VCE Mathematical Methods (CAS) Units 3&4 Solutions Manual

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A note about copying

We get it. We were there ourselves no more than a few years ago. It's tough being a VCE student. You want to do the best you can, and to do so you need to have the best materials. Everyone else seems to have all the resources that you don't have. You can't afford to buy them all, and you don't want to put that pressure on your parents to buy more books for you, when they already work hard enough to send you to school. If you have friends who cannot afford this book but would benefit from its contents, nothing we can do will stop you from letting them copy it without paying for it, and we are honestly happy that they are going to benefit from this wonderful resource.

But if you can afford it, we ask that you kindly spare a thought for the people who wrote this book. We're students who took considerable time and effort, some to the detriment of other commitments, to collate our wisdom and expertise in these subjects and provide them in an easily-accessible book form. Most of the profits from these books go to students like yourselves, who have written this book to help make ends meet. They need your support. Please be considerate of the talented authors who wrote this book, and do the right thing.

Legal jargon

Preface

This guide has been written to help students understand the concepts that appear in the VCE Mathematical Methods (CAS) 3&4 Study design. This book is intended to run the gamut of all the types of questions that can appear on the exam at the end of the year.

Book 1 is divided into two sections, the first comprising topic tests for the four main areas of the Mathematical Methods course; Functions and Graphs, Algebra, Calculus and finally Probability. Each section has two Exam 1-standard and two Exam 2-standard topic tests, which when combined is equivalent to a practice Exam 1 and an Exam 2 for each area. This section is designed to be completed throughout the year as students learn the course material, and aims to teach the key concepts and skills that the student will be able to use immediately in SACs, and also apply it later on when they move onto exams. The second section comprises 3 full sets of practice examinations (3 exam 1's and 3 exam 2's) which should be attempted after students have completed the course. These exams have been designed to expose students to what they are likely to face in exams. They are of a fair standard, but also contain numerous questions that have subtle tricks to them. These types of questions are the ones VCAA (Victorian Curriculum and Assessment Authority) will invariably use to determine the top students sitting each exam.

The main feature that separates this study guide from the rest on the market is the Detailed Solutions section in Book 2. While model solutions will show the student what they need to do to get marks for a given question, they do not always tell the student why they are doing a particular task or why to take a particular approach. Some students learn how to get through problems but not actually understand **why** they are undertaking the process. Instead of blindly applying $f'(x)$ to find stationary points, the detailed solutions go through the theory behind the concept so that the student has a better understanding of the course. These solutions also go through the best way to approach a question, and give more than one option in places where neither method is better or worse.

Thought processes in interpreting a question are also explored, as sometimes students have the skills and the knowledge required to complete a question, but have trouble in actually understanding what the question is trying to ask them, and what they need to apply. For example, an area of study that students tend to struggle with is probability. Most students know the tools (probability methods) they need to complete a given question, but cannot work out what type of probability (tool) they need to apply to the situation, so the detailed solutions pick apart the information and wording of the question so that they should be able to interpret most of the problems that they will face. While the goal was to maintain a particular standard for the questions that have been written, our many writers have managed to come up with some excellent challenging questions that cater to the above-average student. The standard of VCAA exams in the past few years has been trending steadily upwards in difficulty, so these will serve as good practice for the inevitable curveball you will face in the exam proper. As always, the solutions will walk you through every small part of the question so that even a struggling student should be able to understand how to do the question from the solutions.

Throughout the solutions, calculator screenshots and tips for the TI-nspire CX have been included to try and help students learn when it is best to use the calculator and when it is best to work through the problem by hand. Throughout the year, it is advised that the student doesn't use the calculator unless they actually need to, as this will help develop algebraic skills that will benefit them in exam 1, the tech-free exam. This way, the student can still jump to the calculator when needed in an exam situation, but not become reliant on the calculator from working with it too much throughout the year. Those students who do exceptionally well are generally those who don't rely on the calculator (or their bound reference, for that matter).

On the team we have three writers that scored a perfect study score of 50 (of whom two achieved this impressive feat in year 11), two that scored a near perfect 49 and one with a 45. We also had our brilliant editor William, who also scored a 50. We are students who excelled in this system. There is nothing mythical or unattainable about what we did. We were in the same position as you were, 1 year, 2 years etc. ago. We intend through this book to communicate to you all of the thought processes that we learned to ace this subject, from a perspective that we know all too well (and so recently) as bewildered students. As such, the language of the detailed solutions can be quite colloquial at times, preferring plain English over needlessly wordy and long-winded explanations. We also hope you enjoy our (attempted) humour, as we try to ease the pain of year 12 for you, one bad maths joke at a time. You may notice that the writing styles vary throughout the book. This is because different authors were writing different sections. This is a natural outcome of having such a varied team of writers. Through our editing processes, we hope that the quality of the book is roughly uniform throughout, but naturally some sections may be more detailed than others.

If you are having difficulty understanding any part of the book, find any errors or want to discuss the book with others, then feel free to log onto www.atarnotes.com and visit the forum. Follow the links to the ExamPro Units 3&4 Mathematical Methods (CAS) Study Guide messageboard, and post on the errata thread if you find an error, or start a new thread if you have a different query.

Best of luck with your studies,

Trevor Batty

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Section 1: Model Solutions and Marking Schemes

FUNCTIONS AND GRAPHS

TECH-FREE TEST 1

MODEL SOLUTIONS AND MARKING SCHEME

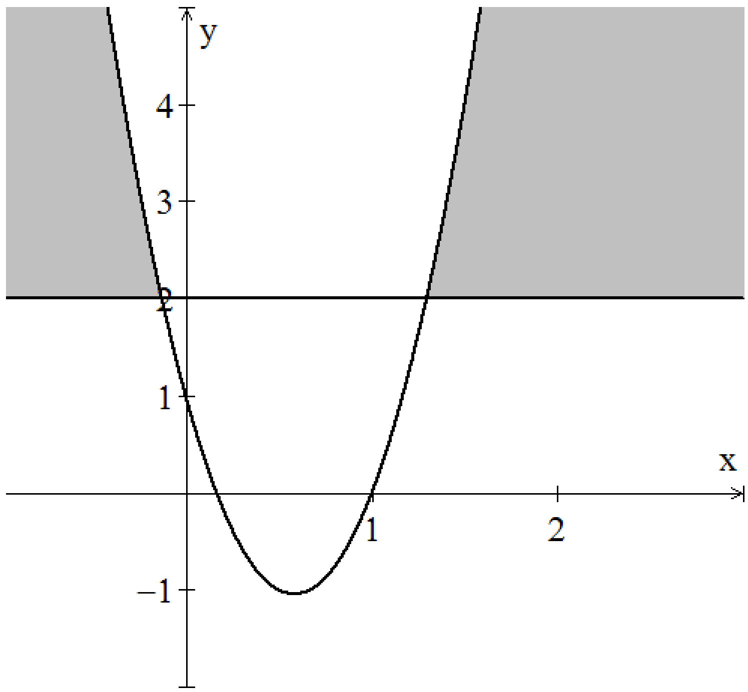
Question 1

a.

$$\begin{aligned}
6x^2 - 7x + 1 &\geq 2 \\
6x^2 - 7x - 1 &\geq 0 \\
6(x^2 - \frac{7}{6}x - \frac{1}{6}) &\geq 0 \\
(x - \frac{7}{12})^2 - \frac{49}{144} - \frac{1}{6} &\geq 0 \\
(x - \frac{7}{12})^2 - \frac{73}{144} &\geq 0 \\
(x - \frac{7}{12})^2 &\geq \frac{73}{144} \\
x - \frac{7}{12} \geq \frac{\sqrt{73}}{12} \quad &or \quad x - \frac{7}{12} \leq -\frac{\sqrt{73}}{12} \\
x \geq \frac{7 + \sqrt{73}}{12} \quad &or \quad x \leq \frac{7 - \sqrt{73}}{12}
\end{aligned}$$

- 1 method mark for completing the square
- 1 method mark for taking the square root at $(x - \frac{7}{12})^2 \geq \frac{73}{144}$ with the correct inequality signs
- 1 answer mark for both answers $\frac{7+\sqrt{73}}{12}$ and $\frac{7-\sqrt{73}}{12}$

b.



The axis intercepts are at (0,1), $(\frac{1}{6},0)$ and (1,0). The intersections of the line $y = 2$ and the function, $f(x)$ are $\left(\frac{7+\sqrt{73}}{12},2\right)$ and $\left(\frac{7-\sqrt{73}}{12},2\right)$.

- 1 answer mark for the correct shape of $f(x)$
- 1 answer mark for correct shading of $2 \le y \le f(x)$
- 1 answer mark for correct axial intercepts
- 1 answer mark for indicating the points at $\left(\frac{7+\sqrt{73}}{12},2\right)$ and $\left(\frac{7-\sqrt{73}}{12},2\right)$

Question 2

For the linear portion:
For $x \in (-6,-3]$, gradient $= 0$, horizontal line described by

$$y = 2$$

For $x \in (2,a]$, the known points are $(2,1)$ and $(3,0)$.

$$\begin{aligned} \text{gradient} &= \frac{0-1}{3-2} \\ &= -1 \\ y-1 &= -1(x-2) \\ y &= -x+3 \end{aligned}$$

$x = a$ when $y = -6$

$$\begin{aligned} -6 &= -a+3 \\ a &= 9 \end{aligned}$$

Hence, this line has a domain of $(2, 9]$

For the quadratic portion:
Known points: $(-3, 2)$, $(0, -1)$ and $(2, 1)$
Substitute $(0, -1)$ into general form of quadratic

$$\begin{aligned} -1 &= a(0)^2 + b(0) + c \\ c &= -1 \end{aligned}$$

Substitute $(-3, 2)$

$$2 = a(-3)^2 + b(-3) - 1$$

Substitute $(2, 1)$

$$1 = a(2)^2 + b(2) - 1$$

Therefore

$$\begin{aligned} 3 &= 9a - 3b \qquad (1) \\ 2 &= 4a + 2b \qquad (2) \end{aligned}$$

By simultaneous equations, (2)+$\frac{2}{3}$(1)

$$\begin{aligned} 2 + \frac{2}{3}(3) &= 4a + 2b + \frac{2}{3}(9a) - \frac{2}{3}(3b) \\ 4 &= 10a \\ a &= \frac{2}{5} \end{aligned}$$

Substitute this value of a into (2)

$$\begin{aligned} 2 &= \frac{2}{5}(4) + 2b \\ \frac{2}{5} &= 2b \\ b &= \frac{1}{5} \end{aligned}$$

Hence, the quadratic is

$$y = \frac{2}{5}x^2 + \frac{1}{5}x - 1$$

Therefore, the hybrid function is as follows

$$y = \begin{cases} 2, & -6 < x \leq -3 \\ \frac{2}{5}x^2 + \frac{1}{5}x - 1, & -3 < x < 2 \\ 3 - x & 2 < x \leq 9 \end{cases}$$

- 1 method mark for utilising some rule of finding straight lines, e.g. $y - y_1 = m(x - x_1)$
- 1 method mark for substituting $(0, -1)$ to find c in quadratic portion
- 1 method mark for using simultaneous equations to find a and b in quadratic form
- 2 answer marks for the correct rule of the linear portion
- 1 answer mark for correct rule of the quadratic portion
- 1 answer mark for the correct domains of the relation

Question 3

The period of $2\cos(\frac{x}{2}+\pi)$ is $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$

The period of $\sin(\frac{2x}{3})$ is $\dfrac{2\pi}{\frac{2}{3}} = 3\pi$

The period of $f(x)$ is the lowest common multiple of periods from the functions being added together. LCM of 4π and 3π is 12π.

Hence the period of $f(x)$ is 12π.

- 1 method mark for finding the period of $2\cos(\frac{x}{2}+\pi)$ and $\sin(\frac{2x}{3})$
- 1 method mark for recognising the use of lowest common multiple of periods
- 1 answer mark for correctly writing down the lowest common multiple, 12π

Question 4

$$\begin{aligned}
\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \end{bmatrix} \\
\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 2x+4 \\ 3y-1 \end{bmatrix}
\end{aligned}$$

$$x' = 2x+4 \quad \textit{and} \quad y' = 3y-1$$

$$x = \frac{x'-4}{2} \quad \textit{and} \quad y = \frac{y'+1}{3}$$

Substitute the result into the pre-image.

$$\begin{aligned}
\frac{y'+1}{3} &= 3(\frac{x'-4}{2})^2 + 1 \\
y' &= \frac{9x'^2}{4} - 18x' + 38
\end{aligned}$$

Hence, $a = \frac{9}{4}$, $b = -18$ and $c = 38$

- 1 method mark for correct matrix arithmetic
- 1 method mark for substituting matrix result into pre-image
- 1 answer mark for correctly writing down the values of a, b, and c as $\frac{9}{4}$, -18, and 38 respectively.

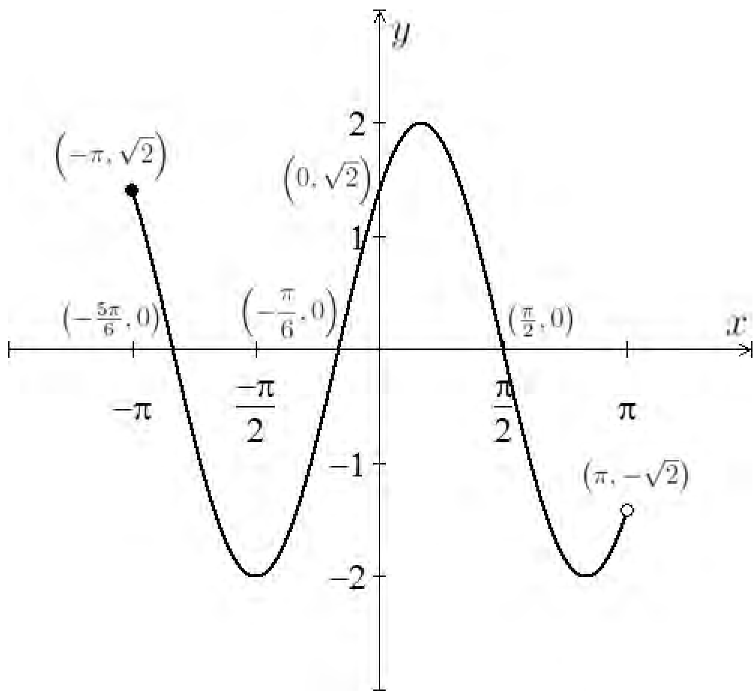
FUNCTIONS AND GRAPHS

TECH-FREE TEST 2

MODEL SOLUTIONS AND MARKING SCHEME

Question 1

a.



- 1 answer mark for correct shape
- 1 answer mark for correct axial intercepts
- 1 answer mark for correct endpoints

b.

Period of original function: $\dfrac{2\pi}{\frac{3}{2}} = \frac{4\pi}{3}$.

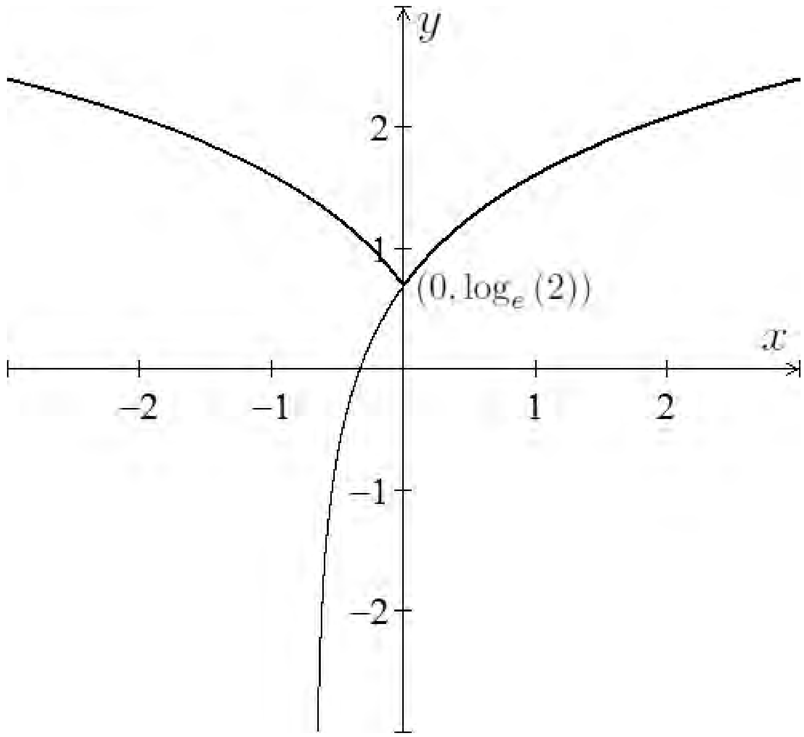
A dilation of factor a from the y-axis is required to provide a period of 24.

$$\begin{aligned} a \times \frac{4\pi}{3} &= 24 \\ a &= \frac{18}{\pi} \end{aligned}$$

Therefore, a dilation of factor $\frac{18}{\pi}$ from the y-axis is required

- 1 method mark for finding the period of the original function
- 1 answer mark for the correct transformation stated

Question 2



$$f(x) = \begin{cases} \log_e(3x+2) & x \geq 0 \\ \log_e(-3x+2) & x < 0 \end{cases}$$

- 1 answer mark for a correct sketch showing a reflection in the y-axis for all positive x-values.
- 1 answer mark for correct hybrid function
- 1 answer mark for correct domains in the hybrid function

Question 3

a.

1. Dilation by factor of $\frac{1}{3}$ from the x-axis
2. Translation of 2 units in the positive direction of the x-axis
3. Translation of $\frac{4}{3}$ units in the negative direction of the y-axis
4. Reflection in the line of $y = x$

- 1 answer mark for each correct transformation (Total 4 marks)

b.

For the original function to have an inverse function, it must be one-to-one. The required domain is $(0, \infty)$. $(-\infty, 0)$ is incorrect since $y = \frac{1}{\sqrt{x+1}} + 2$ only arises from the positive x-values in the original function.

- 1 answer mark for $(0, \infty)$.

Question 4

a.

When $t = 0$, $v = 0$, therefore $0 = a + b$ (1)

When $t = 10$, $v = 108(\frac{1000}{3600}) = 30$, therefore $30 = ae^{-\frac{3}{4}\times 10} = ae^{-\frac{15}{2}} + b$ (2)

Using simultaneous equations, (2) – (1),

$$\begin{aligned} 30 &= ae^{-\frac{15}{2}} - a \\ 30 &= a(e^{-\frac{15}{2}} - 1) \\ a &= \frac{30}{e^{-\frac{15}{2}} - 1} \end{aligned}$$

From equation (1)

$$\begin{aligned} a + b &= 0 \\ b &= -a \\ b &= -\frac{30}{e^{-\frac{15}{2}} - 1} \end{aligned}$$

- 1 method mark for correctly setting up both equations from the information given
- 1 answer mark for a correct exact value for a
- 1 answer mark for a correct exact value for b

b.

He will survive if speed is less than 40 m/s.

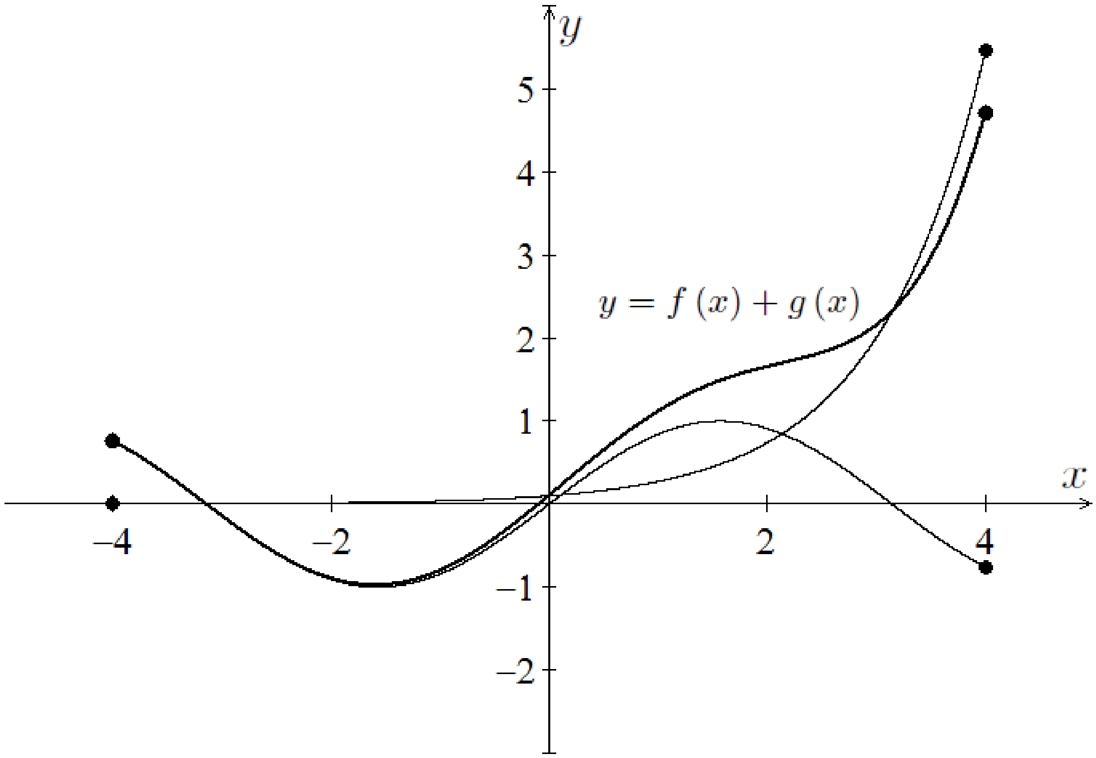
$$\begin{aligned} 40 &= \frac{30}{e^{-\frac{15}{2}} - 1}e^{-\frac{3}{4}t} - \frac{30}{e^{-\frac{15}{2}} - 1} \\ 40 &= \frac{30}{e^{-\frac{15}{2}} - 1}(e^{-\frac{3}{4}t} - 1) \\ \frac{4}{3}(e^{-\frac{15}{2}} - 1) &= e^{-\frac{3}{4}t} - 1 \\ e^{-\frac{3}{40}t} &= \frac{1}{3}(4e^{-\frac{15}{2}} - 1) \\ -\frac{3}{4}t &= \log_e(\frac{1}{3}(4e^{-\frac{15}{2}} - 1)) \\ t &= -\frac{4}{3}\log_e(\frac{1}{3}(4e^{-\frac{15}{2}} - 1)) \end{aligned}$$

The base jumper will survive if he lands before $-\frac{4}{3}\log_e(\frac{1}{3}(4e^{-\frac{15}{2}} - 1))$ seconds

- 1 method mark for correctly setting up the initial equation for solving
- 1 answer mark for a correct exact value of t

Question 5

a.



- 1 answer mark for the correct shape
- 1 answer mark for correct endpoints

b.

The correct domain is [-4,2]

- 1 answer mark for correct domain

FUNCTIONS AND GRAPHS

TECH-ACTIVE TEST 1

MODEL SOLUTIONS AND MARKING SCHEME

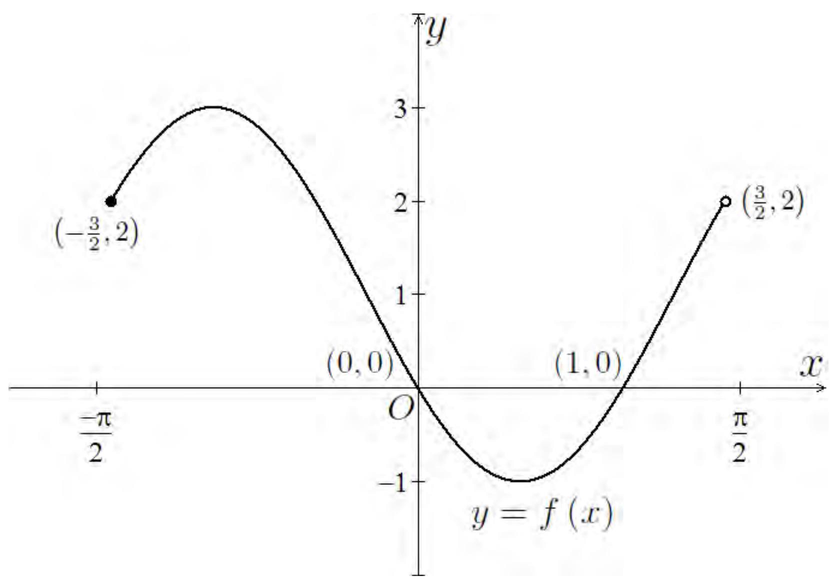
SECTION 1 - Multiple-Choice Questions

Question	1	2	3	4	5	6	7	8	9	10	11
Answer	B	C	E	D	D	B	E	A	C	B	A

SECTION 2 - Extended-Response Questions

Question 1

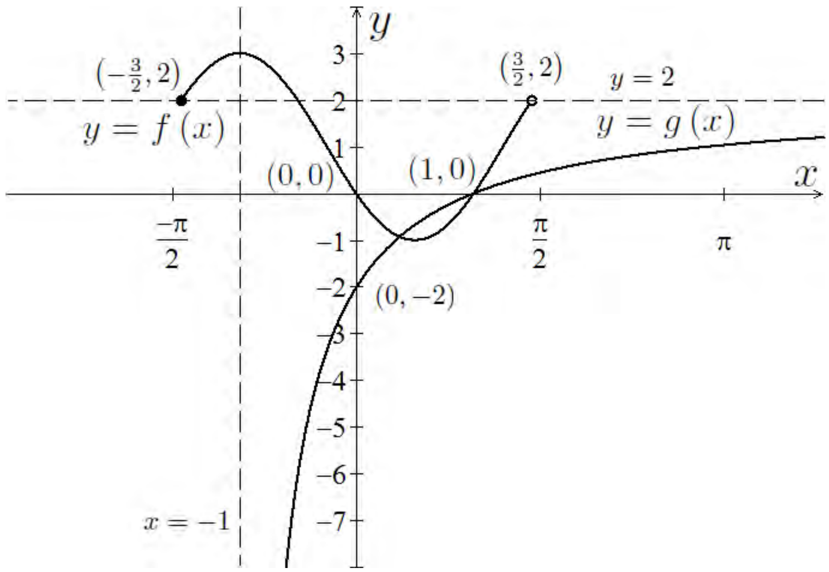
a. **i.**



The period is 3, the amplitude is 2.

- 1 answer mark for the correct shape of a sinusoidal curve, including appropriate application of the transformations
- 1 answer mark awarded for providing the correct period and amplitude
- 1 answer mark awarded for the correct intercepts
- 1 answer mark awarded for the correct endpoints (**including** the correct usage of open and closed circles), and therefore also for sketching the graph over the correct domain

ii.



- 1 answer mark for the correct shape of a hyperbola (**only** the relevant half!)
- 1 answer mark for the correct asymptotes **stated as equations**
- 1 answer mark for the correct intercepts

iii. Using a CAS calculator:
$(0.368, -0.924)$ and $(1, 0)$

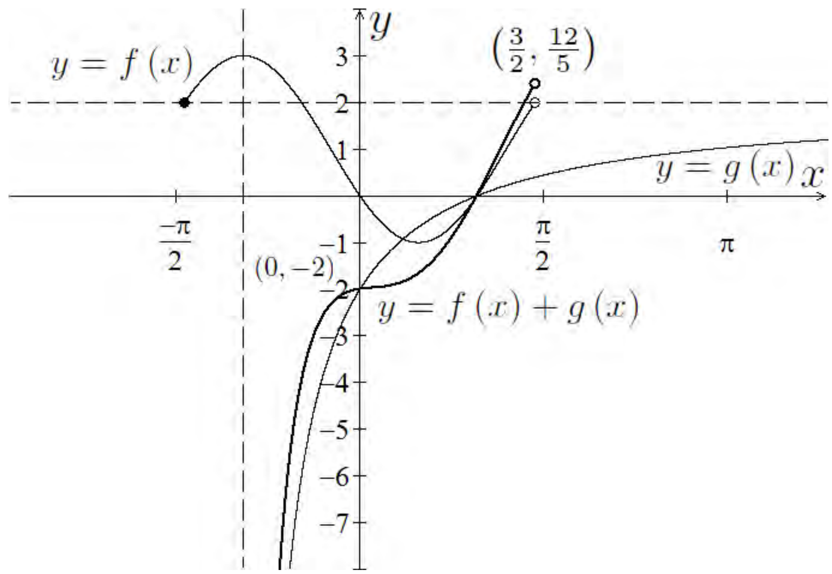
- 1 answer mark awarded for the correct points of intersection

iv.

$$\begin{aligned} D_{f+g} &= D_f \cap D_g \\ &= \left[\frac{-3}{2}, \frac{3}{2}\right) \cap (-1, \infty) \\ &= \left(-1, \frac{3}{2}\right) \end{aligned}$$

- 1 answer mark awarded for stating the correct implied domain

v.



- 1 answer mark for a shape that is roughly correct according to addition of ordinates
- 1 answer mark for the graph tending towards the asymptote $x = -1$ and for the correct endpoint **with an open circle** (i.e. the graph must be sketched over the correct domain)

b. **i.** Let $y = f(x)$ (the original equation), and let $y' = h(x')$ (the image)
Now, rewrite $y = f(x)$:

$$\begin{aligned} y &= -2\cos\left(\frac{\pi}{3}(2x-1)\right)+1 \\ y-1 &= -2\cos\left(\frac{\pi}{3}(2x-1)\right) \\ \frac{y-1}{-2} &= \cos\left(\frac{\pi}{3}(2x-1)\right) \\ \frac{y-1}{-2} &= \cos\left(\frac{2\pi}{3}x-\frac{\pi}{3}\right) \end{aligned}$$

For the image:

$$\begin{aligned} y' &= h(x') \\ \therefore y' &= \cos(x') \end{aligned}$$

Hence, we can state that:

$$y' = \frac{y-1}{-2}$$

And that

$$x' = \frac{2\pi}{3}x - \frac{\pi}{3}$$

Thus, the transformation $(x, y) \to \left(\frac{2\pi}{3}x - \frac{\pi}{3}, \frac{y-1}{-2}\right)$ will obtain the image graph.

To account for the new domain:
The endpoint $\left(-\frac{3}{2}, 2\right)$ will become $\left(-\frac{4\pi}{3}, -\frac{1}{2}\right)$. But we want the graph to begin at $\left(\frac{2\pi}{3}, -\frac{1}{2}\right)$. So shift everything 2π units to the right.
Hence the transformation $(x, y) \to \left(\frac{2\pi}{3}x + \frac{5\pi}{3}, \frac{y-1}{-2}\right)$ will obtain the image graph.
This transformation can thus be described as (in the following order)

1. A dilation from the y-axis by a factor of $\frac{2\pi}{3}$
2. A translation of $\frac{5\pi}{3}$ units to the right (i.e. in the positive direction of the x-axis)
3. A translation of 1 unit downwards (i.e. in the negative direction of the y-axis)
4. A dilation by factor of $\frac{1}{2}$ from the x-axis
5. A reflection in the x-axis

Note that there are other possible sequences and different orders that will also obtain the correct image graph with the correct domain, and would also be awarded full marks.

- 1 method mark is awarded for evidence of using a suitable method (such as using x' and y') to determine the transformations
- 1 answer mark is awarded for correctly obtaining x' and y' in terms of x and y (or an equivalent statement using another method)
- 1 method mark is awarded for consideration of the new domain in determining translations
- 1 answer mark is awarded for a correct sequence of transformations (consequential marks from earlier mistakes apply here)

ii. Translations are applied first according to the rule, so we must rearrange our values of x' and y':

$$\begin{aligned} (x, y) &\to \left(\frac{2\pi}{3}x + \frac{5\pi}{3}, \frac{y-1}{-2}\right) \\ \therefore (x, y) &\to \left(\frac{2\pi}{3}(x + \frac{5}{2}), -\frac{1}{2}(y-1)\right) \end{aligned}$$

Hence $\mathbf{B} = \begin{bmatrix} \frac{5}{2} \\ -1 \end{bmatrix}$ and $\mathbf{T} = \begin{bmatrix} \frac{2\pi}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$.

- 1 answer mark for the correct values of $\mathbf{B}$ and $\mathbf{T}$ (consequential marks apply)

Question 2

a. **i**

$$[f(t)]^2 = f(2t) + 2$$

Sub in $f(t) = e^t + e^{-t}$ into the identity:

$$\begin{aligned} LHS &= [f(t)]^2 \\ &= \left(e^t + e^{-t}\right)^2 \\ &= \left(e^t\right)^2 + 2(e^t)(e^{-t}) + (e^{-t})^2 \\ &= e^{2t} + 2e^0 + e^{-2t} \\ &= \left(e^{2t} + e^{-2t}\right) + 2 \\ &= f(2t) + 2 \\ &= RHS \end{aligned}$$

As required.

- 1 answer mark for showing that the identity is obeyed, in particular including the correct expansion of $\left(e^t + e^{-t}\right)^2$ (correct convention **must** be used for a "show" question)

ii.

The asymptote is given by $y = -1$, $\therefore n = -1$
Subbing in the two points:
$(16, 0) \Rightarrow 0 = ke^{16m} - 1$
$\therefore 1 = ke^{16m}$——-(1)
$(10, e^6 - 1) \Rightarrow e^6 - 1 = ke^{10m} - 1$
$\therefore e^6 = ke^{10m}$ ——-(2)
(2)÷(1):
$e^6 = \dfrac{ke^{10m}}{ke^{16m}}$
$\therefore e^{-6m} = e^6$
$\therefore m = -1$
$\Rightarrow ke^{-16} = 1$
$\therefore k = e^{16}$

- 1 answer mark for deducing $n = -1$ due to the asymptote
- 1 method mark for establishing a system of simultaneous equations using the two points and attempting to solve for k and m
- 1 answer mark awarded for the values of **both** k and m

b.

i.

The function $y = h(t)$ must be continuous - hence, to find a, let:

$$\begin{aligned} t\sin(t) + 5 &= f(t-11) \\ \therefore t\sin(t) + 5 &= e^{t-11} + e^{-(t-11)} \\ \therefore t &= 8.5671, 13.9198 \end{aligned}$$

As $a < 10$, $a = 8.57$ (to two decimal places)
To find b, let

$$\begin{aligned} f(t-11) &= \mid g(t) \mid - \frac{1}{2} \\ \therefore e^{t-11} + e^{-(t-11)} &= \mid ke^{mt} + n \mid - \frac{1}{2} \\ &= \mid e^{16} \times e^{-t} - 1 \mid - \frac{1}{2} \\ &= \mid e^{16-t} - 1 \mid - \frac{1}{2} \\ \therefore t &= 13.4349 \end{aligned}$$

Hence $b = 13.43$ (to two decimal places).

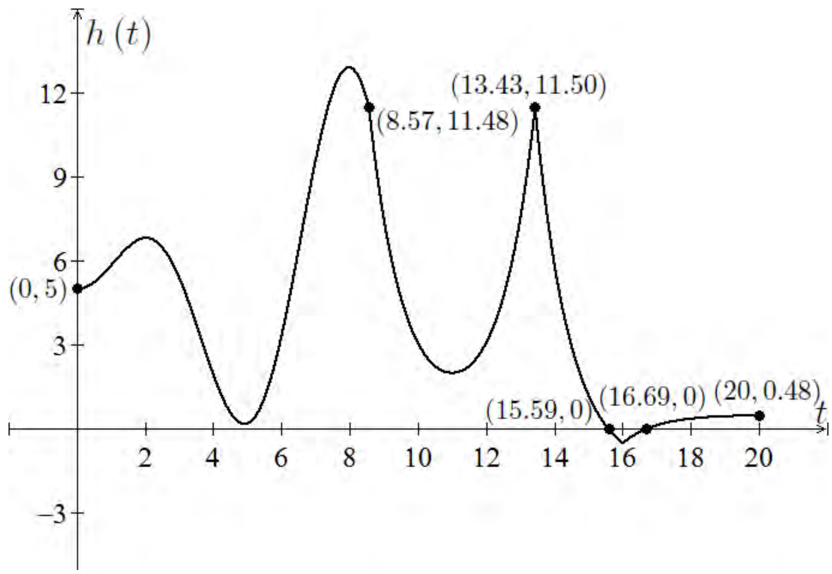
- 1 method mark for writing down two equations that will yield the values of a and b (no attempt to solve the equations is required)
- 1 answer mark for writing down the correct values of **both** a and b, to two decimal places

ii.

$$h(t) = \begin{cases} t\sin(t) + 5, & 0 \leq t \leq 8.57 \\ e^{t-11} + e^{11-t}, & 8.57 < t < 13.43 \\ \mid e^{16-t} - 1 \mid - \frac{1}{2}, & 13.43 \leq t \leq 20 \end{cases}$$

- 1 answer mark awarded for writing the correct and full hybrid equation of $y = h(t)$

c.



- 1 answer mark awarded for the correct shape of all the parts of the hybrid
- 1 answer mark awarded for the function being continuous, but with distinct junctions at the points $t = a$ and $t = b$ (which must be labelled with coordinates to two decimal places)
- 1 answer mark for the correct coordinates of the endpoints and intercepts, to the correct number of decimal places

3 marks

d.

The only t-intercepts occur during the interval $13.43 \leq t \leq 20$

So let:

$$\begin{aligned}
|\, e^{16-t} - 1 \,| - \frac{1}{2} &= 0 \\
|\, e^{16-t} - 1 \,| &= \frac{1}{2} \\
\therefore e^{16-t} - 1 &= \pm\frac{1}{2} \\
e^{16-t} &= 1 \pm \frac{1}{2} \\
\Rightarrow e^{16-t} &= \frac{1}{2}, \frac{3}{2} \\
\therefore 16 - t &= \log_e\left(\frac{1}{2}\right), \log_e\left(\frac{3}{2}\right) \\
\therefore t &= 16 - \log_e\left(\frac{1}{2}\right), 16 - \log_e\left(\frac{3}{2}\right)
\end{aligned}$$

Corey will therefore be underwater for

$$16 - \log_e\left(\frac{1}{2}\right) - \left[16 - \log_e\left(\frac{3}{2}\right)\right]$$
$$= 16 - \log_e\left(\frac{1}{2}\right) - 16 + \log_e\left(\frac{3}{2}\right)$$
$$= \log_e\left(\frac{3}{2}\right) - \log_e\left(\frac{1}{2}\right)$$
$$= \log_e\left(\frac{3}{2} \times 2\right)$$
$$= \log_e(3) \text{ seconds}$$

- 1 method mark for setting up the original equation to find the **exact** value of the t-intercepts
- 1 method mark for attempting to solve the equation to obtain the correct values for the t-intercepts
- 1 answer mark awarded for the correct answer including units

FUNCTIONS AND GRAPHS

TECH-ACTIVE TEST 2

MODEL SOLUTIONS AND MARKING SCHEME

SECTION 1 - Multiple-Choice Questions

Question	1	2	3	4	5	6	7	8	9	10	11
Answer	D	C	B	C	E	A	A	E	B	C	C

SECTION 2 - Extended-Response Questions

Question 1

a. **i.**

Let $u(x) = ax^4 + bx^3 + cx^2 + dx + f$ (using any other variables would be acceptable as well). Subbing in the points given:

$$\begin{aligned} 8 &= a + b + c + d + f \\ 2 &= 16a - 8b + 4c - 2d + f \\ 18 &= f \\ 72 &= 81a + 27b + 9c + 3d + f \\ 338 &= 256 - 64b + 16c - 4d + f \end{aligned}$$

Hence in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 16 & -8 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 256 & -64 & 16 & -4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ f \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 18 \\ 72 \\ 338 \end{bmatrix}$$

- 1 answer mark for the matrix equation (the middle matrix can have any pro numerals as long as they are all different and none are x).

ii.

$$u(x) = 2x^4 - 12x^2 + 18$$

NB: other forms of the same equation would be acceptable too.

- 1 answer mark for the correct equations (consequential marks from **part i.** apply).

iii.

$$
\begin{aligned}
y &= f \circ g(x) \\
&= f(g(x)) \\
&= 2\left[g(x)\right]^2
\end{aligned}
$$

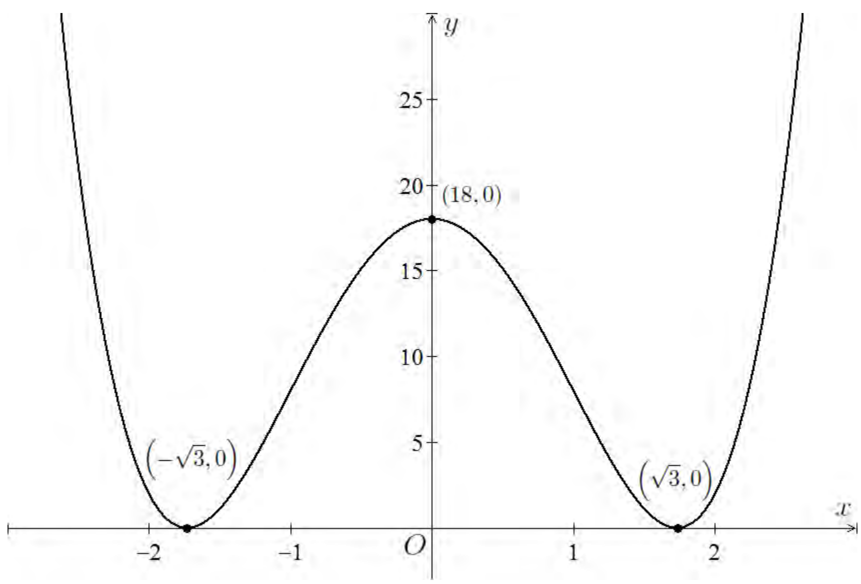
Now,

$$
\begin{aligned}
u(x) &= 2x^4 - 12x^2 + 18 \\
&= 2\left(x^4 - 6x^2 + 9\right) \\
&= 2(x^2 - 3)^2
\end{aligned}
$$

Hence, $g(x) = x^2 - 3$

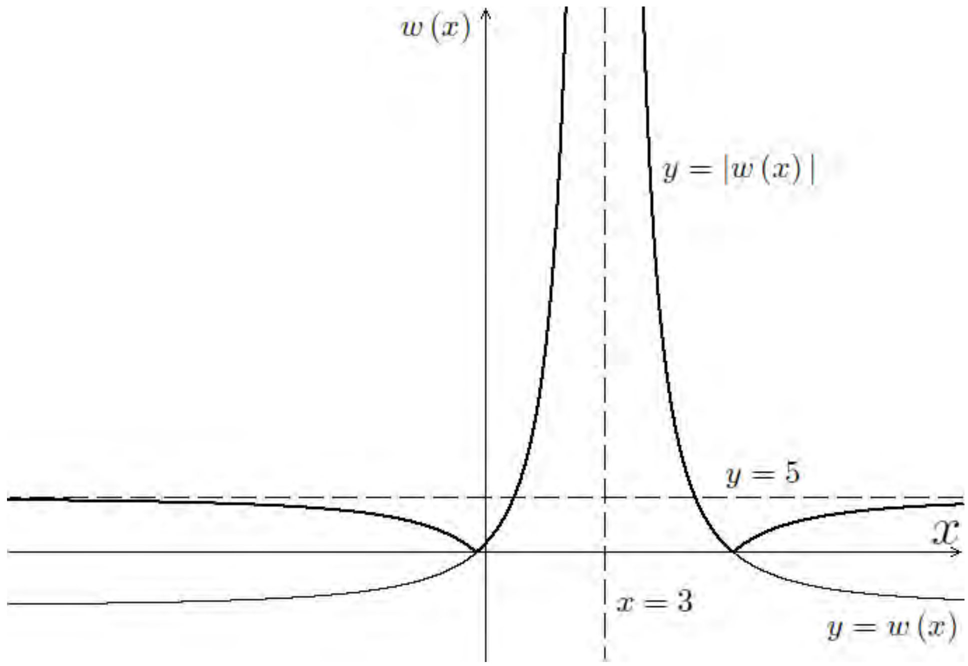
- 1 method mark awarded for the correct factorisation of $u(x)$.
- 1 answer mark for the recognition that $g(x) = x^2 - 3$.

iv.



- 1 mark for the correct shape of the graph (a positive **and symmetrical** quartic with three turning points).
- 1 mark for the correct intercepts - the intercepts must also be turning points to receive this mark.

b. **i.**



- 1 mark for the correct shape of the graph - the parts of the graph that are the same as $y = w(x)$ must clearly be drawn over.
- 1 mark for including the correct asymptotes ($y = 5$ and $x = 3$).

ii.

From the graph: $b = -3$ and $c = -5$.

- 1 answer mark for the correct values of b and c.

iii.

$$\begin{aligned}(p, 8) \implies 8 &= \frac{a}{(p-3)^2} - 5 \\ \implies \frac{a}{(p-3)^2} &= 13 \\ \implies a &= 13(p-3)^2\end{aligned}$$

- 1 answer mark for a correct expression of a in terms of p.

iv.

Subbing $(p, 8)$ into $u(x)$,

$$\begin{aligned}(p,8) \implies 8 &= 2(p^2-3)^2 \\ \implies (p^2-3)^2 &= 4 \\ \implies p^2-3 &= \pm 2 \\ \implies p^2 &= 3 \pm 2\end{aligned}$$

So $p^2 = 5$ or $p^2 = 1$.
Thus $p = \pm\sqrt{5}$ or $p = \pm 1$.
But $p < 2$, so eliminate $p = \sqrt{5}$.
Also, p is to the right of the y-axis, hence $p > 0$ - i.e. $p \neq -1$ and $p \neq -\sqrt{5}$. Hence $p = 1$.
Thus

$$\begin{aligned}a &= 13(1-3)^2 \\ &= 13(-2)^2 \\ &= 13 \times 4 \\ &= 52\end{aligned}$$

- 1 method mark for attempting to find p by either subbing $(p, 8)$ into $u(x)$ or by equating $u(x)$ and $w(x)$.
- 1 answer mark for finding the value of p; values of p discarded in the process **must** be accompanied by a brief explanation as to why they are not valid.
- 1 answer mark awarded for stating that $a = 52$.

Question 2

a. **i.**

$$\begin{aligned} m &= \frac{R_2 - R_1}{w_2 - w_1} \\ &= \frac{67-5}{12-1} \\ &= \frac{62}{11} \end{aligned}$$

$$\begin{aligned} R - R_1 &= m(w - w_1) \\ R - 5 &= \frac{62}{11}(w-1) \\ \therefore R(w) &= \frac{62}{11}w - \frac{7}{11} \end{aligned}$$

- 1 answer mark awarded for the correct rule for $R(w)$ with **exact** values.

ii.

$$R(w) = 5.14w + 7.82$$

- 1 answer mark awarded for the correct rule for $R(w)$, with coefficients correct to **two** decimal places.

b.

$$R(w) = 0.077x^3 - 1.770x^2 + 16.618x - 9.553$$

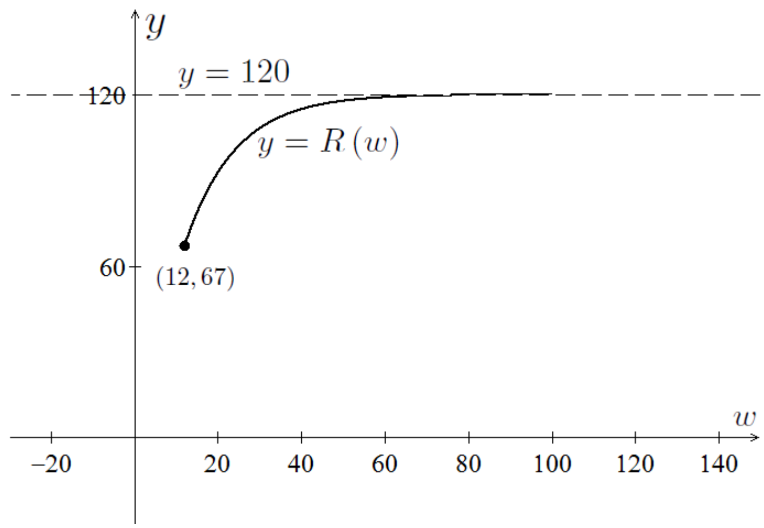
- 1 answer mark awarded for the correct rule for $R(w)$, with coefficients correct to **three** decimal places.

c. **i.**

The domain of $R(w)$ is $[12, \infty)$. The range of $R(w)$ is $[67, 120)$.

- 1 answer mark awarded for the correct domain **and** range.

ii.



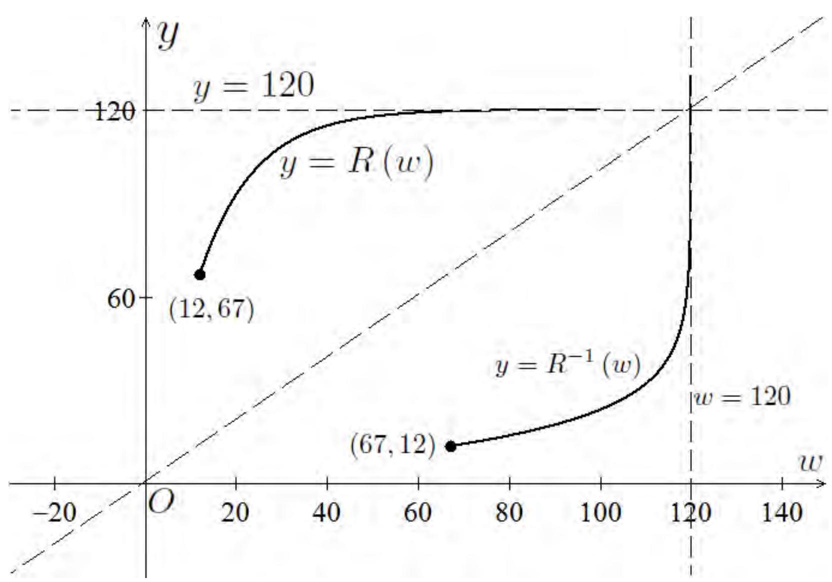
- 1 mark for the correct shape of an exponential graph (which has been reflected in both axes).
- 1 mark for the correct asymptote (with equation: $y = 120$) and for the correct endpoint $(12, 67)$. Do not award this mark if the graph goes past the endpoint and/or if the graph touches the asymptote. Consequential marks from **part i.** apply.

iii.

120 rooms

- 1 answer mark for correctly stating the number of rooms.

iv.



- 1 mark for the correct shape (i.e. a reflection of the original graph across the line $y = x$).
- 1 answer mark for the correct new asymptote and endpoint ($x = 120$ and $(67, 12)$).

v.

$$(119.993, 119.993)$$

- 1 answer mark for giving the point of intersection using **coordinates**.

d. i.

$(w, R) \to (w-15, R) \to (w-15, R-7) \to \left(w-15, \frac{1}{2}(R-7)\right) \to \left(3(w-15), \frac{1}{2}(R-7)\right) \to \left(3(15-w), \frac{1}{2}(R-7)\right) = (w', R')$

$$\begin{aligned} \implies w' &= 3(15-w) \\ \implies 15-w &= \frac{w'}{3} \\ \implies w &= 15 - \frac{w'}{3} \end{aligned}$$

And

$$\begin{aligned} \implies R' &= \frac{1}{2}(R-7) \\ \implies R-7 &= 2R' \\ \implies R &= 2R'+7 \end{aligned}$$

Hence,

$$\begin{aligned} 2R'+7 &= -53e^{\left(1-\frac{1}{12}\left(15-\frac{w'}{3}\right)\right)} + 120 \\ 2R' &= -53e^{\left(1-\frac{5}{4}+\frac{w'}{36}\right)} + 113 \\ R' &= -\frac{53}{2}e^{\left(\frac{w'}{36}-\frac{1}{4}\right)} + \frac{113}{2} \end{aligned}$$

Hence the new rule will be $R(w) = -\frac{53}{2}e^{\left(\frac{w}{36}-\frac{1}{4}\right)} + \frac{113}{2}$.

- 1 method mark is awarded for evidence of correctly using mapping, or another method for interpreting transformations.
- 1 method mark for correctly obtaining R and w in terms of R' and w'; if a different method is used, this mark is to be awarded for correctly applying each transformation in the correct order.
- 1 answer mark for the correct new rule for $R(w)$.

ii.

To reverse Ben's transformations, we would need the following matrix transformation (i.e. apply exactly the opposite of each transformation in the reverse order):

$$
\begin{aligned}
T\left(\begin{bmatrix} w \\ y \end{bmatrix}\right) &= \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 15 \\ 7 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 15 \\ 7 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 15 \\ 7 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 15 \\ 7 \end{bmatrix}\right) \\
&= \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -45 \\ \frac{7}{2} \end{bmatrix}\right)
\end{aligned}
$$

Hence $\mathbf{T} = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -45 \\ \frac{7}{2} \end{bmatrix}$.

- 1 method mark for using any suitable method to find these matrices (e.g. using the reverse of Ben's transformations, as above, or by using mapping from the second equation to the original).
- 1 answer mark for the correct $\mathbf{T}$ matrix.
- 1 answer mark for the correct $\mathbf{B}$ matrix.

ALGEBRA

TECH-FREE TEST 1

MODEL SOLUTIONS AND MARKING SCHEME

Question 1

a.

Gradient of first line is $-\frac{2m}{3}$
Gradient of second line is $-\frac{1}{m}$
For lines to have the same gradient,

$$\begin{aligned} -\frac{2m}{3} &= -\frac{1}{m} \\ m^2 &= \frac{3}{2} \\ m &= \pm\sqrt{\frac{3}{2}} \end{aligned}$$

- 1 method mark for correctly writing the gradients of both lines
- 1 answer mark for correctly writing down both solutions for m

b. i.

There are no solutions when the lines are parallel (but are not the same line).

Since the y-intercept of the first line is $\frac{8}{3}$ and for the second line is 1 for any value of m, the y-intercepts cannot be the same.

Hence, there is no solution for $m \in \left\{-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right\}$

b. ii.

There is a unique solution when the gradients of two lines are different.

Hence, there is a unique solution for $m \in \mathbb{R} \setminus \left\{-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right\}$

- 1 method mark for recognising that there is no solution when the lines are parallel and unique
- 1 method mark for recognising that there is a unique solution when the lines are not parallel
- 1 answer mark for correctly finding the values of m for which there is no solution
- 1 answer mark for correctly finding the values of m for which there is a unique solution

Question 2

$$\begin{aligned} 4\sin(x) + (2-2\sqrt{3})\sin(x) &= \sqrt{3} \\ 4\sin^2(x) + (2-2\sqrt{3})\sin(x) - \sqrt{3} &= 0 \end{aligned}$$

Let $\sin(x) = a$

$$\begin{aligned} 4a^2 + (2-2\sqrt{3})a - \sqrt{3} &= 0 \\ (2a+1)(2a-\sqrt{3}) &= 0 \\ 2a+1=0 \quad &or \quad 2a-\sqrt{3}=0 \\ a = -\frac{1}{2} \quad &or \quad a = \frac{\sqrt{3}}{2} \\ \sin(x) = -\frac{1}{2} \quad &or \quad \sin(x) = \frac{\sqrt{3}}{2} \end{aligned}$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

- 1 method mark for recognising that the equation is in a quadratic form
- 1 method mark for correctly factorising/completing the square/using the quadratic formula
- 1 answer mark for correctly calculating $\sin^{-1}(-\frac{1}{2})$, and $\sin^{-1}(\frac{\sqrt{3}}{2})$
- 1 answer mark for correctly writing the solutions for x in the appropriate domain

Question 3

$$
\begin{aligned}
RHS &= \frac{1}{g(x)} - g(x) \\
&= \frac{1}{\cos(x)} - \cos(x) \\
LHS &= f(x)g(x) \\
&= \tan^2(x)\cos(x) \\
&= \frac{\sin^2(x)}{\cos^2(x)}\cos(x) \\
&= \frac{\sin^2(x)}{\cos(x)} \\
&= \frac{1-\cos^2(x)}{\cos(x)} \\
&= \frac{1}{\cos(x)} - \frac{\cos^2(x)}{\cos(x)} \\
&= \frac{1}{\cos(x)} - \cos(x) \\
&= RHS
\end{aligned}
$$

This is defined for $\cos(x) \neq 0$

$$
\begin{aligned}
\cos(x) &\neq 0 \\
x &\neq 2k\pi \pm \frac{\pi}{2},\ k \in Z \\
x &\neq \frac{\pi}{2} + k\pi,\ k \in Z
\end{aligned}
$$

As such $f(x)g(x) = \frac{1}{g(x)} - g(x)$ for $x \in R \setminus \left\{\frac{\pi}{2} + k\pi,\ k \in Z\right\}$

- 1 method mark for the use of an identity (e.g. Pythagorean identity)
- 1 method mark for restricting the values of x based on $\cos(x) \neq 0$
- 1 method mark for solving $\cos(x) = 0$ to obtain reference angle of 0
- 1 answer mark for correctly stating the full general solution

Question 4

$$
\begin{aligned}
|\log_e(2x+1)| &= 9
\end{aligned}
$$

$$
\begin{cases}\log_e(2x+1) = 9 \\ \log_e(2x+1) = -9\end{cases}
$$

$$
\begin{cases}2x+1 = e^9, \\ 2x+1 = e^{-9},\end{cases}
$$

$$
\begin{cases}x = \frac{e^9-1}{2}, \\ x = \frac{e^{-9}-1}{2},\end{cases}
$$

- 1 method mark for expanding the absolute value correctly
- 1 answer mark for correctly writing the solutions for x

Question 5

a.

$$
\begin{aligned}
g(f(x)) &= 3(e^{x^2})^2 + 1 \\
&= 3e^{2x^2} + 1
\end{aligned}
$$

- 1 answer mark for correctly writing the rule

b.

Let $y = 3e^{2x^2} + 1$. To find the inverse, we reflect in the $y = x$ line.
Therefore,

$$
\begin{aligned}
x &= 3e^{2y^2} + 1 \\
e^{2y^2} &= \frac{x-1}{3} \\
y^2 &= \frac{\log_e(\frac{x-1}{3})}{2} \\
y &= \pm\sqrt{\frac{\log_e(\frac{x-1}{3})}{2}}
\end{aligned}
$$

Thus, $g(f(x))^{-1} = \pm\sqrt{\frac{\log_e(\frac{x-1}{3})}{2}}$

- 1 method mark for reflecting in the $y = x$ line
- 1 answer mark for correct expression for inverse

c.

