magnitude g m/s^2, where $g = 9.8$.

All written responses should be in English.

Question 1

Consider the function $f: R^+ \to R$ where $f(x) = \dfrac{6}{x} - 6 + 3\log_e x$.

a. Find

i. $f'(x)$

1 mark

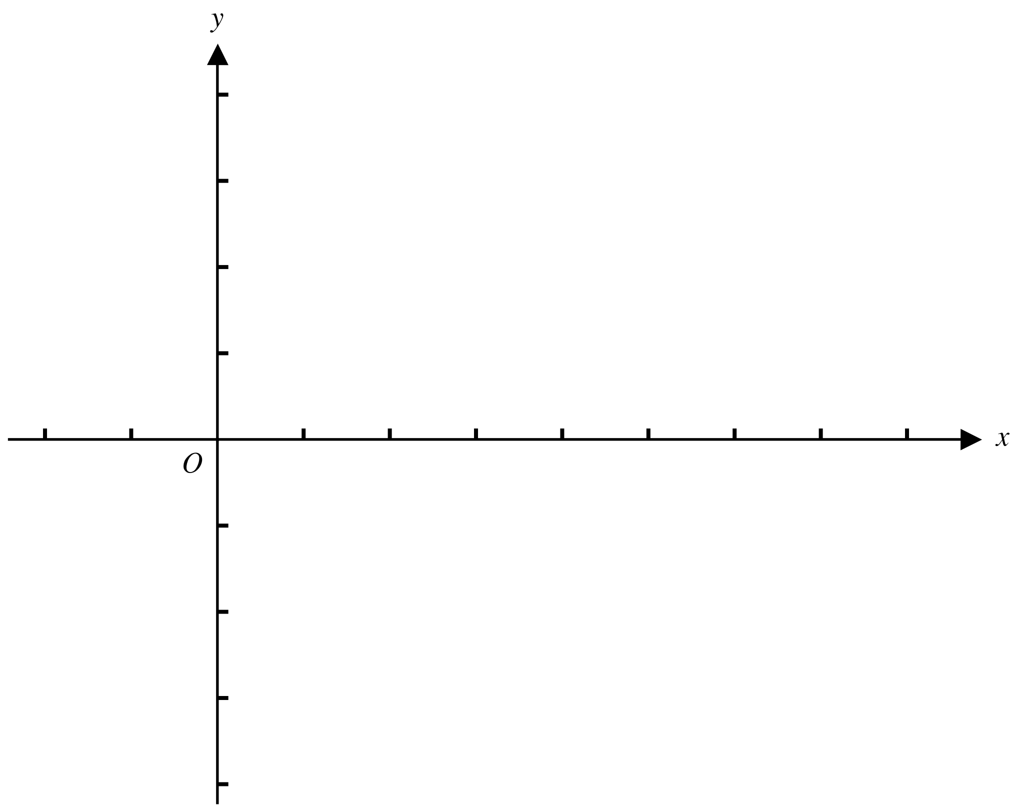
ii. $f''(x)$

1 mark

b. Verify that the graph of $y = f(x)$ has an x-intercept at $x = 1$.

1 mark

c. **i.** Sketch the graph of $y = f(x)$ on the axes below.



2 marks

Question 1 – continued

ii. Find the exact coordinates of any stationary points.

2 marks

iii. Find, correct to two decimal places, any x-intercepts other than $x = 1$.

1 mark

d. The graph in **c.i.** has a maximum positive gradient. Find its exact value.

2 marks

e. Carefully sketch and label, on the axes in **c.i.**, a curve with equation $y = F(x)$ where F is an antiderivative of f.

2 marks

Total 12 marks

Question 2

Kris wants to move a box of mass 50 kg which is resting on a rough, level floor.

a. If Kris pulls on the box with a horizontal force of 140 N, he can just make the box slide across the floor. Find the coefficient of friction between the box and the floor.

2 marks

b. If Kris pulls on the box with a force of P N acting at an angle of 15° to the horizontal, what is the least value of P that will just make the box slide across the floor? Give your answer correct to the nearest integer.