Domain of inverse: $[4, \infty)$

- 1 answer mark for correct domain

ALGEBRA

TECH-FREE TEST 2

MODEL SOLUTIONS AND MARKING SCHEME

Question 1

a.

$$
\begin{aligned}
f(x) &= \frac{x-3}{x+1} \\
&= \frac{(x+1)-4}{x+1} \\
&= 1-\frac{4}{x+1} \\
f(f^{-1}(x)) &= x \\
\implies 1-\frac{4}{f^{-1}(x)+1} &= x \\
\therefore f^{-1}(x) &= -\frac{4}{x-1}-1 \\
&= \frac{4}{1-x}-1
\end{aligned}
$$

- 1 method mark for converting the fraction into a proper fraction (either initially or after introducing the inverse)
- 1 method mark for reflecting in the $y = x$ line and/or stating $f(f^{-1}(x)) = x$ to find the inverse
- 1 answer mark for the correct inverse function

b.

The domain of the inverse function is the range of the original function, $x \in \mathbb{R} \setminus \{1\}$

- 1 answer mark for the correct domain

Question 2

$$\begin{aligned}
\log_3(x) &= 4\log_x(3) \\
\log_3(x) &= 4\frac{\log_3(3)}{\log_3(x)} \\
(\log_3(x))^2 &= 4 \\
\log_3(x) = 2 \quad &\textit{or} \quad \log_3(x) = -2 \\
x = 3^2 \quad &\textit{or} \quad x = 3^{-2} \\
x = 9 \quad &\textit{or} \quad x = \frac{1}{9}
\end{aligned}$$

x-values are limited to all real numbers larger than 0. Hence, both solutions are correct.

- 1 method mark for a correct change of base
- 1 answer mark for $x = 9$
- 1 answer mark for $x = \frac{1}{9}$
- 1 answer mark for verification of solutions

Question 3

$f(x)$ and $g(x)$ intersect when,

$$\begin{aligned}
f(x) &= g(x) \\
\frac{1}{x} - 3 &= -ax \\
ax + \frac{1}{x} - 3 &= 0 \\
ax^2 - 3x + 1 &= 0 \\
\triangle &= (-3)^2 - 4(a)(1) \\
&= 9 - 4a
\end{aligned}$$

There is no solution when $\triangle < 0$

$$\begin{aligned}
\triangle &< 0 \\
9 - 4a &< 0 \\
a &> \frac{9}{4}
\end{aligned}$$

- 1 method mark for setting up the correct equations to reach a quadratic
- 1 method mark for finding the discriminant
- 1 method mark for understanding that there is no solution when$\triangle < 0$
- 1 answer mark for the correct values for a

Question 4

$$\begin{aligned}(a+3b)^2-12ab &= a^2+6ab+9b^2-12ab\\ &= a^2-6ab+9b^2\\ &= (a-3b)^2\end{aligned}$$

If we consider the function $y=x^2$, we know that for any x, $y \geq 0$
Hence, $(a-3b)^2 \geq 0$
Therefore, $(a+3b)^2-12ab \geq 0$ as required.

- 1 method mark for expanding perfect square, $(a+3b)^2$ and factorising $a^2-6ab+9b^2$
- 1 method mark for justifying that $(a-3b)^2 \geq 0$
- 1 answer mark for a logical solution with the required result

Question 5

$$\begin{aligned}|x-1|^2-2|x-3| &\geq 3\\ (x-1)^2-2|x-3| &\geq 3\end{aligned}$$

For $x-3 \geq 0$,

$$\begin{aligned}(x-1)^2-2(x-3) &\geq 3\\ x^2-2x+1-2x+6-3 &\geq 0\\ x^2-4x+4 &\geq 0\\ (x-2)^2 &\geq 0\\ x &\in \mathbb{R}\end{aligned}$$

The intersection of $x \in \mathbb{R}$ and $x \geq 3$ is $x \geq 3$

For $x-3<0$,

$$\begin{aligned}(x-1)^2-2(-(x-3)) &\geq 3\\ x^2-2x+1+2x-6-3 &\geq 0\\ x^2-8 &\geq 0\\ x^2 &\geq 8\\ x \geq 2\sqrt{2} \quad &\text{and} \quad x \leq -2\sqrt{2}\end{aligned}$$

The intersection of this result and $x<3$ is $x \leq -2\sqrt{2}$ or $2\sqrt{2} \leq x < 3$

Hence, the values of x that satisfy the equation are $x \geq 2\sqrt{2}$ and $x \leq -2\sqrt{2}$.

- 1 method mark for showing that $|x-1|^2=(x-1)^2$
- 1 method mark for appropriately splitting up $|x-3|$ with the correct signs
- 1 method mark for restricting x according to the intersections of different sets
- 1 answer mark for $x \geq 2\sqrt{2}$
- 1 answer mark for $x \leq -2\sqrt{2}$

ALGEBRA

TECH-ACTIVE TEST 1

MODEL SOLUTIONS AND MARKING SCHEME

SECTION 1 - Multiple-Choice Questions

Question	1	2	3	4	5	6	7	8	9	10	11
Answer	E	E	D	E	A	D	B	C	C	D	E

SECTION 2 - Extended-Response Questions

Question 1

Part a.

- 1 mark is awarded for stating that the minimum number of stationary points is 1.
- 1 mark is awarded for explaining that the derivative function will always have at least one x-intercept **and** that it is possible to have only one x-intercept by increasing or decreasing p to extreme values so that the derivative does not have more than one x-intercept.

Part b.

$$f'(x) = 4x^3 - 9x^2 - 12x + p$$

Let $a(x) = f'(x)$. Then

$$a(x) = 4x^3 - 9x^2 - 12x + p$$

To find the values of p such that $f(x)$ has only one stationary point (i.e. $a(x) = f'(x) = 0$ for exactly one value of x), we need to find the values of p such that there is only one x-intercept. Let us figure out the x-coordinates of the stationary points of $y = a(x)$.

$$\begin{aligned} a'(x) &= 12x^2 - 18x - 12 = 0 \\ x &= -\frac{1}{2}, 2 \end{aligned}$$

Now, considering the shape of the graph $y = g(x)$, this will only have one x-intercept when:

- Case 1: both of the stationary points are above the x-axis
- Case 2: both of the stationary points are below the x-axis

Case 1: This means $a\left(-\frac{1}{2}\right) > 0$ and $a(2) > 0$ - in other words, $a(2) > 0$.

$$\begin{aligned} a(2) &> 0 \\ \implies 32 - 36 - 24 + p &> 0 \\ \implies p - 28 &> 0 \\ \implies p &> 28 \end{aligned}$$

Case 2: This means $a\left(-\frac{1}{2}\right) < 0$ and $a(2) < 0$ - in other words, $a\left(-\frac{1}{2}\right) < 0$.

$$\begin{aligned} a\left(-\frac{1}{2}\right) &< 0 \\ \implies -\frac{1}{2} - \frac{9}{4} + 6 + p &< 0 \\ \implies \frac{13}{4} + p &< 0 \\ \implies p &< -\frac{13}{4} \end{aligned}$$

Hence, we can say that $p \in \left(-\infty, -\frac{13}{4}\right) \cup (28, \infty)$. Also note that the number of stationary points is independent of q. Hence, (p, q) can be any ordered pair such that $p \in \left(-\infty, -\frac{13}{4}\right) \cup (28, \infty)$ and $q \in \mathbb{R}$.

- 1 method mark for determining $f'(x)$
- 1 method mark for determining the x coordinates of the stationary points of $f'(x)$
- 1 method mark for determining that $f'(2) > 0$ and $g\left(-\frac{1}{2}\right) < 0$
- 1 answer mark is awarded for stating that $p \in \left(-\infty, -\frac{13}{4}\right) \cup (28, \infty)$ **and** $q \in \mathbb{R}$

Part c.

From the shape of the graph itself, it is evident that a is negative. Now,

$$g'(x) = 3ax^2 + 12x - c$$

We can see that there are no stationary points, hence $g'(x)$ must have no real solutions.

$$\begin{aligned} \Delta &= 144 + 12ac < 0 \\ ac &< -12 \\ c &> -\frac{12}{a} \text{ (given a < 0)} \end{aligned}$$

Now, since $a < 0$ and $c > -\frac{12}{a}$, $c \in \mathbb{R}^+$. Given these deductions are independent of the value of d, $d \in \mathbb{R}$.

- 1 method mark is awarded for determining the discriminant of $g'(x)$
- 1 method mark is awarded for setting the discriminant being less than 0
- 1 answer mark is awarded for stating $a < 0$, $c > 0$ and $d \in \mathbb{R}$
- 1 answer mark is awarded for stating $c > -\frac{12}{a}$

Question 2
Part a.

$$
\begin{aligned}
p(x) &= 5^x \\
&= (6^{\log_6(5)})^x \\
&= 6^{(\log_6(5))x} \\
&= q((\log_6(5))x)
\end{aligned}
$$

- 1 method mark for converting from base 5 to base 6
- 1 method mark for using the index law $(a^b)^c = a^{bc}$

Part b.

$$
\begin{aligned}
g(x) &= 2e^{x+3} - 5 \\
&= 2 \times (2^{\log_2(e)})^{x+3} - 5 \\
&= 2 \times 2^{(\log_2(e))(x+3)} - 5
\end{aligned}
$$

We can derive a transformation from here:

$$2^{x+4} \to \quad 2^x \to \quad 2^{(\log_e(e))x} \to 2^{(\log_2(e))(x+3)} \to 2 \times 2^{(\log_2(e))(x+3)} \to 2 \times 2^{(\log_2(e))(x+3)} - 5$$

This corresponds to the following series of transformations:

- translation 4 units in the positive direction of the x-axis
- dilation by factor $\dfrac{1}{\log_2(e)}$ from the y-axis
- translation 3 units in the negative direction of the x-axis
- dilation by factor 2 from the x-axis
- translation 5 units in the negative direction of the x-axis

- 1 method mark is awarded for algebraically manipulating the two expressions by applying a series of transformations to find a pathway between them
- 1 method mark for converting from base e to base 2
- 2 answer marks for sequence of transformations in order (award 1 mark if it appears the student has managed to go halfway through the transformations or if the transformations were mostly correct with only one mistake)

Part c.

$$
\begin{aligned}
s(x) &= 2e^{x-3} \\
&= e^{\log_e(2)} \times e^{x-3} \\
&= e^{(x-3+\log_e(2))} \\
&= r\left(x - 7 + \log_e(2)\right)
\end{aligned}
$$

The translation is $7 - \log_e(2)$ units in the positive direction of the x-axis.

- 1 method mark for using an index law to convert 2 to base e
- 1 answer mark for the transformation

Part d.

$$\begin{aligned} b(x) &= \log_e\left(\frac{x^6}{3}\right) \\ &= 6\log_e(x) - \log_e(3) \\ &= 6a(x) - \log_e(3) \end{aligned}$$

The sequence of transformations is:

- dilation by factor 6 from x-axis
- translation $\log_e(3)$ units in the negative direction of the y-axis

- 1 method mark for using logarithm laws to manipulate $b(x)$
- 1 answer mark for both transformations

Part e.
The mark is awarded for the following answer:

$$S = (0, \infty)$$

Question 3
Part a.

$$\begin{aligned} |\sin(\log_e(2x+5))| &= \frac{\sqrt{3}}{2} \\ \sin(\log_e(2x+5)) &= \pm\frac{\sqrt{3}}{2} \\ \log_e(2x+5) &= n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z} \\ 2x+5 &= e^{n\pi \pm \frac{\pi}{3}}, n \in \mathbb{Z} \\ x &= \frac{1}{2}\left(e^{n\pi \pm \frac{\pi}{3}} - 5\right), n \in \mathbb{Z} \end{aligned}$$

- 1 mark awarded for each of the last four lines in the solution

Part b.

$$\begin{aligned} 2x+5 &> 0 \\ x &> -\frac{5}{2} \\ \text{dom} &= \left(-\frac{5}{2}, \infty\right) \end{aligned}$$

- 1 method mark awarded for determining that $2x+5>0$
- 1 answer mark awarded for determining the domain

Part c.

To the left. The gradient of $y = \log_e(2x + 5)$ decreases as x increases. Consider $u = \log_e(2x + 5)$, so we are considering $y = \sin(u)$. Let $u_1 = \log_e(2x_1 + 5)$ and $u_2 = \log_e(2x_2 + 5)$. Note that between x-intercepts the stationary point of $y = \sin(x)$ is the same as $\left(\frac{x_1 + x_2}{2}, f\left(\frac{x_1 + x_2}{2}\right)\right)$ if the variables were defined the same way. However, for $y = \sin(u)$, since the gradient of $y = \log_e(2x + 5)$ decreases as x increases, we can determine visually that $u = \frac{u_1 + u_2}{2}$ when $x < \frac{x_1 + x_2}{2}$. At $u = \frac{u_1 + u_2}{2}$, there would be a stationary point (given the graph is $y = \sin(u)$). This means that the stationary point would be when $x < \frac{x_1 + x_2}{2}$.

- 1 mark awarded for indicating that the stationary point is to the left
- 1 mark awarded for a valid and rigorous explanation

ALGEBRA

TECH-ACTIVE TEST 2

MODEL SOLUTIONS AND MARKING SCHEME

SECTION 1 - Multiple-Choice Questions

Question	1	2	3	4	5	6	7	8	9	10	11
Answer	E	D	E	C	D	E	B	E	A	C	A

SECTION 2 - Extended-Response Questions

Question 1

Part a. i.

$$\begin{aligned} f(x) &= \frac{1}{3}x^3 - x^2 - 3x \\ &= \frac{1}{3}x(x^2 - 3x - 9) \end{aligned}$$

Calculate the discriminant of the quadratic:

$$\Delta = (-3)^2 - 4(1)(-9) = 45$$

The quadratic will have two solutions, the linear factor will contribute one more solution, thus, three solutions - i.e. three x-intercepts.

- 1 mark for factorising out the linear factor.
- 1 mark for showing that the quadratic factor has two solutions, either through the discriminant or actually solving.

Part a. ii.

$$f'(x) = x^2 - 2x - 3 = 0$$

$$(x+1)(x-3) = 0$$

$$x = -1 \text{ or } x = 3$$

- 1 mark for the derivative.
- 1 mark for each solution.

Part b. i.

There will be one real solution if:

$$c < -\frac{5}{3} \text{ or } c > 9$$

- 1 mark for each of the bounds.

Part b. ii.

There are three distinct real solutions for:

$$-\frac{5}{3} < c < 9$$

- 1 mark for each of the bounds.

Part b. iii.

There are two distinct positive real solutions for:

$$0 < c < 9$$

- 1 mark for each of the bounds.

Part c. i.

$$\begin{aligned} g(x) &= d \\ \frac{1}{3} \times 2^{3x} - 2^{2x} - 3 \times 2^x + d &= d \\ \frac{1}{3} \times 2^{3x} - 2^{2x} - 3 \times 2^x &= 0 \end{aligned}$$

Let $A = 2^x$,

$$\frac{1}{3}A^3 - A^2 - 3A = 0$$

$$\frac{1}{3}A(A^2 - 3A - 9) = 0$$

$$A = 0 \text{ or } A = \frac{3 \pm 3\sqrt{5}}{2}$$

Since we need to take the logarithm of A, only $A = \dfrac{3 + 3\sqrt{5}}{2}$ is a possible solution, and so:

$$x = \log_2\left(\frac{3 + 3\sqrt{5}}{2}\right)$$

- 1 mark for the three solutions for A.
- 1 mark for recognising that $A > 0$.
- 1 mark for the final answer.

Part c. ii.

$$0 < d < 9$$

- 1 mark for each of the bounds.

Question 2

Part a.

$$\begin{aligned} x - y - 2z &= -3......(1) \\ tx + y - z &= 3t........(2) \\ x + 3y + tz &= 13.......(3) \end{aligned}$$

Let $(4) = (1) + (2)$ and $(5) = (3) + 3 \times (1)$:

$$\begin{aligned} (t+1)x - 3z &= 3t - 3......(4) \\ 4x + (t-6)z &= 4.............(5) \end{aligned}$$

Multiplying equation (4) by $\dfrac{t-6}{3}$, we get:

$$\frac{1}{3}(t+1)(t-6)x - (t-6)z = \frac{3(t-1)(t-6)}{3} = (t-1)(t-6)........(6)$$

Adding equations (5) and (6) gives:

$$\begin{aligned} \left(\frac{1}{3}(t+1)(t-6) + 4\right)x &= (t-1)(t-6) + 4 \\ ((t+1)(t-6) + 12)x &= 3(t-1)(t-6) + 12 \\ (t^2 - 5t - 6 + 12)x &= 3(t^2 - 7t + 6) + 12 \\ (t^2 - 5t + 6)x &= 3t^2 - 21t + 18 + 12 \\ &= 3t^2 - 21t + 30 \end{aligned}$$

Hence,

$$x = \frac{3t^2 - 21t + 30}{t^2 - 5t + 6}$$

- 1 mark for correctly eliminating y from the system of equations.
- 1 mark for correctly eliminating z from the system of equations.
- 1 mark for isolating x.

Part b.

$$\begin{aligned} t^2 - 5t + 6 &= 0 \\ (t-3)(t-2) &= 0 \\ t &= 2, 3 \end{aligned}$$

- 1 mark for recognising the quadratic in the denominator is critical to this question.
- 1 mark for equating it to zero and an attempt to solve.
- 1 mark for the correct final answer.

Part c.

If $t = 3$, then equations (5) and (6) will be:

$$\begin{aligned} 4x - 3z &= 4..........(5) \\ -4x + 3z &= -6........(6) \end{aligned}$$

Equation (6), multiplied by -1, yields

$$4x - 3z = 6$$

Clearly $4 \neq 6$, so when $t = 3$, there would be no solutions.

- 1 mark for substituting $t = 3$ ($t = 2$ is not necessary!) into an intermediate equation (such as equation (6)).
- 1 mark for manipulating the two equations to ensure the LHSs are the same.
- 1 mark for correctly showing RHSs of the final equations are not the same, hence $t = 3$ is the answer.

Part d.

$$\begin{aligned} x - y - 2z &= -3 \\ tx + y - z &= 3t \\ x + 3y + tz &= 13 \end{aligned}$$

If $t = 0$,

$$\begin{aligned} x - y - 2z &= -3 \\ y - z &= 0 \\ x + 3y &= 13 \end{aligned}$$

By CAS,

$$\begin{aligned} x &= 5 \\ y &= \frac{8}{3} \\ z &= \frac{8}{3} \end{aligned}$$

- 1 mark for correctly substituting $t = 0$ into the three equations.
- 1 mark for correct answer for x.
- 1 mark for correct answer for y.
- 1 mark for correct answer for z.

CALCULUS

TECH-FREE TEST 1

MODEL SOLUTIONS AND MARKING SCHEME

Question 1

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{\cos(x) \times \cos(x) - \sin(x) \times -\sin(x)}{(\cos(x))^2} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \\ &= \sec^2(x) \end{aligned}$$

- 1 method mark is awarded for correct use of quotient rule
- 1 answer mark is awarded for use of Pythagorean identity ($\cos^2(x) + \sin^2(x) = 1$)

Question 2

a.
By the product rule,

$$\begin{aligned} \frac{d}{dx}(x \log_e(x^2)) &= x\left(\frac{2x}{x^2}\right) + 1(\log_e(x^2)) \\ &= 2 + \log_e(x^2) \end{aligned}$$

- 1 method mark is awarded for correct use of product rule
- 1 answer mark for the correct derivative, $2 + \log_e(x^2)$

b.

$$\begin{aligned} \frac{d}{dx}(x \log_e(x^2)) &= 2 + \log_e(x^2) \\ \implies x \log_e(x^2) &= \int \left(2 + \log_e(x^2)\right) dx + c_1 \\ x \log_e(x^2) &= \int 2dx + \int \log_e(x^2) dx + c_1 \\ \int \log_e(x^2) dx &= -2x + x \log_e(x^2) + c_2 \end{aligned}$$

where c is a constant.

- 1 method mark is awarded for $x \log_e(x^2) = \int 2 + \log_e(x^2) dx$, as a result of part a)
- 1 method mark for the recognition that the sum of integrals is equal to the integral of sums
- 1 answer mark for the correct anti-derivative including the constant, $-2x + x \log_e(x^2) + c$

Question 3

Let $f(r) = SA = 4\pi r^2$

Therefore, $f'(r) = 8\pi r$
The radius when $f(r) = 64\pi$ is,

$$\begin{aligned} 64\pi &= 4\pi r^2 \\ r^2 &= 16 \\ r &= 4 \end{aligned}$$

Approximate change is given by$h$ in the formula $f(r+h) = f(r) + hf'(r)$

$$\begin{aligned} h &= \frac{f(r+h) - f(x)}{f'(r)} \\ &= \frac{65\pi - 64\pi}{8\pi(4)} \\ &= \frac{1}{32} \end{aligned}$$

The radius has approximately changed by $\frac{1}{32}$ units

- 1 method mark for calculating the radius of the sphere with $SA = 64\pi$
- 1 method mark for finding derivative of surface area function
- 1 method mark for correctly substituting values into rule for linear approximation
- 1 answer mark for the correct approximate change in radius, $\frac{1}{32}$ units (deduct a mark if "units" not included)

Alternate Solution:
We can also define the original function as the radius in terms of the surface area, A.

$$\begin{aligned} A &= 4\pi r^2 \\ r &= \sqrt{\frac{A}{4\pi}} \end{aligned}$$

Hence, let $f(A) = \sqrt{\frac{A}{4\pi}}$, $x = 64\pi$ and $h = \pi$. $f'(A) = \frac{1}{4\sqrt{\pi A}}$
In this case, the approximate change in radius is given by $hf'(A)$

$$\begin{aligned} hf'(A) &= \pi \times \frac{1}{4\sqrt{\pi(64\pi)}} \\ &= \frac{\pi}{4\sqrt{64\pi^2}} \\ &= \frac{\pi}{4(8)\pi} \\ &= \frac{1}{32} \end{aligned}$$

The radius has approximately changed by $\frac{1}{32}$ units

- 1 method mark for defining the radius function in terms of surface area
- 1 method mark for finding derivative of radius function
- 1 method mark for correctly substituting values into rule for linear approximation
- 1 answer mark for the correct approximate change in radius, $\frac{1}{32}$ units

Question 4

The $\tan(-\frac{x}{3})$ function and linear function are always decreasing.
However, there is an asymptote due to the tan function at $x = \frac{3\pi}{2}$
Hence, the largest subset for which $f(x)$ is strictly decreasing is $(\frac{3\pi}{2}, 6\pi]$

- 1 method mark for showing an understanding of a decreasing function
- 1 method mark for stating the equation of the asymptote
- 1 answer mark for the correct largest subset for which $f(x)$ is strictly decreasing

Question 5

a.

If we cut the cylinder from one corner to another, the result is a right angled triangle with a hypotenuse of 12 cm due to the shape of the sphere.

$$\begin{aligned} (2r)^2 + h^2 &= 12^2 \\ 4r^2 &= 144 - h^2 \\ r &= \frac{\sqrt{144 - h^2}}{2} \end{aligned}$$

- 1 method mark for using the Pythagorean theorem
- 1 answer mark for the correct value of r in terms of h

b.

$$\begin{aligned} V &= \pi r^2 h \\ V &= \pi h \left(\frac{144 - h^2}{4}\right) \\ V &= 36\pi h - \frac{\pi h^3}{4} \end{aligned}$$

- 1 answer mark for the correct rule of V

c.

$$\frac{dV}{dh} = 36\pi - \frac{3\pi h^2}{4}$$

Let $\frac{dV}{dh} = 0$

$$\begin{aligned} \frac{dV}{dh} &= 0 \\ 36\pi - \frac{3\pi h^2}{4} &= 0 \\ \frac{3\pi h^2}{4} &= 36\pi \\ h^2 &= 48 \\ h &= 4\sqrt{3} \end{aligned}$$

Ignoring the $h = -4\sqrt{3}$ solution, substitute this h into V

$$\begin{aligned} V &= 36\pi(4\sqrt{3}) - \frac{\pi(4\sqrt{3})^3}{4} \\ &= 96\sqrt{3}\pi \end{aligned}$$

Hence, the maximum volume of the cylinder is $96\sqrt{3}\pi$ cubic units

- 1 method mark for finding $\frac{dV}{dh}$ and letting it equal 0.
- 1 method mark for finding the value of h that corresponds to the maximum volume of the cylinder
- 1 answer mark for the correct maximum volume of the cylinder

CALCULUS

TECH-FREE TEST 2

MODEL SOLUTIONS AND MARKING SCHEME

Question 1

a.

$$\begin{aligned}\int_0^t \left(e^{-x}+x\right)dx &= \left[-e^{-x}+\frac{x^2}{2}\right]_0^t\\ &= (-e^{-t}+\frac{t^2}{2})-(-1+0)\\ &= -e^{-t}+\frac{t^2}{2}+1\end{aligned}$$

- 1 method mark for anti-differentiation
- 1 answer mark for correct evaluation of definite integral

b.

This answer is a function in terms of t, that represents the net signed area under the curve $y = e^{-x} + x$ between $x = 0$ and $x = t$

- 1 answer mark for a reasonable response that mentions "signed area" between $x = 0$ and $x = t$

Question 2

a.

$$\begin{aligned}\text{Gradient of line} &= 2\\ \text{Gradient of normal} &= -\frac{1}{2}\end{aligned}$$

- 1 answer mark for correct gradient of normal, $-\frac{1}{2}$

b.

Substitute general point (x_a, y_a) into general form of a line with gradient $-\frac{1}{2}$

$$\begin{aligned}y-y_a &= -\frac{1}{2}(x-x_a)\\ y &= -\frac{1}{2}x+\frac{1}{2}x_a+y_a\end{aligned}$$

- 1 method mark for substitution of (x_a, y_a) into the general form of a straight line
- 1 answer mark for correct general normal to the line

c.

Note: This question may otherwise be answered by defining a function that describes the distance between the general point on $y = 2x - 3$ and point A, then finding the minimum of this function. For the purpose of following the previous parts to this question, the normal is used in the solution below.

The shortest distance can be found by a line that connects the point to the original line at a right angle, i.e. the normal.

We can substitute point A into the general normal,

$$\begin{aligned} 2 &= -\frac{1}{2}(-4) + \frac{1}{2}x_a + y_a \\ y_a &= -\frac{x_a}{2} \end{aligned}$$

Hence, the point on the line at which the normal connects to point A is $\left(x_a, -\frac{x_a}{2}\right)$. If we substitute this result into the original line, we can get the coordinates of the point at which the right angle is formed

$$\begin{aligned} -\frac{x_a}{2} &= 2x_a - 3 \\ x_a &= \frac{6}{5} \\ y_a &= -\frac{3}{5} \end{aligned}$$

Hence the closest point on the line to point A is $(\frac{6}{5}, -\frac{3}{5})$.

$$\begin{aligned} \text{Distance} &= \sqrt{(\frac{6}{5} - (-4))^2 + (\frac{-3}{5} - 2)^2} \\ &= \sqrt{\frac{676}{25} + \frac{169}{25}} \\ &= \frac{13\sqrt{5}}{5} \text{ units} \end{aligned}$$

The shortest distance between the line and point A is $\dfrac{13\sqrt{5}}{5}$ units

- 1 method mark for recognising that the shortest distance forms a line that is perpendicular to $y = 2x - 3$
- 1 method mark for substitution of point A into normal found in part b)
- 1 method mark for finding the point on the line closest to point A
- 1 method mark for using the distance formula for two points
- 1 answer mark for the correct shortest distance, $\frac{13\sqrt{5}}{5}$ units

Question 3

$$f(x+h) \approx f(x) + hf'(x)$$

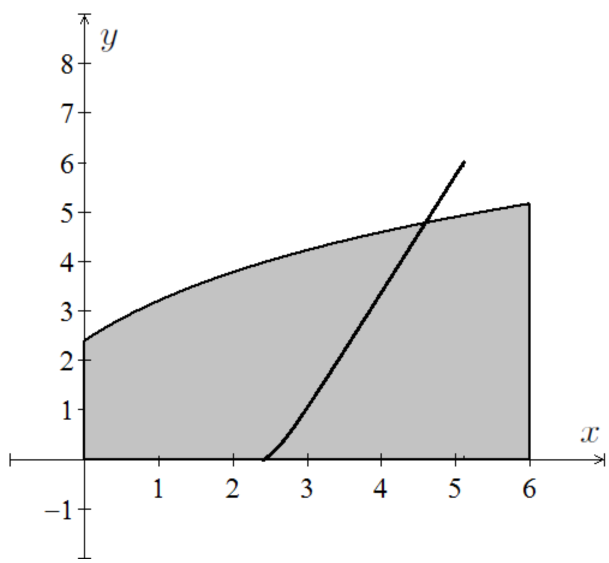
Let $f(x) = \dfrac{1}{\sqrt[3]{x}}$, $x = 27$, and $h = 1$

$$\begin{aligned} f'(x) &= -\frac{1}{3}x^{-\frac{4}{3}} \\ f(x+h) &\approx \frac{1}{\sqrt[3]{27}} + 1(-\frac{1}{3})(27^{-\frac{4}{3}}) \\ &= \frac{1}{3} - \frac{1}{243} \\ &= \frac{80}{243} \end{aligned}$$

- 1 method mark for using linear approximation formula
- 1 method mark for finding $f'(x)$
- 1 answer mark for the correct approximation

Question 4

a.



$$\begin{aligned} f(f^{-1}(x)) &= x \\ 2\log_e(f^{-1}(x)+2)+1 &= x \\ f^{-1}(x) &= e^{\frac{x-1}{2}} - 2 \end{aligned}$$

- 1 answer mark for correctly shading the field
- 1 answer mark for a correct sketch
- 1 answer mark for the correct inverse function

b.

From the diagram, the area between the y-axis and the inverse from 0 to $y = 6$ is the same as the area required from the original function.

$f^{-1}(x)$ cuts the x-axis when $f(x)$ cuts the y-axis, i.e. $2\log_e(0+2)+1 = 2\log_e(2)+1$

We now need to find the x-value for when $f^{-1}(x) = 6$

$$\begin{aligned} 6 &= e^{\frac{x-1}{2}} - 2 \\ x &= 2\log_e(8) + 1 \end{aligned}$$

Hence, we can find the area underneath the inverse between the x-intercept and $2log_e(8)+1$

$$\begin{aligned} \int_{2\log_e(2)+1}^{2\log_e(8)+1} \left(e^{\frac{x-1}{2}} - 2\right) dx &= \left[2e^{\frac{x-1}{2}} - 2x\right]_{2\log_e(2)+1}^{2\log_e(8)+1} \\ &= 12 - 8\log_e(2) \end{aligned}$$

If we subtract this area from the square formed by the axes, $y = 6$ and $x = 2\log_e(8)+1$ we get the required area.

$$6 \times (2\log_e(8)+1) - (12 - 8\log_e(2)) = 44\log_e(2) - 6$$

Hence the area of Farmer Bebbington's field is $44\log_e(2) - 6$ square units

- 1 method mark for finding the correct value of when $f^{-1}(x) = 6$ and x-intercept of this inverse
- 1 method mark for calculating the correct area under the inverse function
- 1 answer mark for the correct area of the field

CALCULUS

TECH-ACTIVE TEST 1

MODEL SOLUTIONS AND MARKING SCHEME

SECTION 1 - Multiple-Choice Questions

Question	1	2	3	4	5	6	7	8	9	10	11
Answer	B	C	B	A	A	D	C	E	E	E	A

SECTION 2 - Extended-Response Questions

Question 1

a.

$$\begin{aligned}\frac{4}{r} &= \frac{15}{h}\\ r &= \frac{4h}{15}\end{aligned}$$

- 1 answer mark is awarded for the correct expression for r in terms of h

b.

$$\begin{aligned}V &= \frac{1}{3}\pi r^2 h\\ &= \frac{1}{3}\pi\left(\frac{4h}{15}\right)^2 h\\ &= \frac{16\pi}{675}h^3 \text{ cm}^3\end{aligned}$$

- 1 answer mark for the correction expression for the volume V in terms of the height h

c.

$$\begin{aligned}
V_{max} &= \frac{16\pi}{675}(15)^3 \\
&= 80\pi \text{ cm}^3 \\
\frac{1}{2}V_{max} &= 40\pi \text{ cm}^3 \\
\frac{4\pi}{75}h^3 &= 40\pi \\
h^3 &= 1687.5 \\
h &= 11.91 \text{ cm}
\end{aligned}$$

- 1 answer mark for the correct maximum volume, $V_{max} = 80\pi \text{ cm}^3$
- 1 method mark for equating the expression for the volume in terms of h to half the maximum volume
- 1 answer mark for the correct value of h, 11.91 cm

d.

$$\begin{aligned}
\frac{dV}{dt} &= 20\, cm^3/s \\
\frac{dh}{dt} &= \frac{dh}{dV}\frac{dV}{dt} \\
V &= \frac{16\pi}{675}h^3 \\
\frac{dV}{dh} &= \frac{16}{225}\pi h^2 \\
\text{When } h = 5 \text{ cm} & \\
\frac{dh}{dt} &= \frac{225}{16\pi \times 5^2} \times 20 \\
&= \frac{45}{4\pi} \text{ cm/s}
\end{aligned}$$

- 1 method mark for the correct application of the chain rule to find $\frac{dh}{dt}$
- 1 answer mark for the correct expression for V in terms of h, $V = \dfrac{16\pi}{225}h^2$
- 1 answer mark for the value of $\frac{dh}{dt}$ when $h = 5$ cm, $\frac{45}{4\pi}$ cm/s

e.

$$\begin{aligned}
\frac{dV}{dt} &= 20t \\
\int \frac{dV}{dt}dt &= \int 20t dt \\
V(t) &= 10t^2 + C \\
\text{When } V = 0\,\text{cm}^3, t = 0\,s & \\
\therefore C &= 0 \\
V(t) &= 10t^2 \\
V &= 80\pi \\
10t^2 &= 80\pi \\
t^2 &= 8\pi\ s \\
t &= \pm 2\sqrt{2\pi}\ s \\
t > 0 & \\
\therefore t &= 2\sqrt{2\pi}\ s
\end{aligned}$$

- 1 method mark for correct integration to find a family of anti-derivatives for $\frac{dV}{dt}$
- 1 answer mark for the correct expression for $V(t)$, that requires the constant of integration C to be found
- 1 answer mark for correct value of t, $2\sqrt{2\pi}\ s$

Question 2

a.

$$\begin{aligned}
f(4) &= 300 \\
4096a + 1024b - 7060 &= 300 \\
f'(x) &= 6ax^5 + 5bx^4 - \frac{705}{2}x^3 + \frac{4635}{4}x^2 - 1215x \\
f'(4) &= 0 \\
6144a + 1280b - 8880 &= 0 \\
\therefore a = -\frac{5}{16}, b = \frac{135}{16} &
\end{aligned}$$

- 1 method mark for obtaining the equation from the point $(4, 300)$
- 1 answer mark for finding the derivative $f'(x)$
- 1 answer mark for correctly writing down $a = -\frac{5}{16}$ and $b = \frac{135}{16}$

bi.

$$\begin{aligned}
x=0 \quad f(0) = \quad & 500 \\
& (0,500) \\
y=0 \quad & 0 = -\frac{5}{16}x^6 + \frac{135}{16}x^5 - \frac{705}{8}x^4 + \frac{1545}{4}x^3 - \frac{1215}{2}x^2 + 500 \\
& x = -0.73,\ 2,\ 5 \\
& \text{But } 0 \leq x \leq 5 \\
& \therefore x = 2,\ 5 \\
& (2,0) \text{ and } (5,0)
\end{aligned}$$

- 1 answer mark for writing down the y-intercept $(0,500)$
- 1 answer mark for writing down the two x-intercepts $(2,0)$ and $(5,0)$

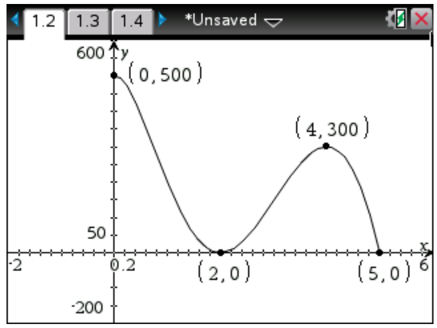
bii.

$$\begin{aligned}
f'(x) &= 0 \\
x &= 0,\ 2,\ 4 \\
f(0) &= 500 \\
& \quad (0,500) \\
f(2) &= 0 \\
& \quad (2,0) \\
f(4) &= 300 \\
& \quad (4,300)
\end{aligned}$$

$\therefore$The stationary points are $(0,500)$, $(2,0)$ and $(4,300)$

- 1 answer mark for writing down the x values of the turning points, $x = 0,\ 2,\ 4$
- 1 answer mark for writing down the coordinates of the turning points, $(0,500)$, $(2,0)$ and $(4,300)$

biii.



- 1 answer mark for the correctly labelled x and y intercepts, $(2,0)$, $(5,0)$ and $(0,500)$
- 1 answer mark for the correctly labelled minor peak turning point $(4,300)$
- 1 answer mark for the correct shape and appropriate domain $[0,5]$

- **Note**: Endpoints must be closed (deduct a mark if not closed)

c.

$$-\frac{5}{16}x^6+\frac{135}{16}x^5-\frac{705}{8}x^4+\frac{1545}{4}x^3-\frac{1215}{2}x^2+500=100$$
$$x=-0.665,\ 1.300,\ 2.795,\ 4.831$$

But $0<x<4$ for our intersection

$$\therefore x=1.300,\ 2.795$$
$$1000\int_{1.300}^{2.795}\left(100-\left(-\frac{5}{16}x^6+\frac{135}{16}x^5-\frac{705}{8}x^4+\frac{1545}{4}x^3-\frac{1215}{2}x^2+500\right)\right)dx = 97,882.752\,\mathrm{m}^2$$
$$\text{Volume}=97,882.752\times 1000$$
$$=97,883,000\,\mathrm{m}^3$$

- 1 answer mark for the correct values of x for the intersection of the two functions, rejecting the solutions that are not within the desired region
- 1 method mark for the correct terminals and integral
- 1 answer mark for writing down the volume correct to the nearest $1000\,\mathrm{m}^3$, $V=97,883,000\,\mathrm{m}^3$

d.

$$\begin{aligned} x &= 2.795 \\ f'(2.795) &= 214.343\,\mathbf{m/km} \\ &= \frac{214.343}{1000}\,\mathbf{m/m} \\ &= 0.214 \end{aligned}$$

Let θ be the angle between the water surface and Nina

$$\begin{aligned} \tan(\theta) &= 0.214 \\ \theta &= \tan^{-1}(0.214) \\ &= 12.08^o \end{aligned}$$

$\therefore$ Nina hits the water at 12.08^o

- 1 answer mark for writing down the value of the gradient at the point she hits the water, $f'(2.795) = 214.343\,\mathrm{m/km}$
- 1 answer mark for writing down the angle she hits the water at, $\theta = 12.08^o$

e.

$$\begin{aligned} -\frac{5}{16}x^6 + \frac{135}{16}x^5 - \frac{705}{8}x^4 + \frac{1545}{4}x^3 - \frac{1215}{2}x^2 + 500 &= 300 \\ x &= -0.493,\ 0.738,\ 4 \end{aligned}$$

But for out intersection $0 < x < 4$

$$1000\int_{0.738}^{4}\left(300 - \left(-\frac{5}{16}x^6 + \frac{135}{16}x^5 - \frac{705}{8}x^4 + \frac{1545}{4}x^3 - \frac{1215}{2}x^2 + 500\right)\right)dx = 550641.708\,\mathrm{m}^2$$

$$\begin{aligned} \text{Volume} &= 550641.708 \times 1000 \\ &= 550,642,000\,\mathrm{m}^3 \end{aligned}$$

The minimum volume needed to overflow the dam is $550,642,000\,\mathrm{m}^3$.

- 1 answer mark for the correct values of x for the intersection of the two functions, rejecting the solutions that are not within the desired region
- 1 answer mark for the correct the integral with the correct terminals
- 1 answer mark for writing down the correct volume of water, $550,642,000\,\mathrm{m}^3$

CALCULUS
TECH-ACTIVE TEST 2
MODEL SOLUTIONS AND MARKING SCHEME

SECTION 1 - Multiple-choice questions

Question	1	2	3	4	5	6	7	8	9	10	11
Answer	B	B	B	A	E	D	C	C	E	A	C

SECTION 2 - Extended Response questions

Question 1

a.

$$\begin{aligned}\text{Cross-sectional Area} &= -\frac{1}{50}\int_{-25}^{25}\left(x^2-625\right)dx \\ &= -\frac{1}{50}\left[\frac{x^3}{3}-625x\right]_{-25}^{25} \\ &= \frac{1250}{3}\ \mathrm{m}^2\end{aligned}$$

- 1 method mark for the correct integral for the cross-sectional area
- 1 answer mark for the correct area

b.

$$\begin{aligned}\frac{1}{50}\left(x^2-625\right) &= -\sqrt{\frac{3}{25}\left(625-x^2\right)} \\ x &= \pm 25,\ \pm 5\sqrt{13} \\ \therefore a &= 5\sqrt{13}\end{aligned}$$

- 1 answer mark for the correct value of a

c.

$$\begin{aligned}\text{Area}_{\text{sediment}} &= \int_{-5\sqrt{13}}^{5\sqrt{13}}\left(-\sqrt{\frac{3}{25}\left(625-x^2\right)}-\frac{1}{50}\left(x^2-625\right)\right)dx \\ &= 90\,\mathrm{m}^2\end{aligned}$$

- 1 method mark for the correct integral for the area
- 1 answer mark for the correct area of the sediment

d.

$$
\begin{aligned}
g\left(\sqrt{6}-3\right) &= \frac{13}{2}-\sqrt{6}\\
-\frac{3}{a+\sqrt{6}-3}-b\left(\sqrt{6}-3\right) &= \frac{3}{2}-\sqrt{6}....[1]\\
g'(x) &= \frac{3}{(x+a)^2}-b\\
g'\left(\sqrt{6}-3\right) &= 0\\
\frac{3}{\left(a+\sqrt{6}-3\right)^2} &= b....[2]\\
\therefore a=3,\ b=\frac{1}{2}
\end{aligned}
$$

- 1 method mark for the equation from the original point
- 1 method mark for the equation from the derivative of g
- 1 answer mark for the correct values of a and b

e.

Let d be the distance between the Nuclear Power Plant and the river.

$$
\begin{aligned}
d &= \sqrt{x^2+y^2}\\
&= \sqrt{x^2+\left(\frac{1}{3}x+2\right)^2}\\
&= \frac{1}{3}\sqrt{2\left(5x^2+6x+18\right)}\\
\frac{dd}{dx} &= \frac{1}{3}\times\frac{1}{2}\left(2\left(5x^2+6x+18\right)\right)^{-\frac{1}{2}}\times(10x+6)\\
&= \frac{\sqrt{2}\,(5x+3)}{3\sqrt{5x^2+6x+18}}\\
\frac{\sqrt{2}\,(5x+3)}{3\sqrt{5x^2+6x+18}} &= 0\\
5x+3 &= 0\\
\therefore x &= -\frac{3}{5}\\
y &= \frac{1}{3}\left(-\frac{3}{5}\right)+2\\
\therefore y &= \frac{9}{5}\\
&\left(-\frac{3}{5},\frac{9}{5}\right)\\
d_{min} &= \frac{1}{3}\sqrt{2\left(5\left(-\frac{3}{5}\right)^2+6\left(-\frac{3}{5}\right)+18\right)}\\
&= \frac{3\sqrt{10}}{5}
\end{aligned}
$$

The minimum distance between the Nuclear Power Plant and the canal is $\frac{3\sqrt{10}}{5}$ metres and this occurs at the point $\left(-\frac{3}{5},\frac{9}{5}\right)$ on the canal.

- 1 answer mark for the correct expression for the distance d
- 1 answer mark for the correct expression for the derivative of d with respect to x
- 1 answer mark for the correct co-ordinate
- 1 answer mark for the correct minimum value of d

Alternate solution

$$\begin{aligned}
h(x) &= \frac{1}{3}x + 2 \\
h'(x) &= \frac{1}{3} \\
m_1 m_2 &= -1 \\
m_1 &= \frac{1}{3} \\
m_2 &= -3 \\
y &= -3x \\
\frac{1}{3}x + 2 &= -3x \\
\frac{10}{3}x &= -2 \\
\therefore x &= -\frac{3}{5} \\
y &= \frac{1}{3}\left(-\frac{3}{5}\right) + 2 \\
&= \frac{9}{5} \\
& \left(-\frac{3}{5}, \frac{9}{5}\right) \\
d_{min} &= \sqrt{\left(-\frac{3}{5} - 0\right)^2 + \left(\frac{9}{5} - 0\right)^2} \\
&= \frac{3\sqrt{10}}{5}\, m
\end{aligned}$$

The minimum distance between the Nuclear Power Plant and the canal is $\frac{3\sqrt{10}}{5}$ metres and this occurs at the point $\left(-\frac{3}{5}, \frac{9}{5}\right)$ on the canal.

- 1 answer mark for the correct derivative of h, $\frac{1}{3}$
- 1 method mark for finding the equation of the line of minimum distance
- 1 answer mark for the correct co-ordinate
- 1 answer mark for the correct minimum value of d

f.

$$\begin{aligned}
V(t) &= \int \left(e^{-\frac{1}{15}(t-30)} + \cos\left(\frac{1}{15}t\right)\right) dt \\
&= -15e^{-\frac{1}{15}(t-30)} + 15\sin\left(\frac{1}{15}t\right) + C \\
V(0) &= 20 \\
20 &= -15e^{2} + C \\
\therefore C &= 15e^{2} + 20 \\
\therefore V(t) &= 15e^{2}\left(1 - e^{-\frac{1}{15}t}\right) + 15\sin\left(\frac{1}{15}t\right) + 20 \\
V : [0, 180] \to R,\ V(t) &= 15e^{2}\left(1 - e^{-\frac{1}{15}t}\right) + 15\sin\left(\frac{1}{15}t\right) + 20
\end{aligned}$$

- 1 method mark for the correct integration of $\frac{dv}{dt}$
- 1 answer mark for finding the correct value of C
- 1 answer mark for the correct domain and function

Question 2
ai.

$$\begin{aligned}
V &= \frac{1}{3}\pi r^{2}h \\
\frac{5}{r} &= \frac{15}{15-h} \\
r &= 5 - \frac{1}{3}h ...[2]
\end{aligned}$$

- 1 method mark for finding the expression for r in terms of h

aii.

$$\begin{aligned}
V &= \frac{1}{3}\pi\left(5 - \frac{1}{3}h\right)^{2} \\
h &= \frac{1}{27}\pi h\,(15-h)^{2}
\end{aligned}$$

$$0 < h < 15$$

$$V : (0, 15) \to R,\ V(h) = \frac{1}{27}\pi h\,(15-h)^{2}$$

- 1 answer mark for the correct expression for $V(h)$
- 1 answer mark for the correction domain for $V(h)$ and the function written out using function notation

b.

$$\begin{aligned}\frac{dV}{dh} &= \frac{\pi}{9}(h-15)(h-5)\\ \frac{dV}{dh} &= 0\\ \frac{\pi}{9}(h-15)(h-5) &= 0\\ h=5 \quad &\text{or} \quad h=15\end{aligned}$$

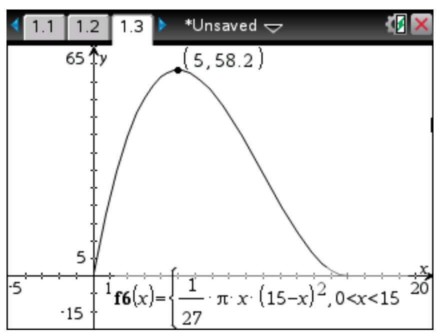
But $0 < h < 15$

$$\begin{aligned}\therefore h &= 5\,cm\\ V(5) &= \frac{1}{27}\pi(5)(15-5)^2\\ &= 58.18\,cm^3\end{aligned}$$

$\therefore V_{max} = 58.18\,cm^3$ when $h = 5\,cm$

- 1 method mark for the correct expression for the derivative of V with respect to h
- 1 answer mark for the correct value of h, and why the second value was rejected
- 1 answer mark for the correct values of V_{max}

c.



- 1 answer mark for the correct shape and domain
- 1 answer mark for the correct labelled turning point

Question 3

a.

$$\begin{aligned}
f'(x) &= 2a\cos(ax) \\
-2a\cos(ax) &= 0 \\
\cos(ax) &= 0 \\
ax &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2} \\
x &= \frac{\pi}{2a}, \frac{3\pi}{2a}, \frac{5\pi}{2a}, \frac{7\pi}{2a}, \frac{9\pi}{2a}, \frac{11\pi}{2a} \\
5^{th} \text{ solution} \quad x &= \frac{9\pi}{2a} \\
\frac{9\pi}{2a} &< 2\pi \\
a &> \frac{9}{4} \\
6^{th} \text{ solution} \quad x &= \frac{11\pi}{2a} \\
\frac{11\pi}{2a} &\geq 2\pi \\
a &\leq \frac{11}{4} \\
\therefore \frac{9}{4} < a \leq \frac{11}{4}
\end{aligned}$$

- 1 method mark for the correct derivative $f'(x)$
- 1 answer mark for the correct values of a

b.

$$\begin{aligned}
\frac{dy}{dx} &= 3ax^2 + 4ax + 3a \\
&= a\left(3x^2 + 4x + 3\right) \quad (\text{NOTE} : a \neq 0) \\
\frac{dy}{dx} &= 0 \\
\triangle &= (4)^2 - 4(3)(3) \\
&= -20
\end{aligned}$$

$\therefore$There are no solutions to $\frac{dy}{dx} = 0$. Therefore the possible values of n is 0.

- 1 method mark for the correct derivative $\frac{dy}{dx}$
- 1 answer mark for the number of solutions

c.

$$\begin{aligned}
\frac{1}{b-0}\int_0^b \left(2\cos\left(\frac{1}{2}x\right) + 1\right) dx &= \frac{5}{2} \\
b &= 2.55
\end{aligned}$$

- 1 method mark for the correct integral for the average value of the function
- 1 answer mark for the correct value of b correct to 2 decimal places

PROBABILITY

TECH-FREE TEST 1

MODEL SOLUTIONS AND MARKING SCHEME

Question 1

$$\begin{aligned}
\sum_x p(x) &= 1 \\
2p + p^2 + \frac{p^2}{2} + \frac{p+2}{3} + p^2 &= 1 \\
\frac{5}{2}p^2 + \frac{7p}{3} - \frac{1}{3} &= 0 \\
15p^2 + 14p - 2 &= 0
\end{aligned}$$

Using the quadratic formula,

$$\begin{aligned}
p &= \frac{-14 \pm \sqrt{14^2 - 4(15)(-2)}}{2(15)} \\
&= \frac{-14 \pm \sqrt{316}}{30} \\
&= \frac{-14 \pm 2\sqrt{79}}{30} \\
&= \frac{-7 \pm \sqrt{79}}{15} \\
&= \frac{-7 + \sqrt{79}}{15} \text{ since } p \geq 0
\end{aligned}$$

- 1 method mark for writing that the sum of probabilities is 1
- 1 method mark for use of quadratic formula/completing the square/factorising the quadratic
- 1 answer mark for the correct value of p

Question 2

a.
Let $X \sim \text{Bi}(20, 0.2)$

$$\begin{aligned}
\text{Var}(X) &= np(1-p) \\
&= 20(0.2)(0.8) \\
&= 3.2 \\
\sigma &= \sqrt{\text{Var}(x)} \\
&= \sqrt{\frac{16}{5}} \\
&= \frac{4\sqrt{5}}{5}
\end{aligned}$$

- 1 method mark for calculating $\text{Var}(X)$
- 1 answer mark for the correct value of the standard deviation

b.
To score 95% in this test, he must score 19 or more.

$$\begin{aligned}
\Pr(X = 19, 20) &= \Pr(19) + \Pr(20) \\
&= \frac{20!}{19!1!} 0.2^{19} \left(0.8\right)^{1} + \frac{20!}{20!0!} 0.2^{20} \left(0.8\right)^{0} \\
&= 20(0.2)^{19}(0.8) + 0.2^{20} \\
&= (16.2)0.2^{19}
\end{aligned}$$

- 1 method mark for the correct expansion of binomial probabilities
- 1 answer mark for the required probability in the correct form

Question 3

a.

If A and B are mutually exclusive, then they share no common outcome, therefore $\Pr(A \cap B) = 0$

$$\begin{aligned}
\Pr(A \mid B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\
&= 0
\end{aligned}$$

- 1 method mark for the recognising that $\Pr(A \cap B) = 0$
- 1 answer mark for the correct value of $\Pr(A \mid B)$

b.

$\Pr(C|D) = \Pr(C)$ since C and D are independent events.

$$\begin{aligned}
\Pr(C|D) &= \frac{\Pr(C \cap D)}{\Pr(D)} \\
\Pr(C) &= \frac{\Pr(C \cap D)}{\Pr(D)} \\
\Pr(C \cap D) &= \Pr(C)\Pr(D) \text{ as required}
\end{aligned}$$

- 1 method mark for the recognising that $\Pr(C|D) = \Pr(C)$
- 1 answer mark for achieving the required result in a logical manner

Question 4

a.

For a probability density function, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\begin{aligned}
\int_0^2 kx\,dx + \int_2^4 5k(x-2)\,dx &= 1 \\
\left[\frac{kx^2}{2}\right]_0^2 + \left[\frac{5kx^2}{2} - 10kx\right]_2^4 &= 1 \\
2k + 10k &= 1 \\
12k &= 1 \\
k &= \frac{1}{12}
\end{aligned}$$

- 1 method mark for recognising that $\int_{-\infty}^{\infty} f(x)\,dx = 1$
- 1 method mark for the correct anti-derivative
- 1 answer mark for writing the correct value of k

b.

$\mu = \int_{-\infty}^{\infty} xf(x)\,dx$

$$\begin{aligned}
\mu &= \frac{1}{12}\int_0^2 x^2\,dx + \int_2^{54} 5x(x-2)\,dx \\
&= \frac{1}{12}\left(\left[\frac{x^3}{3}\right]_0^2 + \left[\frac{5x^3}{3} - 5x^2\right]_2^4\right) \\
&= \frac{1}{12}(\frac{8}{3} + \frac{100}{3}) \\
&= 3
\end{aligned}$$

- 1 method mark for the recognising that $\mu = \int_{-\infty}^{\infty} xf(x)\,dx$
- 1 method mark for the correct anti-derivative
- 1 answer mark for writing the correct value of μ

c.

$$\begin{aligned}
\Pr(\mid X - \mu \mid < 1) &= \Pr(\mu - 1 < X < \mu + 1) \\
&= \Pr(2 < X < 4) \\
&= \int_2^4 5k(x-2)\,dx \\
&= \left[\frac{5kx^2}{2} - 10kx\right]_2^4 \\
&= k\,[0 - (-10)] \\
&= \frac{5}{6}
\end{aligned}$$

- 1 method mark for expanding the absolute value correctly
- 1 method mark for the correct definite integrals that give the required probability
- 1 answer mark for writing the correct value of $\Pr(\mid X - \mu \mid < 1)$

PROBABILITY

TECH-FREE TEST 2

MODEL SOLUTIONS AND MARKING SCHEME

Question 1

a.

Let C represent the event that Mr. Li drives in his car, and$B$ is the event that he rides his bike

$$\begin{aligned}\Pr(C, C, C) &= 1 \times 0.3 \times 0.3 \\ &= 0.09\end{aligned}$$

- 1 answer mark for correct probability required

b.

$S_3 = T^2 S_1$, $T = \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}$, and $S_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ since we know he drives his car on the first day

$$\begin{aligned}S_3 &= T^2 S_1 \\ &= \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.37 & 0.36 \\ 0.63 & 0.64 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.37 \\ 0.63 \end{bmatrix}\end{aligned}$$

Hence, the probability that he rides his bike on day 3 is 0.63

- 1 method mark for writing the transition matrix
- 1 method mark for the expression $S_3 = T^2 S_1$ in some form
- 1 answer mark for the correct probability required

Question 2

a.

$$
\begin{aligned}
\text{Total Area of Dartboard} &= \pi \times 30^2 \\
&= 900\pi \text{ square units}
\end{aligned}
$$

$$
\begin{aligned}
\text{Area of inner circle} &= \pi \times 3^2 \\
&= 9\pi \text{ square units} \\
\text{Portion of whole board} &= \frac{9\pi}{900\pi} \\
&= 0.01
\end{aligned}
$$

$$
\begin{aligned}
\text{Area of outer region} &= \text{Area of Board-Circle of radius 15 cm} \\
&= 900\pi - \pi \times 15^2 \\
&= 675\pi \text{ square units} \\
\text{Portion of whole board} &= \frac{675\pi}{900\pi} \\
&= 0.75
\end{aligned}
$$

$$
\begin{aligned}
\text{Area of middle region} &= \text{Area of board - Area of outer region - Area of inner circle} \\
&= 900\pi - 675\pi - 9\pi \\
&= 216\pi \text{ square units} \\
\text{Portion of whole board} &= \frac{216\pi}{900\pi} \\
&= 0.24
\end{aligned}
$$

Note: once you find two regions (in whichever order), you can find the 3rd region by using the complement (as there are only 3 probabilities). i.e. Pr(3rd region) $= 1 -$ Pr(1st region) $-$ Pr(2nd region)

Hence, the probability distribution is:

x	10	25	100
$\Pr(X = x)$	0.75	0.24	0.01

, X is the score for one throw.

- 1 method mark for finding the area of each region
- 1 method mark for finding the proportional area of each region
- 1 answer mark for the correct probability distribution

b.

The possible throws to score more than 50 points are: 100,10; 10,100; 100,25; 25,100; 100,100.

$$
\begin{aligned}
\Pr(Score > 50) &= 0.75 \times 0.01 + 0.01 \times 0.75 + 0.24 \times 0.01 + 0.01 \times 0.24 + 0.01 \times 0.01 \\
&= 2 \times 0.0075 + 2 \times 0.0024 + 0.0001 \\
&= 0.015 + 0.0048 + 0.0001 \\
&= 0.0199
\end{aligned}
$$

- 1 method mark for stating the possible throws to score more than 50 points
- 1 method mark for showing ability to find the scoring probability of two throws
- 1 answer mark for the correct probability required

Question 3

a.

Let $X \sim \mathrm{N}(75, 3^2)$

$$\begin{aligned} \Pr(X > 81) &= \Pr(X > \mu + 2\sigma) \\ &= \frac{1}{2}(0.05) \\ &= 0.025 \end{aligned}$$

- 1 method mark for recognising that $81 = \mu + 2\sigma$
- 1 answer mark for the correct probability required

b.

$$\begin{aligned} \Pr(X > 75) &= \Pr(X > \mu) \\ &= 0.5 \end{aligned}$$

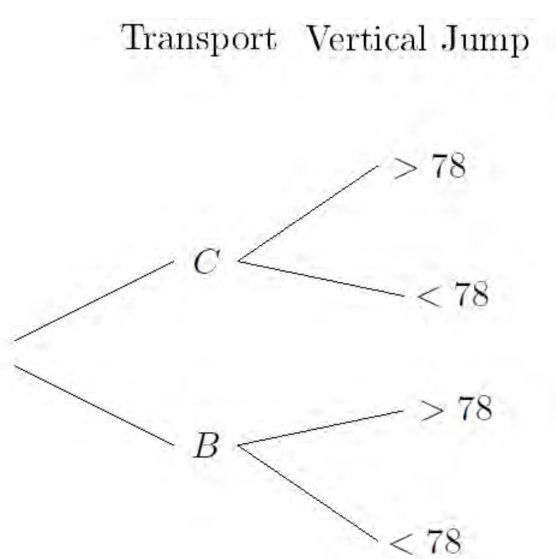
Let $Y \sim \mathrm{Bi}(7, 0.5)$

$$\begin{aligned} \Pr(Y = 5) &= \frac{7!}{5!2!} 0.5^5 \, (0.5)^2 \\ &= 21(0.5)^5(1-0.5)^2 \end{aligned}$$

- 1 method mark for finding the probability he will jump over 75 cm
- 1 method mark for the use of the binomial distribution and correct expansion
- 1 answer mark for the correct probability required in the correct form asked for

Question 4

The information above can be split into the following tree diagram:



Probabilities of whether he takes the car or bike on day 2 (using the result from **Question 1 b.**)

$$\begin{aligned} S_2 &= TS_1 \\ &= \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} \end{aligned}$$

For the case that he drives his car, the probability of winning is:

$$\begin{aligned} \Pr(\text{Win}_{\text{car}}) &= 0.3 \times \Pr(X > 78) \times 1 \\ &= 0.3 \times \frac{1}{2}(1 - 0.68) \\ &= 0.048 \end{aligned}$$

Let $A \sim \mathrm{N}(74, 2)$, the distribution that describes his vertical jump after he rides his bike
Hence the probability of winning when he rides his bike is:

$$\begin{aligned} \Pr(\text{Win}_{\text{bike}}) &= 0.7 \times \Pr(A > 78) \times 1 \\ &= 0.7 \times \Pr(A > \mu + 2\sigma) \\ &= 0.7 \times \frac{1}{2}(1 - 0.95) \\ &= 0.0175 \end{aligned}$$

Therefore, the probability of winning is:

$$\begin{aligned} \Pr(\text{Win}) &= \Pr(\text{Win}_{\text{car}}) + \Pr(\text{Win}_{\text{bike}}) \\ &= 0.048 + 0.0175 \\ &= 0.0655 \\ &\sim 0.066 \end{aligned}$$

- 1 method mark for stating the possible paths to winning the event on day 2
- 1 method mark for finding the probability of either riding his bike or driving his car on day 2
- 1 method mark for finding the probability of jumping over 78 cm after riding his bike
- 1 method mark for finding the probabilities of winning for both driving or riding
- 1 answer mark for the correct probability required to 3 decimal places

PROBABILITY

TECH-ACTIVE TEST 1

MODEL SOLUTIONS AND MARKING SCHEME

SECTION 1 - Multiple-choice questions

Question	1	2	3	4	5	6	7	8	9	10	11
Answer	E	B	A	C	B	D	A	B	E	A	C

SECTION 2 - Extended Response questions

Question 1

ai.

$$\begin{aligned} \mu &= 600 \\ \sigma &= 3 \\ X &\sim \mathrm{N}(600, 9) \\ \Pr(X \geq 603) &= 0.1587 \end{aligned}$$

- 1 answer mark is awarded for the correct value of $\Pr(X \geq 603)$ correct to 4 decimal places

aii.

$$\begin{aligned} \Pr(597 \leq Y \leq 606) &= \int_{597}^{606} \left(\frac{3y^2 - 3600y + 1080064}{2048} \right) dy \\ &= \left[\frac{1}{2048} \left(y^3 - 1800y^2 + 1080064y \right) \right]_{597}^{606} \\ &= 0.3999 \end{aligned}$$

- 1 method mark for the correct integral and terminals of $f(y)$
- 1 answer mark for the correct value of $\Pr(597 \leq Y \leq 606)$ correct to 4 decimal places

b.

$$\begin{aligned}
\mathrm{E}(Y) &= \int_{592}^{608} \left(y \times \frac{3y^2 - 3600y + 1080064}{2048} \right) dy \\
&= \left[\frac{y^2}{8192} \left(3y^2 - 4800y + 2160128 \right) \right]_{592}^{608} \\
&= 600 \text{ ml}
\end{aligned}$$

- 1 method mark for the correct integral and terminals of $y \times f(y)$
- 1 answer mark for the correct value of $\mathrm{E}(Y) = 600$ ml

c.

$$\begin{aligned}
\Pr(Y \leq a) &= 0.85 \\
\int_{592}^{a} \left(\frac{3y^2 - 3600y + 1080064}{2048} \right) dy &= 0.85 \\
\left[\frac{1}{2048} \left(y^3 - 1800y^2 + 1080064y \right) \right]_{592}^{a} &= 0.85 \\
a &= 606.64
\end{aligned}$$

- 1 method mark for the correct integral and terminals of $f(y)$
- 1 answer mark for the correct value of $a = 606.64$ ml

d.

Let A be the event that the bottle contains less than 595 ml.

$$\begin{aligned}
X &\sim \mathrm{N}(600, 9) \\
\Pr(X \leq 595) &= 0.04779 \\
n &= 10 \\
p &= 0.04779 \\
A &\sim \mathrm{Bi}(10, 0.04779) \\
\therefore \Pr(A \geq 2) &= 0.0796
\end{aligned}$$

- 1 answer mark for the correct value for $\Pr(X \leq 595)$, 0.04779
- 1 method mark for correctly identifying the use of a binomial distribution and the associated parameters, $A \sim \mathrm{Bi}(10, 0.04779)$
- 1 answer mark for correct value of $\Pr(A \geq 2)$, 0.0796 correct to 4 decimal places

e.

$$
\begin{aligned}
\Pr(Y \le 595) &= \int_{592}^{595} \left(\frac{3y^2 - 3600y + 1080064}{2048}\right) dy \\
&= \left[\frac{1}{2048}\left(y^3 - 1800y^2 + 1080064y\right)\right]_{592}^{595} \\
&= 0.282715 \\
\Pr(Y \le 600) &= \int_{592}^{600} \left(\frac{3y^2 - 3600y + 1080064}{2048}\right) dy \\
&= \left[\frac{1}{2048}\left(y^3 - 1800y^2 + 1080064y\right)\right]_{592}^{600} \\
&= 0.5000 \\
\Pr(Y \le 595 | Y \le 600) &= \frac{\Pr(Y \le 595 \cap Y \le 600)}{\Pr(Y \le 600)} \\
&= \frac{\Pr(Y \le 595)}{\Pr(Y \le 600)} \\
&= \frac{0.282715}{0.5} \\
&= 0.5654
\end{aligned}
$$

- 1 answer mark for correct values of $\Pr(Y \le 595) = 0.5654$ and $\Pr(Y \le 600) = 0.5000$
- 1 method mark for finding the equivalent expression $\Pr(Y \le 595 | Y \le 600) = \dfrac{\Pr(Y \le 595)}{\Pr(Y \le 600)}$
- 1 answer mark for correct value $\Pr(Y \le 595 | Y \le 600)$, 0.5654

f.

Let O be the event that an Orange flavoured bottle of soft drink is picked.
Let V be the amount of liquid in the bottle selected.

$$
\begin{aligned}
\Pr(O | V \le 595) &= \frac{\Pr(O \cap V \le 595)}{\Pr(V \le 595)} \\
\Pr(X \le 595) &= 0.04779 \\
\Pr(Y \le 595) &= 0.282715 \\
\Pr(V \le 595) &= \left(\frac{20}{30} \times 0.04779\right) + \left(\frac{10}{30} \times 0.282715\right) \\
&= 0.12610 \\
\Pr(O \cap V \le 595) &= \frac{20}{30} \times 0.04779 \\
&= 0.03186 \\
\Pr(O | V \le 595) &= \frac{0.03186}{0.12610} \\
&= 0.2527
\end{aligned}
$$

- 1 method mark for correct use and expressions for conditional probability
- 1 answer mark for correct value of $\Pr(V \le 595)$, 0.12610
- 1 answer mark for correct value of $\Pr(O | V \le 595)$, 0.2527

Question 2

a.

Let A be the event that Will scores on his previous shot.

Let B be the event that he scores on the current shot.

$$\begin{aligned}
\mathrm{T} &= \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix} \\
S_0 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
S_n &= \mathrm{T}^n \times S_0 \\
S_4 &= \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix}^4 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.6016 \\ 0.3984 \end{bmatrix}
\end{aligned}$$

The probability that Will does not score on his fifth attempt is 0.3984.

- 1 answer mark for obtaining the correct transition matrix
- 1 answer mark for stating the correct probability that Will does not score on his fifth attempt, 0.3984.

NOTE: If you do not specify which probability is the answer from S_4, then you will not get the mark.

b.

Let S be the event that Will scores a goal.

$$\begin{aligned}
\Pr(SSS|S') &= 0.45 \times 0.7 \times 0.7 \\
&= 0.2205
\end{aligned}$$

- 1 answer mark for the correct value of 0.2205

c.

$$\begin{aligned}
\Pr(S_\infty) &= \frac{0.45}{0.30 + 0.45} \\
&= \frac{3}{5}
\end{aligned}$$

So the long term probability that Will scores a goal is $\frac{3}{5}$.

- 1 answer mark for writing down the correct long term probability of scoring a goal, $\frac{3}{5}$

d.

There is two ways to approach this question, with the first being quicker and less tedious than the second.

The first being by matrix methods.

$$
\begin{aligned}
\mathrm{E}\left(\begin{bmatrix} X_0 \\ X_1 \end{bmatrix}\right) &= T^1 S_0 + T^2 S_0 + T^3 S_0 \\
&= \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix}^2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix}^3\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1.93125 \\ 1.06875 \end{bmatrix} \\
\therefore \mathrm{E}(X) &= 1.93
\end{aligned}
$$

- 1 method mark for the correct expressions for the expected value
- 1 answer mark for the correct matrix $\begin{bmatrix} 1.93125 \\ 1.06875 \end{bmatrix}$
- 1 answer mark for the correct expected number of goals, 1.93 correct to 2 decimal places.

The second method is a lot more tedious.
Let F be the event that he does not score a goal.
Let x be the number of goals Will scores.

$$
\begin{aligned}
\Pr(FFF) &= 0.30 \times 0.55 \times 0.55 \\
&= 0.09075 \\
\therefore \Pr(X = 0) &= 0.09075
\end{aligned}
$$

$$
\begin{aligned}
\Pr(FFS) &= 0.30 \times 0.55 \times 0.45 \\
&= 0.07425 \\
\Pr(FSF) &= 0.30 \times 0.45 \times 0.30 \\
&= 0.0405 \\
\Pr(SFF) &= 0.70 \times 0.30 \times 0.55 \\
&= 0.1155 \\
\therefore \Pr(X = 1) &= 0.07425 + 0.0405 + 0.1155 \\
&= 0.23025
\end{aligned}
$$

$$
\begin{aligned}
\Pr(SSF) &= 0.70 \times 0.70 \times 0.30 \\
&= 0.1470 \\
\Pr(SFS) &= 0.70 \times 0.30 \times 0.45 \\
&= 0.0945 \\
\Pr(FSS) &= 0.30 \times 0.45 \times 0.70 \\
&= 0.0945 \\
\therefore \Pr(X = 2) &= 0.1470 + 0.0945 + 0.0945 \\
&= 0.3360
\end{aligned}
$$

$$\begin{aligned}\Pr(SSS) &= 0.70 \times 0.70 \times 0.70 \\ &= 0.3430 \\ \therefore \Pr(X=3) &= 0.3430\end{aligned}$$

x	0	1	2	3
$\Pr(X=x)$	0.09075	0.23025	0.3360	0.3430

$$\begin{aligned}\mathrm{E}(X) &= (0 \times 0.09075) + (1 \times 0.23025) + (2 \times 0.3360) + (3 \times 0.3430) \\ &= 1.93\end{aligned}$$

So the expected number of goals scored will be 1.93 goals.

- 1 working mark for calculating the probabilities for each value of x
- 1 answer mark for the correct discrete probability distribution
- 1 answer mark for the correct expected number of goals, 1.93 correct to 2 decimal places.

e.

Let Y be the number of goals Michael blocks.

$$\begin{aligned}Y &\sim \mathrm{Bi}(5, 0.4) \\ \Pr(Y=2|Y\geq 1) &= \frac{\Pr(Y=2 \cap Y \geq 1)}{\Pr(Y \geq 1)} \\ &= \frac{\Pr(Y=2)}{\Pr(Y\geq 1)} \\ &= \frac{0.34560}{0.92224} \\ &= 0.3747\end{aligned}$$

- 1 method mark for the correct use of conditional probability
- 1 answer mark for the correct values of $\Pr(Y=2)$ and $\Pr(Y \geq 1)$
- 1 answer mark for the correct value of $\Pr(Y=2|Y\geq 1)$, 0.3747

f.

$$\begin{aligned}\Pr(Y \geq 2) &> 0.7 \\ 1 - \Pr(Y<2) &> 0.7 \\ \Pr(Y<2) &< 0.3 \\ \Pr(Y=0) + \Pr(Y=1) &< 0.3 \\ Y &\sim \mathrm{Bi}(n, 0.4) \\ \frac{n!}{0!(n-0)!}(0.4)^0(0.6)^{n-0} + \frac{n!}{1!(n-1)!}(0.4)^1(0.6)^{n-1} &< 0.3 \\ 0.6^n + 0.4 \times 0.6^{n-1}n &< 0.3 \\ n &= 5.32213 \\ \therefore n &= 6\end{aligned}$$

At least 6 shots are required so that the probability of blocking at least 2 shots is greater than 0.7.

- 1 method mark manipulating the expression into $\Pr(Y=0) + \Pr(Y=1) < 0.3$
- 1 method mark for the correct use of a binomial distribution
- 1 answer mark for the correct value of n, 6

PROBABILITY

TECH-ACTIVE TEST 2

MODEL SOLUTIONS AND MARKING SCHEME

SECTION 1 - Multiple-choice questions

Question	1	2	3	4	5	6	7	8	9	10	11
Answer	C	A	A	E	D	D	C	D	B	E	D

SECTION 2 - Extended Response questions

Question 1

a.

$$\begin{aligned}
X &\sim \mathrm{Bi}(25, p)\\
\Pr(X > 22) &= 4\Pr(X = 25)\\
\Pr(X = 23) + \Pr(X = 24) + \Pr(X = 25) &= 4\Pr(X = 25)\\
\Pr(X = 23) + \Pr(X = 24) &= 3\Pr(X = 25)\\
\frac{25!}{23!2!}p^{23}(1-p)^2 + \frac{25!}{24!1!}p^{24}(1-p)^1 &= 3 \times \frac{25!}{25!0!}p^{25}(1-p)^0\\
300p^{23}(1-p)^2 + 25p^{24}(1-p) &= 3p^{25}\\
p^{23}\left(272p^2 - 575p + 300\right) &= 0\\
p^{23}(16p-15)(17p-20) &= 0\\
p = 0 \quad \text{or} \quad 16p - 15 = 0 &\quad \text{or} \quad 17p - 20 = 0\\
p = 0 \quad \text{or} \quad p = \frac{15}{16} &\quad \text{or} \quad p = \frac{20}{17}\\
\text{But } 0 < p \le 1 &\\
\therefore p &= \frac{15}{16}
\end{aligned}$$

- 1 method mark is awarded for correct application of a binomial distribution
- 1 method mark is awarded for arriving at an expression in terms of p and applying the null factor law
- 1 answer mark is awarded for the correct value of p, $\frac{15}{16}$

bi

$$\begin{aligned}
23 &= 25p \\
\therefore p &= \frac{23}{25} \\
\operatorname{Var}(X) &= 25 \times \frac{23}{25}\left(1 - \frac{23}{25}\right) \\
&= \frac{46}{25} \\
\operatorname{sd}(X) &= \sqrt{\frac{46}{25}} \\
&= 1.36
\end{aligned}$$

- 1 method mark for the manipulation of $\mathrm{E}(X) = np$ and $\operatorname{Var}(X) = np(1-p)$
- 1 answer mark for the correct value of $\operatorname{sd}(X)$, 1.36

bii.

$$\begin{aligned}
X &\sim \operatorname{Bi}(25, 0.92) \\
Y &\sim \mathrm{N}\left(\mu, \sigma^2\right) \\
\Pr(X = 25) &= 0.124364 \\
\Pr(X \geq 22) &= 0.864908 \\
\Pr(Y \leq 34.2) &= 0.124364 \\
\Pr(Y \geq 34.5) &= 0.864908 \\
\Pr\left(Z \leq \frac{34.2 - \mu}{\sigma}\right) &= 0.124364 \\
\frac{34.2 - \mu}{\sigma} &= -1.15344....[1] \\
\Pr\left(Z \geq \frac{34.5 - \mu}{\sigma}\right) &= 0.864908 \\
1 - \Pr\left(Z \leq \frac{34.5 - \mu}{\sigma}\right) &= 0.864908 \\
\Pr\left(Z \leq \frac{34.5 - \mu}{\sigma}\right) &= 0.135092 \\
\frac{34.5 - \mu}{\sigma} &= -1.10264....[2] \\
\therefore \mu &= 41.0117 \\
\therefore \sigma &= 5.9055
\end{aligned}$$

- 1 answer mark for defining the two distributions, $X \sim \operatorname{Bi}(25, 0.92)$ and $Y \sim \mathrm{N}\left(\mu, \sigma^2\right)$
- 1 answer mark for the two values of $\Pr(Y \leq 34.2) = 0.124364$ and $\Pr(Y \geq 34.5) = 0.864908$
- 1 method mark for transformation to a standard normal distribution
- 1 answer mark for the correct values of μ and σ, that is 41.0117 and 5.9055 respectively, correct to 4 decimal places

biii.

$$\begin{aligned}
Y &\sim \mathrm{N}(41.0117, 34.8751) \\
\Pr(Y < 30) &= 0.03112 \\
\Pr(Y < 34) &= 0.11755 \\
\Pr(Y < 30 | Y < 34) &= \frac{\Pr(Y < 30 \cap Y < 34)}{\Pr(Y < 34)} \\
&= \frac{\Pr(Y < 30)}{\Pr(Y < 34)} \\
&= \frac{0.03112}{0.11755} \\
&= 0.2647
\end{aligned}$$

So the probability that the journey takes less than 30 minutes, given that the train runs on time is 0.2647.

- 1 answer mark for the correct values of $\Pr(Y < 30)$ and $\Pr(Y \leq 34)$, 0.03112 and 0.11755 respectively.
- 1 method mark for the correct application of conditional probability
- 1 answer mark for the correct value of $\Pr(Y < 30 | Y < 34)$, 0.2647

Question 2

a.

$$\begin{aligned}
T &= \begin{bmatrix} p & p + \frac{3}{20} \\ 1 - p & \frac{17}{20} - p \end{bmatrix} \\
\begin{bmatrix} \frac{p+\frac{3}{20}}{1-p+p+\frac{3}{20}} \\ \frac{1-p}{1-p+p+\frac{3}{20}} \end{bmatrix} &= \begin{bmatrix} \frac{7}{23} \\ \frac{16}{23} \end{bmatrix} \\
\frac{p + \frac{3}{20}}{1 - p + p + \frac{3}{20}} &= \frac{7}{23} \\
p + \frac{3}{20} &= \frac{7}{23} \times \frac{23}{20} \\
p &= \frac{7 - 3}{20} \\
\therefore p &= \frac{1}{5} \text{ as required}
\end{aligned}$$

OR (continued from line 3)

$$
\begin{aligned}
\frac{1-p}{1-p+p+\frac{3}{20}} &= \frac{16}{23} \\
1-p &= \frac{16}{23} \times \frac{23}{20} \\
p &= \frac{20-16}{20} \\
\therefore p &= \frac{1}{5} \text{ as required}
\end{aligned}
$$

- 1 answer mark for the correct transition matrix in terms of p
- 1 method mark for applying the steady state formula to the transition matrix
- 1 method mark for working coming to the correct result for p, $\frac{1}{5}$

b.

$$
\begin{aligned}
S_0 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
T &= \begin{bmatrix} \frac{1}{5} & \frac{1}{5}+\frac{3}{20} \\ 1-\frac{1}{5} & \frac{17}{20}-\frac{1}{5} \end{bmatrix} \\
\therefore T &= \begin{bmatrix} \frac{1}{5} & \frac{7}{20} \\ \frac{4}{5} & \frac{13}{20} \end{bmatrix} \\
S_{14} &= T^{14} \times S_0 \\
&= \begin{bmatrix} \frac{1}{5} & \frac{7}{20} \\ \frac{4}{5} & \frac{13}{20} \end{bmatrix}^{14} \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0.304348 \\ 0.695652 \end{bmatrix}
\end{aligned}
$$

So the probability that Jake attends the 15^{th} event given that he didn't attend the first is 0.3043, correct to 4 decimal places.

- 1 answer mark for the correct transition matrix T
- 1 method mark for the application of $S_n = T^n \times S_0$
- 1 answer mark for the correct value of 0.3043

c.

$$\begin{aligned}
S_3 &= \begin{bmatrix}0.3020\\0.6980\end{bmatrix}\\
S_3 &= T^3 \times S_0\\
\begin{bmatrix}0.3020\\0.6980\end{bmatrix} &= \begin{bmatrix}\frac{1}{5} & \frac{7}{20}\\ \frac{4}{5} & \frac{13}{20}\end{bmatrix}^3 \times S_0\\
S_0 &= \begin{bmatrix}1\\0\end{bmatrix}
\end{aligned}$$

So Jake did attend the first event.

- 1 method mark for the correct use of $S_3 = T^3 \times S_0$ and the associated values
- 1 answer mark for $S_0 = \begin{bmatrix}1\\0\end{bmatrix}$ and the interpretation, stating that Jake did attend the first event.

Question 3

a.

$$\begin{aligned}
1 &= 0.3 + a + 0.2 + 0.05 + b\\
a + b &= 0.45....[1]\\
\mathrm{E}(X) &= (0 \times 0.3) + (1 \times a) + (2 \times 0.2) + (3 \times 0.05) + (4 \times b)\\
&= a + 4b + 0.55\\
\mathrm{E}\left(X^2\right) &= \left(0^2 \times 0.3\right) + \left(1^2 \times a\right) + \left(2^2 \times 0.2\right) + \left(3^2 \times 0.05\right) + \left(4^2 \times b\right)\\
&= a + 16b + 1.25\\
(1.06184)^2 &= a + 16b + 1.25 - (a + 4b + 0.55)^2\\
0.1800 &= -a^2 - 16b^2 - a(8b + 0.1) + 11.6b....[2]\\
a = -0.5 \text{ and } b = 0.95 &\text{ or } a = 0.4 \text{ and } b = 0.05
\end{aligned}$$

But $0 \leq a \leq 1$ and $0 \leq b \leq 1$

$$\begin{aligned}
\therefore a &= 0.4\\
\therefore b &= 0.05
\end{aligned}$$

- 1 answer mark for the correct expression for $\mathrm{E}\left(X^2\right)$ in terms of a and b
- 1 method mark finding two equations in terms of a and b
- 1 answer mark for finding the values of a and b to be 0.20 and 0.05 respectively, correct to 2 decimal places

b.

$$\begin{aligned}
X = 1,\ Y &= 2.00 \\
X = 2,\ Y &= 4.00 \\
X = 3,\ Y &= 4.50 \\
X = 4,\ Y &= 6.00 \\
\Pr(Y = y | X \geq 1) &= \frac{\Pr(Y = y \cap X \geq 1)}{\Pr(X \geq 1)} \\
\Pr(Y = 2.00 | X \geq 1) &= \frac{0.4}{0.4 + 0.2 + 0.05 + 0.05} \\
&= \frac{4}{7} \\
\Pr(Y = 4.00 | X \geq 1) &= \frac{0.2}{0.4 + 0.2 + 0.05 + 0.05} \\
&= \frac{2}{7} \\
\Pr(Y = 4.50 | X \geq 1) &= \frac{0.05}{0.4 + 0.2 + 0.05 + 0.05} \\
&= \frac{1}{14} \\
\Pr(Y = 6.00 | X \geq 1) &= \frac{0.05}{0.4 + 0.2 + 0.05 + 0.05} \\
&= \frac{1}{14}
\end{aligned}$$

y	2	4	4.5	6
$\Pr(Y = y)$	$\frac{4}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	$\frac{1}{14}$

- 1 answer mark for the correct values of Y corresponding to the values of X
- 1 method mark for the use of conditional probability, and reducing the sample space
- 2 answers mark for the correct values of y and $\Pr(Y = y)$ set out in a discrete probability distribution table

c.

$$\begin{aligned}
\mathrm{E}(Y) &= \left(2 \times \frac{4}{7}\right) + \left(4 \times \frac{2}{7}\right) + \left(4.5 \times \frac{1}{14}\right) + \left(6 \times \frac{1}{14}\right) \\
&= \$3.04 \\
m &= \$2.00
\end{aligned}$$

The expected price and median price are \$3.04 and \$2.00 respectively.

- 1 answer mark for the correct value of $\mathrm{E}(Y)$, \$3.04 correct to 2 decimal places
- 1 answer mark for the correct value of m, \$2.00 correct to 2 decimal places

SET 1 EXAM 1

MODEL SOLUTIONS AND MARKING SCHEME

Question 1

a.

$$\begin{aligned}\frac{d}{dx}\left(2\cos\left(2x+3\right)\right) &= -2\sin\left(2x+3\right)\times 2\\ &= -4\sin\left(2x+3\right)\end{aligned}$$

- 1 answer mark is awarded for the correct derivative, $-4\sin\left(2x+3\right)$

b.

$$\begin{aligned}f'\left(x\right) &= uv' + vu'\\ &= \left(3x^3\times\frac{1}{x}\right)+\left(\left(\log_e\left(2x\right)-1\right)\times 9x^2\right)\\ &= 3x^2+9x^2\left(\log_e\left(2x\right)-1\right)\\ &= 3x^2\left(3\log_e\left(2x\right)-2\right)\\ f'\left(2\right) &= 3\left(2\right)^2\left(3\log_e\left(2\times 2\right)-2\right)\\ &= 12\left(3\log_e\left(4\right)-2\right)\\ &= 72\log_e\left(2\right)-24\end{aligned}$$

- 1 method mark is awarded for the use of the product rule arriving at the expression $3x^2\left(3\log_e\left(2x\right)-2\right)$ or equivalent
- 1 answer mark is awarded for the correct value of $f'\left(2\right)$, $72\log_e\left(2\right)-24$ or equivalent

c.

$$\begin{aligned}3x^3\log_e\left(2x\right)-3x^3 &= \int\left(9x^2\log_e\left(2x\right)-6x^2\right)dx+C_1\\ 3x^3\log_e\left(2x\right)-3x^3 &= 9\int\left(x^2\log_e\left(2x\right)\right)dx-6\int\left(x^2\right)dx+C_1\\ 9\int\left(x^2\log_e\left(2x\right)\right)dx &= 3x^3\log_e\left(2x\right)-3x^3+6\int\left(x^2\right)dx+C_2\\ \therefore\int\left(x^2\right)\log_e\left(2x\right)dx &= \frac{1}{9}x^3\left(3\log_e\left(2x\right)-1\right)+C_3\end{aligned}$$

- 1 method mark is awarded for the use of $f\left(x\right)=\int f'\left(x\right)dx$
- 1 answer mark is awarded for the correct anti-derivative of $x^2\log_e\left(2x\right)$, $\frac{1}{9}x^3\left(3\log_e\left(2x\right)-1\right)$ or equivalent, the $+C$ is not necessary, a mark is award for expressions with or without the $+C$

Question 2

a.

$$\int \left(\frac{1}{2x-1}+4\right) dx \quad = \quad \frac{1}{2}\log_e\left(|2x-1|\right)+4x+C$$

Where C is not necessary.

- 1 answer mark is awarded for the correct anti-derivative, $\frac{1}{2}\log_e\left(|2x-1\right)+4x+C$. The mark is still awarded regardless if the given expression contains the $+C$ or not.

b.

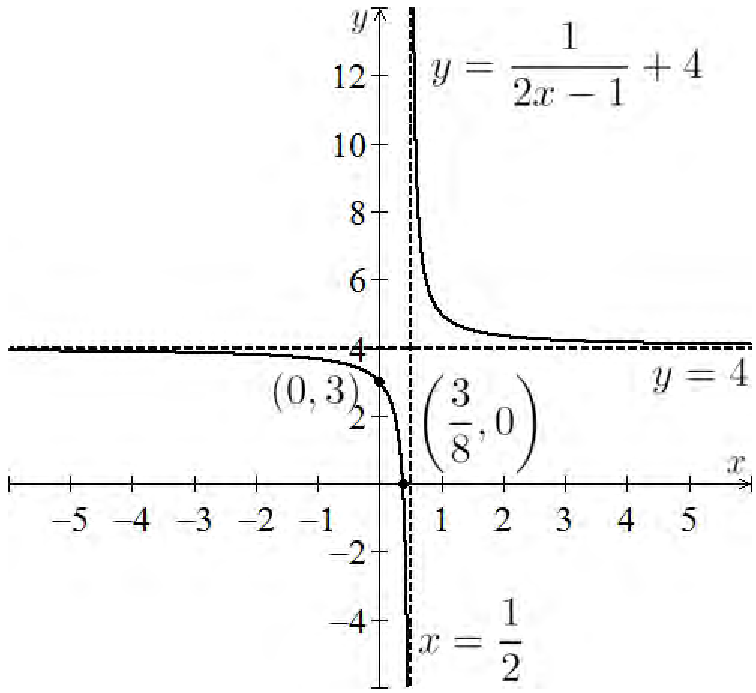
$$\begin{aligned}
\int_1^3 \left(\frac{1}{2x-1}+4\right) dx &= \left[\frac{1}{2}\log_e\left(|2x-1|\right)+4x\right]_1^3 \\
&= \frac{1}{2}\log_e\left(|6-1|\right)+12-\left(\frac{1}{2}\log_e\left(|2-1|\right)+4\right) \\
&= \frac{1}{2}\log_e(5)+8 \\
&= \frac{1}{2}\log_e\sqrt{5}+8 \\
\log_e(a)+b &= \log_e\sqrt{5}+8 \\
\therefore a &= \sqrt{5} \\
\therefore b &= 8
\end{aligned}$$

- 1 method mark is awarded for the use of log laws
- 1 answer mark is awarded for the correct values of a and b, $\sqrt{5}$ and 8 respectively

c.

$$\begin{aligned}
x=0 \quad y &= 3 \\
&(0,3) \\
y=0 & \\
0 &= \frac{1}{2x-1}+4 \\
\frac{1}{2x-1} &= -4 \\
-8x+4 &= 1 \\
\therefore x &= \frac{3}{8} \\
&\left(\frac{3}{8},0\right)
\end{aligned}$$

Asymptotes at $x=\frac{1}{2}$ and $y=4$



- 1 answer mark is awarded for the correct shape of the curve
- 1 answer mark is awarded for the correctly labeled intercepts of the curve, $(0,3)$ and $\left(\dfrac{3}{8}, 0\right)$
- 1 answer mark is awarded for the correctly labeled dotted asymptotes, $x = \dfrac{1}{2}$ and $y = 4$

Question 3

$$
\begin{aligned}
\text{Let } a &= 3^x \\
2a^2 - 48a - 162 &= 0 \\
2\left(a^2 - 24a - 81\right) &= 0 \\
2(a+3)(a-27) &= 0 \\
a + 3 = 0 \quad &\text{or} \quad a - 27 = 0 \\
a = -3 \quad &\text{or} \quad a = 27 \\
3^x = -3 \quad &\text{or} \quad 3^x = 27 \\
\text{no solution} \quad & \quad 3^x = 3^3 \\
& \quad \therefore x = 3
\end{aligned}
$$

- 1 method mark is awarded for letting $a = 3^x$
- 1 method mark is awarded for correctly factorising to $2(a+3)(a-27) = 0$ (or equivalent) and applying the null factor law
- 1 answer mark is awarded for finding the only solution to be $x = 3$

Question 4

$$\begin{aligned}
&\text{For inverse, swap } x \text{and } y \\
x &= 2e^{-3y}-2 \\
\frac{x+2}{2} &= e^{-3y} \\
\log_e\left(\frac{x+2}{2}\right) &= -3y \\
\therefore y &= -\frac{1}{3}\log_e\left(\frac{x+2}{2}\right) \\
\therefore f^{-1}(x) &= -\frac{1}{3}\log_e\left(\frac{x+2}{2}\right) \\
f(-1) &= 2e^3-2 \\
f\left(\frac{4}{3}\log_e(2)\right) &= 2e^{-3\times\frac{4}{3}\log_e(2)}-2 \\
&= 2e^{\log_e\left(\frac{1}{2^4}\right)}-2 \\
&= \frac{2^1}{2^4}-2 \\
&= -\frac{15}{8} \\
\text{Ran } f &= \left[-\frac{15}{8}, 2e^3-2\right) \\
\text{Dom } f^{-1} &= \text{Ran } f \\
\therefore \text{Dom } f^{-1} &= \left[-\frac{15}{8}, 2e^3-2\right) \\
f^{-1}: \left[-\frac{15}{8}, 2e^3-2\right) \to R, \quad f^{-1}(x) &= -\frac{1}{3}\log_e\left(\frac{x+2}{2}\right)
\end{aligned}$$

- 1 method mark is awarded for swapping x and y and solving for x
- 1 answer mark is awarded for finding the correct expression for f^{-1}, $f^{-1}(x) = -\frac{1}{3}\log_e\left(\frac{x+2}{2}\right)$
- 1 answer mark is awarded for finding the correct domain for f^{-1}, $\left[-\frac{15}{8}, 2e^3-2\right)$ with the correct brackets

Question 5

a.

$$\begin{aligned}
\text{Ran } f &= [2-5, 2+5] \\
\text{Ran } f &= [-3, 7] \\
\text{Period} &= 2\pi \div \frac{\pi}{2} \\
&= 4
\end{aligned}$$

- 1 answer mark is awarded for the correct range, $[-3, 7]$
- 1 answer mark is awarded for the correct period, 4

b.

$$\begin{aligned}
0 &\le x \le && 2\pi \\
0 &\le 2x \le && 4\pi \\
0-\frac{\pi}{2} &\le 2x-\tfrac{\pi}{2} \le && 4\pi-\frac{\pi}{2} \\
-\frac{\pi}{2} &\le 2x-\tfrac{\pi}{2} \le && \frac{7\pi}{2} \\
2\sin\left(2x-\frac{\pi}{2}\right)+3 &= && 2 \\
\sin\left(2x-\frac{\pi}{2}\right) &= && -\frac{1}{2} \\
2x-\frac{\pi}{2} &= && -\frac{5\pi}{6},\ -\frac{\pi}{6},\ -\frac{5\pi}{6}+2\pi,\ -\frac{\pi}{6}+2\pi,\ -\frac{5\pi}{6}+4\pi,\ -\frac{\pi}{6}+4\pi \\
&= && -\frac{5\pi}{6},\ -\frac{\pi}{6},\ \frac{7\pi}{6},\ \frac{11\pi}{6},\ \frac{19\pi}{6},\ \frac{23\pi}{6} \\
2x &= && -\frac{\pi}{6}+\frac{3\pi}{6},\ \frac{7\pi}{6}+\frac{3\pi}{6},\ \frac{11\pi}{6}+\frac{3\pi}{6},\ \frac{19\pi}{6}+\frac{3\pi}{6} \\
2x &= && \frac{\pi}{3},\ \frac{5\pi}{3},\ \frac{7\pi}{3},\ \frac{22\pi}{3} \\
\therefore x &= && \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{7\pi}{6},\ \frac{11\pi}{6}
\end{aligned}$$

- 1 method mark is awarded for finding the correct reference angle, $-\frac{\pi}{6}$ or equivalent
- 1 answer mark is awarded for the four solutions for x, $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Question 6

a.

$$f(x) = \begin{cases} kx(x-1)(x-3) & 0 \le x \le 1 \\ -kx(x-1)(x-3) & 1 \le x \le 3 \end{cases}$$

$$\begin{aligned}
\int_0^1 (kx(x-1)(x-3))\,dx - \int_1^3 kx(x-1)(x-3)\,dx &= 1 \\
k\int_0^1 \left(x^3-4x^2+3x\right)dx - k\int_1^3 \left(x^3-4x^2+3x\right)dx &= 1 \\
k\left[\frac{1}{4}x^4-\frac{4}{3}x^3+\frac{3}{2}x^2\right]_0^1 - k\left[\frac{1}{4}x^4-\frac{4}{3}x^3+\frac{3}{2}x^2\right]_1^3 &= 1 \\
k\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}-0\right) - k\left(\frac{81}{4}-\frac{108}{3}+\frac{27}{2}-\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}\right)\right) &= 1 \\
k\left(2\left(\frac{3-16+18}{12}\right)-\left(\frac{81+54}{4}-36\right)\right) &= 1 \\
k\left(\frac{5}{6}+\frac{9}{4}\right) &= 1 \\
\frac{37}{12}k &= 1 \\
\therefore k &= \frac{12}{37}
\end{aligned}$$

- 1 method mark is awarded for setting up the correct integrals and terminals and equating to 1
- 1 answer mark is awarded for the correct value of k, $k = \frac{12}{37}$

b.

$$\begin{aligned}
\Pr(X \le 2) &= \int_0^1 \frac{12}{37}(x(x-1)(x-3))\,dx - \int_1^2 \frac{12}{37}(x(x-1)(x-3))\,dx \\
&= \frac{12}{37}\left(\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}-0\right)-\left(\frac{1}{4}(16)-\frac{4}{3}(8)+\frac{3}{2}(4)-\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}\right)\right)\right) \\
&= \frac{12}{37}\left(\frac{5}{6}-\frac{30-32}{3}\right) \\
&= \frac{12}{37}\left(\frac{9}{6}\right) \\
\therefore \Pr(X \le 2) &= \frac{18}{37}
\end{aligned}$$

- 1 method mark is awarded for the correct integrals and terminals
- 1 answer mark is awarded for the value of $\Pr(X \le 2)$, $\frac{18}{37}$

c.

$$\begin{aligned}
\Pr(X \le 2) + \Pr(X \ge 2) &= 1 \\
\Pr(X \ge 2) &= 1 - \Pr(X \le 2) \\
&= \frac{19}{37} \\
\Pr(X \ge 1) &= -\int_1^3 \frac{12}{37}\left(x^3 - 4x^2 + 3x\right)dx \\
&= -\frac{12}{37}\int_1^3 \left(x^3 - 4x^2 + 3x\right)dx \\
&= -\frac{12}{37}\left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_1^3 \\
&= -\frac{12}{37}\left(\frac{81}{4} - 36 + \frac{27}{2} - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right)\right) \\
&= -\frac{12}{37}\left(\frac{135}{4} - \frac{5}{12} - 36\right) \\
&= \frac{12}{37}\left(\frac{100}{3} - 36\right) \\
&= \frac{32}{37}
\end{aligned}$$

-OR-

$$\begin{aligned}
\Pr(1 \le X \le 3) &= \frac{8}{3} \div \left(\frac{5}{12} + \frac{8}{3}\right) \\
&= \frac{8}{3} \times \frac{12}{37} \\
&= \frac{32}{37}
\end{aligned}$$

Returning to previous working

$$\begin{aligned}
\Pr(X \ge 2 | X \ge 1) &= \frac{\Pr(X \ge 2 \cap X \ge 1)}{\Pr(X \ge 1)} \\
&= \frac{\Pr(X \ge 2)}{\Pr(X \ge 1)} \\
&= \frac{19}{37} \div \frac{32}{37} \\
&= \frac{19}{37} \times \frac{37}{32} \\
&= \frac{19}{32}
\end{aligned}$$

- 1 method mark is awarded for the correct integrals and terminals or the use of area ratios from **part a**
- 1 method mark is awarded for the use of conditional probability
- 1 answer mark is awarded for the value of $\Pr(X \ge 2 | X \ge 1)$, $\frac{19}{32}$

Question 7

Method 1

$$\begin{aligned}
mx + y &= m - 2 \\
\therefore y &= -mx + (m-2) \\
6x + (m-1)\,y &= 12 \\
\therefore y &= -\frac{6}{m-1}x + \frac{12}{m-1} \\
-m &= -\frac{6}{m-1} \\
m^2 - m - 6 &= 0 \\
(m-3)\,(m+2) &= 0 \\
m - 3 = 0 \quad &\text{or} \quad m + 2 = 0 \\
\therefore m &= 3,\ -2 \\
m - 2 &\neq \frac{12}{m-1} \\
(m-2)\,(m-1) &\neq 12 \\
m^2 - 3m + 2 &\neq 12 \\
m^2 - 3m - 10 &\neq 0 \\
(m-5)\,(m+2) &\neq 0 \\
m - 5 \neq 0 \quad &\text{or} \quad m + 2 \neq 0 \\
\therefore m &\neq 5,\ -2 \\
\\
\therefore m &= 3
\end{aligned}$$

The two simultaneous equations have no solutions only when $m = 3$.

- 1 method mark is awarded for equating the gradients and y intercepts
- 1 method mark is awarded for disregarding the $m = -2$ solution
- 1 answer mark is awarded for the value of $m = 3$ only (this mark will not be awarded if $m = -2$ is not rejected)

Method 2

$$
\begin{aligned}
\begin{bmatrix} m & 1 \\ 6 & m-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 \\ 12 \end{bmatrix} \\
m\left(m-1\right) - \left(1 \times 6\right) &= 0 \\
m^2 - m - 6 &= 0 \\
\left(m-3\right)\left(m+2\right) &= 0 \\
m - 3 = 0 \quad &\text{or} \quad m + 2 = 0 \\
\therefore m &= -2,\ 3 \\
\text{If } m = -2 & \\
-2x + y &= -4 \\
6x - 3y &= 12 \\
\therefore m = -2 \text{ is not a solution} & \\
\text{If } m = 3 & \\
3x + y &= 3 \\
6x + 2y &= 12 \\
y &= 3 - 3x \\
y &= 6 - 3x \\
\therefore m &= 3
\end{aligned}
$$

The two simultaneous equations have no solutions only when $m = 3$.

- 1 method mark is awarded for writing out the matrix equation
- 1 method mark is awarded for disregarding the $m = -2$ solution
- 1 answer mark is awarded for the value of $m = 3$ only (this mark will not be awarded if $m = -2$ is not rejected)

Question 8

$$\begin{aligned}
V &= \pi r^2 h \\
&= 3\pi r^2 \\
\frac{dV}{dt} &= 5\pi \text{ mm}^3/\text{s} \\
\frac{dV}{dr} &= 6\pi r \\
\frac{dr}{dV} &= \frac{1}{6\pi r} \\
\frac{dr}{dt} &= \frac{dr}{dV} \times \frac{dV}{dt} \\
&= \frac{1}{6\pi \times 10} \times 5\pi \\
&= \frac{1}{12} \text{ mm/s}
\end{aligned}$$

The radius of the puddle is increasing at $\frac{1}{12}$ mm/s.

- 1 method mark is awarded for the use of the chain rule
- 1 answer mark is awarded for the correct rate for $\frac{dr}{dt}$, $\frac{1}{12}$ mm/s

Question 9

a.

$$\begin{aligned}
\cos(\theta) &= \frac{AB}{x} \\
\therefore AB &= x\cos(\theta) \\
\sin(\theta) &= \frac{AF}{x} \\
\therefore AF &= x\sin(\theta)
\end{aligned}$$

- 1 answer mark is awarded for the correct expression for AB
- 1 answer mark is awarded for the correct expression for AF

b.

$$\begin{aligned}
V &= (x)(x)(x\sin(\theta)) + 2 \times \frac{1}{2}(x\cos(\theta))(x\sin(\theta))(x) \\
&= x^3 \sin\theta + x^3 \cos(\theta)\sin(\theta)
\end{aligned}$$

- 1 answer mark is awarded for the correct expression, $V = x^3 \sin\theta + x^3 \cos(\theta)\sin(\theta)$

c.

$$\begin{aligned}
\frac{dV}{d\theta} &= x^3\cos(\theta) + x^3\left(\cos(\theta)\times\cos(\theta) + \sin(\theta)\times -\sin(\theta)\right) \\
&= x^3\cos(\theta) + x^3\left(\cos^2(\theta) - \sin^2(\theta)\right) \\
&= x^3\left(\cos^2(\theta) + \cos(\theta) - \left(1-\cos^2(\theta)\right)\right) \\
&= x^3\left(2\cos^2(\theta) + \cos(\theta) - 1\right) \\
x^3\left(2\cos^2(\theta) + \cos(\theta) - 1\right) &= 0 \\
x^3 = 0 \quad &\text{or} \quad 2\cos^2(\theta) + \cos(\theta) - 1 = 0 \\
\left(2\cos(\theta) - 1\right)\left(\cos(\theta) + 1\right) &= 0 \\
2\cos(\theta) - 1 = 0 \quad &\text{or} \quad \cos(\theta) + 1 = 0 \\
\cos(\theta) = \frac{1}{2} \quad &\text{or} \quad \cos(\theta) = -1 \\
\text{But } 0 < \theta < \frac{\pi}{2} & \\
\therefore \theta &= \frac{\pi}{3}
\end{aligned}$$

- 1 answer mark is awarded for the derivative, $\frac{dV}{d\theta}$
- 1 method mark is awarded for factorising to the expression $\left(2\cos(\theta) - 1\right)\left(\cos(\theta) + 1\right) = 0$ and applying the null factor law
- 1 answer mark is awarded for the correct expression the correct value of θ, $\frac{\pi}{3}$

d.

$$\begin{aligned}
V &= x^3\sin(\theta) + \frac{1}{2}x^3\cos(\theta)\sin(\theta) \\
x = 5\,, \theta = \frac{\pi}{3} & \\
V_{max} &= (5)^3\sin\left(\frac{\pi}{3}\right) + \frac{1}{2}(5)^3\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) \\
&= \frac{125\sqrt{3}}{2} + 125\times\frac{1}{2}\times\frac{\sqrt{3}}{2} \\
&= \frac{375\sqrt{3}}{4}\,m^3
\end{aligned}$$

The maximum volume of the structure when $x = 5$ is $\frac{375\sqrt{3}}{4}\,m^3$.

- 1 answer mark is awarded for the maximum volume, $\frac{375\sqrt{3}}{4}\,m^3$ or equivalent

SET 1 EXAM 2

MODEL SOLUTIONS AND MARKING SCHEME

SECTION 1 - Multiple-choice questions

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
Answer	C	E	A	E	D	A	A	D	B	A	A	B	B	C	D	C	E	D	E	B	D	C

SECTION 2 - Extended Response questions

Question 1

a.

$$\begin{aligned}
f'(x) &= \frac{1}{16}\Big((3x-a)^2 \times 2b(bx-2) + (bx-2)^2 \times 6(3x-a)\Big) \\
&= \frac{1}{16}(3x-a)(bx-2)\Big(2b(3x-a)+6(bx-2)\Big) \\
&= \frac{1}{16}(3x-a)(bx-2)\Big(12bx-2ab-12\Big) \\
&= \frac{1}{8}(3x-a)(bx-2)(6bx-ab-6) \\
f'(x) &= 0 \\
\frac{1}{8}(3x-a)(bx-2)(6bx-ab-6) &= 0
\end{aligned}$$

$$\begin{aligned}
3x-a=0 \text{ or } \quad bx-2=0 \quad &\text{or } 6bx-ab-6=0 \\
x=\frac{a}{3} \text{ or } \quad x=\tfrac{2}{b} \quad &\text{or } x=\frac{ab+6}{6b},\ b\neq 0
\end{aligned}$$

$$\begin{aligned}
f\left(\frac{a}{3}\right) &= \frac{1}{16}\left(3\times\frac{a}{3}-a\right)^2\left(b\times\frac{a}{3}-2\right)^2-4 \\
&= -4 \\
f\left(\frac{2}{b}\right) &= \frac{1}{16}\left(3\times\frac{2}{b}-a\right)^2\left(b\times\frac{2x}{b}-2\right)^2-4 \\
&= -4 \\
f\left(\frac{ab+6}{6b}\right) &= \frac{1}{16}\left(3\times\frac{ab+6}{6b}-a\right)^2\left(b\times\frac{ab+6}{6b}-2\right)^2-4 \\
&= \frac{1}{16}\left(\frac{6-ab}{2b}\right)^2\left(\frac{ab-6}{6}\right)^2-4 \\
&= \frac{1}{2304b^2}(ab-6)^4-4
\end{aligned}$$

The stationary points are at $\left(\frac{a}{3},-4\right)$, $\left(\frac{2}{b},-4\right)$ and $\left(\frac{ab+6}{6b}, \frac{1}{2304b^2}(ab-6)^4-4\right)$ for $b\neq 0$.

- 1 method mark is awarded for the correct use of the product rule for differentiation
- 1 answer mark is awarded for the correct derivative, $f'(x)$ in terms of a and b
- 1 answer mark is awarded for the correct values of x required in terms of a and b
- 1 answer mark is awarded for the correct coordinated in terms of a and b

bi.

$$\begin{aligned}
b = 2 & & \\
f'(x) &= \frac{1}{8}(3x-a)(2x-2)(6\times 2x - 2\times a - 6) \\
&= \frac{1}{2}(3x-a)(x-1)(6x-a-3) \\
&= 0 \\
\text{For } n = 3 & & \\
3x - a = 0 &\quad \text{or} \quad 6x - a - 3 = 0 \ \text{ or } \ x - 1 = 0 \\
x = \frac{a}{3} &\quad \text{or} \quad x = \frac{a+3}{6} \ \text{ or } \ x = 1 \\
\text{For } n = 2 & & \\
\frac{a}{3} &= \frac{a+3}{6} \\
2a &= a + 3 \\
\therefore a &= 3 \\
x &= \frac{3}{3} \\
\therefore x &= 1
\end{aligned}$$

Only 1 solution when $a = 3$, $\therefore n \neq 2$

$$\therefore n = 1,\ 3$$

- 1 method mark is awarded for a reason rejecting $n = 2$
- 1 answer mark is awarded for the correct values of n, 1, 3

bii.

$$\begin{aligned}
\frac{a}{3} &\neq \frac{a+3}{6} \\
2a &\neq a + 3 \\
a &\neq 3 \\
\therefore a &\in R\backslash\{3\}
\end{aligned}$$

- 1 answer mark is awarded for $a \in R\backslash\{3\}$

biii.

$$
\begin{aligned}
\left(\frac{a}{3},-4\right) &= (3,-4)\\
\left(\frac{2}{b},-4\right) &= (1,-4)\\
\left(\frac{ab+6}{6b},\frac{1}{2304b^2}(ab-6)^4-4\right) &= \left(\frac{18+6}{6\times 2},\frac{1}{2304\times 2^2}(18-6)^4-4\right)\\
&= \left(2,-\frac{7}{4}\right)\\
f'(0) &= 9(0-3)(0-1)(0-2)\\
&= -ve\\
f'\left(\frac{3}{2}\right) &= 9\left(\frac{3}{2}-3\right)\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)\\
&= +ve\\
f'\left(\frac{5}{2}\right) &= 9\left(\frac{5}{2}-3\right)\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right)\\
&= -ve\\
f'(4) &= 9(4-3)(4-1)(4-2)\\
&= +ve
\end{aligned}
$$

x		1		2		3	
$f'(x)$	$-ve$	0	$+ve$	0	$-ve$	0	$+ve$
Shape	$\searrow$	—	$\nearrow$	—	$\searrow$	—	$\nearrow$
		Local Minimum		Local Maximum		Local Minimum	

There are local minimums at $(1,-4)$ and $(3,-4)$ and a local maximum at $\left(2,-\frac{7}{4}\right)$.

- 1 answer mark is awarded for the correct values of x for which there are stationary points, 1, 2, 3
- 1 method mark is awarded for using a sign test table or equivalent
- 1 answer mark is awarded for the three points, $(1,-4)$, $(3,-4)$ and $\left(2,-\frac{7}{4}\right)$.

ci.

$$\begin{aligned}
a = 3,\ b = 2 & \\
f(x) &= \frac{1}{16}(3x-3)^2(2x-2)^2 - 4 \\
&= \frac{9}{4}(x-1)^4 - 4 \\
f'(x) &= 9(x-1)^3 \\
9(x-1)^3 &= 0 \\
x-1 &= 0 \\
\therefore x &= 1 \\
\therefore c &= 1
\end{aligned}$$

- 1 answer mark is awarded for correct value of c, $c = 1$

cii.

For inverse, swap x and y

$$\begin{aligned}
x &= \frac{9}{4}(y-1)^4 - 4 \\
\frac{4}{9}(x+4) &= (y-1)^4 \\
y-1 &= \pm\left(\frac{4}{9}\right)^{\frac{1}{4}}(x+4)^{\frac{1}{4}} \\
y \geq 1, & \quad \text{take positive root} \\
y &= \left(\frac{4}{9}\right)^{\frac{1}{4}}(x+4)^{\frac{1}{4}} + 1 \\
f(1) &= -4 \\
\text{Ran } f &= \left[-4, \infty\right) \\
\text{Dom } f^{-1} &= \text{Ran } f \\
\therefore \text{Dom } f^{-1} &= \left[-4, \infty\right) \\
f^{-1} : \left[-4, \infty\right) \to R,\ f^{-1}(x) &= \left(\frac{4}{9}\right)^{\frac{1}{4}}(x+4)^{\frac{1}{4}} + 1
\end{aligned}$$

- 1 answer mark is awarded for the correct rule for $f^{-1}(x)$, $\left(\frac{4}{9}\right)^{\frac{1}{4}}(x+4)^{\frac{1}{4}} + 1$
- 1 answer mark is awarded for correct domain of $f^{-1}(x)$, $\left[-4, \infty\right)$

ciii.

$$\begin{aligned} f(x) &= x \\ \frac{9}{4}(x-1)^4 - 4 &= x \\ \therefore x &= 2.29 \\ y &= x \\ &= 2.29 \end{aligned}$$

- 1 method mark is awarded for using one of $f(x) = f^{-1}(x)$, $f(x) = x$ or $f^{-1}(x) = x$ to solve for x
- 1 answer mark is awarded for the correct point of intersection, $(2.29, 2.29)$

Question 2

a.

$$\begin{aligned} \frac{5}{r} &= \frac{5\sqrt{3}}{5\sqrt{3}-h} \\ 5\sqrt{3} - h &= \sqrt{3}r \\ h &= \sqrt{3}(5-r) \end{aligned}$$

- 1 method mark is awarded for the correct use of the ratio of the side lengths using similar triangles
- 1 answer mark is awarded for the correct expression for h in terms of r, $h = \sqrt{3}(5-r)$

b.

$$\begin{aligned} V &= \sqrt{3}\pi r^2 (5-r) \\ & 0 < r < 5 \\ V : (0,5) \to R,\ V(r) &= \sqrt{3}\pi r^2 (5-r) \end{aligned}$$

- 1 answer mark is awarded for the correct expression of the volume of the cylinder in terms of the radius, that is $V = \sqrt{3}\pi r^2 (5-r)$
- 1 answer mark is awarded for the correct domain for r, $0 < r < 5$

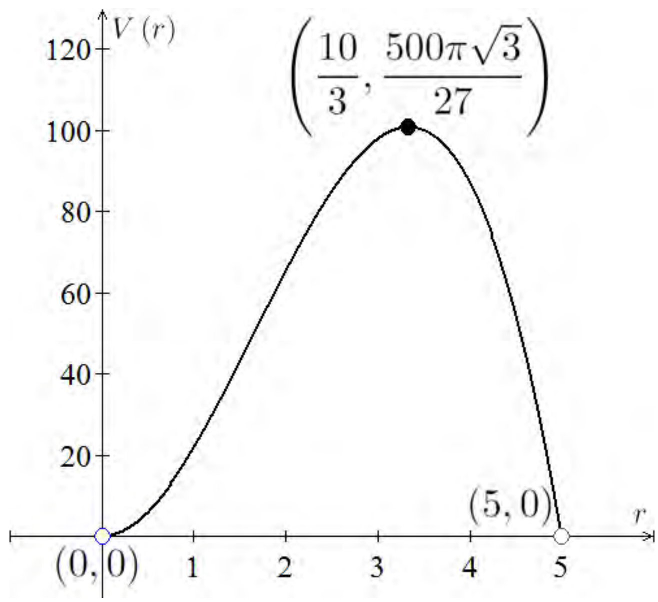
c.

$$\begin{aligned}
V &= \sqrt{3}\pi r^2 (5-r) \\
\frac{dV}{dr} &= \sqrt{3}\pi r^2 \times -1 + (5-r) \times 2\sqrt{3}\pi r \\
&= \sqrt{3}\pi r (2(5-r) - r) \\
&= \sqrt{3}\pi r (10 - 3r) \\
\frac{dV}{dr} &= 0 \\
\sqrt{3}\pi r (10-3r) &= 0 \\
r = 0 \quad &\text{or} \quad 10 - 3r = 0 \\
& \quad\; r = \frac{10}{3} \\
\text{But } 0 < r < 5 & \\
\therefore r &= \frac{10}{3} \text{ cm} \\
V_{max} &= V\left(\frac{10}{3}\right) \\
&= \sqrt{3}\pi \left(\frac{10}{3}\right)^2 \left(5 - \frac{10}{3}\right) \\
&= \frac{500\sqrt{3}\pi}{27} \text{ cm}^3
\end{aligned}$$

The maximum volume of the cylinder is $\frac{500\sqrt{3}\pi}{27}$ cm^3 which occurs when $r = \frac{10}{3}$ cm.