4 marks

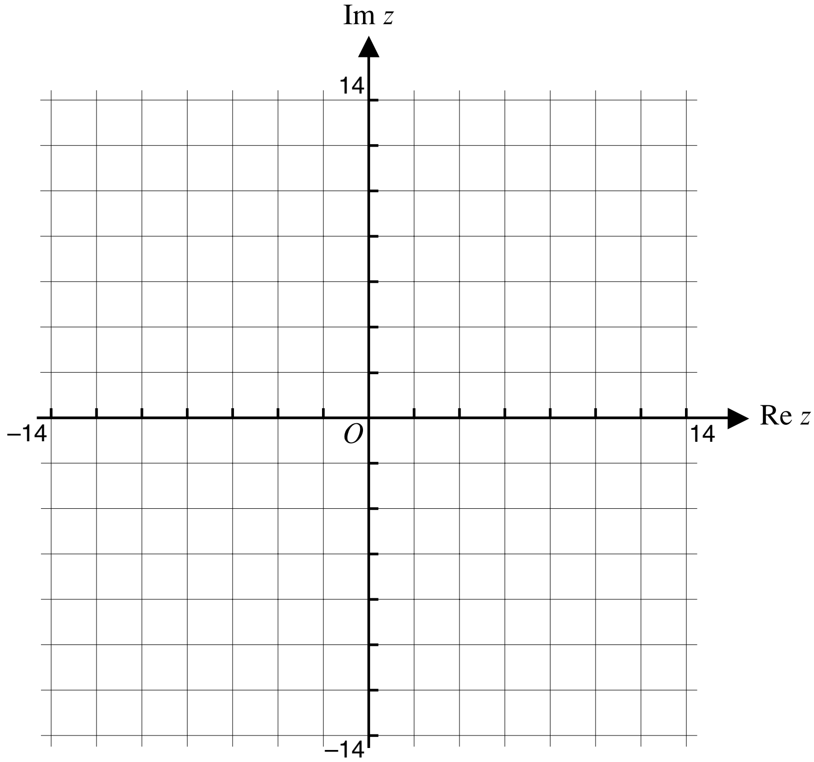
Total 6 marks

Working space

Question 3

Let $v = 6 + 8i$ and $w = 7 + i$.

a. Plot the points corresponding to v and w on the Argand diagram below, labelling them as V and W respectively.



1 mark

b. Let S be defined by $S = \{z: |z| = 10, z \in C\}$.

i. Verify that $v \in S$.

1 mark

ii. Sketch S on the Argand diagram in **a.**

1 mark

c. Let u be such that $u + i\,\overline{w} = \overline{w}$. Find u in cartesian form.

2 marks

d. Sketch, on the Argand diagram in **a.**, $T = \{z: |z| \leq 10\} \cap \{z: |z - u| = |z - v|\}$.

2 marks

e. Use a vector method to prove that $\angle OWV$ is a right angle.

3 marks

Total 10 marks

TURN OVER

Question 4

a. **i.** Find $\frac{d}{dx}(x\cos x)$.

1 mark

ii. Hence show that an antiderivative of $x \sin x$ is $(\sin x - x \cos x)$.

1 mark

The Seafresh Company grows and harvests a type of shellfish. The bay in which the shellfish are farmed is very calm so that the water surface is flat. The company has partitioned the bay into pens using nets parallel to each other and 150 metres apart. The nets extend 60 metres from the straight shoreline and are perpendicular to it. In Figure 1, *OB* is one such net.

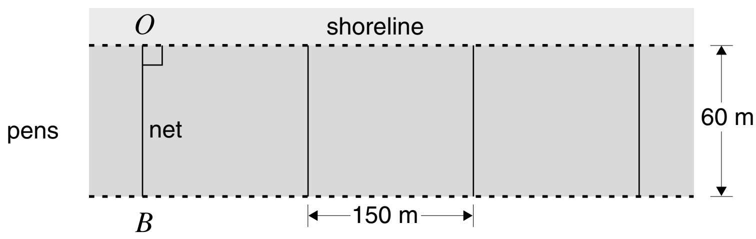


Figure 1

The shape of the seabed can be modelled by $y = -\frac{x}{50}(8 + \sin x)$, where x metres is the perpendicular distance from the shoreline and y metres is the vertical displacement. The vertical cross-section of the water parallel to the net *OB* is the shape of the shaded region in Figure 2.

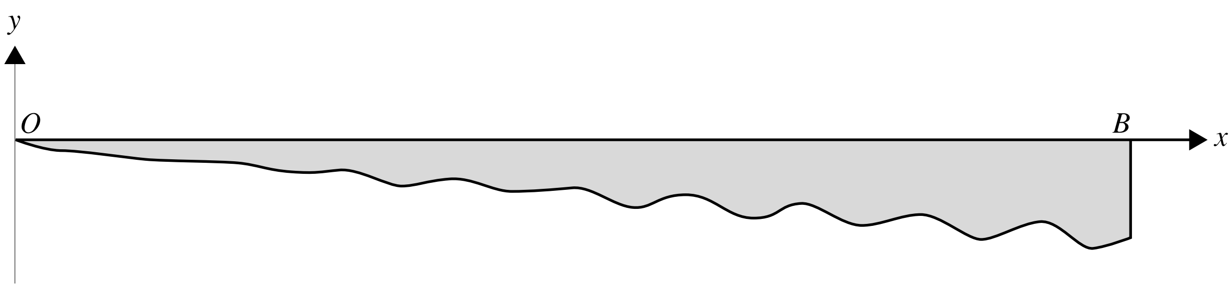


Figure 2

Question 4 – continued

b. **i.** How deep is the water 10 metres from the shore? Give your answer correct to the nearest tenth of a metre.