- 1 answer mark is awarded for the correct derivative of V with respect to r, that is $\dfrac{dV}{dr} = \sqrt{3}\pi r (10-3r)$
- 1 method mark is awarded for equating the derivative to 0 and finding value(s) or r
- 1 answer mark is awarded for the correct maximum volume and value of r for which this occurs, $\dfrac{500\sqrt{3}\pi}{27}$ cm^3 and $\dfrac{10}{3}$ cm respectively.

d.



- 1 answer mark is awarded for the correct shape of the function, that includes the gradient being zero at $r = 0$
- 1 answer mark is awarded for the correct correctly labeled endpoints with open circles across the correct domain
- 1 answer mark is awarded for correctly labeling the point for which the volume is a maximum, at the local maximum turning point at $\left(\frac{10}{3}, \frac{500\sqrt{3}\pi}{27}\right)$

If the above three marks is satisfied but the axis are labeled x and y instead of $V(r)$ and r then two marks will be awarded instead of three.

e.

$$\begin{aligned}
\frac{x}{2} \div 5 &= \frac{h}{5\sqrt{3}} \\
x &= \frac{2\sqrt{3}}{3}h \\
V_p &= \frac{1}{3}x^2 h \\
&= \frac{1}{3}\left(\frac{2\sqrt{3}}{3}h\right)^2 h \\
&= \frac{4}{9}h^3 \\
\frac{dV_p}{dh} &= \frac{4 \times 3}{9}h^2 \\
&= \frac{4}{3}h^2 \\
\therefore \frac{dh}{dV_p} &= \frac{3}{4h^2},\ h \neq 0 \\
\frac{dh}{dt} &= \frac{3}{4 \times 5^2} \times 10 \\
&= \frac{3}{10}\ \text{cm/s}
\end{aligned}$$

The rate that the height of the water in the pyramid is increasing when $h = 5$ cm is $\frac{3}{10}$ cm/s.

- 1 answer mark is awarded for obtaining the correct expression for x in terms of h, $x = \dfrac{2\sqrt{3}}{3}h$
- 1 answer mark is awarded for obtaining the correct expression for V_p in terms of h, $V(h) = \dfrac{4}{9}h^3$
- 1 method mark is awarded for applying the chain rule and finding the derivative $\frac{dV_p}{dh}$
- 1 answer mark is awarded for obtaining the correct rate of increasing of h with respect to t, $\dfrac{dh}{dt} = \dfrac{3}{10}$ cm/s. 0.3 cm/s would also be acceptable as this is an exact decimal.

Question 3

ai.

$$\begin{aligned}
T &= \begin{bmatrix} 0.75 & 0.6 \\ 0.25 & 0.4 \end{bmatrix} \\
S_0 &= \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \\
S_4 &= T^4 \times S_0 \\
&= \begin{bmatrix} 0.75 & 0.6 \\ 0.25 & 0.4 \end{bmatrix}^4 \times \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \\
&= \begin{bmatrix} 0.7057 \\ 0.2942 \end{bmatrix}
\end{aligned}$$

The probability that Rohit plays the game on the fifth day is 0.7057.

- 1 answer mark is awarded for the correct transition matrix, $\begin{bmatrix} 0.75 & 0.6 \\ 0.25 & 0.4 \end{bmatrix}$
- 1 method mark is awarded for the use of a Markov chain using T^4
- 1 answer mark is awarded for the correct value of 0.7057

aii.

$$\begin{aligned}
\Pr('\text{Success}') &= \frac{0.60}{0.25 + 0.60} \\
&= \frac{12}{17}
\end{aligned}$$

So the long run probability that Rohit plays the game is $\dfrac{12}{17}$.

- 1 answer mark is awarded for the correct value of $\dfrac{12}{17}$

bi.

$$\begin{aligned} X &\sim \mathrm{Bi}(5, 0.75) \\ \Pr(X=4|X\geq 3) &= \frac{\Pr(X=4\cap X\geq 3)}{\Pr(X\geq 3)} \\ &= \frac{\Pr(X=4)}{\Pr(X\geq 3)} \\ &= \frac{0.395508}{0.896484} \\ &= 0.4412 \end{aligned}$$

So the probability that Steph plays on 4 days given that she played on at least 3 days is 0.4412.

- 1 answer mark for stating the appropriate distribution $X \sim \mathrm{Bi}(5, 0.75)$
- 1 method mark for the use of conditional probability
- 1 answer mark for the value of $\Pr(X=4|X\geq 3)$, 0.4412 correct to 4 decimal places

bii.

$$\begin{aligned} X &\sim \mathrm{Bi}(n, 0.75) \\ \Pr(X\geq 2) &> 0.75 \\ 1-\Pr(X<2) &> 0.75 \\ \Pr(X<2) &< 0.25 \\ \frac{n!}{0!n!}(0.75)^0(0.25)^n + \frac{n!}{1!(n-1)!}(0.75)^1(0.25)^{n-1} &< 0.25 \\ 0.25^n + 0.75\times 025^{n-1}n &< 0.25 \\ \therefore n &= 3 \end{aligned}$$

- 1 method mark for the use of the binomial theorem
- 1 method mark for arriving at the expression $0.25^n + 0.75\times 025^{n-1}n < 0.25$
- 1 answer mark for the correct value for n, 3

ci.

$$\begin{aligned} \int_1^a \frac{1}{50}\log_e(x)\,dx &= 1 \\ \therefore a &= 22.9629 \end{aligned}$$

- 1 method mark for the correct integral and terminals, and equating to 1
- 1 answer mark for arriving at the correct value of a, 22.9629

cii.

$$
\begin{aligned}
\mu &= \int_{1}^{22.9629} \left(x \times \frac{1}{50}\log_e(x)\right) dx \\
&= 13.8933 \\
\mathrm{E}\left(X^2\right) &= \int_{-\infty}^{\infty} \left(x^2 f(x)\right) dx \\
&= \int_{1}^{22.9629} \left(x^2 \times \frac{1}{50}\log_e(x)\right) dx \\
&= 226.0640 \\
\sigma^2 &= 226.0640 - 13.8933^2 \\
&= 33.0425 \\
\sigma &= \sqrt{33.0425} \\
&= 5.7483
\end{aligned}
$$

- 1 answer mark for the correct value for the mean, $\mu = 13.89$
- 1 method mark for finding $\mathrm{E}\left(X^2\right)$, 226.06
- 1 answer mark for the correct value of the standard deviation, 5.75

Question 4

a.

$$\begin{aligned}\int f'(x)\,dx &= \int \frac{1}{30}x\left(4x^2-3(c+10)x+20c\right)dx\\ &= \frac{1}{30}\left(x^4-(c+10)x^3+10cx^2\right)+C\\ f(0) &= 0\\ \therefore C &= 0\\ \therefore f(x) &= \frac{1}{30}\left(x^4-(c+10)x^3+10cx^2\right)\end{aligned}$$

- 1 answer mark is awarded for the correct anti-derivative of $f'(x)$
- 1 answer mark is awarded for the correct expression for $f(x)$ in terms of c, $f(x)=\frac{1}{30}\left(x^4-(c+10)x^3+10cx^2\right)$ with $C=0$

b.

$$\begin{aligned}\frac{1}{30}\left(x^4-(c+10)x^3+10cx^2\right) &= 0\\ x^2\left(x^2-(c+10)x+10x\right) &= 0\\ x^2(x-10)(x-c) &= 0\end{aligned}$$

$$\begin{array}{lll} x=0 \quad \text{or} & x-10=0 \;\text{ or} & x-c=0\\ x=0 & x=10 & x=c\end{array}$$

$$\begin{aligned}\frac{1}{30}\int_0^c\left(x^4-(c+10)x^3+10cx^2\right)dx &= -\frac{1}{30}\int_c^{10}\left(x^4-(c+10)x^3+10cx^2\right)dx\\ \left[\frac{1}{5}x^5-\left(\frac{c+10}{4}\right)x^4+\frac{10}{3}cx^3\right]_0^c &= -\left[\frac{1}{5}x^5-\left(\frac{c+10}{4}\right)x^4+\frac{10}{3}cx^3\right]_c^{10}\\ \frac{1}{5}c^5-\frac{1}{4}(c+10)c^4+\frac{10}{3}c^4-0 &= -\left(\frac{1}{5}(10)^5-\frac{10^4}{4}(c+10)+\frac{10^4}{3}c-\left(\frac{1}{5}c^5-\frac{1}{4}(c+10)c^4+\frac{10}{3}c^4\right)\right)\\ \therefore c &= 6\end{aligned}$$

- 1 answer mark is awarded for finding the correct value of the intercept, $x=c$
- 1 method mark is awarded for equating the two integrals with the correct terminals, the second with a negative coefficient, $\int_0^c f(x)\,dx=-\int_c^{10} f(x)\,dx$
- 1 answer mark is awarded for showing the correct value of c, $c=6$, given that there is sufficient working leading to this statement.

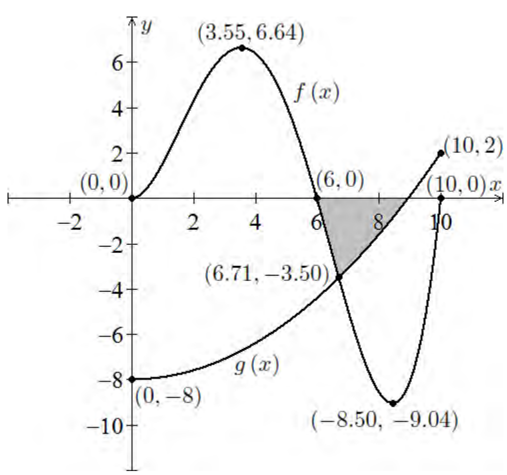
c.

$$
\begin{aligned}
\frac{1}{30}x^2(x-10)(x-6) &= \frac{1}{10}x^2-8 \\
x^4-16x^3+57x^2+240 &= 0 \\
\therefore x &= 6.7087 \\
\frac{1}{10}x^2-8 &= 0 \\
x &= 4\sqrt{5} \\
\text{Area} &= -\int_6^{6.7087}\frac{1}{30}\left(x^4-16x^3+60x^2\right)dx-\int_{6.7087}^{4\sqrt{5}}\frac{1}{10}\left(x^2-8\right)dx \\
&= 1.2349+4.0977 \\
&= 5.3326 \\
\text{A}_2 &= -\int_6^{10}\frac{1}{30}\left(x^4-16x^3+60x^2\right)dx \\
&= 23.0400 \\
\%\ \text{Area} &= \frac{5.3326}{23.0400}\times 100 \\
&= 23.15\%
\end{aligned}
$$

They can fire into 23.15% of the enemies territory.

- 1 answer mark is awarded for finding the intersection of the two curves, $x = 6.71$
- 1 method mark is awarded for the correct integrals and terminals for the area of the required region, $-\int_6^{6.7087} f(x)\,dx-\int_{6.7087}^{4\sqrt{5}} g(x)\,dx$
- 1 answer mark is awarded for the correct area percentage, 23.15%

d.



- 1 answer mark is awarded for the correct shape of $f(x)$ and $g(x)$. This includes showing that the gradient at $x = 0$ is zero for both curves
- 1 answer mark is awarded for correctly labeled intercepts, endpoints and stationary points of $f(x)$ and $g(x)$
- 1 answer mark is awarded for shaded the correct region between $f(x)$, $g(x)$ and the x-axis

e.

$$\begin{aligned}
xf'(x) &= f(x) \\
\frac{2}{15}x\left(x^2-12x+30\right) &= \frac{1}{30}x^2(x-10)(x-6) \\
\therefore x &= 0,\ 2.4274,\ 8.2393 \\
f(x) &= \frac{1}{30}(2.4274)^2(4.4274-10)(4.4274-6) \\
&= 5.3136 \\
&\ (2.43, 5.31) \\
\text{distance} &= \sqrt{2.43^2+5.31^2} \\
&= 5.84\,\text{km}
\end{aligned}$$

The peasants travel a distance of 5.84 km to the point $(2.43, 5.31)$

- 1 method mark is awarded for some use of $xf'(x) = f(x)$
- 1 answer mark is awarded for finding the correct point, $(2.43, 5.31)$ correct to 2 decimal places
- 1 answer mark is awarded for correct distance, 5.84 km correct to 2 decimal places

SET 2 EXAM 1

MODEL SOLUTIONS AND MARKING SCHEME

Question 1
Part a.

$$\frac{d}{dx}(x\log_e(x)) = (x)\left(\frac{1}{x}\right) + (1)(\log_e(x)) = \log_e(x) + 1$$

$$\frac{d}{dx}(\cos(x\log_e(x)) = -\sin(x\log_e(x))(\log_e(x) + 1)$$

- 1 mark awarded for an attempt to use the chain rule
- 1 mark awarded for the correct answer

Part b.

$$y = 2\sin^3(x) + 4x + 11$$

$$x = \pi,\ y = 4\pi + 11$$

$$\frac{dy}{dx} = 6\sin^2(x)\cos(x) + 4$$

$$x = \pi,\ \frac{dy}{dx} = 4$$

$$m_N \times \frac{dy}{dx} = -1$$

$$\therefore m_N = -\frac{1}{4}$$

$$y - 4\pi - 11 = -\frac{1}{4}(x - \pi)$$

$$y = -\frac{1}{4}x + \frac{17\pi}{4} + 11$$

- 1 mark for correctly finding the derivative
- 1 mark for correctly finding the gradient of the normal
- 1 mark for the correct answer

Question 2

$$\begin{aligned} g'(x) &= \cos(2x) - 2\sin(x) \\ g(x) &= \int (\cos(2x) - 2\sin(x))dx \\ &= \frac{1}{2}\sin(2x) + 2\cos(x) + c \end{aligned}$$

$$g(0) = 1$$

$$\therefore 1 = 2 + c$$

$$\therefore c = -1$$

$$g(x) = \frac{1}{2}\sin(2x) + 2\cos(x) - 1$$

- 1 mark for the correct integral statement, including the dx.
- 1 mark for correctly finding the anti-derivative.
- 1 mark for finding c.

Question 3
Part a.

$$\text{Let } y = f(x) = 2\log_e(5x) - 2$$

$$\text{For inverse, swap } x \text{ and } y$$

$$\begin{aligned} x &= 2\log_e(5y) - 2 \\ \frac{x+2}{2} &= \log_e(5y) \\ e^{\frac{x+2}{2}} &= 5y \\ \therefore y &= \frac{1}{5}e^{\frac{x+2}{2}} \\ f^{-1}(x) &= \frac{1}{5}e^{\frac{x+2}{2}} \end{aligned}$$

- 1 mark for swapping x and y to find the inverse.
- 1 mark for the correct answer.

Part b.
The domain of f^{-1} is $x \in \mathbb{R}$.

- 1 mark for the correct answer.

Part c.

1 mark for each of the following transformations:

- Dilation by a factor of $\frac{1}{5}$ away from the x-axis.
- Dilation by a factor of 2 away from the y-axis.
- Translation of 2 units horizontally to the left.

Question 4

$$\begin{aligned}
2\sin^2(x) - 1 &= \cos(x) \\
2(1 - \cos^2(x)) - 1 &= \cos(x) \\
1 - 2\cos^2(x) &= \cos(x) \\
2\cos^2(x) + \cos(x) - 1 &= 0
\end{aligned}$$

$$\text{Let } A = \cos(x)$$

$$\begin{aligned}
2A^2 + A - 1 &= 0 \\
(2A - 1)(A + 1) &= 0 \\
A &= \frac{1}{2} \text{ or } -1
\end{aligned}$$

$$\cos(x) = \frac{1}{2} \implies x = \pm\frac{\pi}{3} + 2k\pi \ \ k \in \mathbb{Z}$$

$$\cos(x) = -1 \implies x = (2k+1)\pi \ \ k \in \mathbb{Z}$$

$$\therefore x = 2k\pi \pm \frac{\pi}{3} \text{ or } (2k+1)\pi \ \ k \in \mathbb{Z}$$

- 1 mark for correctly solving the quadratic for $\cos(x)$.
- 1 mark for each of the final solutions.

Question 5
Part a.

$$\text{Area under PDF} = 1$$

$$\begin{aligned}
\text{Area} &= \int_0^{\pi} (a\sin(x))dx \\
&= \left[-a\cos(x)\right]_0^{\pi} \\
&= a(-\cos(\pi) + \cos(0)) \\
&= a(1+1) \\
\therefore 2a &= 1 \\
\implies a &= \frac{1}{2}
\end{aligned}$$

- 1 mark for correctly finding the anti-derivative.
- 1 mark for correctly substituting in.
- 1 mark for the correct final answer.

Part b.

$$\begin{aligned}
\Pr\left(X < \frac{\pi}{3}\right) &= \int_0^{\frac{\pi}{3}} \frac{1}{2}\sin(x)dx \\
&= \frac{1}{2}\left(-\cos\left(\frac{\pi}{3}\right) + \cos(0)\right) \\
&= \frac{1}{4}
\end{aligned}$$

- 1 mark for correctly recognising the bounds of the integral are $\frac{\pi}{3}$ and 0.
- 1 mark for the correct final answer.

Question 6
Part a.

$$\frac{dy}{dx} = \frac{1}{x}\cos(\log_e(x))$$

- 1 mark for the correct answer.

Part b.

$$\begin{aligned}
& \int \frac{1}{x}\sin\left(\frac{\pi}{2}-\log_e(x)\right)dx \\
= & \int \frac{1}{x}\cos(\log_e(x))dx \\
= & \sin(\log_e(x))+c
\end{aligned}$$

- 1 mark for being able to change the sin into a cos.
- 1 mark for the correct answer.

Part c.

$$\begin{aligned}
\text{Avg} &= \frac{1}{\pi-1}\int_1^{\pi}\frac{1}{x}\sin\left(\frac{\pi}{2}-\log_e(x)\right)dx \\
&= \frac{1}{\pi-1}\left[\sin(\log_e(x))\right]_1^{\pi} \\
&= \frac{\sin(\log_e(\pi))}{\pi-1}
\end{aligned}$$

- 1 mark for the formula for the average value of a function.
- 1 mark for the correct answer.

Question 7
Find the gradient of the line:

$$m=\frac{aq^2-ap^2}{2aq-2ap}=\frac{a(q+p)(q-p)}{2a(q-p)}=\frac{q+p}{2}$$

Find the equation of the line PQ:

$$y-y_1=m(x-x_1)$$

$$y-ap^2=\frac{(q+p)(x-2ap)}{2}$$

Substitute the point $(0,a)$ into the equation:

$$\begin{aligned}
a-ap^2 &= \frac{(q+p)(-2ap)}{2} \\
a(1-p^2) &= -apq-ap^2 \\
1-p^2 &= -pq-p^2 \\
pq &= -1
\end{aligned}$$

- 1 mark for finding the gradient of PQ.
- 1 mark for finding the rule of PQ.
- 1 mark for the correct answer.

Question 8
Consider the Hettige Bakery:

$$Z = \frac{X-\mu}{\sigma} = \frac{750-760}{7} = -\frac{10}{7}$$

$$\therefore \Pr(X > 750) = \Pr\left(Z > -\frac{10}{7}\right)$$

Consider the Levy Bakery:

$$Z = \frac{X-\mu}{\sigma} = \frac{750-768}{14} = -\frac{9}{7}$$

$$\therefore \Pr(X > 750) = \Pr\left(Z > -\frac{9}{7}\right)$$

The probability that the mass of the produced loaf is greater than 750 g is greater for the Hettige bakery, because:

$$\Pr\left(Z > -\frac{10}{7}\right) > \Pr\left(Z > -\frac{9}{7}\right)$$

- 1 mark for the correct z-value for the Hettige Bakery.
- 1 mark for the correct z-value for the Levy Bakery.
- 1 mark for the correct answer, stating that the probability for the Hettige Bakery is greater.

Question 9
There are two ways that Trevor can drive to university exactly once on the next two days. Drive on Tuesday and train on Wednesday or train on Tuesday and drive on Wednesday.

$$\Pr(DTD) = 1 \times 0.20 \times 0.75 = 0.15$$

$$\Pr(DDT) = 1 \times 0.80 \times 0.20 = 0.16$$

$$\text{Total Probability} = 0.31$$

- 1 mark for $\Pr(DTD)$
- 1 mark for $\Pr(DDT)$
- 1 mark for the correct answer.

Question 10
Part a.

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 160\pi r$$

- 1 mark for the correct answer.

Part b.

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = 0.5\,\text{mm/hr}$$

$$\frac{dS}{dr} = 4\pi r + 160\pi$$

$$\frac{dS}{dt} = 0.5(4\pi r + 160\pi) = 2\pi r + 80\pi$$

$$r = 10,\ \frac{dS}{dt} = 2\pi \times 10 + 80\pi = 100\pi\,\text{mm}^2/\text{hr}$$

- 1 mark for $\dfrac{dS}{dr}$.
- 1 mark for $\dfrac{dS}{dt}$.
- 1 mark for correct answer.

SET 2 EXAM 2

MODEL SOLUTIONS AND MARKING SCHEME

SECTION 1 - Multiple-Choice Questions

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
Answer	D	E	C	D	D	B	D	E	D	D	C	C	C	B	E	C	D	D	B	B	B	B

SECTION 2 - Extended-Response Questions

Question 1

a.

When $x = 1.5$, $y = 0$

$$0 = \frac{a}{1.5^2} + c \ (1)$$

When $x = 0.350$, $y = 1.3$

$$1.3 = \frac{a}{0.350^2} + c \ (2)$$

$(2) - (1)$

$$\begin{aligned} 1.3 - 0 &= \frac{a}{0.350^2} - \frac{a}{1.5^2} \\ a &= 0.168 \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &= \frac{0.168}{1.5^2} + c \\ c &= -0.075 \end{aligned}$$

Hence the function required is $y = \dfrac{0.168}{x^2} - 0.075$ for $x \in [-1.5, 1.5] \setminus (-0.350, 0.350)$

- 1 method mark for correctly identifying the points $(1.5, 0)$ or $(-1.5, 0)$ and $(0.350, 1.3)$
- 1 method mark for correct substitution and solving to find a and c
- 1 answer mark for the correct function required
- 1 answer mark for the correct specified domain

b. **i.**

The "radius" of the lava is the distance "x" from the y-axis. Hence, $\frac{dx}{dt} = -1$ m/s.

Since "y" is the depth of the lava, we require $\frac{dy}{dt} = \frac{dx}{dt} \times \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-0.337}{x^3}$$

$$\begin{aligned} \text{Therefore, } \frac{dy}{dt} &= -0.001 \times \frac{-0.337}{x^3} \\ &= \frac{0.337 \times 10^{-3}}{x^3} \text{ km/s} \end{aligned}$$

- 1 method mark for correctly interpreting the line "the radius of the lava decreases is 1 m/s" as $\frac{dx}{dt}$
- 1 method mark for finding the derivative of the volcano function
- 1 method mark for setting up the solution as $\frac{dy}{dt} = \frac{dx}{dt} \times \frac{dy}{dx}$
- 1 answer mark for the correct rate of change in terms of x

ii.

Since $\frac{dy}{dt}$ is given in terms of x, it is necessary to find the x-values of when the lava is at the floor and when it is at the ledge.

When $y = 0$, $x = \pm 1.5$

When $y = 1.295$

$$\begin{aligned} 1.295 &= \frac{0.168}{x^2} - 0.075 \\ x &= \pm 0.351 \end{aligned}$$

$$\begin{aligned} \text{Average value of the vate of change in lava depth} &= \frac{1}{1.5 - 0.351} \int_{0.351}^{1.5} \frac{0.337 \times 10^{-3}}{x^3} \, dx \\ &= 0.00113 \text{ km/s} \end{aligned}$$

- 1 method mark for finding the x-values that correspond to the y-values of the floor and ledge
- 1 method mark for correct use of "average value"
- 1 answer mark for the correct average rate of change

iii.

At 5 cm/s, Mr. Williams must travel 5 m, hence the time it takes him to get out is

$$\begin{aligned} t &= \frac{d}{s} \\ &= \frac{5}{0.05} \\ &= 100\,\text{seconds} \end{aligned}$$

We know that $\dfrac{dx}{dt} = -0.001$, therefore

$$\begin{aligned} x &= \int -0.001\,dt \\ &= -0.001t + c \end{aligned}$$

At $t = 0$, $y = 0$ and $x = -1.5$ (left hand of volcano required since as t increases, x moves towards 0). Therefore

$$\begin{aligned} -1.5 &= 0 + c \\ c &= 1.5 \\ \text{Therefore, } x &= -0.001t + 1.5 \end{aligned}$$

The lava reaches the top once $x = -0.350$, i.e. the radius of the crater
The time once the reaches the top can thus be found by

$$\begin{aligned} -0.350 &= -0.001t + 1.5 \\ t &= 1850\,\text{seconds} \end{aligned}$$

Hence the lava takes longer to reach the top of the crater and Mr. Williams survives.

- 1 method mark for utilising $t = \dfrac{d}{s}$
- 1 answer mark for stating the time it takes for Mr. Williams to get out of the volcano
- 1 method mark for finding the function of x in terms of t
- 1 answer mark for stating the time it takes for the lava to reach the top of the volcano
- 1 answer mark for the correctly stating or implying that Mr. Williams reaches the top of the crater first

Question 2

a.

$$\begin{aligned} g(x) &= -f(2x-1)+3 \\ &= -[((2x-1)-1)(2x-1)^2-k)]+3 \\ &= -8x^3+16x^2+(2k-10)x-2k+2 \end{aligned}$$

- 1 method mark for substituting the transformations on $f(x)$
- 1 answer mark for the $g(x)$ in the required form

b. **i.**

$$\begin{aligned} f'(x) &= 3x^2-2x-k \\ f'(1) &= 1-k \end{aligned}$$

$$f(1)=0$$

Therefore the tangent at $x=1$ is

$$\begin{aligned} y-0 &= (1-k)(x-1) \\ y &= (1-k)(x-1) \\ &= (1-k)x+k-1 \end{aligned}$$

- 1 method mark for finding the derivative of $f'(x)$
- 1 method mark for finding $f'(1)$ and $f(1)$
- 1 answer mark for the correct tangent required

ii.

$$\begin{aligned} g'(x) &= -24x^2+32x+2bk-10 \\ g'(3) &= 2k-130 \end{aligned}$$

Hence the gradient of the normal is $-\dfrac{1}{2k-130}$

When $x=3$

$$g(3)=4k-100$$

Therefore, the normal at $x=3$ is

$$\begin{aligned} y-(4k-100) &= \frac{-1}{2k-130}(x-3) \\ y &= \frac{-1}{2k-130}(x-3)+4k-100 \end{aligned}$$

- 1 method mark for finding the derivative of $g'(x)$
- 1 method mark for finding the gradient of the normal
- 1 answer mark for the correct normal required

c.

No intersection when the gradients of the two lines are the same and the y-intercepts are different

Gradients are equal when

$$\begin{aligned} (1-k) &= \frac{-1}{2k-130} \\ k &= \frac{66\pm\sqrt{4098}}{2} \end{aligned}$$

y-intercept of the tangent: $y = k - 1$

y-intercept of the normal: $y = \dfrac{3}{2k - 130} + 4k - 100$

The y-intercepts will only be equal when,

$$\begin{aligned} k - 1 &= \frac{3}{2k - 130} + 4k - 100 \\ k &= \frac{98 \pm \sqrt{1022}}{2} \end{aligned}$$

These k values are not equal to either of k values when the gradients are the same. Hence, the original k values found do not produce infinite solutions.

Hence, the k values that produce no intersection are $k = \dfrac{66 + \sqrt{4098}}{2}$ and $k = \dfrac{66 - \sqrt{4098}}{2}$

- 1 method mark for recognising that there is no solution when the lines are unique i.e. have the same gradient but different y-intercepts
- 1 method mark for finding the values that y-intercepts are the same
- 1 answer mark for justifying that the k-values found do not produce infinite solutions
- 1 answer mark for the correct values of k

Question 3

a. i.

$S_4 = T^3 S_1$, $T = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$, and $S_1 = \begin{bmatrix} 150 \\ 60 \end{bmatrix}$

$$\begin{aligned} S_4 &= T^3 S_1 \\ &= \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}^3 \begin{bmatrix} 150 \\ 60 \end{bmatrix} \\ &= \begin{bmatrix} 0.781 & 0.438 \\ 0.219 & 0.562 \end{bmatrix} \begin{bmatrix} 150 \\ 60 \end{bmatrix} \\ &= \begin{bmatrix} 143.43 \\ 66.57 \end{bmatrix} \end{aligned}$$

Hence, Clare will need to stock 143 watches and 67 wallets in April.

- 1 method mark for defining the transition matrix
- 1 method mark for defining the initial state matrix
- 1 method mark for setting up the equation $S_4 = T^3 S_1$
- 1 answer mark for the correct number of watches and wallets she will have to stock in April

ii.

$$\begin{aligned} \text{Sales} &= 143 \times \$80 + 67 \times \$40 \\ &= \$14120 \end{aligned}$$

- 1 method mark for multiplying number of watches by price
- 1 answer mark for the correct number amount of money made in April through sales

b.

In the long run, Clare will sell wallets and watches according to

$$T^{t \to \infty} S_1 = \begin{bmatrix} 140 \\ 70 \end{bmatrix}$$

Alternatively, proportion of watch buyers in the long term

$$\frac{0.2}{0.2 + 0.1} = \frac{2}{3}$$

Hence, the number of watch buyers in the town in the long run (out of 210) is $\frac{2}{3}(210) = 140$.
Therefore, the number of wallet buyers in the long run is $210 - 140 = 70$.

Hence, in the long term, she will sell 140 watches and 70 wallets every month.
Therefore, per month she will make

$$\begin{aligned} \text{Sales} &= 140 \times \$80 + 70 \times \$40 \\ &= \$14000 \end{aligned}$$

- 1 method mark for understanding the long term involves $T^{t \to \infty}$
- 1 method mark for finding the number of sales of watches and wallets every month in the long term
- 1 answer mark for the correct amount of money made per month in the long run

c.

Let $X \sim \mathrm{N}(\mu, \sigma^2)$

$$\begin{aligned} \Pr(X < -15) &= 0.03 \\ \Pr(X > 15) &= 0.02 \end{aligned}$$

$$\begin{aligned} \text{For} \quad \Pr(Z &< c_1) = 0.03 \\ & c_1 = -1.88 \\ \text{Therefore, } \frac{-15-\mu}{\sigma} &= -1.88 \ (1) \end{aligned}$$

$$\begin{aligned} \text{For} \quad \Pr(Z &> c_2) = 0.02 \\ & c_2 = 2.05 \\ \text{Therefore, } \frac{15-\mu}{\sigma} &= 2.05 \ (2) \end{aligned}$$

$$\begin{aligned} \frac{(2)}{(1)} \quad \frac{2.05}{-1.88} &= \frac{\frac{15-\mu}{\sigma}}{\frac{-15-\mu}{\sigma}} \\ -1.09 &= \frac{\mu - 15}{\mu + 15} \\ \mu &= -0.66 \end{aligned}$$

- 1 method mark for stating that $\Pr(X < -15) = 0.03$ and $\Pr(X > 15) = 0.02$
- 1 method mark for finding the z-scores for $X = -15$ and $X = 15$
- 1 method mark for solving the simultaneous equations formed by standardisation
- 1 answer mark for the correct average of the distribution

Question 4

a. **i.**

$$\begin{aligned} AC &= \sqrt{(5-1)^2 + (2-3)^2} \\ &= \sqrt{17}\text{ units} \end{aligned}$$

- 1 method mark for use of the distance formula
- 1 answer mark for the correct distance of AC

ii.

$$\begin{aligned} \text{Gradient of } AC &= \frac{2-3}{5-1} \\ &= -\frac{1}{4} \\ \text{Gradient of Normal to } AC &= -\frac{1}{-\frac{1}{4}} \\ &= 4 \end{aligned}$$

Substitute point B into general form of a straight line,

$$\begin{aligned} y-5 &= 4(x-3) \\ y &= 4x-7 \end{aligned}$$

- 1 method mark for the gradient of the normal required
- 1 method mark for substituting point B in order to find normal
- 1 answer mark for the correct normal to AC

iii.

(Diagram of info found already)

The line AC:

$$\begin{aligned} y-3 &= -\frac{1}{4}(x-1) \\ y &= -\frac{1}{4}x + \frac{13}{4} \end{aligned}$$

Normal intersects with the line AC when

$$\begin{aligned} 4x-7 &= \frac{-1}{4}x + \frac{13}{4} \\ x &= \frac{41}{17} \\ \text{and } y &= 4\left(\frac{41}{17}\right) - 7 \\ &= \frac{45}{17} \end{aligned}$$

Point of intersection is $\left(\frac{41}{17}, \frac{45}{17}\right)$. Let us call this point D.

$$\begin{aligned} \text{Height of Triangle} &= DB \\ &= \sqrt{\left(3-\frac{41}{17}\right)^2 + \left(5-\frac{45}{17}\right)^2} \\ &= \frac{10\sqrt{17}}{17}\text{ units} \end{aligned}$$

Consider the diagram; the base is AC.
Hence,

$$\begin{aligned}\text{Area of } \Delta ABC &= \frac{\text{Height} \times \text{Base}}{2} \\ &= \frac{\frac{10\sqrt{17}}{17} \times \sqrt{17}}{2} \\ &= 5\,\text{units}\end{aligned}$$

- 1 method mark for finding the rule for the line AC
- 1 method mark for finding the intersection of the normal with line AC
- 1 method mark for finding the height of the triangle as DB
- 1 answer mark for the correct area of ΔABC

b. i.

When $t = 0$, $v = 0$. Hence,

$$\begin{aligned}0 &= -2e^0 + b \\ b &= 2\end{aligned}$$

- 1 method mark for substituting the point $(0, 0)$
- 1 answer mark for the correct value of b

ii.

$$
\begin{aligned}
x(t) &= \int v(t)\, dt \\
&= \int -2e^t + 2\, dt \\
&= -2e^t + 2t + c
\end{aligned}
$$

When $t = 0$, $x = 0$
Therefore,

$$
\begin{aligned}
0 &= -2 + 0 + c \\
c &= 2
\end{aligned}
$$

Thus,

$$x(t) = -2e^t + 2t + 2$$

- 1 method mark for recognising $x(t) = \int v(t)\, dt$
- 1 method mark for substituting $(0, 0)$ or equivalent into function to find constant
- 1 answer mark for the correct function for $x(t)$

iii.

The marble hits the bottom of the hole at $x = -3$

$$
\begin{aligned}
-3 &= -2e^t + 2t + 2 \\
t &= 1.35\,\text{seconds}
\end{aligned}
$$

The velocity at $t = 1.35$ is

$$
\begin{aligned}
v(1.35) &= -2e^{1.35} + 2 \\
&= -5.69 \;\; \text{m/s}
\end{aligned}
$$

- 1 method mark for finding the time it takes for the marble to his the bottom of the hole
- 1 method mark for substituting time to hit the bottom into velocity function
- 1 answer mark for the correct velocity that the marble hits the bottom

SET 3 EXAM 1

MODEL SOLUTIONS AND MARKING SCHEME

Question 1

a.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{\sqrt{1-x^2}}\right) &= \frac{d}{dx}\left((1-x^2)^{-\frac{1}{2}}\right)\\ &= (-2x)\times -\frac{1}{2}(1-x^2)^{-\frac{3}{2}}\\ &= \frac{x}{(1-x^2)^{\frac{3}{2}}}\end{aligned}$$

The expression may be left as $x(1-x^2)^{-\frac{3}{2}}$.

- 1 method mark for showing use of the chain rule.
- 1 answer mark for correctly finding the derivative of the expression with respect to x.
- **Note:** Avoid writing "$\frac{dy}{dx} = ...$" or "$f'(x) = ...$" unless you define "$y = ...$" or "$f(x) = ...$" first. It is very unlikely that you will be penalised for omitting this step, but it is better form to include it. The best option is to write "$\frac{d}{dx}(...) = ...$" since it doesn't require the additional line to make mathematical sense.

b. **Method 1:** Using the product rule

$$\begin{aligned}f'(x) &= (2\cos(2x))\times\cos(2x)+\sin(2x)\times(-2\sin(2x))\\ &= 2\left(\cos^2(2x)-\sin^2(2x)\right)\\ \therefore\ f'\left(\frac{\pi}{8}\right) &= 2\left(\cos^2\left(\frac{\pi}{4}\right)-\sin^2\left(\frac{\pi}{4}\right)\right)\\ &= 2\left(\left(\frac{1}{\sqrt{2}}\right)^2-\left(\frac{1}{\sqrt{2}}\right)^2\right)\\ &= 0\end{aligned}$$

$f'(x)$ does not need to be fully simplified before substituting in $x = \frac{\pi}{8}$.

- 1 method mark for showing use of the product rule.
- 1 answer mark for correctly evaluating the derivative at $x = \frac{\pi}{8}$ as 0.

Method 2: Using the identity $\sin(\theta)\cos(\theta) = \dfrac{1}{2}\sin(2\theta)$

Specialist Mathematics students may notice that this identity can be easily used to simplify the expression for $f(x)$, however it should be noted that the **use or memorisation of compound and double angle formulas are NOT required for the Mathematical Methods (CAS) study**. However, the study design for Mathematical Methods (CAS) does allow for their use, so if you happen to remember an appropriate compound or double angle formula it is acceptable to use it as a shortcut, though if you are not sure you remembered the formula correctly it is best not to use it.

$$\begin{aligned}
\sin(2x)\cos(2x) &= \frac{1}{2}\sin(4x) \\
\implies f(x) &= \frac{1}{2}\sin(4x) \\
\implies f'(x) &= 2\cos(4x) \\
\implies f'\left(\frac{\pi}{8}\right) &= 2\cos\left(\frac{\pi}{2}\right) \\
&= 0
\end{aligned}$$

- 1 method mark for showing use of the double angle formula and correctly differentiating $f(x)$.
- 1 answer mark for correctly evaluating the derivative at $x = \dfrac{\pi}{8}$ as 0.

Question 2

a.

$$\begin{aligned}
\int \sin(4-2x)dx &= \frac{1}{(-2)} \times (-\cos(4-2x)) \\
&= \frac{1}{2}\cos(4-2x)
\end{aligned}$$

An equivalent answer is $\dfrac{1}{2}\cos(2x-4)$. A constant of integration may be included, but is not necessary.

- 1 answer mark for finding a correct anti-derivative of $\sin(4-2x)$.

b.

$$\begin{aligned}
\int_0^a \frac{1}{x+a}\,dx &= \left[\log_e\left(|x+a|\right)\right]_0^a \\
&= \log_e\left(|a+a|\right) - \log_e\left(|a|\right) \\
&= \log_e(2a) - \log_e(a) \quad (\text{as } a > 0) \\
&= \log_e\left(\frac{2a}{a}\right) \\
&= \log_e(2)
\end{aligned}$$

The final answer must not contain a, as the instruction was to "evaluate".

- 1 method mark for the correct anti-derivative and evaluation technique.
- 1 answer mark for correctly evaluating the integral as $\log_e(2)$.

Question 3

a. We require that $\log_e(x) > 0$. This is true if and only if $x > 1$, hence $D = (1, \infty)$.

- 1 method mark for recognising that we require $\log_e(x) > 0$.
- 1 answer mark for correctly solving this inequality to give $x > 1$ **and** for stating $D = (1, \infty)$.

b. We require that $\text{ran}(g) \subseteq \text{dom}(f)$, hence:

$$\begin{aligned}
g(x) &> 1 \\
\implies \frac{1}{4}x^2 &> 1 \\
\implies x^2 &> 4 \\
\implies x &\in (-\infty, -2) \cup (2, \infty)
\end{aligned}$$

But the domain of g is given as a single interval, hence $\text{dom}(g) = (2, \infty)$ and therefore $b = 2$ is the smallest possible value of b.

- 1 method mark for setting up and solving the inequality $g(x) > 1$.
- 1 answer mark for stating that $b = 2$ is the smallest possible value of b (it is sufficient to simply write $b = 2$ as your final answer, though it is better form to also state it is the smallest possible value).
- Award these marks consequentially if the answer to **part a.** is incorrect but is used correctly.

Question 4

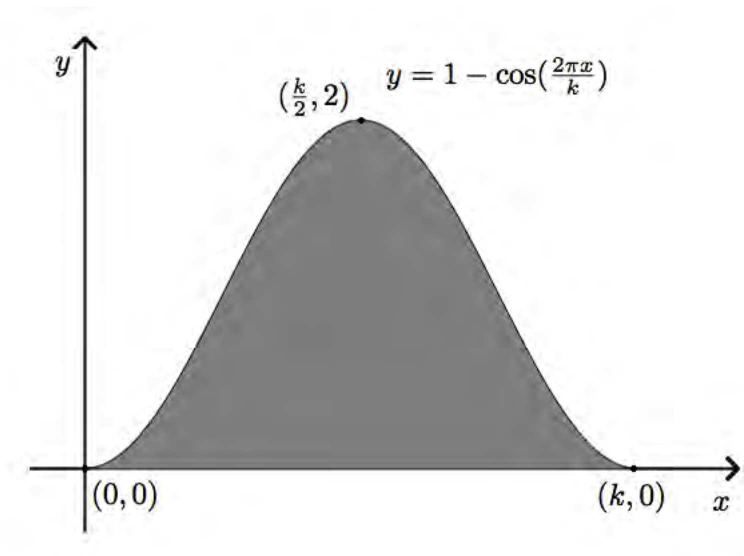
Method 1: Using the definite integral

$$\begin{aligned}
\int_0^k \left(1 - \cos\left(\frac{2\pi x}{k}\right)\right) dx &= 1 \\
\implies \left[x - \frac{k}{2\pi}\sin\left(\frac{2\pi x}{k}\right)\right]_0^k &= 1 \\
\implies k - \frac{k}{2\pi}\sin(2\pi) - 0 + 0 &= 1 \\
\implies k &= 1
\end{aligned}$$

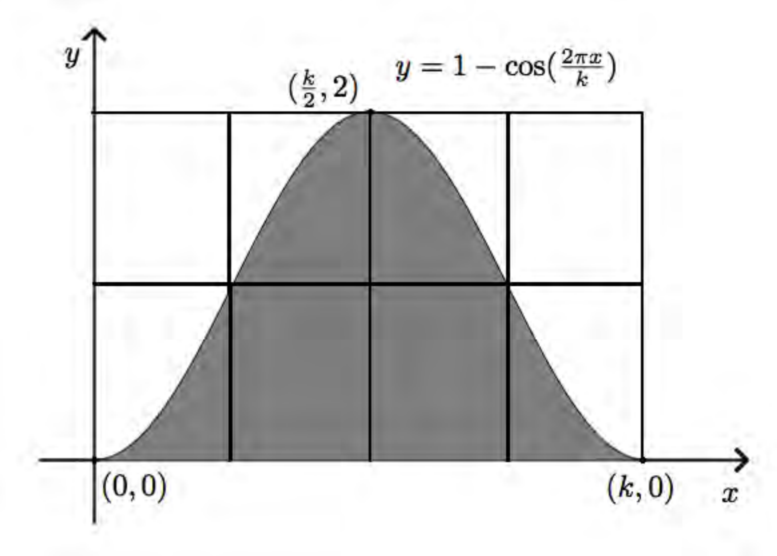
- 1 method mark for setting up the correct definite integral and equating it to 1.
- 1 method mark for the correct anti-derivative and evaluation technique.
- 1 answer mark for correctly finding $k = 1$.

Method 2: Using symmetry

Note that the period of the rule for the function is $\dfrac{2\pi}{\left(\dfrac{2\pi}{k}\right)} = k$, so the non-zero portion of the graph will cover exactly one period. A quick sketch of the graph of the probability density function will give:



The region is encompassed by a rectangle of height 2 and width k, and in fact takes up exactly half of this region, as can be seen in the following diagram:



If the area is equal to 1 and takes up half of a rectangle of dimensions $2 \times k$, then

$$\begin{aligned} \frac{1}{2} \times 2 \times k &= 1 \\ \implies k &= 1 \end{aligned}$$

- 1 method mark for articulating that the area is equal to half of the area of a $2 \times k$ rectangle.
- 1 method mark for correctly using the area to write down an equation for k.
- 1 answer mark for correctly finding $k = 1$.

Question 5

Method 1: Solving for $\tan(x)$

$$\begin{aligned} \sin^2(x) &= \cos^2(x) \\ \implies \frac{\sin^2(x)}{\cos^2(x)} &= 1 \\ \implies \left(\frac{\sin(x)}{\cos(x)}\right)^2 &= 1 \\ \implies \tan^2(x) &= 1 \\ \implies \tan(x) &= \pm 1 \\ \implies x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

- 1 method mark for solving for $\tan(x)$.
- 1 answer mark for finding the four solutions, $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
- Give 1 mark total for this question if the negative solution to $\tan^2(x) = 1$ is missing but the two resulting solutions, $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$, are correct.

Method 2: Using the identity $\sin^2(\theta) + \cos^2(\theta) = 1$

$$\begin{aligned}
\sin^2(x) &= \cos^2(x) \\
\implies \sin^2(x) &= 1 - \sin^2(x) \\
\implies 2\sin^2(x) &= 1 \\
\implies \sin(x) &= \frac{1}{2} \\
\implies \sin(x) &= \pm\frac{1}{\sqrt{2}} \\
\implies x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}
\end{aligned}$$

This can also be done by substituting $1 - \cos^2(x)$ instead of $1 - \sin^2(x)$.

- 1 method mark for correctly using the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and solving for $\sin(x)$ or $\cos(x)$.
- 1 answer mark for finding the four solutions, $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
- Give 1 mark total for this question if the negative solution to $\sin^2(x) = \frac{1}{2}$ or $\cos^2(x) = \frac{1}{2}$ is missing but the two resulting solutions are correct.

Method 3: Using the identity $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$ (see discussion in **Method 2** of **Question 1 b.**)

$$\begin{aligned}
\sin^2(x) &= \cos^2(x) \\
\implies \cos^2(x) - \sin^2(x) &= 0 \\
\implies \cos(2x) &= 0
\end{aligned}$$

Now, $0 \leq x \leq 2\pi \iff 0 \leq 2x \leq 4\pi$, so

$$\begin{aligned}
\cos(2x) &= 0 \\
\implies 2x &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \\
\implies x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}
\end{aligned}$$

- 1 method mark for correctly using the identity $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$.
- 1 answer mark for finding the four solutions, $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Question 6

Let X be the number of consecutive bull's-eyes that James throws. The event of James winning more than \$20 corresponds to $X > 2$.

$\Pr(X = 0) = 0.9$
$\Pr(X = 1) = 0.1 \times 0.9 = 0.09$
$\Pr(X = 2) = 0.1 \times 0.1 \times 0.9 = 0.009$

$\therefore \Pr(X \le 2) = 0.9 + 0.09 + 0.009 = 0.999$, and hence $\Pr(X > 2) = 1 - 0.999 = 0.001 \left(= \dfrac{1}{1000}\right)$.

- 1 method mark for finding the probabilities of $X = 0,\ 1,\ 2$.
- 1 method mark for using the complement method, $\Pr(X > 2) = 1 - \Pr(X \le 2)$.
- 1 answer mark for correctly evaluating the probability as 0.001 or $\dfrac{1}{1000}$.

Question 7

a. Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix}$, where x' and y' are the images of x and y. Then

$$\begin{aligned} x' &= ax + c \\ \text{and} \quad y' &= by + d \end{aligned}$$

If the point $(1, 1)$ is mapped to $(-2, 2)$, then

$$\begin{aligned} -2 &= a + c \quad (1) \\ \text{and} \quad 2 &= b + d \quad (2) \end{aligned}$$

And if the point $(2, 2)$ is mapped to $(3, 1)$, then

$$\begin{aligned} 3 &= 2a + c \quad (3) \\ \text{and} \quad 1 &= 2b + d \quad (4) \end{aligned}$$

Subtracting equation (1) from equation (3) gives $3 - (-2) = a \implies a = 5$, and hence $c = -2 - a = -2 - 5 = -7$.

Subtracting equation (2) from equation (4) gives $1 - 2 = b \implies b = -1$, and hence $d = 2 - b = 2 - (-1) = 3$. Hence, $a = 5$, $b = -1$, $c = -7$ and $d = 3$.

- 1 method mark for correctly setting up the four equations in a, b, c, and d.
- 1 answer mark for finding $a = 5$ and $c = -7$.
- 1 answer mark for finding $b = -1$ and $d = 3$.

b. **Method 1:** Using **part a.**

We have

$$\begin{aligned} x' &= 5x - 7 \\ \text{and} \quad y' &= -y + 3 \end{aligned}$$

Rearranging these gives

$$\begin{aligned} x &= \frac{x'+7}{5} \\ \text{and} \quad y &= -(y'-3) \end{aligned}$$

Hence the equation of the image of the line $y = x$ is

$$\begin{aligned} -(y'-3) &= \frac{x'+7}{5} \\ \iff y' &= -\frac{x'}{5} - \frac{7}{5} + 3 \\ \iff y' &= -\frac{x'}{5} + \frac{8}{5} \end{aligned}$$

- 1 answer mark for finding the equation $y' = -\frac{x'}{5} + \frac{8}{5}$ (you may leave the variables as just x and y). Award this mark consequentially if any of the values from **part a.** are incorrect.

Method 2: Using the transformed coordinates

Note that the line $y = x$ passes through the points $(1, 1)$ and $(2, 2)$. The transformed line must then pass through the points $(-2, 2)$ and $(3, 1)$.

The gradient of this line is $\frac{1-2}{3-(-2)} = -\frac{1}{5}$, and therefore

$$\begin{aligned} \frac{y-2}{x+2} &= -\frac{1}{5} \\ \implies y - 2 &= -\frac{x}{5} - \frac{2}{5} \\ \implies y &= -\frac{x}{5} + \frac{8}{5} \end{aligned}$$

- 1 answer mark for correctly finding the equation $y = -\frac{x}{5} + \frac{8}{5}$ from the transformed points (you may write the variables as x' and y').

Question 8

a.

$$\begin{aligned}\Pr(A) &= \Pr(A \cap B) + \Pr(A \cap B') \\ &= \frac{1}{4} + \frac{5}{8} \\ &= \frac{7}{8}\end{aligned}$$

As A and B are independent, $\Pr(A) \times \Pr(B) = \Pr(A \cap B)$, and therefore

$$\begin{aligned}\Pr(B) &= \frac{\Pr(A \cap B)}{\Pr(A)} \\ &= \frac{\left(\frac{1}{4}\right)}{\left(\frac{7}{8}\right)} \\ &= \frac{2}{7}\end{aligned}$$

- 1 answer mark for finding $\Pr(A) = \frac{7}{8}$.
- 1 method mark for using the relation $\Pr(A) \times \Pr(B) = \Pr(A \cap B)$.
- 1 answer mark for finding $\Pr(B) = \frac{2}{7}$.

b.

$$\begin{aligned}\Pr(B') &= 1 - \Pr(B) \\ &= 1 - \frac{2}{7} \\ &= \frac{5}{7} \\ \therefore\ \Pr(A) \times \Pr(B') &= \frac{7}{8} \times \frac{5}{7} \\ &= \frac{5}{8} \\ &= \Pr(A \cap B')\end{aligned}$$

Therefore A and B' are independent.

- 1 mark for finding $\Pr(B') = \frac{5}{7}$.
- 1 method mark for finding $\Pr(A) \times \Pr(B')$ and showing that it is equal to $\Pr(A \cap B')$.

Question 9

a. **Method 1:** Using the derivative

$f'(x) = 3x^2 - 12x + 8$, so the gradient of f at P is $f'(p) = 3p^2 - 12p + 8$. This is equal to the gradient of the tangent, which can be found using the line segment OP. Therefore:

$$\begin{aligned} \frac{f(p)-0}{p-0} &= f'(p) \\ \implies \frac{p^3-6p^2+8p}{p} &= 3p^2-12p+8 \\ \implies p^2-6p+8 &= 3p^2-12p+8 \\ \implies 2p^2-6p &= 0 \\ \implies p(p-3) &= 0 \\ \implies p &= 3 \quad (\text{as } p > 0) \\ \\ \therefore -a &= f'(3) \\ \implies a &= -(27-36+8) \\ &= 1 \end{aligned}$$

Therefore $a = 1$ and $p = 3$, as required.

A slightly different method can be used where the generalised tangent at P can be found, and letting the y-intercept equal 0 will result in the rest of the method being the same as the one above.

- 1 method mark for correctly finding the derivative of f at p and equating it to the gradient of OP **or** for finding the general tangent at P and letting the y-intercept equal 0.
- 1 method mark for correctly solving the equation for p and rejecting $p = 0$.
- 1 method mark for correctly using $f(3)$ or $f'(3)$ to find a.

Method 2: Using the number of intersections

There are exactly two points of intersection for the graphs of f and g, so there must be two solutions to the equation $f(x) = g(x)$.

$$\begin{aligned} x^3-6x^2+8x &= -ax \\ \iff x^3-6x^2+(8+a)x &= 0 \\ \iff x(x^2-6x+8+a) &= 0 \end{aligned}$$

There are two solutions $\iff 8+a=0$ or $8+a=9 \iff a=-8, 1$, but $a>0$, so $a=1$.

The value of p is the value of the non-zero solution to $x(x-3)^2 = 0$, which is 3. Hence $p = 3$. Therefore $a = 1$ and $p = 3$, as required.

- 1 method mark for articulating that there are two intersections of the graphs and equating the two functions.
- 1 method mark for using the fact that there must be two solutions to show that $a = 1$ (rejecting $a = -8$ does not have to be explicit).
- 1 method mark for using the resulting equation $x(x-3)^2 = 0$ to show that $p = 3$ and rejecting $p = 0$.

b. The area of the shaded region is given by

$$\begin{aligned}
\int_0^3 (f(x) - g(x))\, dx &= \int_0^3 \left(x^3 - 6x^2 + 8x - (-x)\right) dx \\
&= \int_0^3 \left(x^3 - 6x^2 + 9x\right) dx \\
&= \left[\frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2\right]_0^3 \\
&= \frac{3^4}{4} - 2 \times 3^3 + \frac{3^4}{2} \\
&= 3^3\left(\frac{3}{4} - 2 + \frac{3}{2}\right) \\
&= 27\left(\frac{3}{4} - \frac{8}{4} + \frac{6}{4}\right) \\
&= \frac{27}{4}
\end{aligned}$$

- 1 method mark for setting up the correct integral, $\displaystyle\int_0^3 (f(x) - g(x))\, dx$.
- 1 method mark for the correct anti-derivative and evaluation technique.
- 1 answer mark for correctly evaluating the integral as $\dfrac{27}{4}$ (or $\dfrac{27}{4}$ units^2).

Question 10

a. ACD is an equilateral triangle, so

$$\begin{aligned}
\frac{AM}{AC} &= \sin\left(\frac{\pi}{3}\right) \\
\implies \frac{AM}{x} &= \frac{\sqrt{3}}{2} \\
\implies AM &= \frac{\sqrt{3}}{2}x \\
\therefore AO &= \sqrt{AM^2 - OM^2} \\
&= \sqrt{\left(\frac{\sqrt{3}}{2}x\right)^2 - \left(\frac{1}{2}x\right)^2} \\
&= \sqrt{\frac{3}{4}x^2 - \frac{1}{4}x^2} \\
&= \frac{1}{\sqrt{2}}x
\end{aligned}$$

Or alternatively:

$$\begin{aligned}
BD &= \sqrt{2}x \\
\implies OD &= \frac{\sqrt{2}}{2}x \\
&= \frac{1}{\sqrt{2}}x \\
\therefore AO &= \sqrt{AD^2 - OD^2} \\
&= \sqrt{x^2 - \left(\frac{1}{\sqrt{2}}x\right)^2} \\
&= \sqrt{x^2 - \frac{1}{2}x^2} \\
&= \frac{1}{\sqrt{2}}x \\
\therefore AM &= \sqrt{AO^2 + OM^2} \\
&= \sqrt{\left(\frac{1}{\sqrt{2}}x\right)^2 + \left(\frac{1}{2}x\right)^2} \\
&= \sqrt{\frac{1}{2}x^2 + \frac{1}{4}x^2} \\
&= \frac{\sqrt{3}}{2}x
\end{aligned}$$

- 1 mark for using a valid method to find $AM = \dfrac{\sqrt{3}}{2}x$.
- 1 mark for using a valid method to find $AO = \dfrac{1}{\sqrt{2}}x$.

b. $V = \frac{1}{3} \times x^2 \times \left(\frac{1}{\sqrt{2}}x\right) = \frac{1}{3\sqrt{2}}x^3 = \frac{\sqrt{2}}{6}x^3$

Either $\frac{1}{3\sqrt{2}}x^3$ or $\frac{\sqrt{2}}{6}x^3$ are acceptable forms.

- 1 answer mark for finding $V = \frac{1}{3\sqrt{2}}x^3 = \frac{\sqrt{2}}{6}x^3$.

c. **Method 1:** Using similar shapes

OY is perpendicular to AM, and ΔAOM is similar to ΔOYM, so using similar triangles:

$$\begin{aligned} \frac{OY}{OM} &= \frac{AO}{AM} \\ \implies OY &= \frac{\left(\frac{1}{2}x\right) \times \left(\frac{1}{\sqrt{2}}x\right)}{\left(\frac{\sqrt{3}}{2}x\right)} \\ &= \frac{1}{\sqrt{6}}x \end{aligned}$$

Hence, when $x = \frac{\sqrt{30}}{2}$ m, $OY = \frac{1}{\sqrt{6}} \times \frac{\sqrt{30}}{2}$ m $= \frac{\sqrt{5}}{2}$ m.

- 1 method mark for recognising and using the fact that OY is perpendicular to AM.
- 1 method mark for correctly using similar triangles to find OY (either exactly or in terms of x).
- 1 answer mark for correctly finding $OY = \frac{\sqrt{5}}{2}$ (or $\frac{\sqrt{5}}{2}$ m).

Method 2: Using calculus

Let O, A and M be points on the x-y plane, with the origin being O, A being a point on the y-axis, and M being a point on the x-axis.

The length of OA is $\frac{1}{\sqrt{2}} \times \frac{\sqrt{30}}{2} = \frac{\sqrt{15}}{2}$, so A is the point $\left(0, \frac{\sqrt{15}}{2}\right)$.

The length of OM is $\frac{1}{2} \times \frac{\sqrt{30}}{2} = \frac{\sqrt{30}}{4}$, so M is the point $\left(\frac{\sqrt{30}}{4}, 0\right)$.

The gradient of the line through A and M is $\dfrac{\left(0-\dfrac{\sqrt{15}}{2}\right)}{\left(\dfrac{\sqrt{30}}{4}-0\right)} = -\dfrac{\sqrt{15}}{2} \times \dfrac{4}{\sqrt{30}} = -\dfrac{2}{\sqrt{2}} = -\sqrt{2}$, and the y-intercept of the line through A and M is $\left(0, \dfrac{\sqrt{15}}{2}\right)$. Hence, the equation of the line is $y = \dfrac{\sqrt{15}}{2} - \sqrt{2}x$.

Now, let l be the distance of any point on this line from the origin. Then:

$$\begin{aligned}
l^2 &= x^2 + y^2 \\
&= x^2 + \left(\frac{\sqrt{15}}{2} - \sqrt{2}x\right)^2 \\
&= x^2 + \frac{15}{4} - \sqrt{30}x + 2x^2 \\
&= 3x^2 - \sqrt{30}x + \frac{15}{4}
\end{aligned}$$

$$\begin{aligned}
\text{Now, } \frac{d(l^2)}{dx} &= 0 \\
\implies 6x - \sqrt{30} &= 0 \\
\implies x &= \frac{\sqrt{30}}{6}
\end{aligned}$$

Hence, the minimum value of l^2 is

$$\begin{aligned}
l^2 &= 3\left(\frac{\sqrt{30}}{6}\right)^2 - \frac{\sqrt{30} \times \sqrt{30}}{6} + \frac{15}{4} \\
&= \frac{3 \times 30}{36} - \frac{30}{6} + \frac{15}{4} \\
&= \frac{30}{12} - \frac{60}{12} + \frac{45}{12} \\
&= \frac{15}{12} \\
&= \frac{5}{4}
\end{aligned}$$

Therefore the minimum value of l is $\sqrt{\dfrac{5}{4}} = \dfrac{\sqrt{5}}{2}$, so the length of OY is $\dfrac{\sqrt{5}}{2}$ m.

- 1 method mark for finding a correct equation for the line through the points A and M, using the x-y plane.
- 1 method mark for correctly differentiating l or l^2 and equating it to 0.
- 1 answer mark for correctly finding $OY = \dfrac{\sqrt{5}}{2}$ (or $\dfrac{\sqrt{5}}{2}$ m).

SET 3 EXAM 2

MODEL SOLUTIONS AND MARKING SCHEME

SECTION 1 - Multiple-Choice Questions

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
Answer	D	C	E	B	C	D	C	B	D	D	C	E	C	B	D	A	A	E	B	D	E	B

SECTION 2 - Extended-Response Questions

Question 1

a.i.

$$x = \frac{2(3n \pm 1)\pi}{3},\ n \in \mathbb{Z}$$

Or:

$$x = 2n\pi \pm \frac{2\pi}{3},\ n \in \mathbb{Z}$$

- 1 answer mark for a correct solution set in terms of a parameter.
- 1 answer mark for specifying that $n \in \mathbb{Z}$.

a.ii.

$$f'(x) = \cos(x) + \frac{1}{2}$$

$$\begin{aligned} f'(x) &= 0 \\ \implies x &= \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3} \end{aligned}$$

- 1 method mark for a finding the derivative of f and equating it to 0.
- 1 answer mark for the three correct solutions.

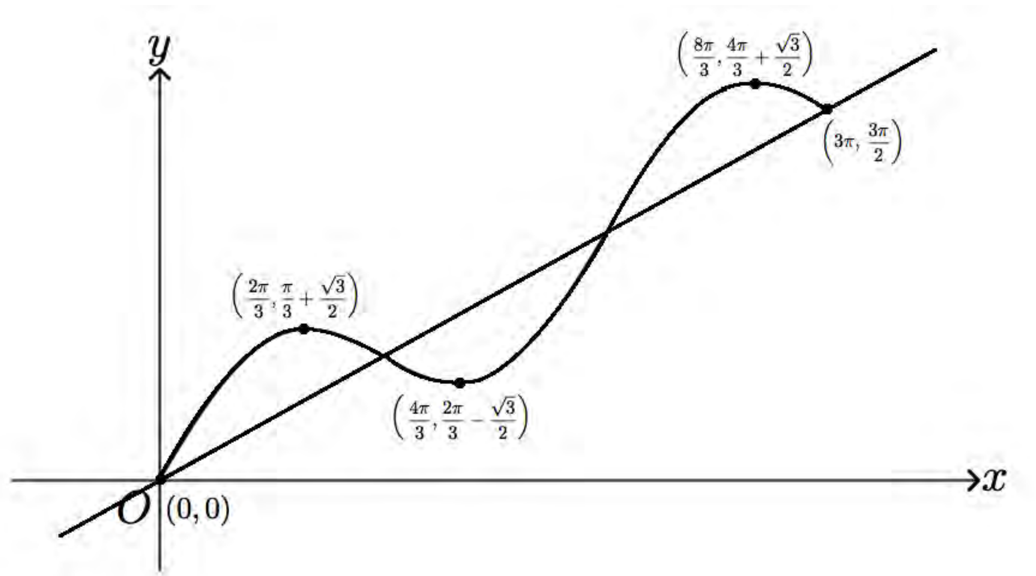
a.iii.

$$\begin{aligned} f\left(\frac{2\pi}{3}\right) &= \frac{\pi}{3}+\frac{\sqrt{3}}{2} \\ f\left(\frac{4\pi}{3}\right) &= \frac{2\pi}{3}-\frac{\sqrt{3}}{2} \\ \text{and} \quad f\left(\frac{8\pi}{3}\right) &= \frac{4\pi}{3}+\frac{\sqrt{3}}{2} \end{aligned}$$

Hence the coordinates are $\left(\frac{2\pi}{3}, \frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$, $\left(\frac{4\pi}{3}, \frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$, and $\left(\frac{8\pi}{3}, \frac{4\pi}{3}+\frac{\sqrt{3}}{2}\right)$

- 1 answer mark for finding the three correct y-values.
- 1 answer mark for writing down the three correct coordinates.

b.



- 1 mark for the correct shape of $y = f(x)$.
- 1 mark for correctly drawing $y = f(x)$ in relation to the line.
- 1 mark for labelling the five required points and for restricting the graph of f to its correct domain.

c.i.

$$\begin{aligned}
m &= \frac{f(b)-f(a)}{b-a} \\
&= \frac{\sin(b)+\dfrac{b}{2}-\sin(a)-\dfrac{a}{2}}{b-a} \\
&= \frac{\sin(b)-\sin(a)}{b-a} + \frac{\dfrac{b}{2}-\dfrac{a}{2}}{b-a} \\
&= \frac{\sin(b)-\sin(a)}{b-a} + \frac{1}{2}\left(\frac{b-a}{b-a}\right) \\
&= \frac{\sin(b)-\sin(a)}{b-a} + \frac{1}{2}, \quad \text{as required}
\end{aligned}$$

- 1 mark for using the two points to find an expression for the gradient, m.
- 1 mark for correctly manipulating the expression to give the required result.

c.ii.

$$b = a + 2\pi$$

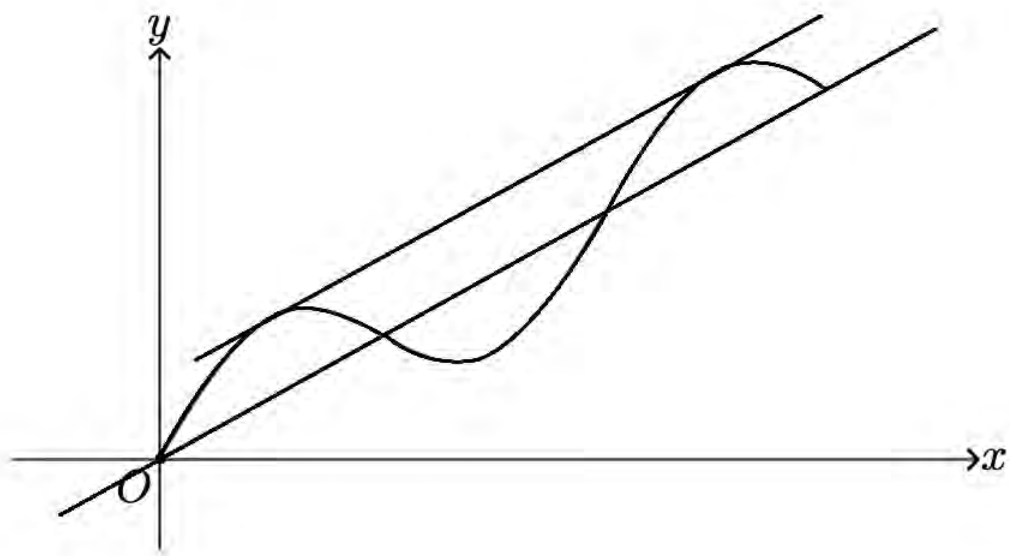
- 1 answer mark for correctly giving b in terms of a.

c.iii. **Method 1:** Using the previous result

$$\begin{aligned}
m &= \frac{\sin(b)-\sin(a)}{b-a} + \frac{1}{2} \\
&= \frac{\sin(a+2\pi)-\sin(a)}{a+2\pi-a} + \frac{1}{2} \qquad (\text{as } b = a+2\pi) \\
&= \frac{\sin(a)-\sin(a)}{2\pi} + \frac{1}{2} \\
&= 0 + \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

- 1 method mark for substituting in the result from **part c.ii.** correctly.
- 1 answer mark for the correct value of m.

Method 2: Analysing the graph



The derivative is periodic in x, and the function itself oscillates about the line $y = \frac{x}{2}$, so the line that is tangent to f at A and B must be parallel to this line. Hence, $m = \frac{1}{2}$.

- 1 method mark for indicating that the line is parallel to the line $y = \frac{x}{2}$.
- 1 answer mark for the correct value of m.

d. Solving $f'(x) = \frac{1}{2}$ gives $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$. Since $a \in [0, \pi]$, and $b \in [2\pi, 3\pi]$, we have $a = \frac{\pi}{2}$ and $b = \frac{5\pi}{2}$.

$f(a) = f\left(\frac{\pi}{2}\right) = 1 + \frac{\pi}{4}$, so the tangent passes through $\left(\frac{\pi}{2}, 1 + \frac{\pi}{4}\right)$. Since $m = \frac{1}{2}$, the equation of the line is:

$$\begin{aligned} y &= \frac{1}{2}\left(x - \frac{\pi}{2}\right) + 1 + \frac{\pi}{4} \\ \iff y &= \frac{1}{2}x - \frac{\pi}{4} + 1 + \frac{\pi}{4} \\ \iff y &= \frac{1}{2}x + 1 \end{aligned}$$

- 1 answer mark for finding a and b by solving $f'(x) = \frac{1}{2}$ and choosing the correct values.
- 1 method mark for using the coordinates of A or B and the value of m to find the equation of the tangent.
- 1 answer mark for the correct equation for the line through A and B.
- Award these marks consequentially if an incorrect value of m is found in the previous part but is used correctly.

Question 2

a.

$$\begin{aligned}\frac{d}{dx}\left(\sqrt{D(x)}\right) &= \frac{d}{dx}\left((D(x))^{\frac{1}{2}}\right)\\ &= \frac{1}{2}\times(D(x))^{-\frac{1}{2}}\times D'(x)\\ &= \frac{D'(x)}{2\sqrt{D(x)}}\end{aligned}$$

When $D'(x)=0$,

$$\begin{aligned}\frac{D'(x)}{2\sqrt{D(x)}} &= \frac{0}{2\sqrt{D(x)}}\\ &= 0\end{aligned}$$

$\therefore\ \frac{d}{dx}\left(\sqrt{D(x)}\right)=0$ when $D'(x)=0$, as required.

- 1 mark for correctly finding $\frac{d}{dx}\left(\sqrt{D(x)}\right)=\frac{D'(x)}{2\sqrt{D(x)}}$.
- 1 mark for showing by **substituting** in $D'(x)=0$ that $\frac{d}{dx}\left(\sqrt{D(x)}\right)=0$.

b.i.

$$\begin{aligned}D(x) &= (5-x)^2+(3-y)^2\\ &= (5-x)^2+(3-(-x^2+4x-1))^2\\ &= x^4-8x^3+25x^2-42x+41\end{aligned}$$

- 1 method mark for using the distance formula correctly.
- 1 answer mark for the expression for $D(x)$ in the correct form.

b.ii.

$$D'(x) = 4x^3-24x^2+50x-42$$

Or:

$$D'(x) = 2(x-3)(2x^2-6x+7)$$

- 1 answer mark for a correct expression for $D'(x)$.

c. Solving $D'(x) = 0$ for x gives $x = 3$, so the y-coordinate of P is given by:

$$\begin{aligned} y &= -(3)^2 + 4(3) - 1 \\ &= -9 + 12 - 1 \\ &= 2 \end{aligned}$$

Hence the y-coordinate of P is 2, as required.

- 1 answer mark for finding the value of x for which $D'(x) = 0$.
- 1 mark for using this value to show that $y = 2$.

d.i. The gradient of the line is given by:

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{3-2}{5-3} \\ &= \frac{1}{2} \end{aligned}$$

Hence, using the point $(3, 2)$, the equation of the line is:

$$\begin{aligned} y &= \frac{1}{2}(x-3) + 2 \\ \iff y &= \frac{1}{2}x + \frac{1}{2} \end{aligned}$$

- 1 answer mark for the correct gradient.
- 1 answer mark for the correct equation.

d.ii. The gradient of the parabola at x is given by

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(-x^2 + 4x - 1) \\ &= -2x + 4 \end{aligned}$$

Therefore, the derivative at P is

$$\begin{aligned} \frac{dy}{dx} &= -2 \times 3 + 4 \\ &= -2 \end{aligned}$$

The gradient of the line is $\frac{1}{2}$, and so

$$\frac{1}{2} \times (-2) = -1$$

Therefore, the line is normal to the parabola at P.

- 1 mark for correctly finding $\frac{dy}{dx}$.
- 1 mark for correctly finding the value of $\frac{dy}{dx}$ at P.

- 1 mark for showing that the product of $\dfrac{dy}{dx}$ at P and the gradient of the line is -1 and giving a conclusion.

d.iii. Solving $-x^2+4x-1=\dfrac{1}{2}x+\dfrac{1}{2}$ for x gives $x=\dfrac{1}{2}, 3$, so the x-coordinate of Q is $\dfrac{1}{2}$. Hence, the required area is given by

$$\begin{aligned}
A &= \int_{\frac{1}{2}}^{3} \left((-x^2+4x-1) - \left(\frac{1}{2}x + \frac{1}{2} \right) \right) dx \\
&= \int_{\frac{1}{2}}^{3} \left(-x^2 + \frac{7}{2}x - \frac{3}{2} \right) dx \\
&= \frac{125}{48}
\end{aligned}$$

- 1 answer mark for correctly finding the x-coordinate of Q.
- 1 method mark for writing down a correct integral that will give the area of the required region.
- 1 answer mark for evaluating the integral to give the exact area.

Question 3

a.

$$E(T) = \frac{k}{2}$$

- 1 answer mark for correctly giving the expected value of T in terms of k.

b.i.

$$\begin{aligned} \text{var}(T) &= \int_0^k \left(t - \frac{k}{2}\right)^2 \cdot \frac{6t}{k^3}(k-t)dt \\ &= \frac{k^2}{20} \end{aligned}$$

Therefore the standard deviation is $\dfrac{k}{\sqrt{20}} = \dfrac{\sqrt{5}k}{10}$.

- 1 method mark for using a correct integral to find the variance of T in terms of k.
- 1 answer mark for giving a correct expression for the standard deviation of T in terms of k.

b.ii.

$$\begin{aligned} \frac{\sqrt{5}k}{10} &= 2 \\ \implies k &= 4\sqrt{5} \end{aligned}$$

- 1 answer mark for the correct value of k in exact form.

b.iii.

$$\begin{aligned} E(T) &= \frac{k}{2} \\ E(T) &= \frac{4\sqrt{5}}{2} \\ &= 2\sqrt{5} \\ &\approx 4.5 \end{aligned}$$

- 1 answer mark for the correct value of $E(T)$ correct to one decimal place.

c. The standard deviation is 80 seconds, which is equal to $\frac{4}{3}$ minutes. Therefore:

$$\begin{aligned} X &\sim \mathrm{N}\left(2, \frac{16}{9}\right) \\ \implies \Pr(X > 5) &\approx 0.0122 \end{aligned}$$

- 1 method mark for using correct parameters for the normal distribution (written notation requires the variance rather than the standard deviation).
- 1 answer mark for the value of $\Pr(X > 5)$ correct to four decimal places.

d.i.

$$1 \times 0.95 \times 0.95 \times 0.95 \approx 0.8574$$

- 1 answer mark for the correct probability given to four decimal places.

d.ii.

$$\begin{bmatrix} 0.95 & 1 \\ 0.05 & 0 \end{bmatrix}^4 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.9524 \\ 0.0476 \end{bmatrix}$$

Hence the probability that he will miss the train on the fifth day is 0.0476, correct to four decimal places.

- 1 method mark for correct transition (raised to the correct power) and initial state matrices.
- 1 answer mark for writing down the required probability correct to four decimal places.

e. He will have to run to work if he arrives after 8:27 am, so we require $\Pr(Y > 21)$, which is given by:

$$\begin{aligned} \Pr(Y > 21) &= \int_{21}^{23} \frac{2}{9}(32 - y)dy \\ &\approx 0.44 \end{aligned}$$

Therefore the probability that he will have to run to work is 0.44, correct to two decimal places.

- 1 method mark for recognising that we require $\Pr(Y > 21)$ and writing down a correct integral.
- 1 answer mark for writing down the required probability correct to two decimal places.

Question 4

a.i.

$$f'(x) = 2\cos(2x) + 2\cos(x)$$

- 1 answer mark for the correct derivative.

a.ii.

$$\begin{aligned} \text{LHS} &= 2\cos(2x) + 2\cos(x) \\ &= 2(2\cos^2(x) - 1) + 2\cos(x) \\ &= 4\cos^2(x) + 2\cos(x) - 2 \end{aligned}$$

$$\begin{aligned} p(x) &= 2(2x-1)(x+1) \\ &= 2(2x^2 + x - 1) \\ &= 4x^2 + 2x - 2 \end{aligned}$$

$$\begin{aligned} \therefore \text{RHS} &= p(g(x)) \\ &= 4(\cos(x))^2 + 2(\cos(x)) - 2 \\ &= 4\cos^2(x) + 2\cos(x) - 2 \\ &= \text{LHS} \end{aligned}$$

Therefore $f'(x) = p(g(x))$, as required.

- 1 mark for correctly substituting in the given identity.
- 1 mark for correctly factorising the quadratic in $\cos(x)$ or expanding $p(x)$.
- 1 mark for clearly showing that $f'(x) = p(g(x))$ using proof conventions.

b.i.

$$\text{ran}(g) = \left(\frac{1}{2}, 1\right]$$

- 1 answer mark for the correct range.

b.ii.

$$x \in (\infty, -1) \cup \left(\frac{1}{2}, \infty\right)$$

- 1 answer mark for the correct range of values, either using inequalities or interval notation.

b.iii.

$$\begin{aligned}
\text{If} \quad x &\in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) = \text{dom}(g), \quad \text{then} \\
g(x) &> \frac{1}{2} \\
\implies p(g(x)) &> 0 \qquad (\text{as } p(x) > 0 \text{ when } x > \frac{1}{2}) \\
\implies f'(x) &> 0 \qquad (\text{as } f'(x) = p(g(x)))
\end{aligned}$$

- 1 mark for using the fact that $g(x) > \frac{1}{2}$ for the given set of x-values.
- 1 mark for using $g(x) > \frac{1}{2}$ to determine that $p(g(x)) > 0$, and therefore $f'(x) > 0$.

c.i.

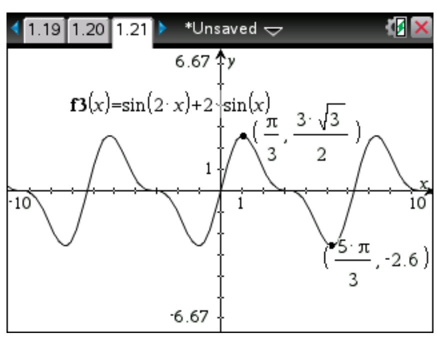
$$x = 0.257$$

- 1 answer mark for the required value of x correct to three decimal places.

c.ii. $f'(x) > 0$ implies that f is one-to-one, so any horizontal line $y = c$ will intersect f exactly once provided that $c \in \text{ran}(f)$.

- 1 mark for articulating that f is one-to-one based on the fact that $f'(x) > 0$.
- 1 mark for using the the fact that f is one-to-one to explain why there is exactly one solution to $f(x) = c$.

d. Using CAS to graph $y = \sin(2x) + 2\sin(x)$ gives:



The x-coordinates of the stationary points are given by:

$$\begin{aligned} 2(2\cos(x)-1)(\cos(x)+1) &= 0 \\ \implies \cos(x) &= \frac{1}{2} \\ \text{or} \quad \cos(x) &= -1 \\ \implies x &= \frac{\pi}{3}, \pi, \frac{5\pi}{3}, \cdots \end{aligned}$$

Hence, h will be one-to-one provided $d \le \dfrac{5\pi}{3}$, and so h will not have an inverse for $d > \dfrac{5\pi}{3}$.

- 1 mark for articulating that h is invertible when it is one-to-one, and is not invertible otherwise.
- 1 mark for identifying the local minimum at $x = \dfrac{5\pi}{3}$.
- 1 mark answer mark for giving the correct range of values, $d > \dfrac{5\pi}{3}$, using the exact value of $\dfrac{5\pi}{3}$.

Section 2: Detailed Solutions

FUNCTIONS AND GRAPHS

TECH-FREE TEST 1

DETAILED SOLUTIONS

Question 1

a.

This question asks us to solve for x the equation of $f(x) \geq 2$. There is generally some confusion when it comes to the use of inequalities - in most cases, it is beneficial for the student to treat them simply as regular equal signs as long as expressions are not randomly swapped between sides. However, it will have to be addressed at some point and in this question, it becomes slightly complicated.

For the meantime, we are simply solving a typical quadratic equation. We will first change it into the form $ax^2 + bx + c = 0$, as usual:

$$\begin{aligned} 6x^2 - 7x + 1 &\geq 2 \\ 6x^2 - 7x - 1 &\geq 0 \end{aligned}$$

The first question to ask is whether there are any solutions - This can revealed if we find the discriminant - $\triangle = (-7)^2 - 4(6)(-1) = 73$. Since the discriminant is greater than 0, there will be two solutions for when the quadratic is equal to 0. Also, since 73 is not a perfect square, its square root will not be a rational number and a quadratic will only factorise into brackets easily when the roots of the equation are rational.

From this point, there are 3 ways to go about solving for x that each student should be aware of:

1) Factorising into the form $(ax - b)(cx - d) = 0$ and using the null factor law.
2) Completing the square in order to isolate the x term so that it can then be solved for.
3) Utilising the quadratic formula to produce the solutions.

Attempting to factorise the expression in its current form is basically futile. It is not easy to intuitively determine what the factors of this quadratic are (as the final result will showcate).

The quadratic formula is most suitable when attempting to produce a specific solution (or two). The problem with this question is the inequality, which means you can potentially get an infinite number of results.

Thus, it seems like completing the square is the only viable approach at the moment.

$$\begin{aligned} 6(x^2 - \frac{7}{6}x - \frac{1}{6}) &\geq 0 \\ (x - \frac{7}{12})^2 - \frac{49}{144} - \frac{1}{6} &\geq 0 \\ (x - \frac{7}{12})^2 - \frac{73}{144} &\geq 0 \\ (x - \frac{7}{12})^2 &\geq \frac{73}{144} \end{aligned}$$

This is a common roadblock for students. Remember that we said above that you can treat questions with inequalities as if they were equal signs. However, you must take care with equations like this that have a square involved. A problem occurs for the same reason $a^2 = 9$ has two solutions for a, 3 and -3. Thus, you must take the same approach in this inequality equation and write the following:

$$\begin{aligned} x - \frac{7}{12} \geq \frac{\sqrt{73}}{12} \quad &or \quad -(x - \frac{7}{12}) \geq \frac{\sqrt{73}}{12} \\ x \geq \frac{7 + \sqrt{73}}{12} \quad &or \quad x \leq \frac{7 - \sqrt{73}}{12} \end{aligned}$$

However, you must be wary of the change in inequality direction for the second set of solutions. At any point where you multiply by a negative number in your working out (in this case, -1), you must reverse your inequality signs. If you were to keep the inequality sign the right way up, but simply used addition and subtraction to get x on the other side, you would get:

$$\begin{aligned} -(x-\frac{7}{12}) &\geq \frac{\sqrt{73}}{12} \\ -x+\frac{7}{12} &\geq \frac{\sqrt{73}}{12} \\ \frac{7}{12}-\frac{\sqrt{73}}{12} &\geq x \end{aligned}$$

This is the same result as above, but with x on the right-hand side now. Inequalities always create these kinds of issues, so take great care in your working out.

b.

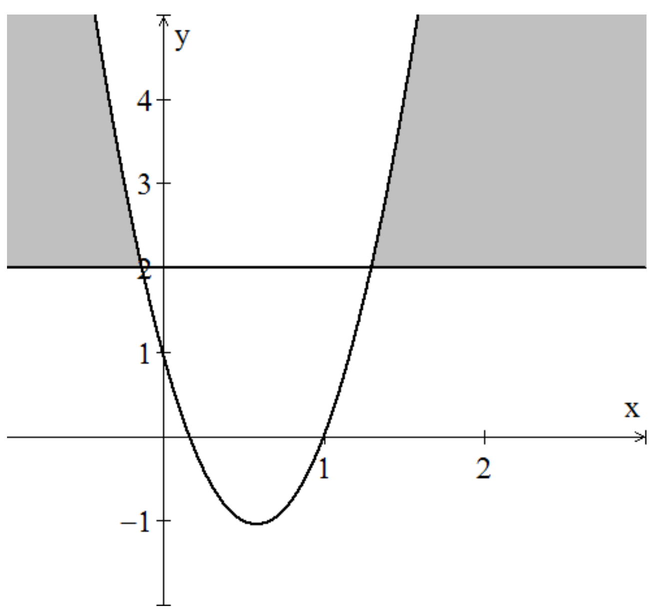
This question is not necessarily easy to interpret, but breaking it down into its various components will help.

The first thing asked for is the sketch of $y = f(x)$, i.e. the quadratic. There are several ways to go about this - the easiest is to recognise that the function can be written as

$$f(x) = (6x-1)(x-1)$$

From this point, you know that the axial intercepts are at $(\frac{1}{6}, 0)$ and $(1, 0)$. And from the form it was previously in, the y-intercept is given as $(0, 1)$. These points are enough to connect together and form the quadratic given in the diagram below.

We are now required to find the y-values which lie between $y = f(x)$ and $y = 2$. We already have $y = f(x)$ so let us draw in the line of $y = 2$. It's clear that if we want the region of all y-values between these two lines, we only have to shade in the area between them where $f(x)$ is above the line $y = 2$.



Note that there are assumed requirements that apply to all graph sketches. For graphs, you should always note the axis-intercepts, any intersections, a scale and the variables used for the axes (in this case x and y). In some cases, you will be expected to note all turning points or stationary points of inflection, but usually the question will indicate if this is necessary (you could write it in anyway to be thorough). VCAA does not always award marks for including these points, but it is generally assumed that they are given and that all students should stay on the safe side and include them anyway. Being aware of mark allocations is always a useful way to structure your answers.

Thus, the following should be noted in your graph, either stated or written as coordinates on your sketch:

The axis intercepts are at (0,1), $(\frac{1}{6}, 0)$ and (1,0).

The intersections of the line $y = 2$ and the function, $f(x)$ are $\left(\frac{7+\sqrt{73}}{12}, 2\right)$ and $\left(\frac{7-\sqrt{73}}{12}, 2\right)$ according to the result in **part a**. It is necessary to make this statement since the question specifically asks you to answer the question "using the result obtained in part a)". This is a requirement that appears in many questions in various forms - other common phrases include "hence" and "as such". There is always a mark given for doing so and most of the time, you have already done all the work so it should be an easy mark to claim.

Question 2

This question is a classic demonstration of how questions can become simple when broken down. The trouble with most function modelling questions is having to pull apart the information given to you and, if the question is a practical application, there are also the restrictions of the real world with which to contend! Fortunately, while this question may seem like it involves a difficult relation, it is simply made up of linear and quadratic relations.

First, let us consider the linear portions (from left to right):

For $x \in (-6, -3]$ the gradient is 0. Hence, it is just a horizontal line. It is not dependent on any particular x value because for any x value, the y-value will always be 2. Thus, for the given domain, the line can be written as $y = 2$.

We cannot specify the domain for the linear relation on the right. For now, let us define the endpoint as $(a, -6)$ and so the domain is for $x \in (2, a]$.

Consider the general form of a straight line as given in the question, "$y = mx + d$". There are two unknowns in this form, and hence we need two equations, or two points in order to solve for m and d. $(2, 1)$ and $(3, 0)$ are the known points given to us. One method of finding the equation of the straight line is to substitute the two points into the general form - the result will be two equations which can be solved simultaneously. However, there is another (perhaps simpler) way in which to find the straight line.

Firstly, we wish to find m, the gradient. The gradient is easily determined in straight lines and can be defined as:

$$\frac{\text{rise}}{\text{run}} = \frac{y_1 - y_2}{x_1 - x_2} \text{ for two known points, } (x_1, y_1) \text{ and } (x_2, y_2)$$

For this question,

$$\begin{aligned}\text{gradient} &= \frac{0-1}{3-2} \\ &= -1\end{aligned}$$

Now we only have to use another general form of the linear relation, $y - y_1 = m(x - x_1)$, which you may realise is very similar to the rule for finding the gradient if you rearrange for m. The simplicity of this rule means that you don't have to solve for "d" in the other general form and then rewrite the rule for y. It may seem like you're only saving a few seconds, but it's still a more efficient method.

$$\begin{aligned}y - 1 &= -1(x-2) \\ y &= -x + 3\end{aligned}$$

Now that we have the rule, let us look to find that value of a we specified before as being the x-value of the end point $(a, -6)$. We only have to substitute this point into the equation of the line we just found.
$x = a$ when $y = -6$. Therefore:

$$\begin{aligned}-6 &= -a + 3 \\ a &= 9\end{aligned}$$

Hence, this line has a domain of $(2, 9]$.

We have now covered two out of three sections! The last portion is the quadratic relation. Finding the rules of quadratic relations are generally simple if you are given the turning point or the x-axis intercepts because there are forms of the quadratic function in each case that will help you. However, in this case, you are given none of them and so you must use the general form provided in the question.

When finding the rules for the linear relations, it is said that for two unknowns, you need two points or equations to find the unknowns. This is true in this case as well, with the obvious difference being that there are now three unknowns and so three points are required. The known points are $(-3, 2)$, $(0, -1)$ and $(2, 1)$.

This may seem like a tedious task to deal with three simultaneous equations - however, remember that the "$+c$" is characteristically known as the y-intercept because when you substitute $x = 0$, the unknowns of a and b are removed from the equation.

Substitute $(0, -1)$ into general form of quadratic:

$$\begin{aligned}-1 &= a(0)^2 + b(0) + c \\ c &= -1\end{aligned}$$

Now we are in a similar position of having two unknowns and two other points to help find the unknowns. We will simply substitute these points into the equation.

Substituting $(-3, 2)$:

$$2 = a(-3)^2 + b(-3) - 1$$

Substituting $(2, 1)$:

$$1 = a(2)^2 + b(2) - 1$$

Therefore, we have the following equations as a result of the substitutions:

$$\begin{aligned}3 = 9a - 3b \quad (1) \\ 2 = 4a + 2b \quad (2)\end{aligned}$$

As per usual with simultaneous equations, we are required to remove one of the unknowns by some arithmetic. For equations which simply involve addition and subtraction of terms, it is easy enough to remove an unknown by subtracting or adding multiples of the equations as appropriate. Below shows the removal of b. It is much easier when the coefficients of an unknown are the same as or multiples of each other, but we can manipulate the equations anyway in a similar fashion.

By simultaneous equations, $(2) + \frac{2}{3}(1)$:

$$\begin{aligned} 2 + \frac{2}{3}(3) &= 4a + 2b + \frac{2}{3}(9a) - \frac{2}{3}(3b) \\ 4 &= 10a \\ a &= \frac{2}{5} \end{aligned}$$

We now have a value for a, so effectively we only have one unknown to find. We shall substitute this value of a into (2) to find b:

$$\begin{aligned} 2 &= \frac{2}{5}(4) + 2b \\ \frac{2}{5} &= 2b \\ b &= \frac{1}{5} \end{aligned}$$

Hence the quadratic is

$$y = \frac{2}{5}x^2 + \frac{1}{5}x - 1$$

We can now put everything together in the form of a hybrid function as required:

$$y = \begin{cases} 2, & -6 < x \leq -3 \\ \frac{2}{5}x^2 + \frac{1}{5}x - 1, & -3 < x < 2 \\ 3 - x, & 2 < x \leq 9 \end{cases}$$

Question 3

This question tests your understanding of a period at a rigorous level beyond simply knowing what the period is, based on the values in a function. Briefly, the period of a function is the interval between its repetitions. A wave will return to its "starting point" after each period.

The question that must be asked is what happens if you have two waves that start at the same point but have different periods - what is the period of these functions when they add on top of each other? If you were to draw the two functions on top of each other, you would find that there are points at which they both return to their starting points at the same time, whereas there are other points where one function returns to its starting point while the other does not.

It should become clear that when both functions reach their starting point at the same time and repeat their cycle, the addition of the functions will also repeat itself. In other words, the period of the addition of the functions is equal to the lowest common multiple (LCM) of the two individual functions' periods.

So first we must find the periods of the two individual functions before we can find the LCM

The period of $2\cos(\frac{x}{2}+\pi)$ is $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$

The period of $\sin(\frac{2x}{3})$ is $\dfrac{2\pi}{\frac{2}{3}} = 3\pi$

We can now find the LCM. This is found by listing the multiples of each period and seeing which value is first seen in both lists.

$$4\pi,\ 8\pi\ 12\pi$$
$$3\pi,\ 6\pi,\ 9\pi,\ 12\pi$$

Hence, the LCM of 4π and 3π is 12π.

Hence the period of $f(x)$ is 12π.

Question 4

Let us first briefly look at what the definition of the transformation given really means. The right-hand side indicates the transformations are being applied to all the points (x, y) in the pre-image function. The result is a whole new set of points given by the left-hand side, (x', y'). Students comfortable with matrix transformations may have learnt what each position of the matrices indicates. We'll come back later to how you can read the transformations straight from the transformation matrices.

For now, we want to find a relationship between the old x and y and the new ones. For a matrix equation, this means we must have an equation that consists of two matrices that have the same dimensions so that we can equate the two. We will simplify the equation as much as we can through matrix arithmetic.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2x+4 \\ 3y-1 \end{bmatrix}$$

Now that we have these equivalent matrices, we can draw the following conclusion:

$$x' = 2x + 4 \quad and \quad y' = 3y - 1$$

So how do we find the transformed function now that we know the relationship between the old and the new? We will have to substitute the new x and y into the given pre-image. The pre-image is given in terms of the old x and y so if we arrange the equations above for x and y, we can substitute those values into the pre-image:

$$x = \frac{x'-4}{2} \ \ and \ \ y = \frac{y'+1}{3}$$

Substituting the result into the pre-image:

$$\begin{aligned}
\frac{y'+1}{3} &= 3(\frac{x'-4}{2})^2+1 \\
\frac{y'+1}{3} &= \frac{3}{4}(x'^2-8x'+16)+1 \\
\frac{y'+1}{3} &= \frac{3}{4}x'^2-6x'+12+1 \\
y'+1 &= \frac{9}{4}x'^2-18x'+39 \\
y' &= \frac{9x'^2}{4}-18x'+38
\end{aligned}$$

The result is a new function in terms of the new x' and y'.

Don't forget to answer the requirements of the question - it asks for the values of a, b and c according to the general quadratic form. Hence, $a=\frac{9}{4}$, $b=-18$ and $c=38$.

FUNCTIONS AND GRAPHS

TECH-FREE TEST 2

DETAILED SOLUTIONS

Question 1

a.

There are many different approaches to this question - perhaps the most straight forward is to calculate the significant points - end points and axis intercepts.
Endpoints:

$$\begin{aligned} f(\pi) &= 2\cos(\frac{3}{2}\pi - \frac{\pi}{4}) \\ &= 2\cos(\frac{5\pi}{4}) \\ &= -\sqrt{2} \end{aligned}$$

$$\begin{aligned} f(-\pi) &= 2\cos(-\frac{3}{2}\pi - \frac{\pi}{4}) \\ &= 2\cos(-\frac{7\pi}{4}) \\ &= \sqrt{2} \end{aligned}$$

The end points are $(-\pi, \sqrt{2})$ and $(\pi, -\sqrt{2})$

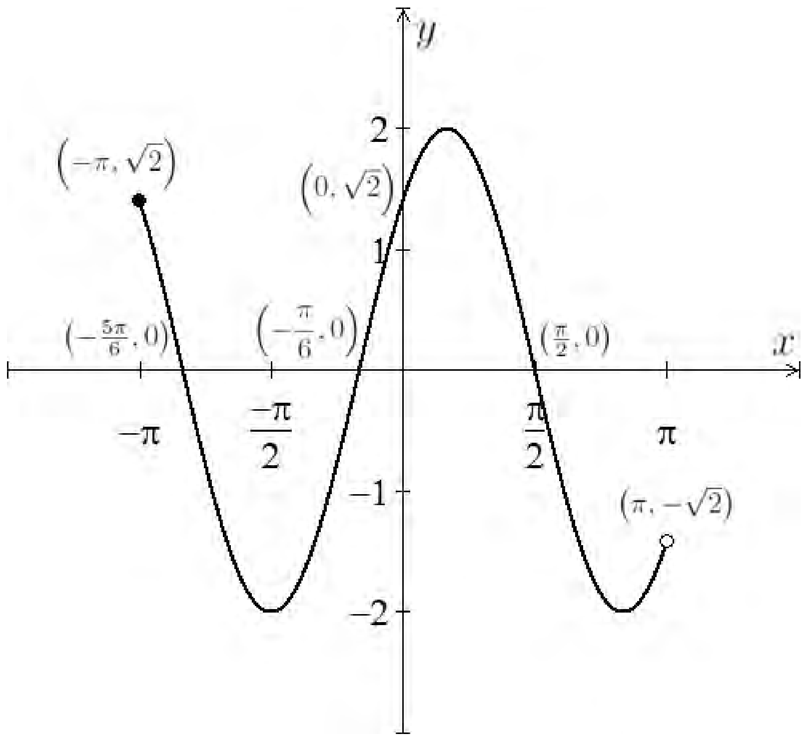
Axis-intercepts:
x-int when $y = 0$

$$\begin{aligned} 0 &= 2\cos(\frac{3}{2}x - \frac{\pi}{4}) \\ \frac{3}{2}x - \frac{\pi}{4} &= \frac{\pi}{2}, -\frac{\pi}{2} \quad -\frac{3\pi}{2} \\ x &= \frac{\pi}{2}, -\frac{\pi}{6} \quad -\frac{5\pi}{6} \end{aligned}$$

y-int when $x = 0$

$$\begin{aligned} y &= 2\cos(0 - \frac{\pi}{4}) \\ &= \sqrt{2} \end{aligned}$$

Now if we were to plot each of these points on the graph the draw the sinusoidal (cosine) curve to fit the points, we achieve our desired result.



Alternatively, some students will endeavour to draw a basic $y = \cos(x)$ and make the appropriate transformations to get the answer. However, this seems unnecessary - we know the shape of the graph is a cosine function; we need to make the calculations to find the required axis-intercepts and endpoints as shown above.

b.

There are two ways to go about this question - 1) Write the function for cosine that has a period of 24 then consider the transformations from the original function to the one needed. Or 2) Understand that periods can be multiplied by a factor equivalent to the factor dilating from the y-axis.

Let us consider the latter method:

Only a dilation from the y-axis can affect the period. If you want to see this result in action, let's say one period is defined from $x = 1$ to $x = 2$, a period of 2. If we dilate by a factor of 2 from the y-axis, then all the x-values will be multiplied by 2. Hence, our transformed period for the given interval is now $x = 2$ to $x = 4$, a period of 4. Therefore, by dilating by a factor of 2 from the y-axis we have multiplied our period by the same factor, 2.

So let us first find the period of the original function, and find its multiplier that results in 24.

For $y = \cos(nx)$, the period is given as $\frac{2\pi}{n}$.

Hence, the period of original function is $\dfrac{2\pi}{\frac{3}{2}} = \frac{4\pi}{3}$.

Now we will multiply this value by "a" in order to get our period of 24. i.e. A dilation of factor a from the y-axis is wanted to provide a period of 24.

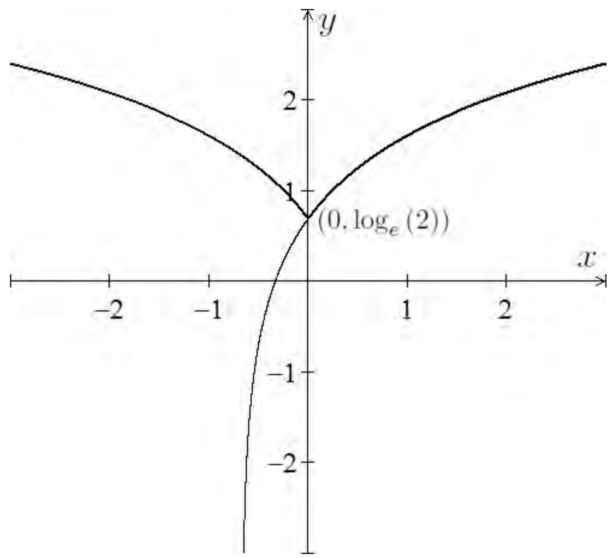
$$\begin{aligned} a \times \frac{4\pi}{3} &= 24 \\ a &= \frac{18}{\pi} \end{aligned}$$

As stated above, if we change our period by a certain multiplier, we would also be dilating from the y-axis by the same multiplier.

Therefore, a dilation of factor $\frac{18}{\pi}$ from the y-axis is required

Question 2

Sketching the function is considerably easier than finding the hybrid function. Let us start with this portion of the question - we don't need to know the exact hybrid function to sketch it. Essentially, by "modding" the x-values, we are taking all the values of y for $x > 0$ and reflecting them in the y-axis as shown in the diagram below.



This reflection in the y-axis for positive values of x is true for all $f(|x|)$ functions.

Y-intercept can be found for $x = 0$:

$$\begin{aligned} y &= \log_e(3(0) + 2) \\ &= \log_e(2) \end{aligned}$$

Now we can consider the hybrid function. In order to split these functions into their components, we use the same method as we would for any absolute value. Generally, students tend to find absolute values of whole functions easier to deal with than just x. This is probably because most questions in textbooks focus on the modulus of a whole function - but it's important to get practice all around.

In general, the modulus function for $|f(x)|$ splits into

$$|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$$

The same is true if we only find the absolute value of x.

$$f(x) = \begin{cases} \log_e(3x + 2) & x \geq 0 \\ \log_e(3(-x) + 2) & x < 0 \end{cases}$$

$$f(x) = \begin{cases} \log_e(3x + 2) & x \geq 0 \\ \log_e(-3x + 2) & x < 0 \end{cases}$$

Question 3

a.

As with most questions that revolve around an order of transformations, there are many reasonable sets of transformations that can take place in order to get the image required. There are certainly functions in which this tends to happen more than others - particularly functions like $\frac{1}{x}$ where a dilation from the x-axis has an easily recognisable alternative dilation in the y-axis (and vice-versa). E.g. for $f(x) = \frac{1}{x}$, a dilation of factor a from the x-axis results in the same function as a dilation of factor a from the y-axis.

In any case, the important thing to recognise is that the path of transformations you choose can be unique, but as long as you are systematic about how you go about it, you will have a reasonable and correct answer.

The most pragmatic way to go about this question is to simply transform the original function back into the basic $y = \frac{1}{x^2}$ function. Then after finding the inverse, apply the appropriate transformation to get the desired image. OR alternatively, find the inverse of the image given, then transform the original function to become this inverse. Let us try the latter method.

The question is how do we "find the inverse" as a transformation?

Most students find the inverse by swapping the y and x and solving for the new y function (though there is perhaps a more logical method I will outline below). But consider what this implies - you are essentially saying that $y = x$ in order for this to occur. And in doing so, you are swapping the x and y value for EVERY point in the original function. As it turns out, the actual result is a reflection in the line $y = x$. Indeed, finding the inverse of a function is akin to reflecting it in the line $y = x$ and students should be aware of this.

So let us find the inverse of our image.

$$\begin{aligned} x &= \frac{1}{\sqrt{y+1}} + 2 \\ x - 2 &= \frac{1}{\sqrt{y+1}} \\ \sqrt{y+1} &= \frac{1}{x-2} \\ y &= \frac{1}{(x-2)^2} - 1 \end{aligned}$$

Now we will attempt to get this function from the original (the pre-image).

1) We need to dilate the original to remove the 3 in the numerator.

- Dilation by factor $\frac{1}{3}$, from the x-axis - $y = \frac{3}{x^2} + 1 \to y = \frac{1}{x^2} + \frac{1}{3}$

2) We need to translate the function from its vertical asymptote at $x = 0$ to $x = 2$

- Translation of 2 units in the positive direction of the x-axis - $y = \frac{1}{x^2} + \frac{1}{3} \to y = \frac{1}{(x-2)^2} + \frac{1}{3}$

3) We need to translate the function to have its horizontal asymptote move from $y = \frac{1}{3}$ to $y = -1$

- Translation of $\frac{4}{3}$ units in the negative direction of the y-axis - $y = \frac{1}{(x-2)^2} + \frac{1}{3} \to y = \frac{1}{(x-2)^2} - 1$

4) We have now reached the desired inverse of the image. So to reach the image we need the transformation that gives us the inverse as discussed above.

- Reflection in the line of $y = x$

b.

For the original function to have an inverse function, it must be one-to-one. Consider the shape of the original truncus - it is symmetrical around its vertical asymptote $x = 0$. How do we know if a function is one-to-one? We do what's called a vertical and horizontal line test. Essentially, a function is one-to-one for a particular domain if you can draw both a vertical and horizontal line through any point within that domain and there is only one y-value to each x-value and vice-versa.

Doing a horizontal and vertical test shows that the truncus function is one-to-one for both halves, i.e. from $(0, \infty)$ and $(-\infty, 0)$.
$(-\infty, 0)$ is incorrect since $y = \frac{1}{\sqrt{x+1}} + 2$ only arises from the positive x-values in the original function.

Therefore, the required domain is $(0, \infty)$.

Question 4

a.

This question is a standard modelling question with a few traps that students should keep an eye out for.

The first thing that should be noted is the use of units. Misuse of units is one of the most common ways for students to lose "silly" marks. Always ensure that you include the units in your working out, especially in questions like this that have a real world practicality. And while you include units, make sure you use the correct ones! There are several units that are considered "standard" in a conventional sense if nothing is specified, e.g. the assumption of m/s. However, it is best for you to ignore standards and simply make the units "units" even if it is a speed/distance/time etc. When the units are specified, you have to make sure that you use the ones in the question - read these carefully! In this question, the function is given in "m/s" but there is also a value of "108 km/hr" given to you. You will either have to convert the function and other points into km/hr or this one value into m/s - the latter is better.

The function has two unknowns "a" and "b", and these are what we are looking for. In general, if there are two unknowns, you are going to need two equations - for functions, these equations are created using two known points. We have to look to the question and find the necessary information that gives us the two points.

When $t = 0$, $v = 0$, therefore $0 = a + b$ (1)
When $t = 10$, $v = 108(\frac{1000}{3600}) = 30$, therefore $30 = ae^{-\frac{3}{4}\times 10} = ae^{-\frac{15}{2}} + b$ (2)

So once we have two equations, we have to somehow put them together in order to solve for the unknowns. This is a tech-free test so calculators are out of the question, and by hand, its easiest to use simultaneous equations. When doing simultaneous solutions, the first thing students should endeavor to do is remove one of the unknowns. You should pick up on the "$+b$" in both solutions and realise that if you were to subtract the two equations, b would be removed.

Hence, using simultaneous equations, (2) – (1),

$$\begin{aligned} 30 &= ae^{-\frac{15}{2}} - a \\ 30 &= a(e^{-\frac{15}{2}} - 1) \\ a &= \frac{30}{e^{-\frac{15}{2}} - 1} \end{aligned}$$

Now that we have a value for one of the unknowns, we can use it to solve for the other. With just one unknown remaining, we can substitute the known value of a into one of the equation.

From equation (1)

$$\begin{aligned} a+b &= 0 \\ b &= -a \\ b &= -\frac{30}{e^{-\frac{15}{2}}-1} \end{aligned}$$

In this way, both unknowns have been found. Read the questions carefully - similar questions may ask you to state the rule of the function and some may only ask for the values of the unknowns. Perhaps if you wish to be cautious, always state the rule of the function - however, in this question, the solution above is more than adequate.

b.

Always think about what a question is asking for. Sometimes, the information is given to you and you must interpret it to reveal a point in a function. In other cases, such as this one, a certain point is required, but which one is it? If we wanted him to survive, he will have to land with a speed less than 40 m/s. Hence, we want the point that describes the time at which he reaches 40 m/s.

Our function has variables of time and velocity, so we only need to know one of them to find the other. We know he survives if he lands before 40 m/s, so let us find the corresponding time by substituting 40 m/s into the function.

$$\begin{aligned} 40 &= \frac{30}{e^{-\frac{15}{2}}-1}e^{-\frac{3}{4}t} - \frac{30}{e^{-\frac{15}{2}}-1} \\ 40 &= \frac{30}{e^{-\frac{15}{2}}-1}(e^{-\frac{3}{4}t}-1) \\ \frac{4}{3}(e^{-\frac{15}{2}}-1) &= e^{-\frac{3}{4}t}-1 \\ e^{-\frac{3}{4}t} &= \frac{1}{3}(4e^{-\frac{15}{2}}-1) \\ -\frac{3}{4}t &= \log_e(\frac{1}{3}(4e^{-\frac{15}{2}}-1)) \\ t &= -\frac{4}{3}\log_e(\frac{1}{3}(4e^{-\frac{15}{2}}-1)) \end{aligned}$$

The base jumper will survive if he lands before $-\frac{4}{3}\log_e(\frac{1}{3}(4e^{-\frac{15}{2}}-1))$ seconds

The solution to this question doesn't suggest the lack of difficulty - in essence, you have only had to substitute a value into a function and solve for t. If you are careful in each step of your solution, there should be no problems.

Question 5

a.

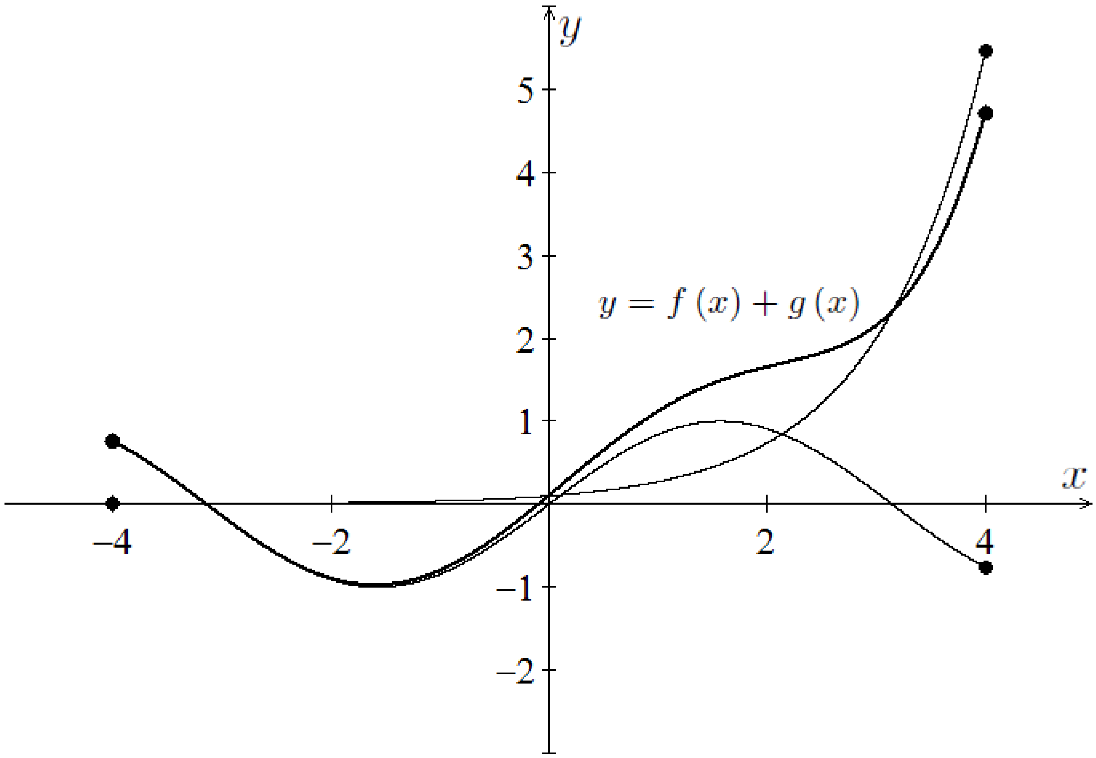
There are many ways to go about this type of question but at its core, the idea is to take the y-value of both functions at a single x-value and add them together (i.e. add up the y-values at every point on both functions). However, there is a systematic way to go about this - certain properties that we can hone in on to make it easier (since technically, there are an infinite number of points for both functions to add up).

Firstly, consider when one function equals to 0. e.g. if $f(x) = 0$ then $h(x) = 0 + g(x) = g(x)$. i.e. the value of y at $f(x) = 0$ is the same value at $g(x)$ so you can mark in a point (with a dot) simply at the same point g is at for that value of x.

Secondly, you can consider points of intersection between the two graphs $f(x)$ and $g(x)$. If $f(x) = g(x)$ then $h(x) = f(x) + f(x) = 2f(x)$, i.e. the y-value will simply be doubled for this x-value. In terms of marking in a point on the graph, it's simply twice the distance from the x-axis as $f(x)$ (or $g(x)$ is).

Thirdly, the behaviour of a graph. If one function is approaching infinity (as e^x does), and the other function does not approach "negative" infinity, then you can expect $h(x)$ to also approach positive infinity.

Similarly, as e^x approaches 0 as $x \to -\infty$ then $h(x) \to g(x)$ since $f(x)$ will approach 0.



b.

The domain for an addition of ordinates function can be found by working out the domains of the functions being added together. To put it simply, the function can only be defined for the values of x that are defined for the individual functions - both functions must be defined at that certain x-value in order to be a defined addition (otherwise it's $y + undefined$, and what is that? Undefined).

$f(x)$ is still defined from $[-4, 4]$ but $g(x)$ is now defined from $[-4-2, 4-2] = [-6, 2]$.

As per the information above, $h(x)$ will only be defined for the intersection of the two domains of the two functions.

Hence, the correct domain is [-4,2]

FUNCTIONS AND GRAPHS

TECH-ACTIVE TEST 1

DETAILED SOLUTIONS

SECTION 1 - Multiple-Choice Questions

Question 1 (B)

The period of the tangent function $y = a\tan(nx - h) + k$ is always given by $\dfrac{\pi}{n}$ - that is, divide π by the coefficient of x. This can often confuse students, as the period of the other circular functions (namely sine and cosine) is given by $\dfrac{2\pi}{n}$, where n is the coefficient of x. So the message here is to be careful when finding the period, particularly of a tan function!

Now, all we need from the equation $f(x) = 7\tan\left(2\left(\frac{2x}{3} - \frac{\pi}{5}\right)\right) + 1$ is the coefficient of x, which is n.
$\therefore n = 2 \times \dfrac{2}{3} = \dfrac{4}{3}$. Note that **both** the 2 outside the bracket and the $\dfrac{2}{3}$ multiply x, hence the coefficient of x is obtained by multiplying both of these numbers.

Hence, the period $= \dfrac{\pi}{\left(\dfrac{4}{3}\right)} = \dfrac{3\pi}{4}$. Thus the answer is **B.**

Question 2 (C)

The terms "image" and "pre-image" can be a bit confusing. Essentially, you should look at functions a bit like machines - there is a certain input (which is usually denoted by x, but really, it could be anything at all), and it gives you an output, in terms of a y value. As such, the term "pre-image" refers to what you are "feeding into" the function, and the term "image" refers to what comes out of it - hence, if you are asked for a "pre-image", give the x value, and if you are asked for an "image", give the y-value.

When we are asked "Consider the function $w(x) = 3\log_e(2x)$. What is the pre-image of $5a$?", you should read this as "what value of x (i.e. a pre-image) will give me a y value (i.e. an image) of $5a$?"
So really, just let $w(x) = 5a$ and solve for x:

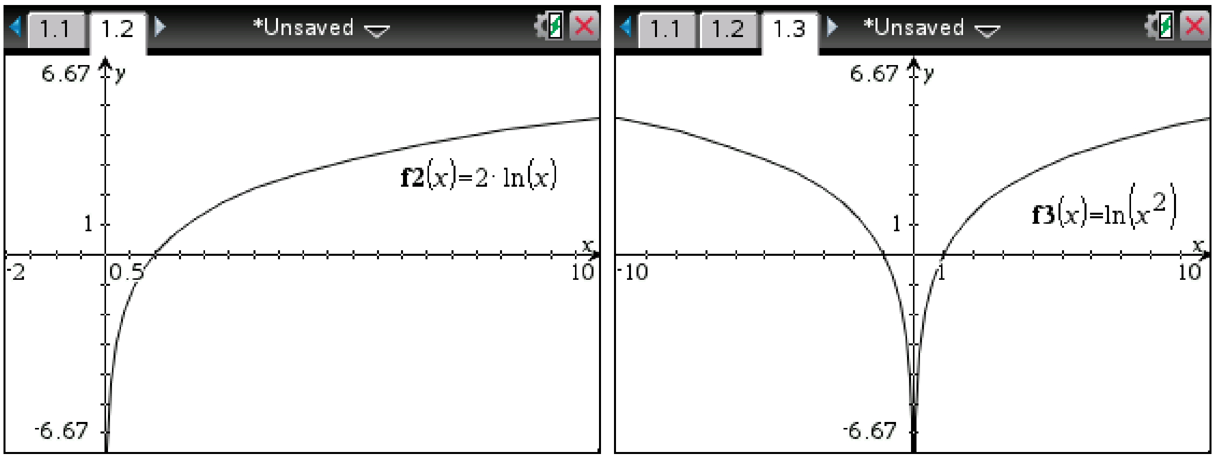
$$\begin{aligned}
3\log_e(2x) &= 5a \\
\log_e(2x) &= \frac{5a}{3} \\
\Rightarrow 2x &= e^{\frac{5a}{3}} \\
x &= \frac{1}{2}e^{\frac{5a}{3}}
\end{aligned}$$

Hence the answer is **C**.