1 mark

ii. Use calculus to find, to the nearest square metre, the area of the shaded region shown in Figure 2.

4 marks

To ensure that the shellfish grow quickly, one unit of a nutrient is added daily to the water for each 500 cubic metres of water in a pen.

c. **i.** To the nearest unit, how many units of nutrient are added daily to each pen?

1 mark

ii. If the shape of the seabed is instead modelled by $y = -\frac{x^2}{1600}(8 + \sin x)$, write an expression for the number of units of nutrient that are added daily to each pen and evaluate it to the nearest unit.

2 marks

Total 10 marks

Working space

Question 5

A javelin is thrown by a competitor on level ground. At time t, the time in seconds measured from the release of the javelin, the position vector $\underset{\sim}{r}(t)$ of the tip of the javelin is given by

$$\underset{\sim}{r}(t) = 19.5t\,\underset{\sim}{i} + \left(\frac{\pi}{2}t - 4\sin\frac{\pi t}{8}\right)\underset{\sim}{j} + \left(2 + 19.5t - 5t^2\right)\underset{\sim}{k}$$

where $\underset{\sim}{i}$ is a unit vector in the east direction, $\underset{\sim}{j}$ is a unit vector in the north direction and $\underset{\sim}{k}$ is a unit vector vertically up. The origin O of the coordinate system is at ground level and displacements are measured in metres.

Let P be the point where the tip of the javelin hits the ground.

a. Show that the tip of the javelin reaches P in 4 seconds.

1 mark

b. How far from O, correct to the nearest centimetre, does the tip of the javelin hit the ground?

2 marks

c. Find the velocity of the tip of the javelin at time $t = 4$.

2 marks

d. At what angle, correct to the nearest tenth of a degree, does the path of the javelin's tip make with the ground at P?

3 marks

Total 8 marks

Question 6

A parachutist drops from a bridge and 'free falls' until reaching a speed of 20 m/s.

a. Neglecting the effect of air resistance, find the time spent in 'free fall' and the distance travelled during this period. Give your answers correct to three significant figures.

2 marks

Having reached the speed of 20 m/s, the parachute opens and the parachutist, whose mass is m kg, is retarded by a variable force of $0.04mgv^2$ N, where v m/s is her velocity t s after the parachute opens and g m/s^2 is the magnitude of the acceleration due to gravity.

b. Taking vertically downwards as positive, write down the equation of motion of the parachutist during this second stage of descent and show that it simplifies to the differential equation

$$\frac{dv}{dt} = -0.04g\left(v^2 - 25\right)$$

2 marks

c. **i.** Solve this differential equation to obtain t in terms of v.

3 marks

ii. Hence show that $v = \dfrac{5\left(1 + 0.6e^{-0.4gt}\right)}{1 - 0.6e^{-0.4gt}}$.