Question 3 (E)

This a trick question (sorry!), as most students too readily assume that the logarithmic property $a\log_e(x) = \log_e(x^a)$ is always true, which is actually **not** always the case. Yes, $a\log_e(3) = \log_e(3^a)$ is always true, but when you starting taking the logarithm of x, you suddenly have to be careful of making such assumptions. But why is that?

On your CAS, sketch the graphs of both $y = 2\log_e(x)$ and $y = \log_e(x^2)$.



What do you notice? The first graph is only defined for $x \in \mathbb{R}^+$, and resembles the logarithm graphs we are used to, whereas the latter has two of these graphs - that is, it is defined for $x \in \mathbb{R}\backslash\{0\}$. For $x > 0$, both graphs are identical, but for $x < 0$, only $y = \log_e(x^2)$ exists. This is because, when you plug in a negative value into $y = 2\log_e(x)$, this will be undefined, but when you do so to $y = \log_e(x^2)$, the negative is squared, disappears, and you end up with a defined value. So, essentially, $y = 2\log_e(x)$ and $y = \log_e(x^2)$ are only the same function for $x > 0$ - that is, they are not always the same function!

So now, looking at the property in **A**:
$2f(x) = 2\log_e(x)$, and $f(x^2) = \log_e(x^2)$. Hence $2f(x) = f(x^2)$ is only true for $x > 0$, and **not** always! So this is not the answer.

Exactly the same logic will show you that **C** is incorrect. $f(xy) = \log_e(xy)$ and $f(x)+f(y) = \log_e(x)+\log_e(y)$. As tempting as it is to obey the logarithm law $\log_e(a) + \log_e(b) = \log_e(ab)$, we are again dealing with variables that could potentially be negative, and not with given numbers. Consequently, since $f(x) + f(y) = \log_e(x) + \log_e(y)$, then $f(x) + f(y)$ is only defined when both x and y are positive. But looking at $f(xy) = \log_e(xy)$, we can see that $f(xy)$ is defined when both x and y are positive, and **also** when both x and y are negative! So once again, the two functions are not always the same.

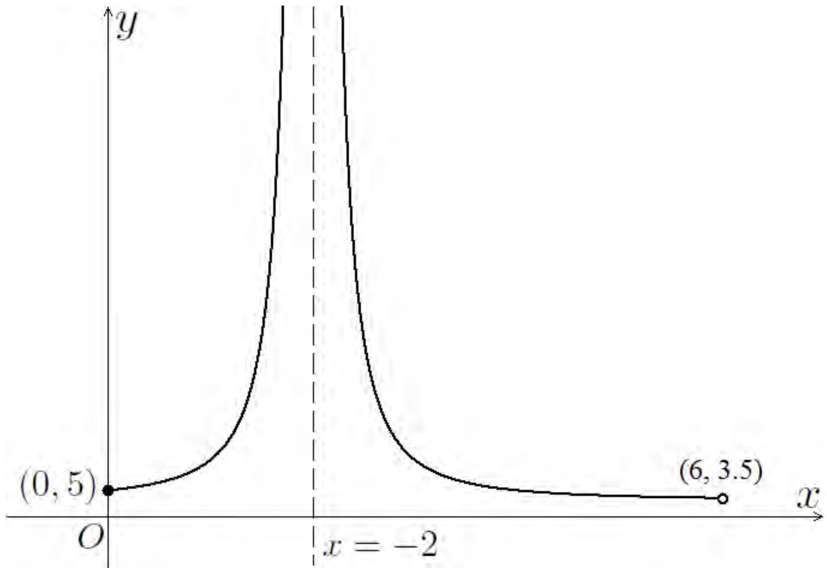
Options **B** and **D** are misinterpretations of logarithm laws. $f(x + y) = f(x)f(y)$ would be true for an exponential function, not for a logarithmic (as $\log_e(x + y) \neq \log_e(x) + \log_e(y)$). And $f(xy) = f(x)f(y)$ is not a property of either.

So now look at the remaining option - is $f(3x) = f(3) + f(x)$ always true?
Well,

$$\begin{aligned} f(3x) &= \log_e(3x) \\ &= \log_e(3) + \log_e(x) \\ &= f(3) + f(x) \end{aligned}$$

This one is **always** true, because if x is positive, then $3x$ must also be positive. If x is negative, then $3x$ will also be negative. And if $x = 0$, then $3x = 0$. So the issue of changing the Domain by applying a logarithm law is not relevant here, meaning we can be certain that **E** is the answer.

Question 4 (D)



As evident from the graph, $y = f(x)$ is part of a truncus - recall that a truncus is any function of the general form $y = \dfrac{a}{(x-b)^2} + c$. Graphically, they resemble hyperbolas (because they really are just a hyperbola to the power of 2), but both parts of the graph are on the same side of the horizontal asymptote.

Now, how do we work out the range? A good general rule is that, for any non-linear functions, you should never ever ever try working out the range without a quick sketch (either by hand or by CAS). The reason is that endpoints do not always signal the highest or lowest points of the graph - there could, for instance, be a turning point in the middle. Hence, questions like this one do commonly pop up in exams to try to trip up the students that rush to find the range without looking at the graph!

Indeed, if we were to sub in $x = 0$ and $x = 6$ to obtain the endpoints, we would obtain $(0, 5)$ and $(6, 3.5)$. But the range is not going to be $(3.5, 5]$ - you can see from the graph that, as we get close to the asymptote, the values of y are much greater than 5. So y will approach positive infinity.

The lower bound of the range, as we can tell from the graph, does indeed occur at the lower of the two endpoints - i.e. at $(6, 3.5)$. Since this is **not** actually part of the function (as 6 is excluded from the domain), the range will therefore be $(3.5, \infty)$, and the answer must be **D**. The message here is, when trying to find the range, you must first take the domain into account, and then look at the graph of the function for that specific domain.

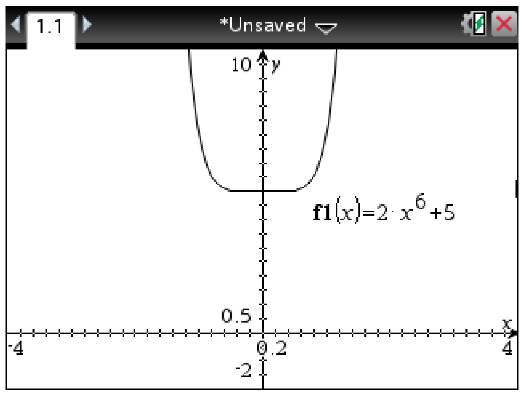
Question 5 (D)

An even function is one that obeys $f(-x) = f(x)$. This may look more complicated than it really is - all this means is that the function is symmetrical about the y-axis. So you can see that a function is even in one of two ways - either sketch its graph (by hand or on the CAS) and see that it is symmetrical about the y-axis, or show algebraically that $f(-x)$ and $f(x)$ boil down to the same thing. And you should also know (although not pertinent to this question) that an odd function is one that obeys $f(-x) = -f(x)$.

So now look at each of the functions we are given individually. Take $f(x) = 2x^6 + 5$. Looking at the algebraic method:

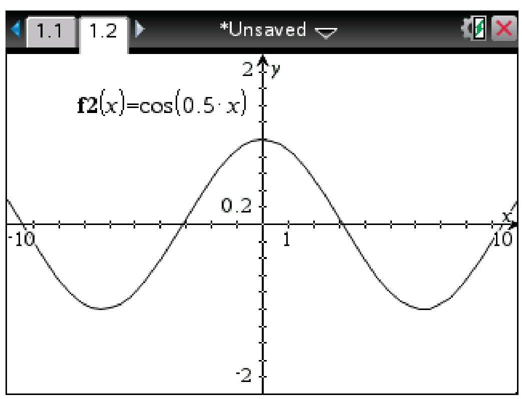
$$\begin{aligned} f(-x) &= 2(-x)^6 + 5 \\ &= 2(-1)^6(x)^6 + 5 \\ &= 2x^6 + 5 \\ &= f(x) \end{aligned}$$

Hence, we can already see that $y = f(x)$ is an even function. Looking at its graph:



We can see that it resembles a flattened parabola lifted upwards 5 units - which is indeed symmetrical about the y-axis.

Now let's consider $g(x) = \cos(\frac{1}{2}x)$. Look first at its graph:



Evidently, this graph is symmetrical about the y-axis, and the function is even. We could use the algebraic method too, because it is a known property that $\cos(-x) = \cos(x)$ (NB: you should know this when you come to the algebra section of the course). That is:

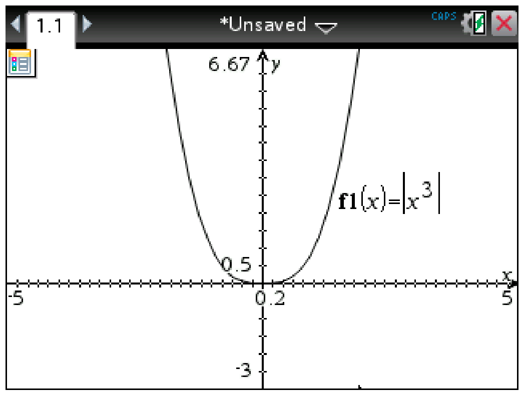
$$\begin{aligned} g(-x) &= \cos(-\frac{1}{2}x) \\ &= \cos(\frac{1}{2}x) \\ &= g(x) \end{aligned}$$

Thus $y = g(x)$ is also an even function.

$h(x) = (x-1)^2$ is a parabola shifted one unit to the right. This means it will **no longer** be even as it will now be symmetrical about $x = 1$. In algebraic terms, we can see this too:

$$\begin{aligned} h(-x) &= (-x-1)^2 \\ &= (-(x+1))^2 \\ &= (x+1)^2 \\ &\neq h(x) \end{aligned}$$

Consider the function $j(x) = |x|^3$. If you are ever unsure of any modulus functions, you should remember that the CAS can sketch the modulus function:



This is the graph of $y = x^3$, but with the part to the right of the y-axis copied and reflected to the left of the x-axis, which is how you sketch any function of the type $y = f(|x|)$. Needless to say, this is symmetrical about the y-axis, and is therefore even!

Algebraically:

$$\begin{aligned} j(-x) &= |x|^3 \\ &= |-1|^3 |x|^3 \\ &= (1)^3 |x|^3 \\ &= |x|^3 \\ &= j(x) \end{aligned}$$

And lastly, $k(x) = 3\tan(x)$ is going to be an odd function. Algebraically this is evident because of another property you should know: $\tan(-x) = -\tan(x)$.

$$\begin{aligned} k(-x) &= 3\tan(-x) \\ &= -3\tan(x) \\ &= -k(x) \end{aligned}$$

Clearly this function is not even, and sketching the graph of any tangent function does show that they are not symmetrical about the y-axis.

So the even functions here are only $f(x)$, $g(x)$ and $j(x)$, and so **D** is the answer.

Question 6 (B)

For these transformation questions, there are again many ways to go about doing it, and you should follow the methods you have been taught. Some people find it easier and quicker to do these questions by observation - say, when you see "Dilation by factor of $\frac{1}{2}$ from the x-axis", you multiply the right-hand side by $\frac{1}{2}$, and so on. I would highly caution **against** using this method though - it is very easy to make mistakes, particularly when you have dilations and translations together, and you have to apply them in the correct order, which is very confusing.

Instead, a much safer and more mathematical method (no pun intended) is to use x' and y'. It takes longer to get the hang of this, but once you do, transformation questions become significantly easier! So I would advise you to try and learn this method, and do plenty of transformation questions between now and the SAC/exam.

What is the first step? We must write out the transformations in a mathematical manner. So take the first transformation (and please, apply them in the given order!), which is "Dilation by factor of $\frac{1}{2}$ from the x-axis". This means that, if we had a point (x, y), it will now become the point $\left(x,\, \dfrac{1}{2}y\right)$.

Why? Because "from the x-axis" means that it affects the y-value, and vice-versa. Be careful though, because "parallel to the x-axis" implies that it affects the x-values! (And yes, the terminology could not possibly get any more confusing!) So we write this as:

$$(x, y) \to \left(x,\, \frac{1}{2}y\right)$$

Now we must keep applying each transformation, in turn, in the given order. The next transformation is a "Translation of 5 units upwards", meaning we add 5 to the y-coordinate.

$$(x, y) \to \left(x,\, \frac{1}{2}y\right) \to \left(x,\, \frac{1}{2}y + 5\right)$$

(Needless to say, you don't need to re-write it every time! You would just do a (rather big) line with all the transformations applied in sequence.)

The next transformation is "Translation of 1 unit to the left", so minus 1 from the x.

$$(x, y) \to \left(x,\, \frac{1}{2}y\right) \to \left(x,\, \frac{1}{2}y + 5\right) \to \left(x - 1,\, \frac{1}{2}y + 5\right)$$

Next, insert a "Dilation by factor of 3 parallel to the x-axis", so multiply the x coordinate by 3. Notice that this time it said "parallel"!

$$(x, y) \to \left(x,\, \frac{1}{2}y\right) \to \left(x,\, \frac{1}{2}y + 5\right) \to \left(x - 1,\, \frac{1}{2}y + 5\right) \to \left(3(x - 1),\, \frac{1}{2}y + 5\right)$$

Observe how the issues of order of dilations takes care of itself with this method - as the dilation was applied second, we can see that the 3 affects both the x and the -1, whereas the dilation in the y only affects y and not the translation, as the translation was second.

Lastly, "The graph is reflected in the y-axis". This should be interpreted as "make x into $-x$", as a reflection in the y-axis is equivalent to swapping the positive and negative sides of the x-axis around. So multiply the x coordinate by -1.

$$(x, y) \to \left(x,\, \frac{1}{2}y\right) \to \left(x,\, \frac{1}{2}y + 5\right) \to \left(x - 1,\, \frac{1}{2}y + 5\right) \to \left(3(x - 1),\, \frac{1}{2}y + 5\right) \to \left(-3(x - 1),\, \frac{1}{2}y + 5\right)$$

We're now (finally!) done with the transformations. The next step is to let the last of the coordinates we have above equal the point (x', y'). So in a single line (as you would write on a SAC/exam), we have

$$(x, y) \to \left(x,\, \frac{1}{2}y\right) \to \left(x,\, \frac{1}{2}y + 5\right) \to \left(x - 1,\, \frac{1}{2}y + 5\right) \to \left(3(x - 1),\, \frac{1}{2}y + 5\right) \to \left(-3(x - 1),\, \frac{1}{2}y + 5\right) = (x', y')$$

We can now build two equations from this:

$$x' = -3\,(x-1) \text{ and } y' = \frac{1}{2}y + 5$$

What to do next then? Well, we need to re-arrange the following equations to get x in terms of x', and y in terms of y'. The reason is, the original equation is in terms of x and y, so we can sub x and y back into the equation and obtain a "new" equation in terms of x' and y' (which is what we need to answer the question). Hence:

$$\begin{aligned} x' = -3\,(x-1) \quad &\text{and} \quad y' = \frac{1}{2}y + 5 \\ x - 1 = -\frac{x'}{3} \quad &\text{and} \quad \frac{1}{2}y = y' - 5 \\ x = 1 - \frac{x'}{3} \quad &\text{and} \quad y = 2y' - 10 \end{aligned}$$

We can substitute these expressions back into the original equation to obtain the image, as follows:

$$\begin{aligned} y &= \sin(x) \\ \implies 2y' - 10 &= \sin(1 - \frac{x'}{3}) \\ 2y' &= \sin(1 - \frac{x'}{3}) + 10 \\ \therefore y' &= \frac{1}{2}\sin(1 - \frac{x'}{3}) + 5 \end{aligned}$$

This is the equation of the image! We can now delete the dashes, to get $y = \frac{1}{2}\sin(1 - \frac{x}{3}) + 5$. But looking at the answers, none of them match up. However, we can easily re-arrange our current answer - by taking out $-\frac{1}{3}$ - to resemble the available answers.

$$\begin{aligned} y &= \frac{1}{2}\sin(1 - \frac{x}{3}) + 5 \\ \therefore y &= \frac{1}{2}\sin(-\frac{1}{3}\,(x-3)) + 5 \end{aligned}$$

This is still not one of our answers. But remember that $\sin(-x) = -\sin(x)$, meaning we can take the negative outside.

$$\therefore y = -\frac{1}{2}\sin(\frac{1}{3}\,(x-3)) + 5$$

And voila! This is the same as option **B**, which must therefore be the answer.

I know this method may look very long and tough right now, particularly if you have never seen it before. But it is significantly quicker when you are doing it yourself as working, and it is a lot more error-proof then other methods. As such, I highly encourage you to persist with trying to use it.

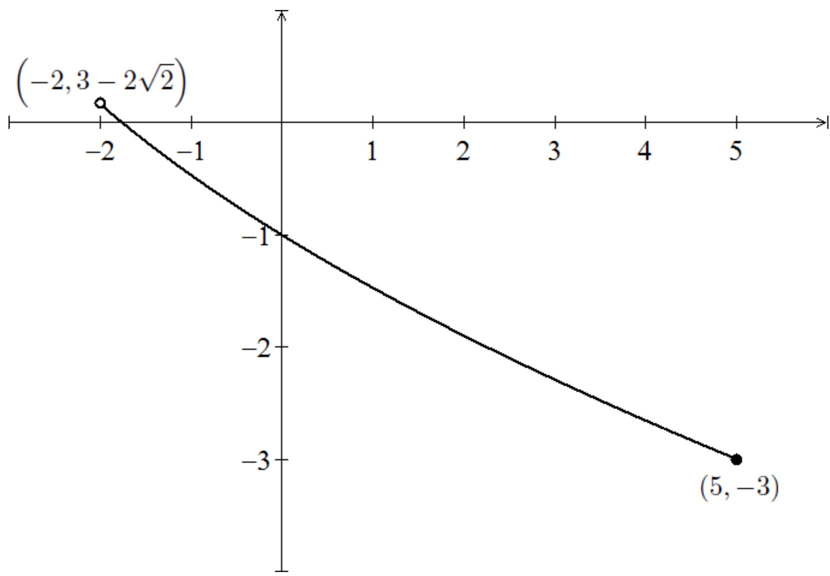
Another point to take note of - when we first applied the transformations to the equation and none of them matched up with the answers, it is easy to assume that an error was made, and perhaps even give up with the question. However, VCAA will often give multiple choice answers that are equivalent to the ones that you obtain with your working, so try to see if your answer can be re-arranged to give one of the options. Moreover, you can always quickly graph the equation you obtained, and then graph options **A**, **B**, etc... until you find one that matches the answer you obtained!

Question 7 (E)

The first piece of information that is vital to this question is knowing that the domain of the original function becomes the range of its inverse, and that the range of the original function becomes the domain of its inverse. So we can (and should) do this question without working out the rule of $y = h^{-1}(x)$.

Now, the domain of $h(x)$ is given in the question itself as $(-2, 5]$ - NB: always pay attention to functions written in this extended form, as the domain must be included! So we already know the range of $h^{-1}(x)$ - it must be $(-2, 5]$.

Now, what is the domain? Well, it will be the range of $h(x)$, meaning we must look at its graph.
$f(-2) = -2\sqrt{4-2} + 3 = -2\sqrt{2} + 3 = 3 - 2\sqrt{2}$. So there is an open circle (as -2 is not actually included in the domain) at this endpoint.
$f(5) = -2\sqrt{9} + 3 = -6 + 3 = -3$. Evidently, this is the lower endpoint, and here we have a closed circle.



As evident from the graph, -3 is the lowest value of $h(x)$, and $3 - 2\sqrt{2}$ is the highest. As $(-2, 3 - 2\sqrt{2})$ is not included in the graph, the range of $h(x)$ must be $\left[-3, 3 - 2\sqrt{2}\right)$. Consequently, the domain of $h^{-1}(x)$ will also be $\left[-3, 3 - 2\sqrt{2}\right)$; therefore, the answer is **E**.

Question 8 (A)

There are several ways of attempting this question. One method you should be familiar with is to expand out the matrix rule - that is, let $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{X}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$. Afterwards, we follow on similarly to the method for **Question 6**.

So:

$$\begin{aligned}
\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & -2 \\ -\frac{1}{3} & 0 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \\
&= \begin{bmatrix} 0 & -2 \\ -\frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} x+1 \\ y+2 \end{bmatrix} \\
&= \begin{bmatrix} -2(y+2) \\ \dfrac{-1}{3}(x+1) \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\therefore x' = -2(y+2) \quad &\text{and} \quad y' = \frac{-1}{3}(x+1) \\
y+2 = \frac{-x'}{2} \quad &\text{and} \quad x+1 = -3y' \\
\therefore y = \frac{-x'}{2} - 2 \quad &\text{and} \quad x = -3y' - 1
\end{aligned}$$

We can now sub this back into the original equation. It is often difficult to remember whether you need to use x and y or x' and y'. As such, it is useful to remember that, if we are looking for the equation of the image, we want x' and y' in the answer. So we need to substitute in x and y (in terms of x' and y') to get this. Transformations can be confusing, but once you do enough questions, they become a lot easier!

Notice also that we have y in terms of x' here, and x in terms of y' here. This may be unfamiliar, but it is completely fine - it means that the matrix we used also contained a transformation to reflect the equation along the line $y = x$ (i.e. swap x and y). This is indeed the case, as transformation matrices of the form $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ will always do this.

So the new equation will be given by:

$$\begin{aligned}
\frac{-x'}{2} - 2 &= \frac{3}{-3y' - 1} \\
\therefore -3y' - 1 &= \frac{3}{\frac{-x'}{2} - 2} \\
&= \frac{3}{\frac{-1}{2}(x'+4)} \\
&= \frac{-6}{(x'+4)} \\
\therefore -3y' &= \frac{-6}{(x'+4)} + 1 \\
y' &= \frac{2}{(x'+4)} - \frac{1}{3}
\end{aligned}$$

We should not need to keep the dashes to express the answer; the equation of the image will therefore be $y = \dfrac{2}{(x+4)} - \dfrac{1}{3}$, and the answer is **A**.

Plenty of other methods for doing this question exist, such as deciphering the matrix equation to see what the actual individual transformations are, and then applying them as you would normally without having to use matrices.

You should do whatever you feel is most comfortable, although for this question, the matrix method outlined above is easiest - you can quickly establish equations for x and y in terms of the image variables (x' and y'). What often trips most people up is knowing what to sub in and when. What you should remember is that x' and y' in terms of x or y will show you what the transformations actually are, whereas x and y in terms of x' and y' allow you to plug the values back in to obtain the image equation.

Question 9 (C)

The way of doing this question is understanding what $y = g \circ f(x)$ really means - that is, understanding the essence of composite functions. Basically, $y = g \circ f(x) = g(f(x))$. Breaking this down step by step, our input into $f(x)$ is x. The output of $f(x)$ will be itself the input into $g(x)$, and then the final output from $g(x)$ will be the value of y. As such, when you are given the graphs of $f(x)$ and $g(x)$, and asked for $y = g \circ f(x)$, you basically need to pick a value of x, see what the corresponding value of y is for $f(x)$, then use that as the x-value of $g(x)$. These new x-values are now inputted into $g(x)$ and outputs a new range, which is $g(f(x))$.

This was probably VERY confusing to read, so let's look at an example from the graphs we have. When $x = 2$, $f(x) = 1$. Consequently, if we input 1 into $g(x)$ - that is, find $g(1)$ - we get $g(1) = -1$ (all of this is visible from the graph). Consequently, the corresponding point in $y = g \circ f(x)$ is going to be $(2, -1)$. Looking at this point alone is enough to obtain the answer as **C**.

Of course, you do not need to look at just specific points - there are multiple ways of doing this question. For instance, the domain of $f(x)$ is $[0, 2]$, meaning that the domain of $y = g \circ f(x)$ can only be $[0, 2]$ or a subset of this, which straight away only leaves you with options **A**, **C**, and **D**. But the range of $f(x)$ is $[-1, 1]$, meaning that all inputs into $g(x)$ must also be within $[-1, 1]$; and the range for this portion of $g(x)$ is $[-2, -1]$. Consequently, the range of $y = g \circ f(x)$ must be $[-2, -1]$, and the only option to obey this is **C**.

Question 10 (B)

What can we tell from the graph of $y = k(x)$? The first part, to the left of $(-3, 1)$, appears to be linear, and with a negative gradient. The second part, in between $(-3, 1)$ and $(-1, 0)$, appears to be an upside-down parabola (or another polynomial of an even degree) - that is, the coefficient of x^2 must be negative. And the third part of this hybrid function appears to be a cubic (or any other polynomial of odd degree) with a point of inflection at the origin.

Now, we cannot exactly work out the equation of the straight line from the single point we are given. But we can certainly reject some options. Any line with a negative gradient that passes through $(-3, 1)$ will satisfy the graph shown, so we should reject any that don't conform to this, and keep the ones that do. From the given options, $-x - 2$ and $-2x - 5$ are both acceptable. And this part of the graph could be for either $x < -3$ or $x \leq -3$ as the function is continuous at -3, meaning that -3 could belong to either the line or the parabola.

Next, we cannot exactly work out the equation of the parabola, but we can narrow our options as above. This is because we only have two points. However, we can see that the turning point is somewhere between -1 and -3. A parabola is symmetrical, so the turning point needs to be closer to -3 than to -1. We can therefore state with certainty that the equation of the parabola is **not** going to be $y = -(x + 2)^2 + 1$. This is also evident by subbing in the point $(-3, 1)$, as the point does not lie on this parabola. Hence we can eliminate options **A** and **C**. NB: elimination is a great strategy for multiple choice questions - you should always read every option (particularly in case a better option comes along, such as "both **A** and **C** seem correct"), and eliminate ones that are clearly wrong when you are not sure which one is correct.

Now we look at the last part. The most obvious equation would be $y = -x^3$, but, as a quintic has a very similar shape to a cubic (as will all functions of the form $y = x^n$ when n is odd), $y = -x^5$ is also a viable option. We can eliminate **C** straight away though, because there is no way this function could be $y = x^3$: it is clearly lying in the wrong quadrants (1st and 3rd instead of 2nd and 4th).

So how do we choose between **B** and **E**? Well, according to the graph, there is an open circle at the end of the parabola, and a closed circle at the "beginning" of the cubic/quintic. This means that $x = -1$ must be included in the domain of the third function, which is only the case in **B**, as in **E** the third part of the function is defined for $x > -1$. Ultimately, **B** is the only option that could possibly fit the given graph, and is therefore the answer.

Question 11 (A)

Questions that give you $y = f(x)$, in either graph or equation form, and ask you for the graph of either $y = |f(x)|$ or $y = f(|x|)$, are very common. Although you need to understand the principles as to why this works, there are very simple ways of obtaining these graphs - for $y = |f(x)|$, any part of the graph lying below the x-axis is reflected in the x-axis (mirrored upwards). As such, you can see that option **B** actually gives the graph of $y = |h(x)|$, because this is simply a reflection of the parts of the graph that have a negative y-value (i.e. below the x-axis) to a positive value (i.e. they are now above the x-axis). This works because the rule $y = |h(x)|$ is essentially stating "for a certain value of x, the value of y will be the absolute value of whatever $h(x)$ is", meaning that y will of course be always positive.

For $y = f(|x|)$ however, which after all is what we actually want to find in this question, the rule is saying that "for a certain value of x, take its absolute value (i.e. make it positive), and then input that into $h(x)$ to get y". So if x is positive, $h(x)$ remains the same, meaning that the part of the graph to the right of the y-axis must remain exactly the same. But when x is negative, we need to make the value of x positive first, and then obtain the value of y. That means the part of the graph to the left of the y-axis will just be an exact copy of the part of the graph to the right of the y-axis! So the answer must be **A.**

SECTION 2 - Extended-Response Questions

Question 1

a. i.

Sketching graphs of circular functions almost always comes up on the exam, and you should be very familiar with it (although it is a bit tedious, do practice them). Since the question tells us to "state the period and amplitude in the lines provided below", let's begin with that, as that will help us with the graph itself.

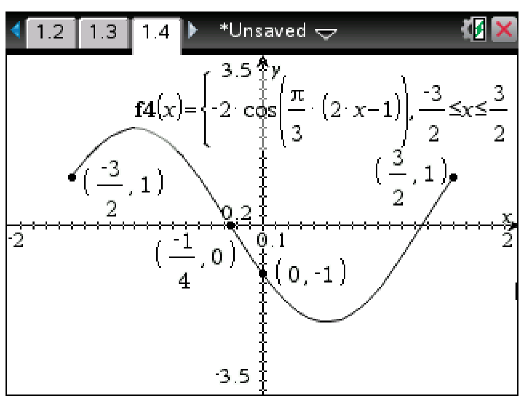
Period is given by $\frac{2\pi}{n}$. Our equation is $f(x) = -2\cos\left(\frac{\pi}{3}(2x-1)\right)+1$. Since n is the coefficient of x, $n = \frac{\pi}{3} \times 2 = \frac{2\pi}{3}$. Hence the period $= 2\pi \div \frac{2\pi}{3} = 3$.

The amplitude is simply the number in front of the cos or sin, as that determines the height of the waves in the graphs of sine and cosine functions. Thus, the amplitude is 2.

What do we do next? Well, seeing as we have a restricted domain, we should find the coordinates of the endpoints (you can, and perhaps should to save time, use the CAS to obtain these points).

When $x = -\frac{3}{2}$, $y = -2\cos\left[\frac{\pi}{3}\left(2\left(-\frac{3}{2}\right)-1\right)\right]+1 = -2\cos\left(-\frac{4\pi}{3}\right)+1 = 1+1 = 2$, so there is a closed circle endpoint at $(-\frac{3}{2}, 2)$.
When $x = \frac{3}{2}$, $y = -2\cos\left[\frac{\pi}{3}\left(2\left(\frac{3}{2}\right)-1\right)\right]+1 = -2\cos\left(\frac{2\pi}{3}\right)+1 = 1+1 = 2$. Remember $x = \frac{3}{2}$ is excluded from the domain, so there is an **open** circle for this endpoint.

Now, you could try to do the remainder of the question by hand - there is nothing that requires the use of technology - but since we have the CAS available, it is obviously a good idea to use this resource. So sketching the graph of $y = f(x)$ on a graph page is certainly very helpful - moreover, using the function graph trace allows you to quickly obtain and double check points.

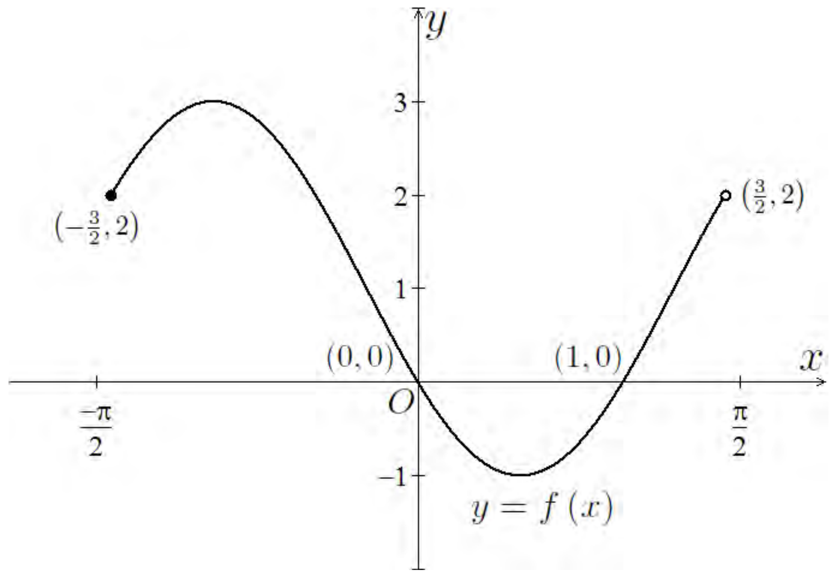


This obviously helps obtain the following graph, giving you a good idea of the shape very quickly, plus you can find the endpoints and intercepts much faster using the CAS (ensure to keep **exact** values though - you might need to use the calculator page for this!)

For the sake of completeness, we will now look at how to obtain intercepts; it is, however, much faster to, on the calculator page, type $f(0)$ for the y-intercept, and "solve($f(x) = 0, x$)" for the intercepts (assuming that you have the function defined as $f(x)$).

y-intercept: $f(0) = -2\cos\left(-\frac{\pi}{3}\right) + 1 = -1 + 1 = 0$. So $(0, 0)$ is the y-intercept.
x-intercept:$-2\cos\left(\frac{\pi}{3}(2x-1)\right) + 1 = 0$.

Solving this with the CAS gives us $x = 0$ and $x = 1$ - you **must** remember to restrict the values of x for this to work, such as by typing "solve($f(x) = 0, x$)$|-\frac{3}{2} \le x \le \frac{3}{2}$". The area of study "Algebra" will concern itself on how to solve these equations by hand. In any case, the coordinates of all the intercepts are $(0, 0)$ and $(1, 0)$, and you should display these on the graph. The graph from the model solutions is shown below.



As such, the marks for this question come down to having the correct shape, and including the information that the question specifically asks for - meaning you can lose a lot of marks for simply not reading the question! The same applies to the domain - you must obey the domain you were given, which includes utilising open and closed circles at endpoints correctly. Make sure to draw all graphs in pencil by the way - you only get one set of axes, and if you get something wrong in pen, it's very difficult to fix up without it looking very messy.

Graphs of this sort can be time consuming, which is why using the CAS to get as many of the crucial features as quickly as possible is a good idea - although, if lines are provided as well as the set of axes, it may mean that some working does need to be shown to the examiner (however, this is more likely in technology-free tests and exams)! Another good idea to not spend too much time is to give all the essentials and quickly draw a graph first, and if you have time at the end of the test/exam, come back and add more detail or perfect the shape of the graph.

ii.

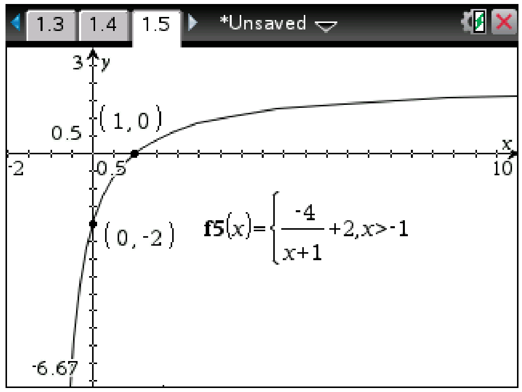
The graph we now need to sketch is of $g : (-1, \infty) \to \mathbb{R}$, $g(x) = -\dfrac{4}{x+1} + 2$. This is a hyperbola, as we have x in the denominator, and we can therefore infer the asymptotes - they must be $x = -1$ and $y = 2$. This is because, if we look at the general equation of a hyperbola, which is $y = \dfrac{a}{x-h} + k$, the asymptotes are always $x = h$ and $y = k$ (the reason behind this is that we can never divide by zero, and because there is no value of x that allows the fraction $\dfrac{a}{x-h}$ to equal zero).

Now, consider the domain we are given: $(1, \infty)$. Since the asymptote is at $x = -1$, this essentially means we only have to sketch the half of the hyperbola that lies to the right of the vertical asymptote.

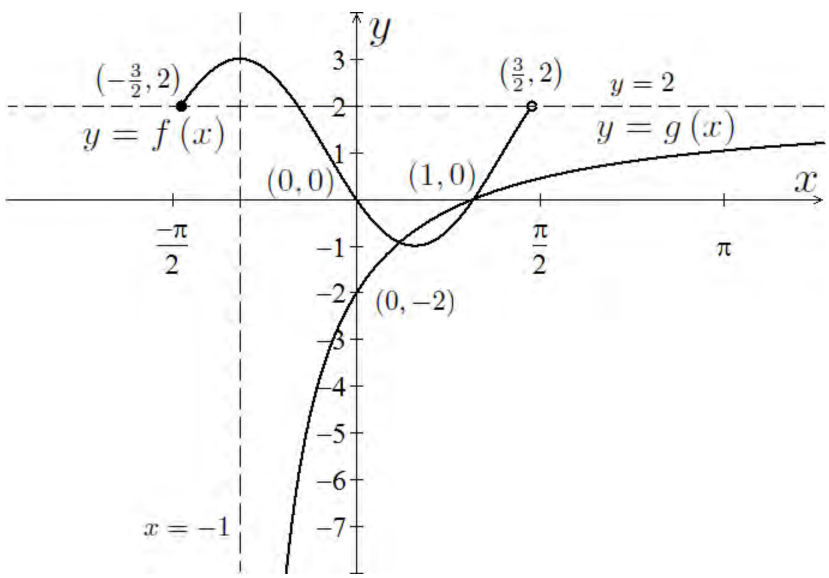
Let's now try to find the intercepts. When $x = 0$, $g(0) = -\frac{4}{1} + 2 = -2$, meaning that the y-intercept is $(0, -2)$.

Let $g(x) = 0$. Then $\dfrac{4}{x+1} = 2 \Rightarrow x + 1 = 2 \Rightarrow x = 1$. So the x-intercept must be $(1, 0)$. Of course, both these intercepts could be found using the graph trace function.

On that note, if you are comfortable enough with sketching hyperbolas, all this information is already enough to go on with: but it does not hurt to sketch it on the CAS to make sure regardless! And if you are not confident with sketching hyperbolas, then practice doing them by hand first, but during SAC/exam conditions, you might as well use the CAS to help you out.



We can therefore go ahead now and sketch this graph on the same axis as the one above, as such:



A few points to be aware of though: first of all, get into the habit of labelling any important points (such as intercepts and endpoints) with coordinates, and of labelling asymptotes as equations; because if the SAC/exam specifies "labelling any intercepts with their coordinates and asymptotes with their equations", you **will** lose easy marks for not having done so! This is just one of those things VCAA is picky about, so get into the habit of always giving points as coordinates, and asymptotes as equations (i.e. not just a floating "-1" next to the asymptote/intercept). And another thing that VCAA is infamous for is that if your graph even slightly nudges the asymptote, you will lose that mark - so make sure that the graph approaches, but does NOT touch, the asymptote!

iii.

Questions worth one mark never require working - that is always stated at the beginning of every exam. You simply need to state the answer, and so you should always (if possible) get the CAS to do it.

Here, all you need to do is equate the two functions - if, for instance, I had previously defined the cosine function as "$f1(x)$", and the hyperbola was saved under "$f2(x)$", all you need to do is go on a calculator screen and type "solve $(f1(x) = f2(x), x) \mid -1 < x < \frac{3}{2}$", and then grab the answers and find "$f1(x)$" for those values of x (or "$f2$"). Alternatively, you could use the graph screen and, under menu → Points & Lines → Intersection Point(s), you could find the points of intersection between the two graphs - but be careful here, as depending on your settings, the graph page might round things up too much! Either way, you must use the CAS - this equation cannot be solved by hand.

It is important, when solving this equation, to be mindful of the domain too - only solutions for $-1 < x < \frac{3}{2}$ qualify, as that is the intersection of the restricted domain of the two graphs.

In any case, when solving this on the CAS for the correct domain, we are given $x = 0.368$ or $x = 1$.

Now, (and you must use the exact value here; using the rounded value is **not** acceptable), $f(0.368) = -0.924$, and $f(1) = 0$.

Hence our two intersection points are: $(0.368, -0.924)$ and $(1, 0)$. And that's all you need to write down for this question, no working needed, just the points, and the first point should be correct to three decimal places.

iv.

Again, this is a one mark question, so if you see the answer straight away, just write it down, as your working will not be looked at.

In any case, the Domain of the function $y = f(x) + g(x)$, which can be written as $y = (f + g)(x)$, is just the intersection of the domains of the two functions being added. This makes sense, as you cannot possibly start adding values that do not exist!

So we want the intersection of two sets, $\left[-\frac{3}{2}, \frac{3}{2}\right)$ and $(-1, \infty)$. This is just going to be the "middle ground", or the values that lie in both sets, which is anything from -1 to $\frac{3}{2}$, not inclusive of either point.

That is:

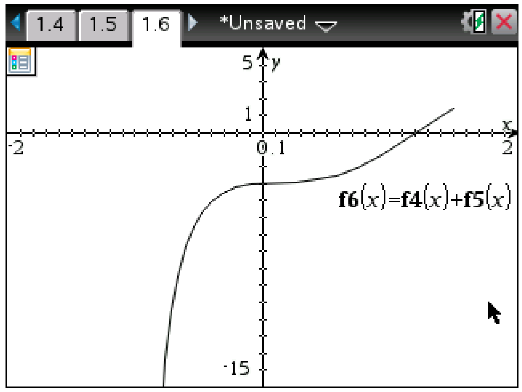
$$\begin{aligned} D_{f+g} &= D_f \cap D_g \\ &= \left[\frac{-3}{2}, \frac{3}{2}\right) \cap (-1, \infty) \\ &= \left(-1, \frac{3}{2}\right) \end{aligned}$$

And all you need to write down to get the mark here is $\left(-1, \frac{3}{2}\right)$.

v.

This is an "addition of ordinates" type question, which could easily come up in a SAC on functions and relations, and could come up on exams too. Essentially, doing this question involves adding the points of both graphs together to obtain the new graph. So, for instance, when $x = 0$, $f(0) = 0$ and $g(0) = -2$. As $0 + (-2) = -2$, we can see that for the graph of $y = (f + g)(x)$, the y-intercept will be the point $(0, -2)$.

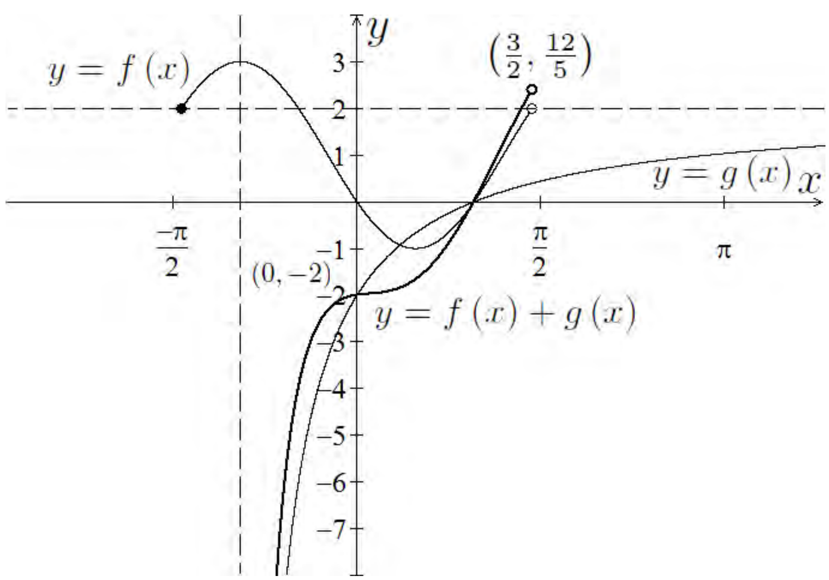
Now, you could do this for a few values to get a rough idea of what the graph of the sum will look like - and, if this was a technology free exam, or if you were not given the equations, this would be your first choice for a method - but here, you can use the CAS to help you. If we had defined the two functions as "$f1(x)$" and "$f2(x)$", we would therefore open a new graph screen (ctrl + I) and type "$f3(x) = f1(x) + f2(x)$", thus allowing us to better visualise the new graph. Note that in the screenshot below, "$f1$", "$f2$" and "$f3$" were already in use, so higher numbers are used.



What now occurs as x approaches both -1 and $\frac{3}{2}$?

The graph of $y = f(x)$ has small, positive values around the vicinity of $x = -1$, whereas the graph of $y = g(x)$ has an asymptote at $x = -1$, and the graph itself is approaching negative infinity - that is, the values of y are becoming very big negative numbers. What happens when you add a huge negative number to a comparatively tiny positive number? You get a huge negative number still, that's only slightly smaller than the negative number we were dealing with before. As such, the graph of $y = f(x) + g(x)$ will also have an asymptote at $x = -1$, and it will be found just above the graph of $y = g(x)$ (**remember:** you are sketching them on the same set of axes, so the new graph must be in the right place relative to the other two, and the graph of $g(x)$ will be slightly more negative than the graph of $y = f(x) + g(x)$).

Now, when $x = \dfrac{3}{2}$, $g\left(\dfrac{3}{2}\right) = -\dfrac{8}{5} + 2 = \dfrac{2}{5}$, which has coordinate $\left(\dfrac{3}{2}, \dfrac{2}{5}\right)$. And for the graph of $f(x)$, there is an open endpoint at $\left(\dfrac{3}{2}, 2\right)$. Adding the y-coordinate of the two points gives us $\left(\dfrac{3}{2}, \dfrac{12}{5}\right)$, which is therefore the coordinate of the endpoint of the graph of $y = f(x) + g(x)$. Note that this must also be an open circle.
We can now certainly go about sketching the graph of $y = f(x) + g(x)$:



Note that the most crucial things are the endpoint and the asymptote, and the general shape. It must look like you added the values of y for both functions together to get the coordinate of the sum. So the shape here is a lot more important than giving specific points. And also, if you had incorrectly sketched one or both of $f(x)$ and $g(x)$, you may be awarded consequential marks if addition of ordinates is correctly done.

b. i.

There is more than one solution to this question, and alternative answers that give the same end-product will certainly be accepted.

The method I would recommend here is using x' and y' to represent the image, like in **question 6** of the multiple choice. The difference here, though, is that we have the two graphs, and we want to know what the transformation was. However, the same process will help us get there:

Let $y = f(x)$ (the original equation), and let $y' = h(x')$ (the image)

Now, we first need to re-write the two equations.

For $y = f(x)$, bring all the numbers on the right-hand side of the equation back to the left. This may seem a tad weird, but the reason is that this allows us to see which transformations actually affected y, as these are "hidden" by being on the RHS when they really affect y, not x. You can see that I also expanded out $\frac{\pi}{3}(2x-1)$, but this is not necessary, it will simply change the order of dilation and translation. The important thing to note is the moving of things to the LHS:

$$\begin{aligned}
y &= -2\cos\left(\frac{\pi}{3}(2x-1)\right)+1 \\
y-1 &= -2\cos\left(\frac{\pi}{3}(2x-1)\right) \\
\frac{y-1}{-2} &= \cos\left(\frac{\pi}{3}(2x-1)\right) \\
\frac{y-1}{-2} &= \cos\left(\frac{2\pi}{3}x-\frac{\pi}{3}\right)
\end{aligned}$$

For the image, we have nothing to re-arrange as it is a basic equation (often this will not be the case! If so, do re-arrange in the same manner as above). However, you should replace x with x', and y with y'. Hence

$$y' = \cos(x')$$

The step after this - the most important one here - is to create two equations. Remember that these questions almost always involve expressing x and y in terms of x' and y', and vice-versa. And here we want to go from $\frac{y-1}{-2} = \cos\left(\frac{2\pi}{3}x-\frac{\pi}{3}\right)$ to $y' = \cos(x')$. We can therefore say that $\frac{y-1}{-2} = y'$, and that $\frac{2\pi}{3}x-\frac{\pi}{3} = x'$.

The next step consists of re-arranging these equations to get them in terms of x' and y', but here we already have that as the image graph was very simple. Hence we can state that:

$$y' = \frac{y-1}{-2} \text{ and } x' = \frac{2\pi}{3}x-\frac{\pi}{3}$$

But how does this help us? Well, recall that we express transformations in the form $(x, y) \to (x', y')$.

Hence the transformation $(x, y) \to \left(\frac{2\pi}{3}x-\frac{\pi}{3}, \frac{y-1}{-2}\right)$ will give us the image equation.

We are not, however, done yet with expressing the transformation. Unfortunately, there is one subtle trick in the question, and it lies in the definition of the image equation. Namely,

$$h : \left[\frac{2\pi}{3}, \frac{8\pi}{3}\right) \to \mathbb{R},\ h(x) = \cos(x)$$

Notice that the image has a specific domain. So while several translations can give us the graph of $y = \cos(x)$, we are now severely limited, as the graph of the image must actually lie in a certain part of the domain.

So how do we account for the new domain? The easiest way is to look at one of the endpoints of $y = f(x)$, which is $\left(-\frac{3}{2}, 2\right)$. Let's apply the transformation $(x, y) \to \left(\frac{2\pi}{3}x - \frac{\pi}{3}, \frac{y-1}{-2}\right)$ to this point.

$$\left(-\frac{3}{2}, 2\right) \to \left(\frac{2\pi}{3} \times -\frac{3}{2} - \frac{\pi}{3}, \frac{2-1}{-2}\right) = \left(-\frac{4\pi}{3}, -\frac{1}{2}\right)$$

So $\left(-\frac{4\pi}{3}, -\frac{1}{2}\right)$ would be the endpoint. But according to the domain, the graph must begin at $x = \frac{2\pi}{3}$. And as $h\left(\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, this point must actually be $\left(\frac{2\pi}{3}, -\frac{1}{2}\right)$.

How do we go from $\left(-\frac{4\pi}{3}, -\frac{1}{2}\right)$ to $\left(\frac{2\pi}{3}, -\frac{1}{2}\right)$? Just add 2π to the x-value! This means that, to get the image graph within the correct domain, we need to shift everything 2π units to the right (because then the endpoints will match up, and so obviously will the rest of the graph).

So the transformation we actually need here is $(x, y) \to \left(\frac{2\pi}{3}x - \frac{\pi}{3} + 2\pi, \frac{y-1}{-2}\right)$

Hence the transformation $(x, y) \to \left(\frac{2\pi}{3}x + \frac{5\pi}{3}, \frac{y-1}{-2}\right)$ will obtain the image graph.

Before we get to describing this in order though, it is important to once again stress that there are many ways of interpreting this transformation. For instance, $\frac{2\pi}{3}x + \frac{5\pi}{3} = \frac{2\pi}{3}\left(x + \frac{5}{2}\right)$, where the form on the right-hand side would be first a translation to the right followed by a dilation, whereas the form on the left-hand side (the one I used here for my solutions) is first a dilation followed by a different translation. And the same holds true for the transformations of the y-coordinate.

So if we were to read the transformation $(x, y) \to \left(\frac{2\pi}{3}x + \frac{5\pi}{3}, \frac{y-1}{-2}\right)$ as it stands, we would obtain:

- First, a dilation from the y-axis by a factor of $\frac{2\pi}{3}$. This is the equivalent to multiplying the x-coordinate by $\frac{2\pi}{3}$ (**remember:** from the y-axis means affecting x!).
- Then a translation of $\frac{5\pi}{3}$ units to the right (i.e. in the positive direction of the x-axis), which is evident in the transformation above by the adding of $\frac{5\pi}{3}$ to the "previously dilated" x. Note that in this method you read according to BODMAS, meaning that something multiplying x is first, and something being added to x is second, unless there is a bracket!
- Next, a translation of 1 unit downwards (i.e. in the negative direction of the y-axis). This is because $\frac{y-1}{-2} = -\frac{1}{2}(y-1)$, meaning that the translation occurs first. If we had expanded this out previously, as I did with x, then obviously we would get a different order and a different translation/dilation.
- Next, we now have a dilation by a factor of $\frac{1}{2}$ from the x-axis (as it is acting in the y-axis).
- Lastly, a reflection in the x-axis. Remember that a reflection in the x axis is always interpreted as making the y-coordinate negative and vice-versa. As the negative outside the y-coordinate here applies to everything, it is the last transformation (however, it is interchangeable with the dilation as both are multiplications).

Note that there are other possible sequences and different orders that will also obtain the correct image graph with the correct domain, and would also be awarded full marks. The domain is the big trick here to be careful of though!

Also, be careful with your descriptions - you do not need to be horribly formal, but what you are trying to say has to be very clear, and you should use the correct terminology, such as "dilation from the x-axis" and "translation of a units to the right/in the positive direction of the y-axis".

ii.

We are now being told to describe the transformations as the matrix rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \mathbf{T}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \mathbf{B}\right)$. Be careful with the brackets, as this here is saying that the translations are applied first (these are denoted by $\mathbf{B}$), and afterwards the dilations are applied (denoted by $\mathbf{T}$). If we were given the rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \mathbf{T}\begin{bmatrix} x \\ y \end{bmatrix} + \mathbf{B}$, it would be the opposite.

In any case, we must now re-arrange our original transformation, which is $(x, y) \to \left(\frac{2\pi}{3}x + \frac{5\pi}{3}, \frac{y-1}{-2}\right)$, so that the dilations are applied first, and translations second. The easiest way of doing this is by factorising the x-coordinate, as I had suggested during the working for part **i.** above.

So we can re-write this sequence of transformations as

$$(x, y) \to \left(\frac{2\pi}{3}(x + \frac{5}{2}), \frac{-1}{2}(y - 1)\right)$$

Now we have two translations first: $\frac{5}{2}$ units to the right and 1 unit down. This is represented by $\mathbf{B} = \begin{bmatrix} \frac{5}{2} \\ -1 \end{bmatrix}$.

The reflection and dilations multiply x and y after the translations have taken place. The matrix $\mathbf{T} = \begin{bmatrix} \frac{2\pi}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ "takes care" of both of them, as $\frac{2\pi}{3}$ will multiply $x + \frac{5}{2}$, and as $-\frac{1}{2}$ will multiply $(y - 1)$ if we expanded the matrix equation out. Hence these are the values of $\mathbf{B}$ and $\mathbf{T}$. And for this question, there is only a single solution, nothing else will obey the rule and correctly transform the graphs.

Since this is a 1 mark question, just writing down the values of the two matrices is all that is necessary for the mark, no working needs to be shown (although re-arranging the transformations is a good step to write down just to help yourself). And if a mistake was made in part **i.** but the working is correct from then onwards, this answer mark could be awarded consequentially.

Question 2

a. **i**

This is a "show that" type of question - a type that is becoming increasingly popular in Methods exams. while it may seem that the hard part (getting the answer) is already done for you, do not be fooled - these questions are deliberately designed to assess better understanding, meaning that markers will be much more rigid regarding working and the use of mathematical conventions. You really need to answer the question as though you do not know what the answer is, and really explain your thinking to the examiner.

So if we look at the property $[f(t)]^2 = f(2t) + 2$, how do we show that it is true? The convention is to take each side of the identity separately and show that one simplifies to something that is actually equal to the other. So here, if we sub in $f(t) = e^t + e^{-t}$ into the identity:

$$\begin{aligned} LHS &= [f(t)]^2 \\ &= \left(e^t + e^{-t}\right)^2 \end{aligned}$$

What to do next? Well the only way it might look like something else is if we expand it out, so using $(a+b)^2 = a^2 + 2ab + b^2$, we get:

$$\begin{aligned} LHS &= \left(e^t\right)^2 + 2(e^t)(e^{-t}) + (e^{-t})^2 \\ &= e^{2t} + 2e^0 + e^{-2t} \\ &= \left(e^{2t} + e^{-2t}\right) + 2 \end{aligned}$$

But now, $RHS = f(2t) + 2 = \left(e^{2t} + e^{-2t}\right) + 2$. So we have managed to re-arrange the LHS into the RHS, so we can say that $LHS = RHS$, meaning that the identity indeed holds true, as required.

Note: although you may not need to give an answer as formally and with as many steps, you do need to obey some conventions. For instance, using LHS and RHS may not be needed, but it does help to make the work clearer, and thus I would recommend it. Likewise, the expansion of $\left(e^t + e^{-t}\right)^2$ is crucial, as simply stating

$$\begin{aligned} LHS &= [f(t)]^2 \\ &= \left(e^t + e^{-t}\right)^2 \\ &= f(2t) + 2 \\ &= RHS \end{aligned}$$

is not enough to warrant a mark. It is vital to show the expansion, and to really demonstrate your thinking to the examiner.

ii.

We are first given the general form of an exponential function, $g(t) = ke^{mt} + n$. Note that here k actually represents translations along the x-axis. Consider that $e^{mt-a} = e^{mt} \times e^{-a} = ke^{mt}$ $(k = e^{-a})$, where a is just an arbitrary number used to show that this works.

The important thing here is that exponential functions have horizontal asymptotes (i.e. $y = b$ form), and that this asymptote is actually determined by the translation along the y-axis, namely n. Hence we can actually say that $y = n$ is the asymptote for this general form. The reason is that, no matter what the value of t is, the expression ke^{mt} will never be zero, so $g(t)$ can consequently never equal n, but will approach it as it gets smaller and smaller.

But we are told that the asymptote is given by $y = -1$, so we know that $n = -1$.
Hence $g(t) = ke^{mt} - 1$ —–(1).

So what is next? We must find the values of both m and k. And what information do we have? Two points. Perfect, 2 unknowns, 2 equations.

Now all we need to do is substitute the two points we are given into the formula for $g(t)$ above, which is (1).

Subbing in the two points:

$$(16, 0) \Rightarrow 0 = ke^{16m} - 1$$

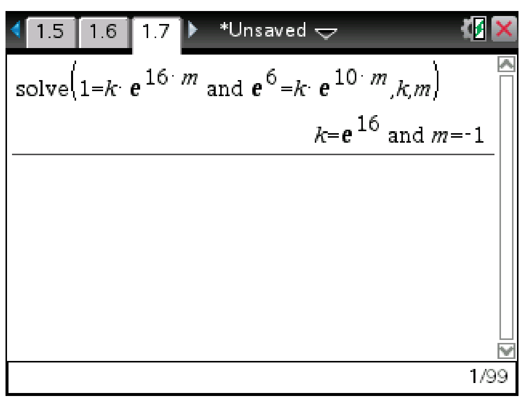
$$\therefore 1 = ke^{16m}\text{——-}(1)$$

$$(10, e^6 - 1) \Rightarrow e^6 - 1 = ke^{10m} - 1$$

$$\therefore e^6 = ke^{10m} \text{ ——-}(2)$$

Because this is a 3 mark question, it is expected that students would, at the very least, write down the two equations above. Do **not** assume that if you are looking for three variables and the question is worth 3 marks, then each answer is worth one mark, because that is not necessarily the case; unless a question is worth one mark, assume that some marks will be method marks!

In any case, we can at this point either solve the simultaneous equation in the CAS (see screenshot below for how to do this), or do so by hand.



Since the question is worth 3 marks, it is a good idea to at least show an attempt to begin solving by hand.

The easiest way to eliminate one of the variables here is to divide equation (2) by equation (1) (or vice-versa), as k cancels out. (2)÷(1):

$$e^6 = \frac{ke^{10m}}{ke^{16m}}$$

$$\therefore e^{-6m} = e^6$$

$$-6m = 6$$

$$\therefore m = -1$$

Substitute this back into equation (1)

$$\Rightarrow ke^{-16} = 1$$

$$\therefore k = e^{16}$$

Ultimately, we end up with $m = -1$ and $k = e^{16}$. One of the dangers of using the CAS here (aside from losing the method marks) is that it may be tempting to give a decimal answer for k. But the question does not specify a certain number of decimal places, meaning you should assume that we need exact values. As such, the answer is $n = -1$, $m = -1$ and $k = e^{16}$.

i.

How can we determine the values of a and b? Well, they are the points where the different parts of the hybrid function $y = h(t)$ connect. Now, a hybrid function is not necessarily continuous at these points, but since $y = h(t)$ is a function of the height of a rollercoaster, it would be impossible for Corey's height above ground level to change so suddenly as to be a discontinuity. That is, since the rollercoaster is continuous, so must be the function of Corey's height!

This means that where one part of the hybrid ends, the other must begin at the same height. So the points a and b can only occur when the respective parts of the function would intersect.

So, to find a, we must solve $t\sin(t) + 5 = f(t-11)$. That is, solve $t\sin(t) + 5 = e^{t-11} + e^{-(t-11)}$.

There is no way to do this by hand, so using the CAS we obtain $t = 8.5671$ or 13.9198. But a will be equal to this value of t, and we are told that $a < 10$.

We can thus conclude that $a = 8.57$ (to two decimal places), as the other solution must be eliminated.

Let's now use a similar process to find b
Let $f(t-11) = |g(t)| - \frac{1}{2}$

We can write this as (although the step above is perfectly acceptable as it indicates the equation you are attempting to solve; so you do not need the next step for the method mark)
$e^{t-11} + e^{-(t-11)} = |ke^{mt} + n| - \frac{1}{2}$

The RHS can be further simplified (although again, this is not required), as we should substitute the recently found values to get
$e^{t-11} + e^{-(t-11)} = \left|e^{16}.e^{-t} - 1\right| - \frac{1}{2}$

Solving this equation with a CAS calculator gives us $t = 13.4349$. Hence, $b = 13.43$ (to two decimal places).

Again, to stress the point, the method mark here is awarded for showing the equations you are trying to solve - again, it is **not** one mark for a and one mark for b. You need to show both equations that you want to solve to tell the examiner where you got your a and b from. And the other mark is for the answer, which is the value of **both** a and b, each correct to two decimal places.

ii.

This question is very simple. All it requires to do is to bring all the information we have gathered so far into a single equation. It is here to help you with the next question, as well as to assess your understanding of how the hybrid function works, and of your ability to slightly modify the functions themselves. So first, just replace a with 8.57 and b with 13.43.

Next, we have to replace $f(t-11)$ with the actual function. Note that this is **not** just the $f(t)$ we used in part **a.** but rather, $f(t-11)$, so be careful here.

Since $f(t) = e^t + e^{-t}$:

$$f(t-11) = e^{(t-11)} + e^{-(t-11)}$$

$$\therefore f(t-11) = e^{t-11} + e^{11-t}$$

This is what we must insert into the second line of the hybrid.

Next, we have $|g(t)| - \dfrac{1}{2}$

Now, we were told that $g(t) = ke^{mt} + n$, and we also found that $n = -1$, $m = -1$ and $k = e^{16}$.

Subbing this back into $g(t)$ gives $g(t) = e^{16} \times e^{-t} - 1$, and using exponential laws (although this is not required for the mark) simplifies the function to $g(t) = e^{16-t} - 1$.

Therefore, we can say that $|g(t)| - \dfrac{1}{2} = \left|e^{16-t} - 1\right| - \dfrac{1}{2}$.

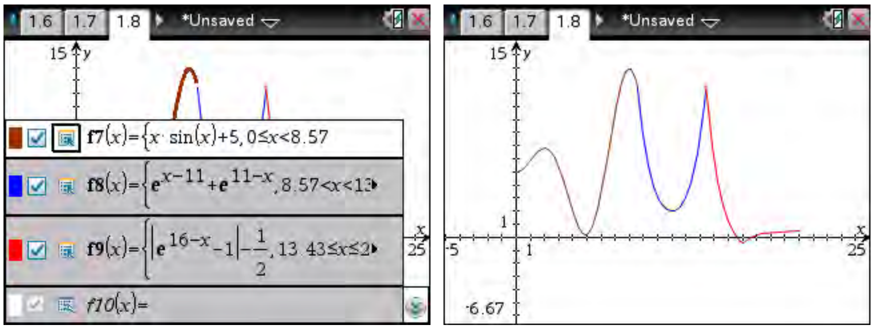
Inserting this into the third line, and copying the rest out from the hybrid function we were given initially, gives:

$$h(t) = \begin{cases} t\sin(t) + 5, & 0 \leq t \leq 8.57 \\ e^{t-11} + e^{11-t}, & 8.57 < t < 13.43 \\ \left|e^{16-t} - 1\right| - \frac{1}{2}, & 13.43 \leq t \leq 20 \end{cases}$$

Again, this is a one mark question, so writing this down is both sufficient and necessary to receive the mark.

c.

This graph may look very ominous before sketching, but you have to remember that it does not have to be drawn perfectly - the shape just needs to look accurate enough - and also that with the CAS (and correct use of the zoom functions), you can get a very good idea of what the graph looks like. This is one of those graphs where I would recommend sketching on the CAS before doing anything else as it is otherwise very difficult to understand the overall shape and know where to start. So let's sketch the hybrid on the CAS - you **can** define hybrid functions on the calculator.

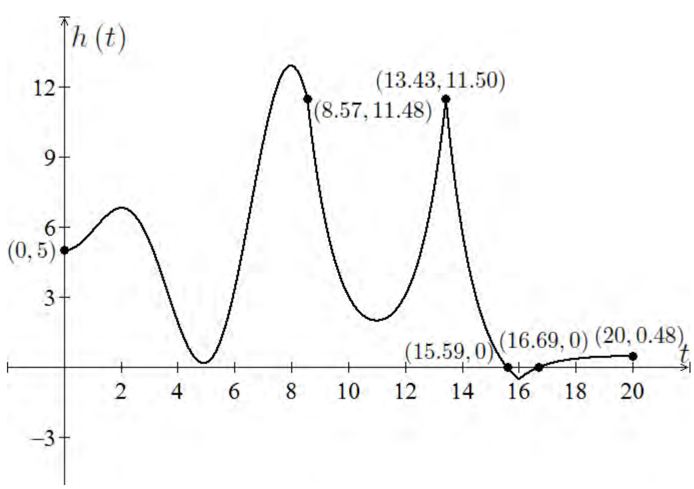


Now, what to do next? This whole question is really a calculator question. Since we are asked to include the coordinates at a and b, plugging $h(a)$ and $h(b)$ into the calculator will give us the y-coordinates - $(8.57, 11.48)$ and $(13.43, 11.50)$. You need to ensure that these points are shown clearly on your sketch. Using either graph trace or the calculator screen, we can also obtain the endpoints - when $t = 0$, we have $h(0) = 0 + 5 = 5$, so $(0, 5)$ is both the y-intercept and one of the endpoints. Moreover, $h(20) = \left|e^{-4} - 1\right| - \dfrac{1}{2} = 0.48$, so the other endpoint is $(20, 0.48)$.

Using graph trace, we see that the only x-intercepts are in the third part of the graph, and they are $(16.69, 0)$ and $(15.59, 0)$.

So once you have all these points, the question becomes a matter of correctly copying out the graph from the CAS. It is important to have an accurate shape, and so you should make sure you have the best zoom setting. It is often a good idea is to zoom out until you can see the majority of the important sections of the graph, and then refine for a better view of the important parts using the window settings. It is also crucial to make it clear that the junctions are not smooth - the different parts of the hybrid are intersecting at the points where $t = a$ and $t = b$. These points must clearly be sharp - if they appear smooth (e.g. like a turning point), then a mark should be deducted.

With all this out of the way, all that remains is the task of actually sketching this function. Again, most of this is just copying from the CAS, so do not be overwhelmed by the size of these functions. And remember to include the modulus for the appropriate part of the third function when typing it out! The modulus function can be found by using the function "abs()" (the command for which is ctrl+×).



d.

As an application question, you first need to take a step back and ask yourself "what is this question really asking me?". We were previously told that "when $h < 0$, the riders are submerged underwater", meaning that Corey will be underwater for the period where $h < 0$. This shows the importance of not only reading carefully, but also underlining and highlighting parts of questions - it may feel like it is a waste of time, but I guarantee you that it is not. Spending an extra second highlighting important bits means you won't miss them later. And in a question like this, where the information is given at the very beginning of a long question but is only needed for the very last part of the question, it could mean the difference between not knowing how to even begin a question and full marks!

So, in essence, we want to know for how long h will be negative, meaning we need to find where the graph crosses the t-axis. Looking at the graph we sketched above (and from the sketch on the CAS), we can see that the graph of $y = h(t)$ only crosses the t-axis for the "third bit" of the hybrid - that is, the only t-intercepts occur during the interval $13.43 \leq t \leq 20$.

So if we want to find these intercepts, we need to let the third part of the hybrid equal zero.
That is, let: $\left|e^{16-t} - 1\right| - \frac{1}{2} = 0$

So we must now proceed to solve this equation. Since this is a three mark question, and as an exact answer is required, we need to show the working to obtain it - even if you do use the CAS to solve this equation as a guide.

The next step is clearly to move the $\frac{1}{2}$ to the right as we are solving for t:

$$\left|e^{16-t} - 1\right| = \frac{1}{2}$$

For the next step, you need to know how to solve equations involving a modulus - that is, of the form $|f(x)| = a$. This is most likely to be the step that would give the method mark in this question, as you need to show the examiner that you understand how to solve equations when there is an absolute value sign.

So if we look at $|f(x)| = a$, we essentially want $f(x)$ to have an absolute value of a, meaning that $f(x)$ could be either $+a$ or $-a$ in "true" value. That is, we can say that $|f(x)| = a \Rightarrow f(x) = \pm a$

Applying this to the equation we want to solve gives: (and on that note, if this seems unfamiliar, you should practice solving equations of this sort, as this is a skill you will need for both technology free and active exams)

$$\left|||\, e^{16-t} - 1\right| = \frac{1}{2} \Rightarrow e^{16-t} - 1 = \pm\frac{1}{2}$$

So now we just need to proceed to solve $e^{16-t} - 1 = \pm\frac{1}{2}$ as you would normally.

Hence we can see that

$$e^{16-t} = 1 \pm \frac{1}{2} \Rightarrow e^{16-t} = \frac{1}{2} \text{ or } \frac{3}{2}$$

Taking now the natural logarithm (i.e. $\log_e$) of both sides gives:

$$16 - t = \log_e\left(\frac{1}{2}\right) \text{ or } 16 - t = \log_e\left(\frac{3}{2}\right)$$

Re-arranging for t now gives:

$$t = 16 - \log_e\left(\frac{1}{2}\right) \text{ or } t = 16 - \log_e\left(\frac{3}{2}\right)$$

So the t-intercepts we have are $16 - \log_e\left(\frac{1}{2}\right)$ and $16 - \log_e\left(\frac{3}{2}\right)$. But this is **not** yet the answer to the question! A good tip, particularly for long questions, is to always look back when you think you have finished, because it is easy to forget what the question is actually asking. We have already done the hard part, so it would be a shame to lose marks for not doing the last few steps. We actually want to know how many seconds Corey spends underwater, not when he hits and/or leaves the water.

This means two things - the first is that **both** the t-values we found above are valid; that is, we do not need to discard any of the values, it just means that Corey goes underwater but then back up again before the end of the ride. The second is that the answer to this question must be the difference of these two t-values.

Now why is this case? Well, we have two values above, and one must be greater than the other. Since it is at these two points in time that $h = 0$, and since the ride begins above ground level (as can be seen from the graph), the time Corey spends underwater must be the larger of the two values (i.e. when he comes back out) minus the smaller of the two values (i.e. when he plunged into the body of water). So the time he spends underwater is indeed the difference of the two intercepts (which can be seen from the graph).

The larger of the two values is $16 - \log_e\left(\dfrac{1}{2}\right)$, so we can say that Corey will therefore be underwater for a total of $16 - \log_e\left(\dfrac{1}{2}\right) - \left[16 - \log_e\left(\dfrac{3}{2}\right)\right]$ seconds.

Simplifying this value gives: (use appropriate logarithm laws here... or a CAS)

$$\begin{aligned} 16 - \log_e\left(\frac{1}{2}\right) - \left[16 - \log_e\left(\frac{3}{2}\right)\right] &= 16 - \log_e\left(\frac{1}{2}\right) - 16 + \log_e\left(\frac{3}{2}\right) \\ &= \log_e\left(\frac{3}{2}\right) - \log_e\left(\frac{1}{2}\right) \\ &= \log_e\left(\frac{3}{2} \times 2\right) \\ &= \log_e(3) \end{aligned}$$

So, once simplified, we see that Corey was underwater for a total of $\log_e(3)$ seconds. This answer must be given as an exact value - it may be tempting, since the rest of the question had so far wanted decimals, to give this answer as a decimal too, particularly after you had to find the t-intercepts as decimals in the previous question. And if is a 3 mark question, you **must** show the relevant working! It is perfectly fine to use the CAS to confirm your answer or to guide you towards it, but simply stating the equation and writing down the intercepts, without attempting to solve it by hand or to at least do some steps by hand, will not get you the marks.

Hence the lesson here is to go slowly, make sure you understand what the question is asking, and always give both adequate working and exact answers, unless explicitly told otherwise!

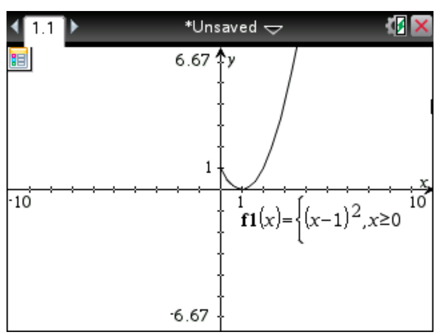
FUNCTIONS AND GRAPHS
TECH-ACTIVE TEST 2
DETAILED SOLUTIONS

SECTION 1 - Multiple-Choice Questions

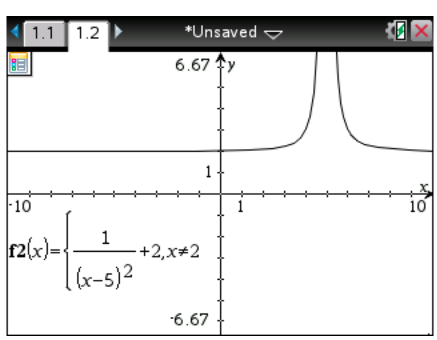
Question 1 (D)

The question "Which of the following functions would have a defined inverse function?" is really just a fancy way of saying "which of the following functions are one-to-one?" The reason is, for an equation to not only be a function itself, but to also have a defined inverse, it must be one-to-one. This is a fact that should be absolutely ingrained into your minds. So we just need to look at each of the graphs and equations, and determine whether they are one-to-one or not.

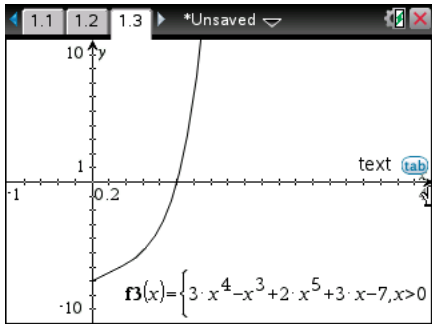
So first, let's consider the graph of $f(x) = (x-1)^2$. This is a parabola, shifted one unit to the right, meaning that the turning point will be at $x = 1$. But if we look at the domain, it is $[0, \infty)$, which means that the graph starts one unit to the left of the turning point, and includes everything to the right of it as well. This clearly will be many-to-one, and you should certainly sketch these graphs on the CAS to quickly visualise them, over the correct domain. You can graph things for a certain domain by pressing the ' | ' button (which means "given") at the end of the equation, and then type the domain (in the form $x \geq 0$) afterwards - e.g. type out $f1(x) = (x-1)^2 | x \geq 0$ in a graph screen.



Now, the graph of $g(x) = \dfrac{1}{(x-5)^2} + 2$ is a truncus, and the domain is $\mathbb{R} \setminus \{5\}$, which is the implied domain of this graph anyway, meaning that there are no further restrictions on the graph. And you should know the shape of a truncus and that it is not one-to-one unless there are further restrictions, meaning that the inverse will certainly not be a function. If you are in doubt though, sketching this graph on the CAS can certainly prove that it is not one of the answers:

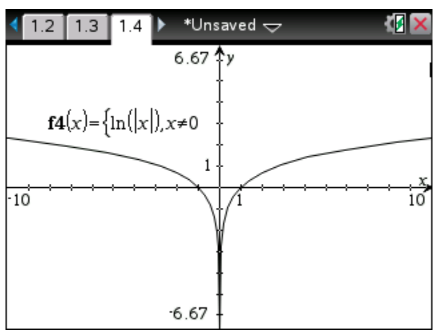


The graph of $h(x) = 3x^4 - x^3 + 2x^5 + 3x - 7$ is a quintic (surprise! it was rearranged with x^4 as the first term to throw off non-careful readers). Sketching just the graph of this quintic would make it appear that it is **not** one-to-one, but the restriction on the domain actually does make it a one-to-one function, since the turning points are to the left of $x = 0$. So if we sketch this function on the CAS for the domain of $\mathbb{R}^+$, which you should type as "$|x > 0$ ", you can see that for this domain, the function is one-to-one, meaning that an inverse would be defined.

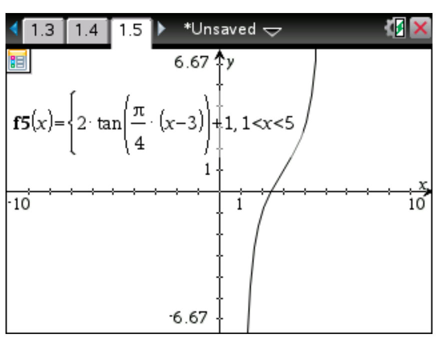


Hence $h(x)$ is one of the functions in our answer.

Now, considering $y = j(x)$: the graph of $y = \log_e(x)$ is a one-to-one function indeed, but the graph of $y = \log_e(|x|)$ is actually not, because this new function, as we can see from sketching it on the CAS, is defined for negative values of x. That is, it looks like the graph of $y = \log_e(x)$, but with the graph "copied" across on the other side of the y-axis. And, as such, it is not one-to-one, because for every value of y, there are now two x-values (one positive and one negative).



The last graph is of $k : (1, 5) \to \mathbb{R}$, $k(x) = 2\tan\left(\frac{\pi}{4}(x - 3)\right) + 1$. This is again a case where the function itself is not normally one-to-one, but the domain on the side changes everything. Again, sketching this one the CAS **for the appropriate domain** will highlight this point.



And even without sketching, since the period of a tangent function is given by $\frac{\pi}{n}$, which here equals $\frac{\pi}{\left(\frac{\pi}{4}\right)} = 4$, and since the domain is 4 units "long", it is clear that we are only going to be sketching a single period of a tangent function, meaning it will be one-to-one. Hence, the one-to-one functions - which are the ones with defined inverses here - must be $h(x)$ and $j(x)$. Thus, the answer is **D**.

Question 2 (C)

There are two ways of doing this question - one is to algebraically deduce what the identity means, and the other is to simply substitute the given functions into the formula to see if the identity holds true. Don't underestimate trial and error as a method for doing multiple-choice questions!

In any case, let's have a closer look at the identity itself: $[g(x)]^2 = g(f(x)) + 2$. Consider first the left-hand side:

$$\begin{aligned}
\text{LHS} &= [g(x)]^2 \\
&= \left(\frac{1}{\sqrt{1+x^2}} + \sqrt{1+x^2}\right)^2 \\
&= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 + 2 \times \frac{1}{\sqrt{1+x^2}} \times \sqrt{1+x^2} + \left(\sqrt{1+x^2}\right)^2 \\
&= \frac{1}{1+x^2} + 2\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} + 1 + x^2 \\
&= \frac{1}{1+x^2} + \left(1+x^2\right) + 2 \times 1 \\
&= \frac{1}{1+x^2} + \left(1+x^2\right) + 2
\end{aligned}$$

And now looking at right-hand side:

$$\begin{aligned}
\text{RHS} &= g(f(x)) + 2 \\
&= \frac{1}{\sqrt{1+[f(x)]^2}} + \sqrt{1+[f(x)]^2} + 2
\end{aligned}$$

So letting LHS = RHS gives

$$\frac{1}{1+x^2} + \left(1+x^2\right) + 2 = \frac{1}{\sqrt{1+[f(x)]^2}} + \sqrt{1+[f(x)]^2} + 2$$

It can thus be seen that

$$\begin{aligned}
1+x^2 &= \sqrt{1+[f(x)]^2} \\
\therefore 1+[f(x)]^2 &= \left(1+x^2\right)^2 \\
&= 1+2x^2+x^4 \\
\implies [f(x)]^2 &= x^4+2x^2 \\
\therefore f(x) &= \sqrt{x^4+2x^2} \\
&= \sqrt{x^2\left(x^2+2\right)} \\
&= x\sqrt{x^2+2}
\end{aligned}$$

Thus we can see that **C** is the answer. This is rather lengthy though, and working out the expanded value of both the LHS and the RHS, and then using trial and error for the right-hand side with the given functions may be much quicker. For instance, if we take option **A** and substitute it into the RHS above:

$$\begin{aligned}
\text{RHS} &= \frac{1}{\sqrt{1+\left[\sqrt{x}\right]^2}} + \sqrt{1+\left[\sqrt{x}\right]^2} + 2 \\
&= \frac{1}{\sqrt{1+x}} + \sqrt{1+x} + 2
\end{aligned}$$

This is clearly not equal to the LHS, and so we can quickly eliminate **A** and so on for the other incorrect candidates. And when the correct option is reached, you can quickly see that it works:

$$\begin{aligned}
\text{RHS} &= \frac{1}{\sqrt{1+\left[x\sqrt{x^2+2}\right]^2}} + \sqrt{1+\left[x\sqrt{x^2+2}\right]^2}+2 \\
&= \frac{1}{\sqrt{1+x^2\left(x^2+2\right)}} + \sqrt{1+x^2\left(x^2+2\right)}+2 \\
&= \frac{1}{\sqrt{x^4+2x^2+1}} + \sqrt{x^4+2x^2+1}+2 \\
&= \frac{1}{\sqrt{\left(x^2+1\right)^2}} + \sqrt{\left(x^2+1\right)^2}+2 \\
&= \frac{1}{x^2+1} + \left(x^2+1\right)+2 \\
&= \text{LHS}
\end{aligned}$$

Question 3 (B)

We will be looking at this question from the x' and y' method of doing transformations - as we saw in the first technology-active test, this method is ultimately much safer and also more accurate and mathematical, than simply looking at the equation to determine transformations. So what we need to do here is first express the two equations in a way that all transformations affecting the y-coordinate are on the left, and then change the equation of the **image** to include x' and y', because x' and y', by definition, are the coordinates of the image.

The original equation, $y=\sqrt{x}$, does not need to be rearranged as it is the "basic" square root graph.

The graph of the image is $y=4\sqrt{3(2-x)}+5$. Here we do need to rearrange it, as follows:

$$\begin{aligned}
y &= 4\sqrt{3(2-x)}+5 \\
\therefore y-5 &= 4\sqrt{3(2-x)} \\
\therefore \frac{y-5}{4} &= \sqrt{3\left(2-x\right)}
\end{aligned}$$

So we can say that the equation of the image, after adding the dashes, must be $\dfrac{y'-5}{4}=\sqrt{3\left(2-x'\right)}$.

In essence, we are trying to go from $y=\sqrt{x}$ to $\dfrac{y'-5}{4}=\sqrt{3\left(2-x'\right)}$. We can therefore create two equations from this (and rearrange them, as you can see below), which allow us to obtain the transformations. These are:

$$\begin{aligned}
y = \frac{y'-5}{4} &\quad \text{and} \quad x=3\left(2-x'\right) \\
\therefore y'-5=4y &\quad \text{and} \quad 2-x'=\frac{x}{3} \\
y'=4y+5 &\quad \text{and} \quad x'=-\frac{x}{3}+2
\end{aligned}$$

The reason for doing this is that all transformations can be expressed in the form $(x,y)\to(x',y')$, and this is the easiest form to deduce transformations.

So we can now say that the transformation we have here is:

$$(x,y)\to\left(-\frac{x}{3}+2, 4y+5\right)$$

Interpreting this transformation, as it stands here, would give us:

- A dilation of factor 4 from the x-axis (**remember** that from the x-axis affects the y-coordinate, and vice-versa!), a dilation of factor $\frac{1}{3}$ from the y-axis, and a reflection in the y-axis (as this results in a multiplication of x by -1). These three could be arranged in any order, as one does not affect the other; followed by:
- A translation of 5 units in the positive direction of the y-axis and a translation of 2 units in the positive direction of the x-axis. These could be in any order, as long as a translation in a specific axis does not precede the dilation on that axis

However, none of the combinations we are provided with in the options matches this perfectly. For instance, in option **A**, the dilation by factor of $\frac{1}{3}$ is applied after the translation. In **E**, the reflection is in the wrong axis. In **D**, the dilation by factor of 4 is applied after the translation by 5 units, which would result in a y-coordinate of $4(y+5)$. And in **C**, the dilation from the y-axis is 3, and not $\frac{1}{3}$.

But why is **B** the answer? Well, we can rearrange the transformations, as follows:

$$
\begin{aligned}
(x, y) &\rightarrow \left(-\frac{x}{3}+2, 4y+5\right) \\
(x, y) &\rightarrow \left(\frac{1}{3}(-x+6), 4y+5\right)
\end{aligned}
$$

This rearrangement shows that dilating by a factor of $\frac{1}{3}$ and then translating 2 units right is exactly the same as translating 6 units right, and then dilating by factor of $\frac{1}{3}$. This is why these questions often have multiple possibilities, and you must know this to get this multiple-choice question right!

Hence, a possible order is dilation by factor of 4 from the x-axis, followed by a reflection in the y-axis, followed by a translation of 5 units in the positive direction of the y-axis, followed by a translation of 6 units in the positive direction of the x-axis, and lastly a dilation by factor of $\frac{1}{3}$ from the y-axis. So by rearranging the transformation, we can see that the sequence of transformations outlined in **B** is indeed a possibility, and consequently this is the answer.

Question 4 (C)

The general equation of the graph of a circle (which is a relation and **not** a function) is $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the centre of the circle, and r is the radius of the circle. Consequently, since we are told that the radius of the circle is 5, we know that the equation of this circle will be of the form $(x-h)^2+(y-k)^2=5^2$, or

$$(x-h)^2+(y-k)^2=25$$

From this, it is already evident that the answer to this question must be either **C** or **E**. But how do we decide which one?

There are a few possible ways of doing it, some being quicker than others. Like I stated in the previous test, though, you should not underestimate the power of trial and error in these situations, particularly when time is a factor (as it often is in Methods 3&4 end-of-year exams). As such, a very simple way of getting to the answer here would be to substitute the $(5,7)$ and $(6,10)$ into the equations given in **C** and **E**, and seeing which one actually goes through these points.

For instance, if we substitute $(5,7)$ into the equation given in **C**: ($(x-10)^2+(y-7)^2=25$)

$$\begin{aligned}\text{LHS} &= (5-10)^2+(7-7)^2\\ &= (-5)^2+(0)^2\\ &= 25\\ &= \text{RHS}\end{aligned}$$

So the point $(5,7)$ certainly lies in this circle. But if we were to substitute $(5,7)$ into the equation given in **E**, which is $(x-1)^2+(y-7)^2=25$, we would get:

$$\begin{aligned}\text{LHS} &= (5-1)^2+(7-7)^2\\ &= (4)^2+(0)^2\\ &= 16\\ &\neq \text{RHS}\end{aligned}$$

So this is clearly false, meaning that the point $(5,7)$ does **not** lie in this circle. This is already enough evidence to show that the answer can only be **C**.

Of course, although this might be the quickest, trial and error of possible multiple choice answers is not the only method available here. We could also try substituting the two points into the general equation $(x-h)^2+(y-k)^2=25$, and using simultaneous equations to work out possible values (as there will be multiple answers) of h and k:

$$\begin{aligned}(5,7) &\implies (5-h)^2+(7-k)^2=25 \quad (1)\\ (6,10) &\implies (6-h)^2+(10-k)^2=25 \quad (2)\end{aligned}$$

Solving this set of equations by hand, though, would not be fun... (and, in fact, would take a VERY long time). Instead, we can use the CAS to solve this set of simultaneous equations, which gives us $h=1$ and $k=10$, or $h=10$ and $k=7$. Consequently, we can see that there are only two possible equations that give a circle with radius 5 and going through the given points. These equations are $(x-1)^2+(y-10)^2=25$, and $(x-10)^2+(y-7)^2=25$. Again, the only equation that matches one of these two is the one given in **C**, and hence this is indeed the answer.

Question 5 (E)

To do this question, we first need to think about what the equation $f(x)+a=0$ actually means. In a sense, $y=f(x)+a$ is a translation by a units upward of the graph of $y=f(x)$, which is the graph we are given. So the answer to this question will be all the values of a that, when applied as a translation in the direction of the y-axis (direction could be negative if a is negative), the graph will still cross the x-axis three times. After all, $f(x)+a=0$ is essentially solving for the x-intercepts of $y=f(x)+a$. Thinking about it like this, if we were to move the graph up, say, 5 units, the minimum turning point would now be above the x-axis. Consequently, there would only be a single x-intercept.

So by this logic, the maximum we could move the graph **upwards** would be 4.16 units. If we moved it up **exactly** 4.16 units though, the minimum turning point would be on the x-axis, and we would have two solutions (try visualising the graph moving up and down). So the maximum value of a is 4.16, but it cannot equal 4.16: hence $a<4.16$.

But how much can we move the graph downwards? Well, the same principles apply: moving the graph down 2.37 units (i.e. $a=-2.37$) would result in the turning point touching the x-axis, and thus two solutions. But if we move it down less than 2.37 units, we will have three intercepts, meaning that $a>-2.37$.

So we know that $a < 4.16$, but also that $a > -2.37$. Hence the answer will be $-2.37 < a < 4.16$, and therefore **E**. Note that some of the other options here are designed to trip up students who used the x values of turning points to determine a, which is a dilation in y, and therefore the y-coordinates have to be used. And the coordinates of the intercepts and the point of inflection given are both irrelevant, as the transformation here is only a translation in y.

Another way of going about this question is to write $f(x) + a = 0$ as $f(x) = -a$. The difference is subtle, but it may make it easier to think about it. Essentially, if we are solving $f(x) = -a$, then we must have $-a$ as a value in between -4.16 and 2.37, because for all these values of y (**not** including -4.16 and 2.37) there are 3 corresponding x-values. so we can see that

$$-4.16 < -a < 2.37$$

Consequently, multiplying all sides by -1 (remember - for inequalities, if you multiply or divide by a negative, you must swap the direction of the greater/less than signs. Thus:

$$4.16 > a > -2.37$$

Meaning that

$$-2.37 < a < 4.16$$

Again, we see that the answer is **E**.

Question 6 (A)

Several methods exist for answering this question, and you should be familiar with some in case you get a similar style of question. I will combine some of these methods into my explanation here, but rest assured that there are plenty of other ways of going about it, and if your method gave you the correct answer and within a reasonable amount of time, it is most likely to be just as valid (well, unless you guessed the answer right...). And elimination is always the way to go in questions where there are multiple possible solutions - because it is much easier to prove that a transformation is wrong, then to prove that it is definitely right!

Let's first compare the two graphs: the original is increasing as x increases (i.e. points upwards), whereas the image is decreasing as x increases. To obtain this shape, we would have to reflect the graph in the x-axis, meaning that - transformation-wise - we need to multiply y by a negative value. So considering the transformations we have, we can already eliminate **B** as this contains a reflection in the y-axis only (as the -3 will multiply x).

Once we have applied this reflection, the next thing to notice is that the graph is moved down and left. In option **D** however, the reflection is applied first, and then followed by a translation to the left (which is actually an incorrect translation, but that's not the most obvious problem) and a translation of 1 unit **up**. This is evidently wrong, as $f(x)$ has to be moved down to obtain $g(x)$, meaning we can eliminate **D**.

Now, we could proceed with eliminating the other options by looking at the graphs to infer the transformations, but that might no longer be the safest and easiest way. Instead, substituting points in is also a good idea, as it can quickly eliminate options. For instance, let us try substituting $(0, 0)$ into the transformation described in **C**:

$$\begin{aligned} T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} 3 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -6 \\ -\frac{1}{5} \end{bmatrix}\right) \\ &= \begin{bmatrix} 3 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}\left(\begin{bmatrix} -6 \\ -\frac{1}{5} \end{bmatrix}\right) \\ &= \begin{bmatrix} -18 \\ \frac{1}{25} \end{bmatrix} \end{aligned}$$

But the new endpoint (which is must be the image of $(0,0)$, as that was the endpoint in the original graph) is $\left(-6, -\frac{1}{5}\right)$, and not $\left(-18, \frac{1}{25}\right)$. Evidently, **C** could not be the answer. We can see that if we have a translation given by $\begin{bmatrix} -6 \\ -\frac{1}{5} \end{bmatrix}$, it has to be applied **after** the dilations, as otherwise you end up with a different endpoint; but it is significantly easier to just sub in the endpoint and let everything work itself out instead! So the message is that there are lots of ways of doing this, and some options are easier to rule out with one, and other options are easy to rule out with another - hence I'm including several ways to think about this question!

Now, how do we go about the two remaining options? Substituting $(0,0)$ will give us the correct endpoint in both cases, so unfortunately this will not help to make the distinction. So what to do instead? There are two ways I would suggest, and we shall look at both. The first is trying to use another point that is not $(0,0)$ - that is because $(0,0)$ is not affected by dilations, meaning that other points might be better.

So looking at the graph of $f(x)$, it appears that $(1,1)$ is a point. We cannot be sure about this, but even if it is not a point, there will be a point awfully close, meaning that is is still useful to give a rough idea of the image. Substituting $(1,1)$ into the transformations outlined in **A**:

$$\begin{aligned}
T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} 3 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 3 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) \\
&= \begin{bmatrix} -3 \\ -\frac{2}{5} \end{bmatrix}
\end{aligned}$$

If we look at $x = -3$ in the graph of $g(x)$, it does actually look like $g(x)$ is very close to -0.4, which is $-\frac{2}{5}$. Hence, we can still not rule out this transformation.

Now, let's substitute $(1,1)$ into the transformations outlined in **E**:

$$\begin{aligned}
T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -6 \\ -\frac{1}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3} \\ -5 \end{bmatrix} + \begin{bmatrix} -6 \\ -\frac{1}{5} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{17}{3} \\ -\frac{26}{5} \end{bmatrix}
\end{aligned}$$

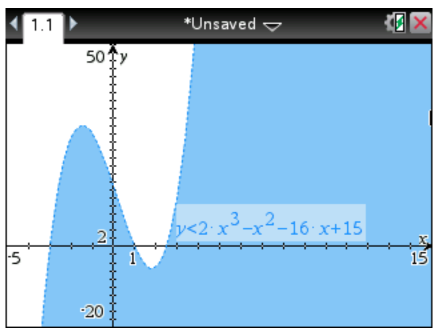
Now, $-\frac{17}{3} \approx -5.67$, and $-\frac{26}{5} = -5.2$. But if we look at the graph of $g(x)$, we notice that around the -5.67 mark, the value of y would be around -0.3. Even though we cannot be sure $(1,1)$ was in $f(x)$ to start with, it was still close enough that the image of it would be, at the very least, near the graph of $g(x)$; but its image is very far away! You can see from the graph of $g(x)$ that it is very far away from reaching y-values around -5, and so we can say with this that the transformation given in **E** is very unlikely to be correct, meaning that our answer must be **A**.

The other way of going about choosing between **A** and **E** is to think about the transformations and see if they match the graph. If we compare $g(x)$ to $f(x)$, the graph appears to be compressed (i.e. "squished up") in a vertical direction - that is, compressed parallel to the y-axis. This means that, if we had a point (x, y), the y-value is likely to be decreased with the mapping - meaning that we probably need a dilation by a factor less than one parallel to the y-axis. And in **A**, we will be multiplying the y-coordinate by $\frac{1}{5}$, which certainly does the job of compressing the graph. However, in **E**, we multiply y by 5, which should do the opposite - the image would be a stretched out graph, and not what we have been given. This is actually the exact same reasoning we used when plugging in $(1, 1)$: the image given by **E** would suggest a stretched out graph that would hit -5 much earlier on.

As such, we can see that the transformations outlined in **E**, through either of these methods, is not suitable. The one outlined in **A** is correct - the translations are applied first, but once the dilations are applied too, you end up with the correct endpoint, even though the numbers 6 and $-\frac{1}{5}$ do not appear in the translation matrix.

Question 7 (A)

This question requires you to sketch an inequality - this is not particularly common in methods exams, but is certainly in the study design, so it is good to be prepared for it just in case. Anyhow, the answer can actually be obtained straight away through the use of the CAS - you **can** sketch inequalities in the CAS! Just replace the $=$ sign with an inequality - in this case, $<$ - and off you go:



However, if you do not use the CAS for this question, there are other simple ways of going about this question too. For starters, the cubic $y = -x^2 + 2x^3 - 16x + 15$ is a positive cubic (notice that the equation is not in order; the term with the highest power is the second one), meaning that it must ultimately travel from negative to positive. With this in mind, we can straight away eliminate options **D** and **E**, as the cubic is travelling the wrong way here. Remember that to sketch an inequality, you first begin by changing the inequality to an equals sign, and sketching the same line/curve as you would.

But when you are sketching inequalities, you must also know that you need to use dotted lines if you have $<$ or $>$, and a solid line when you have $\leq$ and $\geq$. Consequently, we can eliminate option **C** as well, because we need a dotted line for the graph we are sketching.

So now all that remains is choosing between **A** and **B**; this is equivalent to deciding whether the required region (which is the shaded region) of the graph (remember, inequalities are an infinite area bounded by the graph) will be the one above or below the cubic function. Since the region we want is where y is "less than" the cubic, we can infer that the shaded region needs to lie underneath the cubic - and hence the answer is **A**.

Another way of choosing between **A** and **B**, though, is to use a 'test' point, which is sometimes may be the better method (particularly when the LHS is not simply y, but rather an expression). So let's pick the point $(0, 0)$ as it is easy to use, and clearly underneath the graph. Note that any point that does not actually lie on the line (dotted or solid) may be used.

So we now substitute $(0,0)$ into $y < -x^2 + 2x^3 - 16x + 15$:

$$\begin{aligned} 0 &< -0^2 + 2 \times 0^3 - 16 \times 0 + 15 \\ 0 &< 15 \end{aligned}$$

This is clearly a true statement, as 0 is less than 15. As a result, we now know that the point $(0,0)$ lies in the required region, meaning that, indeed, the region below the graph is the required region (which we must shade), and that **A** is therefore our answer.

Question 8 (E)

The fact that the question states 'Which of the following could be its equation' is a hint that elimination is the way to go here - as in, there is not a single equation that is the answer to this question, but instead there are numerous possibilities, meaning that we need to find one possibility amongst the set of options given. Hence, instead of trying to work out an answer solely from the information given, it is best to look at the options themselves and then, working backwards, see if they fit the given information or not. That is, go through every option and ask yourself - is the amplitude 3, the period $\frac{\pi}{5}$, and is there an x-intercept at $x = -\frac{\pi}{6}$?

Considering option **A**, we notice straight away that this is a tangent function - so while the other two 'bits' of information may fit, you have to remember that tan functions do **not** have an amplitude. That is because tangent functions have a range of $\mathbb{R}$, meaning that they are not 'wave-like' as the other two circular functions are. Since the amplitude really is the definition of the size of the 'wave' (it is half the value of the difference between the highest and lowest y-values), we cannot possibly find an amplitude for a tangent function. Instead, the 3 here is simply a dilation factor, and this equation therefore does not fit all the data provided; hence **A** is not the answer.

Looking at **B**, the only problem we encounter is with the period - the other two pieces of information (amplitude and x-intercept) do fit the equation. Remember, that for the general equation

$$y = A \sin(nx - h) + k$$

the amplitude is given by A, which is the value 'just outside' the sin, the period is denoted by $\frac{2\pi}{n}$, and the range will be given by $[k - A, k + A]$. Note also that all of this is exactly the same for $y = A\cos(nx - h)$. Anyhow, for the equation $y = 3\sin\left(\frac{\pi}{5}\left(x + \frac{\pi}{6}\right)\right)$, the value of the period will therefore be $\frac{2\pi}{\left(\frac{\pi}{5}\right)} = 10$, because $n = \frac{\pi}{5}$, as this is the coefficient of x [NB: the brackets make no difference; everything directly multiplying x will make up n]. So the period for this function will be 10, and not $\frac{\pi}{5}$ as we require. So this is not the answer!

Moving on to **C**, you should notice straight away that it fails the amplitude requirement. The number 'just outside' the cos (i.e. A in $y = A\cos(nx - h)$) is 1, not 3; meaning that the amplitude is 1 and not 3. So this is also not the answer you should write down if you want to get a mark! The $+3$ here is an upward translation by 3 units, it is not the amplitude, as it does not affect the size of the waves.

If we now look at **D**: the equation is $y = 3\sin\left(10x + \frac{5\pi}{3}\right) + 1$. The amplitude is clearly correct, so let's now find the period. The coefficient of x is 10 (i.e. $n = 10$), meaning that the period will be $\frac{2\pi}{10} = \frac{\pi}{5}$. So far, this does look very good. But there is one criteria left - is there an x-intercept at $x = -\frac{\pi}{6}$? We can find this by substituting $x = -\frac{\pi}{6}$ back into the equation to see if $y = 0$ (this is so much quicker than trying to find the x-intercepts for this graph).

$$
\begin{aligned}
x = -\frac{\pi}{6} \implies y &= 3\sin\left(10 \times -\frac{\pi}{6} + \frac{5\pi}{3}\right) + 1 \\
&= 3\sin\left(-\frac{5\pi}{3} + \frac{5\pi}{3}\right) + 1 \\
&= 3\sin(0) + 1 \\
&= 1
\end{aligned}
$$

Since $y = 1$ when $x = -\dfrac{\pi}{6}$, there is no x-intercept at this value of x. So it was close, but **D** is not the answer yet again.

So let's now look at **E**: $y = -3\cos\left(10\left(x + \frac{\pi}{5}\right)\right) + \frac{3}{2}$. Is the amplitude correct? Yes, as the number just in front of the cos is 3 (the negative sign does **not** affect amplitude - the amplitude really is $|A|$ since the negative sign just reflects the graph in the x-axis, and does not actually change the size of the waves in any way. So the amplitude equals 3.

Next, what is the value of n? Remember, this is the coefficient of x, meaning that anything multiplying x needs to be considered. So here, $n = 10$. You should note that $\cos\left(10\left(x + \frac{\pi}{5}\right)\right) = \cos\left(10x + 2\,\pi\right)$. So this means that the 10 is very much a coefficient of x, and thus $n = 10$ indeed. So the period will equal $\dfrac{2\pi}{10} = \dfrac{\pi}{5}$, meaning that the period is also correct. So now all that remains is the x-intercept.

Substituting $x = -\dfrac{\pi}{6}$ back into the equation, we get:

$$
\begin{aligned}
x = -\frac{\pi}{6} \implies y &= -3\cos\left(10\left(-\frac{\pi}{6} + \frac{\pi}{5}\right)\right) + \frac{3}{2} \\
&= -3\cos\left(10\left(\frac{\pi}{30}\right)\right) + \frac{3}{2} \\
&= -3\cos\left(\frac{\pi}{3}\right) + \frac{3}{2} \\
&= -3 \times \frac{1}{2} + \frac{3}{2} \\
&= -\frac{3}{2} + \frac{3}{2} \\
&= 0
\end{aligned}
$$

So when $x = -\dfrac{\pi}{6}$, we have $y = 0$. Therefore, there is an x-intercept at $x = -\dfrac{\pi}{6}$, meaning that the equation given in **E** fits all the given data, and is therefore the correct answer!

Question 9 (B)

This is a question where taking a step back to understand what is being asked is a good idea. We are being asked to obtain the graph of $y = 3f\left(\dfrac{x-1}{5}\right) + 2$ from the graph of $y = f(x)$. This means that it is (yet another) transformation question. Are you getting the hint? Transformations are a big deal!!

In any case, the best way to go about it is to rearrange the image so that the transformations are more obvious, as I have suggested in previous questions, and then adding dashes to the equation of the image to get it as x' and y'. So let's rearrange:

$$\begin{aligned} y &= 3f\left(\frac{x-1}{5}\right)+2 \\ \therefore\ y-2 &= 3f\left(\frac{x-1}{5}\right) \\ \frac{y-2}{3} &= f\left(\frac{x-1}{5}\right) \end{aligned}$$

This way, we can clearly see which transformations affect the x-coordinate, and which ones affect y. Now add in the dashes:

$$\frac{y'-2}{3} = f\left(\frac{x'-1}{5}\right)$$

The next step is to equate $\frac{y'-2}{3} = f\left(\frac{x'-1}{5}\right)$ with $y = f(x)$. This will allow us to obtain x' and y' in terms of x and y, which is ultimately what we want as this will make the transformations much more obvious. Hence:

$$\begin{aligned} \frac{y'-2}{3} = y \quad &\text{and} \quad \frac{x'-1}{5} = x \\ \therefore y'-2 = 3y \quad &\text{and} \quad x'-1 = 5x \\ \therefore y' = 3y+2 \quad &\text{and} \quad x' = 5x+1 \end{aligned}$$

We can thus express these transformations as $(x,y) \to (5x+1, 3y+2)$. That is, the graph is dilated by a factor of 5 parallel to the x-axis (or from the y-axis), and by a factor of 3 parallel to the x-axis, and translated 1 unit right and 2 units up. Obviously, there are other ways of describing this same transformation, but you might as well use the simplest one to do this question.

We should now look at each of the graphs in the options, and - by using either specific points, or by thinking about the translations described just above - try to eliminate the options that do not fit these transformations.

Now, notice that our transformation includes moving the graph to the right (adding 1 to the x-coordinate). Since the graph of $f(x)$ was 'centred' in the sense that it was an odd function (and thus the graph goes the same length to either side of the y-axis; i.e. the limits appear to be $\frac{\pi}{2}$ and $-\frac{\pi}{2}$), we can therefore expect the graph of the image to be more to the right: that is, it should look like the 'middle' (which is an inflection, but not a horizontal inflection) is for a value of x greater than zero. And indeed, with our current mapping, the point $(0,0)$ is transformed to $(1,2)$. But now look at option **A**: the graph extends further to the left ($-\frac{5\pi}{2}$) than it does to the right (2π), which is the opposite of what we need. Consequently, we can eliminate option **A**.

Next, notice from our transformations that both dilation factors are greater than 1 - meaning that the graph must be 'stretched out' in both directions, as all x-coordinates are multiplied by 5, and all y-coordinates are to be multiplied by 3. But **D** provides a graph that has been clearly compressed in the direction of the x-axis. This is evident by the scale marking $\frac{\pi}{8}$, which shows that the graph extends from 0 to $\frac{\pi}{4}$, which is a much smaller distance in the x-axis than that 'covered' by the original graph. Again, we are looking for a stretch and not a compression, so **D** is not the answer.

What is most evident from option **C** is how far to the right the graph is - it does not even appear to cross the y-axis. And although we did shift the graph 1 unit right to obtain the image, this shift here evidently too far: for instance, the original graph extend until $-\frac{\pi}{2}$ on the left, which is approximately -1.57. This shows that even after shifting the graph 1 unit right, it would still cross the x-axis, and it would certainly not be as far to the right as it is in **C**, meaning that we can eliminate this as an option. (In fact, since we multiply x by 5 first, and then apply the translation, the graph of the image will go substantially further to the left than approximately -0.57).

There are a few problems with the graph presented in **E**, but the most easily discernible is the fact that this graph is also not very stretched along the x-axis. As in, we require a dilation by factor of 5, which should mean a considerable stretch in x-values, but this graph appears to have been shifted to the right, and then only stretched slightly, as it just crosses the y-axis, and does not extend much further that π. Instead, if the original graph extended from $-\frac{\pi}{2}$ until $\frac{\pi}{2}$, after applying the dilation by factor 5 and the translation one unit right, the image should extend from $1-\frac{5\pi}{2}$ to $1+\frac{5\pi}{2}$. So **E** is not the answer either.

However, **B** does the job perfectly. It reaches slightly more to the right than to the left, and the inflection in the centre also appears to have been moved upwards. Moreover, the x-values are stretched appropriately (indeed, it appears to extend from $1-\frac{5\pi}{2}$ to $1+\frac{5\pi}{2}$), and the y-values are also larger in comparison with the original graph, as a result of the dilation. Hence, we can say that the answer must be **B**.

Another way of doing this question is by realising that $y = f(x)$ is really the graph of $y = \tan(x)$, and then applying the appropriate transformations to the equation, and sketching it on the CAS. A word of caution though - while this may work very well for this question, you cannot assume the equation of a graph when you have not been given the equation; you should only use a method like this as a guide.

Question 10 (C)

This is a question dealing with inverse functions and their definitions. Let's examine each of the properties individually to try and figure out which one is **not** true.

Option **A** states that 'They are reflections of each other along the line $y = x$'. That is indeed a critical definition to know about inverse functions - that is, $y = f^{-1}(x)$ is the mirror image of the function $y = f(x)$ over the line $y = x$. This is because the function $f^{-1}(x)$ is obtained by swapping the x- and y-coordinates of each point around, which is actually equivalent to a reflection along the line $y = x$.

B states that 'The solutions to $f(x) = x$ will be points of intersection between the graphs of $y = f(x)$ and $y = f^{-1}(x)$'. Why would this be the case? As we stated previously, to get the inverse of a function, you must swap x with y for each point in the original function. Consequently, if we have a point (x, y) in $y = f(x)$, then the point (y, x) will be on $y = f^{-1}(x)$. But at an intersection, the points in both functions must equal each other - that is, $(x, y) = (y, x)$, and we can therefore see that $y = x$, and since $y = f(x)$, we can see that any intersections between the functions $f(x)$ and $f^{-1}(x)$ will occur when $f(x) = x$ (there can be intersections at other points, but this option does not say that the solutions to $f(x) = x$ are the **only** points of intersection, so it doesn't matter).

If this seemed a bit confusing, try thinking about it graphically: $f(x)$ and $f^{-1}(x)$ are mirror images along the line $y = x$. This means that, when $f(x)$ crosses the line $y = x$, then $f^{-1}(x)$ will also cross the line $y = x$, and vice versa. This means that any intersections between $f(x)$ and $y = x$ will also be intersections between $y = f(x)$ and $y = f^{-1}(x)$. Consequently, solving $f(x) = x$ (which is, essentially, the same as equating $y = f(x)$ and $y = x$) will give points that are **also** intersections between $f(x)$ and $f^{-1}(x)$. So **B** is also not the answer.

Now, since we said that, if we have a point (x, y) on $y = f(x)$, the point (y, x) must lie in $y = f^{-1}(x)$, you can also see that the domain of $f(x)$ will equal the range of $f^{-1}(x)$, and also that the range of $f(x)$ will equal the domain of $f^{-1}(x)$. This is in fact a crucial property to know about inverse functions - to get domain and range of the inverse, just swap around the domain and range of the original function! This is logical, since we are swapping x and y around for every point in the graph of $f(x)$ to obtain $f^{-1}(x)$. Consequently, the statement 'the range of $y = f(x)$ is equal to the domain of $y = f^{-1}(x)$' is actually spot-on correct, and thus **D** is not the answer.

Now, regarding **E**: '$y = f^{-1}(x)$ will be defined if and only if $y = f(x)$ is a one-to-one function'. This is another key piece of knowledge regarding inverse functions - the inverse of a function will only be itself a function if the original function was one-to-one. Why? Well, we are swapping x and y around, meaning that if $f(x)$ is a many-to-one function, the inverse will be one-to-many, so it is a relation but **not** a function. And if the original graph was one-to-many, it would not be a function itself, meaning that the original would not be a function, and thus we could not label it '$y = f(x)$'. So it is indeed true that for $f^{-1}(x)$ to be defined, $f(x)$ must be one-to-one, and therefore this option is not the answer either.

So now let's look at **C**, which I have left as the last option here purely by coincidence... Anyway, we have seen that to obtain $f^{-1}(x)$ from $f(x)$, we need to swap x and y around for all the points in the original graph. So in mapping notation, we can express this as:

$$(x, y) \to (y, x)$$

Consequently, there obviously exists a transformation that allows you to obtain $y = f^{-1}(x)$ from $y = f(x)$. But the question is, can we express it in matrix notation? And the answer is yes, we can!

Consider the matrix $\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. When you pre-multiply the matrix $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ by $\mathbf{T}$, we get the following:

$$\begin{aligned} \mathbf{TX} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} y \\ x \end{bmatrix} \end{aligned}$$

Hence the transformation T defined by

$$\begin{aligned} T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{X} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} y \\ x \end{bmatrix} \end{aligned}$$

will give the mapping $(x, y) \to (y, x)$, meaning that we actually can express the transformation that allows you to obtain $y = f^{-1}(x)$ from $y = f(x)$ in matrix notation. That is, the matrix $\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ defines a reflection along the line $y = x$, and thus describes the mapping that takes you from a function $f(x)$ to its inverse, $f^{-1}(x)$ - and this is a matrix transformation you must remember! Therefore, the answer to this question is **C**, as the statement it offers is the only one that is incorrect.

Question 11 (C)

One way to go about this question would be to sketch the given functions on the CAS, and eliminate the ones that have the wrong shape. But you should know the families of functions concerned here well enough to be able to eliminate some options even without the CAS calculator.

For instance, option **A** presents $g(x) = 2(4x)^{\frac{2}{3}}$. Firstly, this option should be obviously incorrect, as all x-values here are squared, meaning that $g(x) \geq 0$, as there are no negatives on the LHS; this is clearly not the case according to the graph. Moreover, as can be seen by sketching this graph on the calculator, it looks nothing like the graph we have been given.

Likewise, the graph in **E**, $g(x) = \dfrac{(4x)^{\frac{4}{3}}}{2}$, would also imply that $g(x) \geq 0$, as any negative values of x will be put to the power of 4, and thereby become positive. Indeed, sketching it on the CAS shows that the graph looks like a very sharp and pointed parabola, and again, not at all like the given graph. Hence we can also eliminate this option.

Now, option **D** can also be eliminated. Why? Well, its equation is $g(x) = (2x)^{\frac{3}{2}}$, which means that all values of x will be rooted (remember, $a^{\frac{3}{2}} = \left(a^{\frac{1}{2}}\right)^3 = (\sqrt{a})^3$). Consequently, this implies that the graph of $g(x)$ would only be defined for $x \geq 0$, as you cannot take the square root of a negative. But the graph we have been given is defined for $x \in \mathbb{R}$. Plenty of other reasons would also disqualify this as the answer - indeed, this graph does not at all resemble the given graph.

But shape-wise, the two remaining options could both be correct. The given graph is of a shape that resembles a cubic (or a quintic, etc...) lying on its side - i.e. the inverse function of a cubic or of a quintic, etc. The point is though, it is of the family of functions given by $f(x) = x^{\frac{p}{q}}$, where $q > p$, and both p and q are odd. Hence the function $x^{\frac{1}{3}}$ and a dilation of the function $x^{\frac{3}{5}}$ would both be acceptable shape-wise. So to pick the right answer, we must rely on the second piece of information given - the point $(2, 8)$.

Substituting this point into the equation given in **B** ($g(x) = x^{\frac{1}{3}}$) gives:

$$\begin{aligned} g(2) &= 2^{\frac{1}{3}} \\ &= \sqrt[3]{2} \\ &\neq 8 \end{aligned}$$

This is clearly **not** correct, so the point does not lie in this equation, and we can therefore eliminate option **B**. Now do the same for **C**: ($g(x) = (16x)^{\frac{3}{5}}$)

$$\begin{aligned} g(2) &= (16 \times 2)^{\frac{3}{5}} \\ &= 32^{\frac{3}{5}} \\ &= \left(\sqrt[5]{32}\right)^3 \\ &= 2^3 \\ &= 8 \end{aligned}$$

So $(2, 8)$ indeed lies in this graph. Therefore, the only equation we are given that could possibly describe the graph of $y = g(x)$ is $g(x) = (16x)^{\frac{3}{5}}$. Thus the answer must be **C**.

SECTION 2 - Extended-Response Questions

Question 1

a. i.

We want to find an equation of a polynomial of degree 4 here, and we have been given exactly 5 points. As a general rule, to find the equation of a polynomial of degree w, you need $w+1$ different points - this is only not true when you are given turning points or other additional pieces of information. As such, we have enough information from these five points to deduce the equation of $u(x)$, and the only way of doing this is by subbing in the four points and solving a system of simultaneous equations. This does take a while to do by hand, which is why the question is written in such a way that you only have to establish the system, and then use the CAS to solve it, to receive full marks.

So the first step is to create a 'template' that you can actually substitute these points into. That is, grab the general formula for a quartic, and then substitute the x- and y-coordinates.

So you should begin by stating (although, because this question is worth one mark, you can probably get away without this initial statement and just with the matrix equation):

$$\text{Let } u(x) = ax^4 + bx^3 + cx^2 + dx + f$$

Now, you can use any other pronumerals; they do not have to be a, b, c, d, and f by any means. As long as it is not x, y and $\underline{u}$, any other pronumeral will suffice as a coefficient of an x^n term. So now we must substitute in all of the points above into this template, replacing x and u with the coordinates of each point:

$$\begin{aligned}
(1,8) \implies 8 &= a+b+c+d+f \\
(-2,2) \implies 2 &= 16a-8b+4c-2d+f \\
(0,18) \implies 18 &= 0a+0b+0c+0d+f \\
(3,72) \implies 72 &= 81a+27b+9c+3d+f \\
(-4,338) \implies 338 &= 256-64b+16c-4d+f
\end{aligned}$$

The next step is to rewrite this system in matrix form. Ever since the CAS was introduced into Methods, questions of this sort have become more and more frequent in exams; it is thus very important that you know how to solve systems of equations using a matrix form.