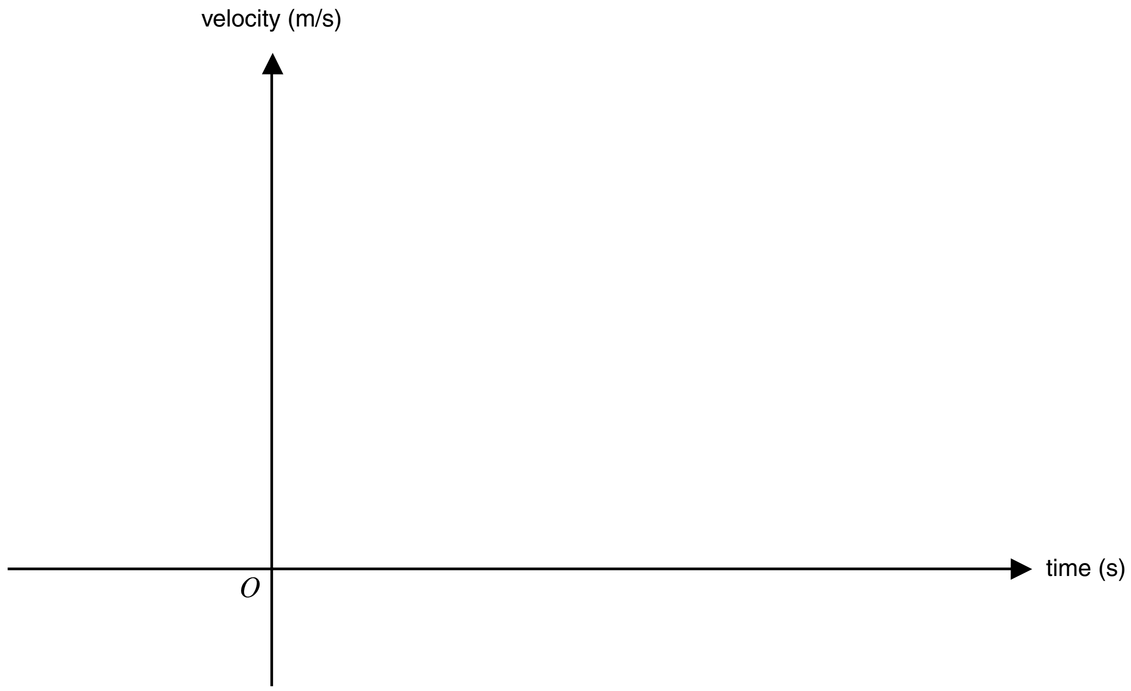
2 marks

The parachutist approaches a limiting velocity before she lands.

d. What is her limiting velocity? Show how you deduce your result.

2 marks

e. Assuming that the parachutist lands 8 seconds after she drops from the bridge, sketch the velocity-time graph of her motion from the time that she drops from the bridge until just after she lands, showing intercepts and key features.



3 marks

Total 14 marks

Working space

END OF QUESTION AND ANSWER BOOK

SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.

Specialist Mathematics Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
cosine rule:	$c^2 = a^2 + b^2 - 2ab\cos C$

Coordinate geometry

ellipse:	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
hyperbola:	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Circular (trigometric) functions

$\cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\cot^2 x + 1 = \operatorname{cosec}^2 x$

$\sin(x+y) = \sin x\cos y + \cos x\sin y$

$\sin(x-y) = \sin x\cos y - \cos x\sin y$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$

$\cos(x-y) = \cos x\cos y + \sin x\sin y$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$

$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\sin 2x = 2\sin x\cos x$

$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

function	Sin^{-1}	Cos^{-1}	Tan^{-1}
domain	$[-1, 1]$	$[-1,1]$	R
range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (Complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$

$|z| = \sqrt{x^2 + y^2} = r$

$-\pi < \operatorname{Arg} z \le \pi$

$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$

$z^n = r^n \operatorname{cis} n\theta$ (de Moivre's theorem)

Calculus

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \qquad \int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax} \qquad \int e^{ax} dx = \frac{1}{a} e^{ax} + c$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x} \qquad \int \frac{1}{x} dx = \log_e x + c, \text{ for } x > 0$$

$$\frac{d}{dx}(\sin ax) = a\cos ax \qquad \int \sin ax \, dx = -\frac{1}{a}\cos ax + c$$

$$\frac{d}{dx}(\cos ax) = -a\sin ax \qquad \int \cos ax \, dx = \frac{1}{a}\sin ax + c$$

$$\frac{d}{dx}(\tan ax) = a\sec^2 ax \qquad \int \sec^2 ax \, dx = \frac{1}{a}\tan ax + c$$

$$\frac{d}{dx}\left(\text{Sin}^{-1}x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{a^2-x^2}} dx = \text{Sin}^{-1}\frac{x}{a} + c, a > 0$$

$$\frac{d}{dx}\left(\text{Cos}^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}} \qquad \int \frac{-1}{\sqrt{a^2-x^2}} dx = \text{Cos}^{-1}\frac{x}{a} + c, a > 0$$

$$\frac{d}{dx}\left(\text{Tan}^{-1}x\right) = \frac{1}{1+x^2} \qquad \int \frac{a}{a^2+x^2} dx = \text{Tan}^{-1}\frac{x}{a} + c$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

mid-point rule: $\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right)$

trapezoidal rule: $\int_a^b f(x)dx \approx \frac{1}{2}(b-a)\left(f(a)+f(b)\right)$

Euler's method: If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h f(x_n)$

acceleration: $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at \qquad s = ut + \frac{1}{2}at^2 \qquad v^2 = u^2 + 2as \qquad s = \frac{1}{2}(u+v)t$

TURN OVER

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$|\underset{\sim}{r}| = \sqrt{x^2 + y^2 + z^2} = r$

$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos\theta = x_1x_2 + y_1y_2 + z_1z_2$

$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$

Mechanics

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

friction: $F \leq \mu N$

END OF FORMULA SHEET