To do this, you need to rewrite the system in the form $\mathbf{AX} = \mathbf{B}$, [NB: a bold, capital letter is always indicatory of a matrix], where $\mathbf{A}$ is the matrix of all the coefficients of x^n (note that each row **must** be in the same order as the others - that is, first the coefficient of a, then the coefficient of b, and so on). $\mathbf{X}$ is a column matrix with the pronumerals you are attempting to find (which, in this case, are the pronumerals a, b, c, d, and f). And $\mathbf{B}$ is a column matrix composed of all the numbers on the other side of the equation - in thise case, the values of $u(x)$.

Hence in matrix form, we can express the system of matrices above as:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 16 & -8 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 256 & -64 & 16 & -4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ f \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 18 \\ 72 \\ 338 \end{bmatrix}$$

If you are still not sure, or if you think you made a mistake somewhere along the way, expanding the left-hand side out will actually give you the system of simultaneous equations we were solving for in the first place, showing you that this is indeed another way of expressing systems of equations.

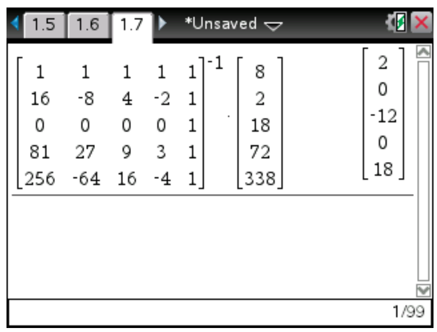
ii.

This question specifically tells you to use the CAS and is worth one mark - these are clear hints that you do not have to do anything more than write down the answer, and that, needless to say, it is extremely not recommended that you try this question by hand!

Using the CAS, we can either solve (under menu $\to$ algebra) the system of equations - in either the normal or matrix forms - for all of the pronumerals, or, alternatively, just type

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 16 & -8 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 256 & -64 & 16 & -4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 2 \\ 18 \\ 72 \\ 338 \end{bmatrix}$$

as the resulting matrix will therefore be equal to $\begin{bmatrix} a \\ b \\ c \\ d \\ f \end{bmatrix}$.



As such, we end up with the result that $\begin{bmatrix} a \\ b \\ c \\ d \\ f \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -12 \\ 0 \\ 18 \end{bmatrix}$.

We can therefore see that $a = 2$, $b = 0$, $c = -12$, $d = 0$ and $f = 18$. Substituting these values back into the general form for the quartic equation, which is $u(x) = ax^4 + bx^3 + cx^2 + dx + f$, we get:

$$u(x) = 2x^4 - 12x^2 + 18$$

And writing down this equation is **all** that is required for the mark! Remember, one-mark questions in methods require no working. Note that factorised forms of this equation are also perfectly acceptable.

One small point to take note of regarding matrix multiplication: whereas with numbers and pronumerals, the order of multiplication does not affect the product, this is NOT the case with matrices. That is,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 16 & -8 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 256 & -64 & 16 & -4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 2 \\ 18 \\ 72 \\ 338 \end{bmatrix} \neq \begin{bmatrix} 8 \\ 2 \\ 18 \\ 72 \\ 338 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 16 & -8 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 256 & -64 & 16 & -4 & 1 \end{bmatrix}^{-1}$$

In fact, the latter is not even defined. As such, it is extremely important that you type/write such equations and expressions out in the correct order, and also that when you are solving $\mathbf{AX} = \mathbf{B}$, you do so by saying that $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ and not by stating that $\mathbf{X} = \mathbf{B}\mathbf{A}^{-1}$. That is, you need to pre-multiply both sides by $\mathbf{A^{-1}}$, and not post-multiply them! [The pre- and post- preffixes refer to where you multiply by the new matrix.]

iii.

This question might seem a bit confusing at first, so it is important to work out what its actually saying. In essence, we are being told that $u(x)$ can be written as a composite function; that is, $u(x) = f \circ g(x)$. And, specifically, that $f(x) = 2x^2$, and that our 'job' in this question is to find the function $g(x)$.

This is one of those questions where what to do is not so obvious, and you need a bit more insight than you would with most other questions.

The first thing you should do is rewrite $f \circ g(x)$ so that $f(x)$ is replaced with $2x^2$:

$$\begin{aligned} u(x) &= f \circ g(x) \\ &= f(g(x)) \\ &= 2\left[g(x)\right]^2 \end{aligned}$$

This is because the notation $y = f \circ g(x)$, used for composite functions, should be read 'f of g of x'. That is, you first insert x into g, and then the result of that is inserted into f, to finally give y. Hence $f \circ g(x) = f(g(x))$. And if we replace $f(x)$ with $2x^2$, we obtain $2\left[g(x)\right]^2$.

Now, we know that $u(x) = 2x^4 - 12x^2 + 18$, but also that $u(x) = 2\left[g(x)\right]^2$, so we can say that:

$$2\left[g(x)\right]^2 = 2x^4 - 12x^2 + 18$$

And dividing both sides by 2 gives:

$$\left[g(x)\right]^2 = x^4 - 6x^2 + 9$$

So how do we get $g(x)$? We need to somehow express the RHS above as the perfect square of an expression of x. But notice that we can factorise the quartic expression above just like we would with a normal quadratic expression, as this is of the form $a^2 + 2ab + b^2$, where $a = x^2$ and $b = 3$. That is:

$$\begin{aligned} x^4 - 6x^2 + 9 &= \left(x^2\right)^2 - 2\left(x^2\right)(3) + (3)^2 \\ &= \left(x^2 - 3\right)^2 \end{aligned}$$

Expand this out if you are not certain, to make sure that it works! (In fact, if ever in doubt about factorising, expanding on the side of the page is a great way to check).

So we can now say that:

$$\left[g(x)\right]^2 = \left(x^2 - 3\right)^2$$

Meaning that:

$$g(x) = x^2 - 3$$

Essentially, the crucial point here is to be able to recognise that you have to factorise $u(x)$, and thus obtain $g(x)$ from there. It is also important to understand the principles of composite functions, and that it really is a function of another function - meaning that you can substitute the unknown $g(x)$ into the formula of $f(x)$.

iv.

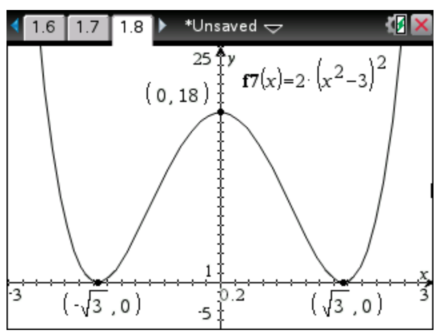
$$u(x) = 2x^4 - 12x^2 + 18 = 2(x^2 - 3)^2$$

This is a graph of the form $y = a(x^2 - h)^2$ - this is a specific form that allows you infer the intercepts and turning points readily from the formula. This is because $\left(x^2 - h\right)$ can be further factorised into $\left(x - \sqrt{h}\right)\left(x + \sqrt{h}\right)$, which shows you the intercepts quite clearly, but also the turning points, as any factor that is squared must be a turning point, and both these factors will be squared. Hence, a graph in this form is not only symmetrical about the y-axis, but also, it has turning points exactly at its intercepts.

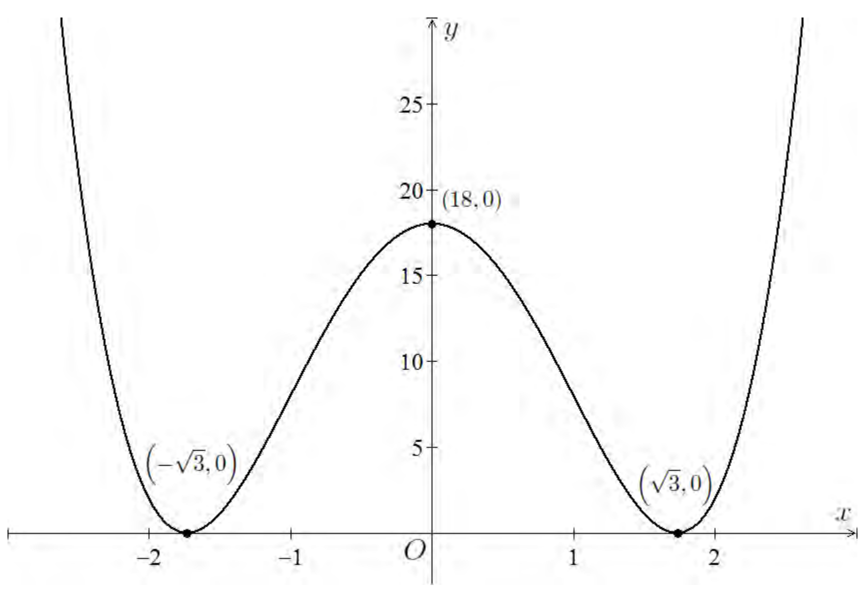
Thus, the graph of $u(x) = 2(x^2 - 3)^2$ [remember that $u(x) = f \circ g(x)$] will have x-intercepts at $x = \pm\sqrt{3}$, and these points will also be turning points. Moreover, you have to label them as coordinates according to the question (and you should get into the habit of always giving important points as coordinates), meaning that they must be labelled as $\left(\sqrt{3}, 0\right)$ and $\left(-\sqrt{3}, 0\right)$. The y-intercept can be found by letting $x = 0$: $f \circ g(0) = 2\left(-3\right)^2 = 18$; hence it is $(0, 18)$.

Now, since $u(x) = f \circ g(x)$, we could go about sketching this graph either via the equation $g(x) = 2x^4 - 12x^2 + 18$ or $g(x) = 2(x^2 - 3)^2$, but the second form does a much better job of showing us crucial points like intercepts and turning points, and is much easier to work with - for this reason, the question actually asks you to sketch the graph of $y = f \circ g(x)$ as opposed to $y = u(x)$, as the second form is not only easier to interpret, but is also a form that you should be familiar with - in fact, the study design explicitly includes an example of graphs of this form, meaning that getting used to this form is not a bad idea!

Anyhow, the working above gives us all the crucial points, so now we just need to sketch the graph - and we can use the CAS to give us an idea of the shape, although you should be able to work this one out on your own.



So without further ado, we can now sketch the graph of $y = f \circ g(x)$:



b. i.

We are being asked to sketch the graph of $y = |w(x)|$ from the graph of $y = w(x)$. Now, if you were really uncertain about how to do this, you could do the rest of the question first and, once the equation is obtained, sketch the graph of $y = |w(x)|$ on the CAS to guide you. However, the point here is to learn how to do these questions without having the equation - but of course, under exam conditions, do whatever you can to get to the answer!

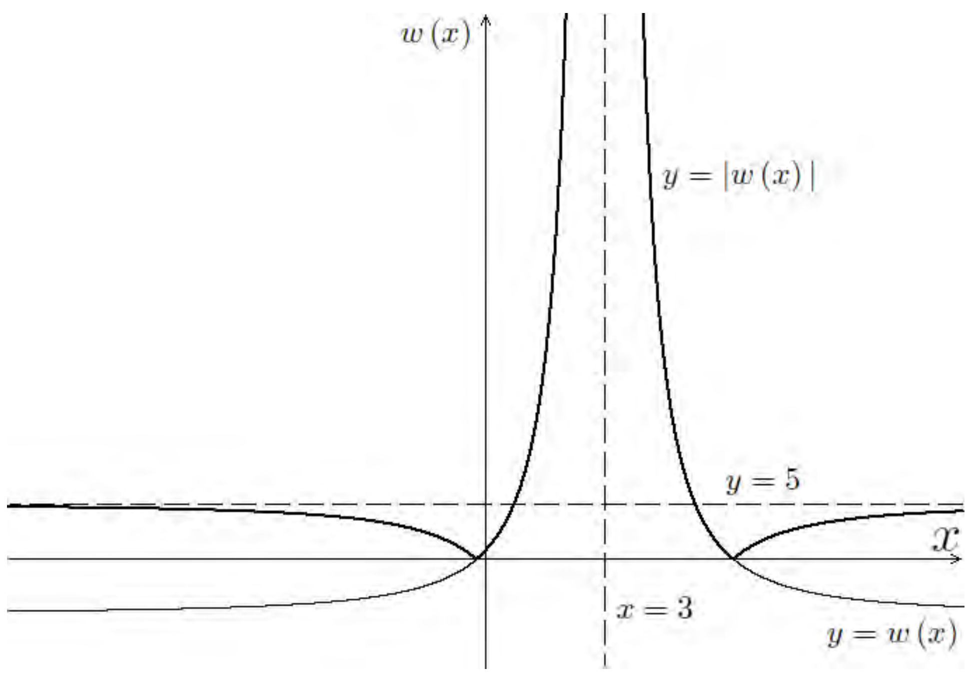
The graph of $y = |w(x)|$ is obtained by reflecting in the x-axis everything in the graph of $y = w(x)$ that is below the x-axis (i.e. where $y < 0$). The parts of the graph that are above the x-axis ($y > 0$), however, remain unchanged. As such, when you are sketching the graph of $y = |w(x)|$, it is essential that, for those bits of the graph, you actually draw over the graph of $y = w(x)$, making it very visible to the marker - I know it may sound silly, but if you are asked for a graph of this sort, you will lose a mark for not drawing over the original graph for the parts that are common to both.

So as far as the shape goes, reflecting the bits of the graph where $y < 0$, and drawing over the other parts, is all you have to do. But this is a two-mark question, and we are also asked for the asymptotes - so we must sketch them, and again, this may involve tracing over an asymptote that is already there, meaning you need to trace right over the line, but it must be very visible.

So what are the asymptotes of $y = |w(x)|$? Well, from the shape it is evident that $x = 3$ will remain an asymptote, as the parts of the graph approaching that asymptote were already on the 'positive' side which remains the same. So we must trace over the asymptote $x = 3$, as it is an asymptote for $y = |w(x)|$.

But does the other asymptote remain the same? The parts of the graph approaching $y = -5$ are clearly below the x-axis, and will be reflected upwards, meaning that $y = -5$ will no longer be an asymptote. Instead, this asymptote must be reflected upwards too, meaning that the graph of $y = |w(x)|$ will have $y = 5$ as its asymptote instead.

So now we can go ahead and actually sketch the graph:



The skill of being able to sketch both $y = f(|x|)$ and $y = |f(x)|$ from the graph of $y = f(x)$ is an important and often-examined one!

ii.

The general form of a truncus we are given here is $w(x) = \dfrac{a}{(x+b)^2} + c$. A truncus is similar in equation to a hyperbola, the only difference being that the denominator is squared. But just like a hyperbola, we have both a horizontal and a vertical asymptote; the difference is that, for a truncus, both 'parts' are on the same side of the horizontal asymptote. In any case, we can use this general rule, and the asymptotes that are evident on the graph we have been given, to work out the values of b and c.

Firstly, for b, the vertical asymptote (i.e. of the form $x = \alpha$), will occur when the denominator is equal to zero, because you can never ever ever ever ever divide by zero. So let $x + b = 0$. This implies $x = -b$ is the vertical asymptote.

But the horizontal asymptote, according to the graph, is $x = 3$, meaning that

$$\begin{aligned} -b &= 3 \\ \implies b &= -3 \end{aligned}$$

Now, the vertical asymptote always occurs at $y = c$ (when c is the translation along the y-axis, as it is in the formula above). Why is this the case? Well, not only can we not divide by zero, but (when $a \neq 0$), there is no value of x whatsoever that will allow the fraction $\dfrac{a}{(x+b)^2}$ equal zero. As $x \to \infty$, this fraction will become smaller and smaller, and approaches zero, but never actually reaches it. As such, as $x \to \infty$, $\dfrac{a}{(x+b)^2} \to 0$, meaning that $y \to c$ (as $y = w(x)$).

So $y = c$ is the horizontal asymptote for the graph of $w(x)$. But the graph clearly shows that the vertical asymptote is $y = -5$. Hence,

$$c = -5$$

Both these values are meant to be inferred directly from the graph - and, as suggested by the fact that this question is worth one mark, no working is necessary.

One thing to be careful of though - the general form we are given has $x + b$, whereas you are probably used to seeing $x - b$. And, with this more normal model, you would be tempted to just straight away state '$b = 3$', and therefore lose a mark that, with a bit more attention being paid, could easily have been obtained. As such, read things carefully, and do not assume the form you were given was indeed the 'standard' one. And thus a tad of working could actually prevent this mistake being made, meaning that just because a question does not require working, it does not mean you should not do any - some working is always a great idea.

iii.

Again, another one-mark question - meaning that just the solution suffices; but yet again, a bit of working does reduce the risk of a careless error. In any case, we now know the values of b and c, so we can rewrite the equation for $w(x)$ as:

$$y = \frac{a}{(x-3)^2} - 5$$

So all that remains to be found is a, and this is what the next questions concern themselves with. Now, seeing as this part of the question specifically states 'find a in terms of p', this is a hint suggesting that the point $(p, 8)$, which is shown on the graph of the truncus, must now be summoned to do its job.

So we must first substitute this point into the equation above, so that we can eventually obtain a in terms of p.

$$(p, 8) \implies 8 = \frac{a}{(p-3)^2} - 5$$

Now, we must rearrange to obtain a by itself (you could use the CAS for this, although it is easy algebra, and it would probably be quicker by hand). So the first step is to move the 5 away from the side where a is:

$$\frac{a}{(p-3)^2} = 13$$

Next, multiply both sides by $(p-3)^2$:

$$a = 13(p-3)^2$$

Which is exactly a in terms of p. Note that expanding this out would also be accepted, but is not necessary.

And although only this last line (which is the answer) is necessary, I would recommend doing all of the working above, so that you do not trip yourself up anywhere.

iv.

We must now try to actually find a, and we are therefore given one more piece of information that will help us out - that is, that the point $(p, 8)$ is also the intersection of $y = w(x)$ with $y = f \circ g(x)$. So what to do now? We could try to equate $w(x)$ with $f \circ g(x)$, but this would be very messy and end up giving us an equation of degree 6, and, quite frankly, who has the time to solve that? Instead, a significantly neater and quicker way is to sub $(p, 8)$ into the equation of $f \circ g(x)$, because this will allow us to calculate possible values of p. This works because, if $(p, 8)$ is the intersection of both graphs, then, needless to say, it must lie in $f \circ g(x)$.

So plug $(p, 8)$ into $f \circ g(x)$, and solve for p (recall that $f \circ g(x) = 2(x^2 - 3)^2$):

$$\begin{aligned} (p, 8) \implies 8 &= 2(p^2-3)^2 \\ \implies (p^2-3)^2 &= 4 \\ \implies p^2 - 3 &= \pm 2 \\ \implies p^2 &= 3 \pm 2 \end{aligned}$$

So solving for p gives you two possibilities already - $p^2 = 5$ or $p^2 = 1$. Remember, you must always take both the positive and negative of the root; you can only eliminate one when you have a substantial reason for doing so.

So $p^2 = 5$ or $p^2 = 1$ implies that $p = \pm\sqrt{5}$ or $p = \pm 1$. We therefore have four possible values for p! But are they all really good enough to meet all the criteria?

Looking back on the graph, since p is the x-coordinate of the point, and since the point is clearly to the right of the x-axis, we can safely say that $p > 0$. As such, we can eliminate the negative solutions - i.e. $p \neq -1$ and $p \neq -\sqrt{5}$. So we only have two possibilities left: 1 and $\sqrt{5}$.

But we were also told that $p < 2$, so eliminate $p = \sqrt{5}$. This is because $\sqrt{5} \approx 2.24$, which is evidently greater than 2. Hence, we know that $p = 1$. Thus, we can plug this into $a = 13(p-3)^2$ and thereby obtain the value of a:

$$\begin{aligned} a &= 13(1-3)^2 \\ &= 13(-2)^2 \\ &= 13 \times 4 \\ &= 52 \end{aligned}$$

So $a = 52$ is the answer. Being a three-mark question, a substantial amount of working is also required here - at the very least, you should state the equation you are solving to find p, write down the possible values of p and **include** reasons for why some are not valid and only $p = 1$ is, and then write down the value of a. The working does need to be in some depth due to the number of marks, and the way in which you obtained $a = 52$ must be made clear to the marker. Moreover, just discarding the other values of p without a reason, or accepting too many values of p, will result in marks being deducted.

Question 2

a. i.

We are being asked for a line between the first and last point that we have been given; that is, the equation of the line connecting $(1,5)$ and $(12,67)$.

There are many ways of calculating the equation of the line, and this is just one of them. We begin by finding the gradient, m, of the line. Since $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, we can plug in these two points to find m; just replace y with R and x with w (remember - this question is in terms of R and w only, so we cannot use x and y like we are used to doing).

$$\begin{aligned} \therefore m &= \frac{R_2 - R_1}{w_2 - w_1} \\ &= \frac{67-5}{12-1} \\ &= \frac{62}{11} \end{aligned}$$

Now that we have the gradient, we can substitute it into the formula $y - y_1 = m(x - x_1)$. If we also plug in one of the points above, as well as changing y and x to R and w, we can use this formula to get the equation of $R(w)$.

$$\begin{aligned} R - R_1 &= m(w - w_1) \\ R - 5 &= \frac{62}{11}(w-1) \\ \therefore R(w) &= \frac{62}{11}w - \frac{7}{11} \end{aligned}$$

Finding the equation of a line from two points should be very basic for methods students, and thus this question is only worth one mark, and therefore only the answer is needed to obtain the mark, although some working is certainly a good idea (unless you decide it is quicker for you to use the CAS the whole way, but that is probably unlikely in such a basic question) as you should definitely not do all the fraction work in your head.

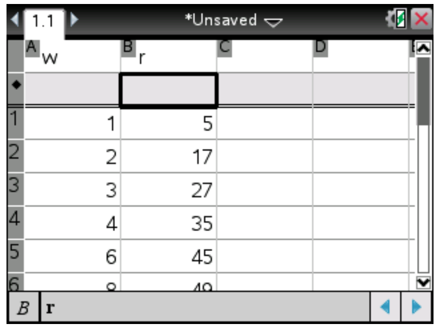
Note that, ideally, the perfect answer would actually be $\therefore R(w) = \dfrac{62}{11}w - \dfrac{7}{11}, 1 \leq w \leq 12$, as that is the domain that the function appears to be defined for. But since this question is worth one mark and does not specifically request the domain, you can get away with not giving it.

ii.

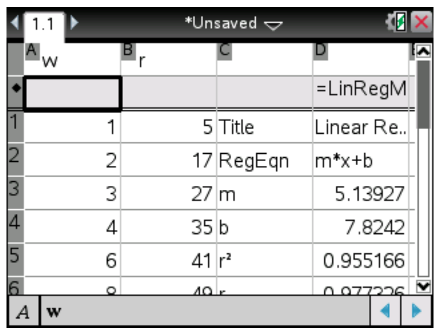
Obtaining a line of best fit should certainly not be attempted by hand! Using the 'linear regression' function of the CAS is certainly the best method of doing this question. But how do we use it?

You must open a new spreadsheet page (yes, the spreadsheets are actually useful for something in Methods!), and begin by labelling the very top rectangles of the first two columns 'w' and 'R' respectively, or anything else that is not x and y. [The reason for this is that, if you do use x and y, you will end up having problems when you try using x and y in graph and calculator pages for the rest of the SAC/exam, which does not sound like a pleasant experience].

Next, copy all the values from the table we have been given into the spreadsheet - obviously all the values for w will go under w in the first column, and likewise for R in the second column, and make sure that the values of one column match that of the next (as in, 1 in w is adjacent to 5 in R).



Now, press Menu $\rightarrow$ Statistics $\rightarrow$ Stat Calculations $\rightarrow$ Linear Regression $(mx+b)$. You will receive a screen with lots of settings - you only need to worry about the first two though! For the first one (X List), you will have to enter 'w (or ' followed by whatever you called the first column), and for the second (Y List), you will have to enter 'R. Press enter, and it should generate a list a few columns down which gives you the equation of the line of best fit.



We can therefore see that $m = 5.14$ and $b = 7.82$ for the line of best fit (each correct to two decimal places). Hence the line of best fit will be:

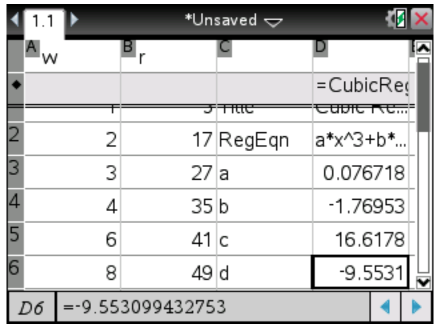
$$R(w) = 5.14w + 7.82$$

This question is worth one mark as all you need to do (and can do, really) is write down the equation. It may seem lengthy to do all that process in the calculator, but you will need it for future questions too.

The importance of knowing how to do regression on the CAS (which is, inserting a table of points and getting the calculator to come up with the function of best fit) is debated - while it is possible to come up on a SAC if you spend a significant amount of time doing it at school, it has never come up on any exams. That being said, since it is part of the study design, there is no reason why it cannot be asked - meaning that it could easily surprise everyone one year, and as such, you should probably be prepared for it by at least being familiar with the process of regression.

b.

This question follows straight on from the previous one. We now need to use the 'Cubic Regression' function, which is under the same menu list where we found Linear Regression. You must now do exactly the same as we had done previously - that is, choose the X and Y Lists as 'w and 'R, and then the cubic function will be given on a column further down again.



Reading off the CAS will show you that $a = 0.077$, $b = -1.770$, $c = 16.618$, $d = -9.553$. Since the 'regular equation' (which the CAS also gives to you) is $y = ax^3 + bx^2 + cx + d$, we can say that:

$$R(w) = 0.077x^3 - 1.770x^2 + 16.618x - 9.553$$

Having done all the preparation work in the last question makes this one much quicker.

One **important** point concerning these last two questions - if you had never heard of regression before, it would have been impossible to do these questions. However, the results of these questions are also not needed for future parts of the question, meaning that you could attempt the rest, even if you were unable to get these questions. What I am trying to say is that, if you come across part of a big question that you cannot do, you should never give up there! Look at the next one (or two), because you may be able to do subsequent parts without having done the previous part. And sometimes, if the results do carry through, its actually not a bad idea to just make up an answer so that you can at least get consequential marks for the future questions!

c. i.

The domain of $R(w)$ will be $[12, \infty)$. This is because the question explicitly states that the model 'is accurate from week 12 onwards.' Consequently, this new model begins at the point $(12, 67)$, and continues from there as w increases.

Note that when you substitute $w = 12$ into the new model, you get

$$\begin{aligned} R(12) &= -53e^{\left(1 - \frac{12}{12}\right)} + 120 \\ &= -53e^0 + 120 \\ &= 120 - 53 \\ &= 67 \end{aligned}$$

Meaning that, indeed, $(12, 67)$ is the endpoint of this graph.

Now, what will the range be? This is an exponential graph - that is, it is of the form $y = ae^{bx} + c$, where $y = c$ is always the equation of the asymptote. This is because, no matter how negative the term bx is, e^{bx} can never equal zero, only approach it and get infinitely closer. As such, as $bx \to -\infty$, $ae^{bx} \to 0$, and thus $y \to c$. So if we look at the equation

$$R(w) = -53e^{\left(1 - \frac{w}{12}\right)} + 120$$

we can see that $c = 120$, meaning that there will be an asymptote given by $y = 120$. And since we know that the graph begins at $(12, 67)$, it will obviously go up towards 120 if that is the asymptote it is tending towards, and therefore the range will have to be $[67, 120)$; 120 cannot actually be included in the range as this value is never reached, hence the curled bracket and not a square bracket. Moreover, $R(w)$ is an exponential function reflected in both the x- and y-axis, meaning that $R(w)$ will increase as w increases (but at a decreasing rate), and thus the graph has to go **upwards** from $R(w) = 67$.

Sketching the graph of $R(w)$, as we shall do in the next question, can also help determine the domain and range.

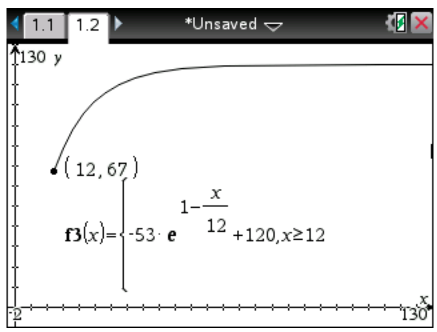
The only real trick in this question is noticing that the graph will not begin at $w = 0$ or 1, but rather when $w = 12$. Other than that, the domain and range should be rather obvious, and sketching the graph in the CAS (as we shall do soon) would certainly help get a good idea of the shape. And being worth a single mark means that you just need to write down the domain and range; no explanations are needed.

ii.

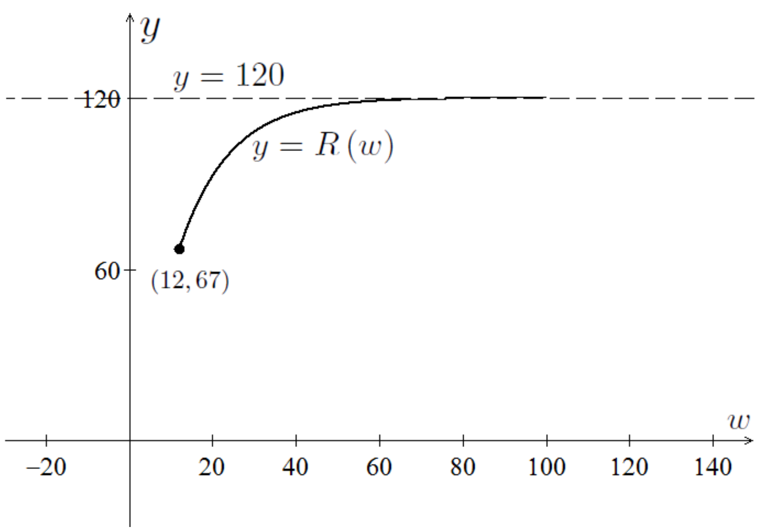
We are trying to sketch the graph of

$$R(w) = -53e^{\left(1 - \frac{w}{12}\right)} + 120$$

As we just saw, the graph of $y = R(w)$ will have an asymptote at $y = 120$, and the graph itself increases towards this asymptote from the endpoint at $(12, 67)$. Although we have also already noted that the graph increases at a decreasing rate (as it is an exponential function reflected in both axis), it is always a good idea to sketch the graph on the CAS first, for two reasons: it ensures that you made no mistakes previously, and gives you a more accurate idea of the exact shape. So let's sketch this graph on the CAS (remember that you will have to replace w with x, so be mindful of this - you do not want to copy down x by accident onto the exam paper in later questions!).



We essentially have all the information we need - the shape, asymptote and endpoint. There are no intercepts, as the graph is increasing from a point already in the first quadrant, so there are no intercepts to find. The question only mentions 'intercepts' for students who have the wrong domain (as consequential marks may be awarded in the case that the domain and range from the previous question were incorrect). So we can now go ahead and sketch the graph of $y = R(w)$, including its asymptote (as an equation!) and the endpoint (as coordinates!):



One thing to take particular note of here - VCAA will penalise you if your graph touches the asymptote, so make sure that the graph only approaches, but does not touch, $y = 120$.

iii.

This question is merely asking you to interpret the graph - if the model is very accurate for a long period of time, and if, at some point in time, the hotel is completely booked out, then the greatest value reached by $R(w)$ must be the number of rooms the hotel has available. And since, as time passes, $R(w)$ becomes infinitely close to 120, the hotel must have a total of 120 rooms. So all you need for the mark in this question is to write down '120 rooms'.

Note that the answer '119 rooms', if accompanied by an explanation saying something along the lines of 'you cannot round up the number of rooms, and $R(w)$ will never actually reach 120', may also be accepted.

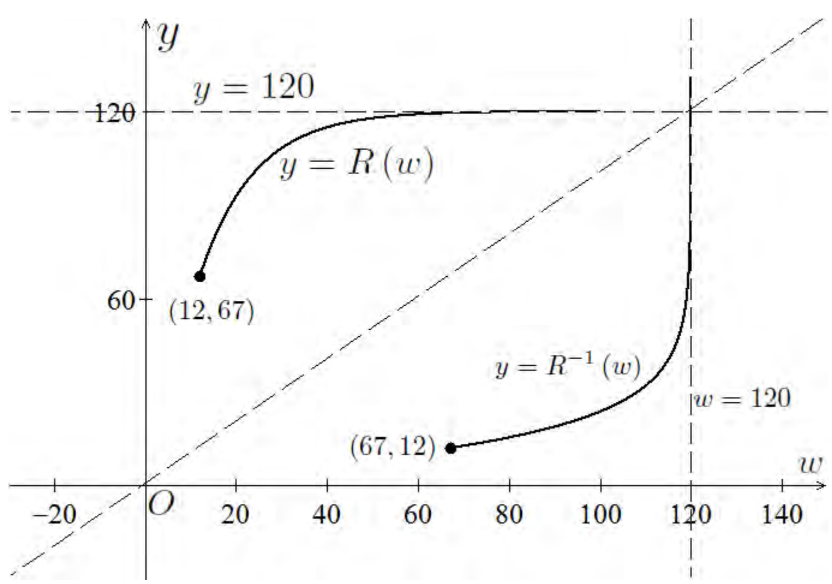
iv.

You should do this question without obtaining the equation of the inverse function (and thus without use of the CAS), particularly as this area of study does not include finding the equation of inverses. Instead, the best idea is to sketch a dotted line $y = w$, as you can reflect the graph across this line to obtain $y = R^{-1}(w)$. Moreover, to give you a more specific idea of what the graph may look like, you can 'reflect' key features and points. For instance, if we are swapping w and y around everywhere, then the asymptote $y = 120$ will become $w = 120$ - that is, it will now be vertical and not horizontal. This makes sense, as reflecting a horizontal line along the 'mirror' $y = w$, will give us a vertical line. So we can begin by sketching the asymptote $w = 120$.

Next, we have an endpoint $(67, 12)$ in the graph $R(w)$; since we swap w and y around for the inverse, the endpoint of the graph $y = R^{-1}(w)$ will therefore be $(12, 67)$. So we can now add a closed circle at that point.

The last part is just the shape - if you are struggling to visualise it, a good strategy is to pick rough points on the original graph, and from there get a rough idea of a coordinate on the graph of the inverse (do not label this point, just use it to help guide you!). Otherwise, you should be able to see that, as a reflection of the original graph, it will increase from the endpoint towards the asymptote - and it will approach, but never touch, the asymptote $w = 120$.

So all in all, this information should be enough to deduce the following graph of $R^{-1}(x)$.



One more point - if the original graph of $R(w)$ was incorrect, consequential marks would apply. If your graph of $R^{-1}(w)$ is correct given the $R(w)$ you had drawn, then full marks will be awarded for this part of the question.

v.

A one-mark question where you are asked to give your answer to a certain number of decimal places is a good hint that you are meant to use the CAS for it - and, indeed, you are. But how are we meant to find any possible points of intersection, since we do not have the equation of $R^{-1}(w)$? Well, you could find the equation of the inverse function, but that would be lengthy and arduous (and it is also within the algebra area of study, not functions and graphs). So instead, refer to what I said back in multiple-choice **question 10** regarding $f(x) = x$ giving points of intersection between the functions $y = f(x)$ and $y = f^{-1}(x)$. That is, instead of finding $y = R^{-1}(w)$ and equating it to $y = R(w)$, we can simply equate $y = R(w)$ with $y = w$; that is, let $R(w) = w$.

One thing you may be wondering is whether **every** point of intersection between a function and its inverse will be along the line $y = x$. Although rare, there are times when this is not the case - for instance, consider the function $f : [0, \infty) \to \mathbb{R}$, $f(x) = 1 - x^2$, and its inverse, $f^{-1}(x) = \sqrt{1 - x}$, which intersect at $(0, 1)$ and at $(1, 0)$, as well as a third time on the line $y = x$. As such, you cannot state that every point of interception will definitely be obtained by equating $y = f(x)$ and $y = x$. But, looking at the graph we have just sketched, it looks like there will be only one point of intersection, and that it will most certainly be on the line $y = x$ (well, $y = w$, since we are using w here instead of our usual good friend x). Subsequently, we can rest assured that the point of intersection we are looking for can be obtained by solving $R(w) = w$. Again, I must emphasise, it is very, very rare that a point of intersection between a function and its inverse will not be on the line $y = x$, so equating as we have done here is almost always fine.

Hence this question really boils down to solving $R(w) = w$. So, using the CAS calculator, we can solve

$$w = -53e^{\left(1 - \frac{w}{12}\right)} + 120$$

And this gives us the solutions $w = 2.440$ or 119.993. But as we have recently seen, $R(w)$ has domain $[12, \infty)$, meaning that we must eliminate $w = 2.440$. So this leaves us with $w = 119.993$, and since we know that this must lie on $y = w$, we know that $w = 119.993$ as well. That is, the solution(s) to $f(x) = x$ always have an x-coordinate that equals the y-coordinate.

So we can say that the point of intersection will be:

$$(119.993, 119.993)$$

Since the question specifically asks for coordinates, we must write these coordinates down to obtain the mark.

d. i.

We must systematically work through these transformations and apply them in a correct order. Again, like I urged you in previous tansformation questions, using mapping notation (x' and y', which here we will call w' and R') is the best way to do these questions. So we will begin by applying the first transformation, which is that 'the graph is translated 15 units in the negative direction of the w-axis'. As such, if we had a point (w, R), it would now become $(w - 15, R)$. So in mapping notation:

$$(w, R) \to (w - 15, R)$$

Next, 'The graph is then translated 7 units in the negative direction of the y-axis'. This means that the R-coordinate (this is the same as saying y-coordinate) must decrease by 7. That is:

$$(w, R) \to (w - 15, R) \to (w - 15, R - 7)$$

After this, the next transformation is that 'the graph is then dilated by a factor of $\frac{1}{2}$ from the w-axis'. This is equivalent to multiplying the entire R-coordinate by $\frac{1}{2}$, (**remember** that 'from the w-axis' will mean that it affects the R-coordinate, and vice versa!) which can be written as:

$$(w, R) \to (w - 15, R) \to (w - 15, R - 7) \to \left(w - 15, \frac{1}{2}(R - 7)\right)$$

Note that the $\dfrac{1}{2}$ multiplies the entire $(R-7)$, because the translation was applied first.

Next, 'the graph is the dilated by a factor of 3 from the y-axis', meaning that we multiply the w-coordinate by 3, i.e.

$$(w,R)\to(w-15,R)\to(w-15,R-7)\to\left(w-15,\,\frac{1}{2}(R-7)\right)\to\left(3(w-15)\,,\,\frac{1}{2}(R-7)\right)$$

Lastly, apply the transformation 'the graph is then reflected in the y-axis'. That is, multiply the w coordinate by -1. We can therefore write this as:

$$(w,R)\to(w-15,R)\to(w-15,R-7)\to\left(w-15,\,\frac{1}{2}(R-7)\right)\to\left(3(w-15)\,,\,\frac{1}{2}(R-7)\right)\to\left(3(15-w)\,,\,\frac{1}{2}(R-7)\right)$$

Note that you obviously do not need to write a new line every time - just the last line containing all the transformations is more than enough!

Now, since (w',R') gives us the coordinates of the image point, we can say that

$$(w',R')=\left(3(15-w)\,,\,\frac{1}{2}(R-7)\right)$$

From this we can create two equations, one by equating the w-coordinates with each other, and the other by equating the R-coordinates with each other.

So first, let's look at w. First establish the equation, and then rearrange to obtain w in terms of w':

$$\begin{aligned} w' &= 3(15-w) \\ \implies 15-w &= \frac{w'}{3} \\ \implies -w &= \frac{w'}{3}-15 \\ \implies w &= 15-\frac{w'}{3} \end{aligned}$$

And now, let's do the same with R:

$$\begin{aligned} R' &= \frac{1}{2}(R-7) \\ \implies R-7 &= 2R' \\ \implies R &= 2R'+7 \end{aligned}$$

Now why did I want to obtain w in terms of w', and R in terms of R'? The reason is because you can only substitute the original variables (that is, w and R) into the equation. That is, we had the equation $R(w)=-53e^{\left(1-\frac{w}{12}\right)}+120$, and we can plug w and R back into this equation, but now in terms of w' and R'. So substitute $w=15-\dfrac{w'}{3}$ and $R=2R'+7$ into $R(w)=-53e^{\left(1-\frac{w}{12}\right)}+120$. This will give us:

$$2R'+7=-53e^{\left(1-\frac{1}{12}\left(15-\frac{w'}{3}\right)\right)}+120$$

Simplifying this rather large expression will give:

$$2R'=-53e^{\left(1-\frac{5}{4}+\frac{w'}{36}\right)}+113$$

And this further simplifies to:

$$R' = -\frac{53}{2}e^{\left(\frac{w'}{36}-\frac{1}{4}\right)} + \frac{113}{2}$$

We now know the equation of the image, and we can thus remove the dashes to give us the new equation for $R(w)$, which will be:

$$R(w) = -\frac{53}{2}e^{\left(\frac{w}{36}-\frac{1}{4}\right)} + \frac{113}{2}$$

This is a long question, worth a total of three marks, and one where the CAS is of little help, other than with simple arithmetic and fraction work if necessary. As such, a good deal of working should be present, particularly in order to avoid mistakes with applying the transformations in an incorrect order. Only one mark goes towards the answer of the new equation for $R(w)$; the rest are for appropriate methods of interpreting and applying transformations - and really, for a question with so many transformations, and with a rule that is already long and complicated to play around with, the safe method is to use mapping notation.

ii.

We want to reverse Ben's transformations here. There are several ways of going about trying to find the transformations that would 'undo' Ben's manipulations. For instance, you could look at the new equation and, by letting the old one be w' and R', reverse the entire process from scratch. But this would take very long, and it would be much quicker to simply try to reverse all of the transformations Ben made.

The following is the order of all the transformations Ben made, given in the question.

- The graph is translated 15 units in the negative direction of the w-axis
- The graph is then translated 7 units in the negative direction of the y-axis
- The graph is then dilated by a factor of $\frac{1}{2}$ from the w-axis
- The graph is then dilated by a factor of 3 from the y-axis
- The graph is then reflected in the y-axis

Accordingly, if we were to reverse this list, we would have to, starting at the last (or most recent) transformation, apply the exact opposite transformation, and then do the same for the second last, and so on, until we reach the first one again.

As such, the following is a list of the transformations we want to make, in order, so that we can return $R(w)$ to what it used to be: Reflect the graph in the y-axis (as two reflections in the y-axis will 'cancel' themselves out, returning the graph to original position)

- Reflect the graph in the y-axis (as two reflections in the y-axis will 'cancel' themselves out)
- Dilate the graph by a factor of $\frac{1}{3}$ from the y-axis (as this reverses the previous dilation by factor 3)
- Dilate the graph by a factor of 2 from the w-axis
- Translate the graph 7 units in the positive direction of the y-axis; and lastly
- Translate the graph 15 units in the positive direction of the w-axis

Obeying this list, in the correct order, will give us the transformation to restore the graph of $R(w)$. And how do we express this is matrix notation?

Since the translations are clearly last, we can express this as

$$T\left(\begin{bmatrix} w \\ y \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} w \\ y \end{bmatrix} + \begin{bmatrix} 15 \\ 7 \end{bmatrix}$$

Why is this the case? Because we are multiplying the w-coordinate by $-\frac{1}{3}$, which is the equivalent of a reflection in the y-axis and a dilation by a factor of $\frac{1}{3}$ from the y-axis. Likewise, multiplying the y-coordinate by 2 is essentially dilating the graph by a factor of 2 from the w-axis. And the translations are evidently being applied after the dilations, and both are in a positive direction.

Unfortunately though, this is not the end of the question - we are asked for a transformation that is specifically of the form:

$$T\left(\begin{bmatrix} w \\ y \end{bmatrix}\right) = \mathbf{T}\left(\begin{bmatrix} w \\ y \end{bmatrix} + \mathbf{B}\right)$$

Which is not the form that our transformation is in. Now, we could go back and try thinking of a different order of transformations (where translations occur first) that will also return $R(w)$ to its original form, but that would take very long, and we are close to the answer. So instead, it is much easier to just rearrange what we have to obtain the correct style of transformations, as follows.

$$\begin{aligned}
T\left(\begin{bmatrix} w \\ y \end{bmatrix}\right) &= \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 15 \\ 7 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 15 \\ 7 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 15 \\ 7 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 15 \\ 7 \end{bmatrix}\right) \\
&= \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -45 \\ \frac{7}{2} \end{bmatrix}\right)
\end{aligned}$$

Remembering that a matrix multiplied by its inverse gives the identity matrix, we are able to factor out $\mathbf{T}$ so that T is in the required form. As such, we now have a matrix transformation of the form $T\left(\begin{bmatrix} w \\ y \end{bmatrix}\right) = \mathbf{T}\left(\begin{bmatrix} w \\ y \end{bmatrix} + \mathbf{B}\right)$, meaning that we can state that:

$$\mathbf{T} = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} -45 \\ \frac{7}{2} \end{bmatrix}$$

So that is the answer to the question! As it is a three-mark question, some working needs to be shown - in particular, how you went about working out the reverse transformations, and the rearranging process. Moreover, the CAS is not very useful here, meaning that most of this indeed has to be done by hand. So, in essence, this question boils down to a good understanding of transformations, but also some insight into what needs to be done to rearrange the system so that translations are applied first is very important.

ALGEBRA

TECH-FREE TEST 1

DETAILED SOLUTIONS

Question 1

a.

In order to find when the gradients of the lines are the same, we will first have to find the gradients of each line. while it's common for "gradient" to be a trigger word for students to find the derivative of functions, it should not be necessary to do so for straight lines. Students should be very aware that for $y = mx + c$, the gradient of this line is m. Be VERY careful in this question - indeed, many students understand that the gradient of a straight line is "m", but in this question, m is just a constant, not necessarily associated with the gradient what so ever. Students tend to get confused when questions include such a value.

Perhaps the easiest way to find the gradient of each line is to rearrange the equations in the form $y = mx + c$.

$$\begin{aligned} y &= \frac{8 - 2mx}{3} \\ y &= \frac{m - x}{m} \end{aligned}$$

Hence, the gradient of first line is $-\frac{2m}{3}$ and the gradient of second line is $-\frac{1}{m}$

For lines to have the same gradient we only have to let them equal each other and solve for m,

$$\begin{aligned} -\frac{2m}{3} &= -\frac{1}{m} \\ m^2 &= \frac{3}{2} \\ m &= \pm\sqrt{\frac{3}{2}} \end{aligned}$$

Ensure that when you take the square root from something like m^2 that both the positive and negative result are included - if you were to take the square of either value of m, you will get the same result, $\frac{3}{2}$. This is opposed to something like $\sqrt{9}$ which ONLY equals 3.

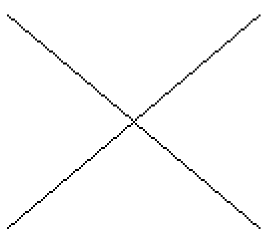
b.

This question tests a student's understanding of the intersection of lines. It's worth drawing two lines and considering how they may intersect. What you will find is that there are three ways in which two lines may intersect i.e. how many solutions produced by the equation. One is simply drawing them such that there is one intersection. But you can also have it so that there are infinite solutions (where the lines overlap) or no intersections because the lines are in the same direction. How do we place this intuitive result in the context of this subject and gradients? The question lies in how to distinguish them.

Let us first consider no solution and the infinite solutions. What's immediately obvious about no solutions is that the lines are parallel - in other words, that means the lines have the same gradient. No matter how far you extend these lines, they will never intersect. However, you'll also notice that the lines that produce infinite solutions are also parallel; they are also the exact same line. If both types of intersections involve lines with the same gradient, how do we tell them apart? The answer lies in $y = mx + c$. We have dealt with the "m", so what about the c? Well it should be obvious that if infinite solutions involve two lines that are the same, then c, the y-intercept, must also be the same in both lines. For no solutions, the y-intercepts are different.

The only thing left to look at is when there is a unique solution. It should be quite clear from the above, that these occur when the lines do not have the same gradient.

Below is a summary of line intersections:



<u>One Intersections</u>	<u>No Intersection</u>	<u>Infinite Intersection</u>
Unique Solution	No Solutions	Infinite Solutions
Gradients not the same	Gradients Equal	Gradients Equal
	y-intercepts different	y-intercepts equal

This question asks you to find when the lines have "no solutions" or a "unique solution".

i.

There are no solutions when the lines are parallel and unique- i.e. The gradients are the same and they do not share a y-intercept.

Refer back to the solutions in part a) above where the equations have been rearranged for y. Since the y-intercept of the first line is $\frac{8}{3}$ and for the second line is 1 for any value of m, the y-intercepts cannot be the same.

Hence, we can rule out the possibility of infinite solutions for the same gradients and therefore, there is no solution for $m \in \left\{-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right\}$

If you are unfamiliar with the notation used about, it is worth doing work on your set notation as symbols such as "$\in$" commonly appear in VCAA exams and is important for your knowledge and answers.

ii.

There is a unique solution when the gradients of two lines are different.

Hence, there is a unique solution for $m \in R \backslash \left\{-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right\}$. That is, the complete reverse of the previous situation.

Question 2

This question makes use of a common trick used to complicate solving equations: hiding a quadratic. Students should always be on the lookout for these - in many instances, it is probably too simple to get students to solve a basic equation, and so VCAA, and other publications will incorporate this type of question to make it more difficult. But when you think about it, it's not difficult at all. You should be very familiar with solving quadratic equations and you should also be able to solve simple equations involving trignometric terms. However, note that quadratics may be hidden in any type of function (logarithmic, exponential, etc). The solution to this question can be applied in all these instances.

Just as you would with recognising any quadratic, you are simply looking for an equation in the form "ax^2+bx+c". The squared term should be your standout clue to at least consider the hidden quadratic. There is a more subtle way to hide a quadratic which is covered at the bottom of this solution.

$$\begin{aligned} 4\sin^2(x) + (2-2\sqrt{3})\sin(x) &= \sqrt{3} \\ 4\sin^2(x) + (2-2\sqrt{3})\sin(x) - \sqrt{3} &= 0 \end{aligned}$$

For these hidden quadratics, the question becomes significantly easier if you can break it down into the root quadratic form by letting the term (in this case $\sin(x)$) equal to a letter such as "a". This is not a necessity but it will make your solutions a lot clearer for both the marker and yourself (and why wouldn't you make things easier for yourself if you could).

Let $\sin(x) = a$

$$4a^2 + (2-2\sqrt{3})a - \sqrt{3} = 0$$

It is from this point that you can tackle the question as a regular quadratic as you have probably done for many years.

First, we factorise (although you can also make use of the quadratic formula or completing the square - though I do not recommend completing the square in this case due to the complications involved in the simplification steps).

$$\begin{aligned}(2a+1)(2a-\sqrt{3}) &= 0 \\ 2a+1=0 \quad \text{or} &\quad 2a-\sqrt{3}=0 \\ a=-\frac{1}{2} \quad \text{or} &\quad a=\frac{\sqrt{3}}{2}\end{aligned}$$

But wait! The question is not completed yet. For one thing, the question asks for the solutions for x, not a so you certainly have not answered the question. Secondly, you must remember that a was fabricated not by VCAA, but by you. Too many students will conclude their solution here and miss out on several marks, including those for the answer.

$$\sin(x) = -\frac{1}{2} \text{ or } \sin(x) = \frac{\sqrt{3}}{2}$$

We've been asked to investigate the solutions in a specific domain, namely $x \in [\pi, 2\pi]$. If we solve $\sin(x) = -\frac{1}{2}$, we obtain reference angles of $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$. Conveniently, both of these values already lie in our required domain. Adding or subtracting 2π to obtain further solutions is futile, as it would clearly take us outside of this domain.

Next, we investigate $\sin(x) = \frac{\sqrt{3}}{2}$ and obtain reference angles of $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$. If we add 2π to these angles, we are clearly taken to a value greater than 2π which is outside our required domain. This means that no values of x can be used from this equation! Don't be confused by this, it is often a trick used to psych students out into thinking "why would they make us solve this if there is no solution?". Sometimes solutions do not make sense when considering a restricted domain. If we had a larger domain (e.g. $[-4\pi, 4\pi]$ then there would be plenty of solutions for this part of the equation.

Side note: I mentioned above that there is a subtle form of hidden quadratic that will trip up many students. These typically arise from an equation similar to $\frac{1}{x} + x = 1$. When you see this question, even if you don't recognise the quadratic, you should automatically realise that you are stumped and look to play around. As you would with any fraction in an equation, you would probably look to remove the denominator by multiplying both sides of the equation with it. If you were to do that in this case, multiplying by x you would end up with $1 + x^2 = x$ and if you rearrange this you get $x^2 - x + 1 = 0$, which is your typical quadratic equation. Keep an eye out for these - they aren't the most common form of hidden quadratic, but they certainly appear. **Question 3**

These types of "show that" proof questions are common in exams and so it's important that students are well versed in it and have a particular approach to them - but remember that it's a personal preference as to which way you would like to do it.

A simple approach is to consider both sides of the expression given. If you state what one side of the expression is equal to, you can start at the other side and try to reach this same result. Why is it important to do this? A common mistake students make with such questions is that they treat it like ane equation to be solved and work on both sides simultaneously. The problem with this is that more often than not, you will end up with $1 = 1$ or some variation of this. while mathematically true, getting to the point of "$1 = 1$" does not constitute a correct answer.

Thus, for the sake of being correct AND being clear to markers, address both sides individually. The following set up is recommended. RHS=Right-hand Side and LHS=Left-hand side.

$$\begin{aligned} RHS &= \frac{1}{g(x)} - g(x) \\ &= \frac{1}{\cos(x)} - \cos(x) \\ LHS &= f(x)g(x) \\ &= \tan^2(x)\cos(x) \end{aligned}$$

Now that both sides have been established, students often question which side they should start from. This is entirely personal and VCAA seems to accept both ways, but some would claim that given the way the expression is written, it is implied that one should start from the left-hand side and conclude with the right-hand side. For those who want to be logical about their decision, you can choose to go from the simpler expression to more complicated, or attempt to simplify the complicated expression. The writer personally finds simplifying something complicated to be the easier method since going from something simple requires you to add terms and such to reach a complexity that wouldn't necessarily be intuitive in any way.

Starting from the left-hand side in this question seems to be the more complicated expression since it has more terms involved.

$$\begin{aligned} LHS &= \tan^2(x)\cos(x) \\ & \frac{\sin^2(x)}{\cos^2(x)}\cos(x) \\ &= \frac{\sin^2(x)}{\cos(x)} \\ &= \frac{1-\cos^2(x)}{\cos(x)} \\ &= \frac{1}{\cos(x)} - \frac{\cos^2(x)}{\cos(x)} \\ &= \frac{1}{\cos(x)} - \cos(x) \\ &= RHS \end{aligned}$$

Take note of the identities being used in the above solution: notably that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and $\sin^2(x) = 1 - \cos^2(x)$. The first is a simple one, and since it is simple, students should take advantage of it because more often than not, manipulating the expression in any way will help you move towards your desired result. Furthermore, you know the final solution only contains the cosine function, so any manipulation to achieve the consine function is a success. And with that said, the purpose in using the second identity above becomes clear. The pythagorean identity is immensely important to remember and is significant in being able to produce many other identities used within the course. And obviously, it is useful in this question in removing the sine function in place of a cosine function as we desire.

But wait! Before you get all jolly over proving the statement, ensure you read the question and make sure you are picking up the rest of the marks. There is a whole mark to be gained for restricting the values of x for which this proof is correct.

As with most domain restrictions, look for reasons why a function would be undefined - the most common ones involve asymptotes. The cosine function does not contain an asymptote, so the restriction must come from elsewhere - and indeed it comes from the term $\dfrac{1}{\cos(x)}$ where obviously the denominator cannot equal 0.
Therefore, the proof is only defined for $\cos(x) \neq 0$

$$\begin{aligned}
\cos(x) &\neq 0 \\
x &\neq 2k\pi \pm \frac{\pi}{2},\ k \in \mathbb{Z} \\
x &\neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}
\end{aligned}$$

How do we arrive at this general solution? General solutions will only be asked in exam 1 if the pattern is easily recognised. $\cos(x) \neq 0$ has reference angle solutions of $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. These are the first two solutions in the cycle of the unit circle. Because the unit circle repeats, we can very clearly see that going forward 2π units (one full rotation, back to the same place) yields another solution of $\frac{\pi}{2} + 2\pi = \frac{5\pi}{2}$. Similarly, adding 2π to $\frac{3\pi}{2}$ gives us $\frac{7\pi}{2}$. The same goes if we reverse the cycle of the unit circle and go in the negative direction. $\frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$, and $\frac{\pi}{2} - 2\pi = -\frac{3\pi}{2}$. Let's put these together, we can clearly see that $\cos(x) \neq 0$ when $x \neq -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$. We can clearly see that the solutions have a clear pattern that they are all separated by π units. It is logical to infer that we can extend this pattern to infinity. We can then pick an "anchoring" solution (usually ideal to use reference angle) and simply add $k\pi$ to it to get every single solution, with the stipulation that k takes on all values from the set of integers, $\mathbb{Z}$. So we can choose k to be any integer, be it 0, 1, -10 or -5823897 and every possible value of k will yield a possible solution. This is why we call it the "general solution", because every single possible solution is accounted for.

Therefore, our final result is that $f(x)g(x) = \frac{1}{g(x)} - g(x)$ for $x \in \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$

Question 4

When dealing with a question that has a modulus function in it, you can be expected to show an ability of splitting them up into hybrid functions - this is especially true for "solve for" questions because there isn't really an easier way to address the question without a calculator.

The easiest way to go about this is to understand what it means to find the absolute value of something. You essentially want every value to be positive - thus for the values of x that a function is negative, we want to reflect the function in the x-axis so that it retains its nominal value, but is now positive.
i.e.

$$|f(x)| = \begin{cases} f(x), & f(x) \geqslant 0 \\ -f(x), & f(x) < 0 \end{cases}$$

Now let us apply this to the question. You will notice that a consideration of when the function is negative is not required when solving for the answer - it is simply implied for now, and once we have solutions, we can check them for whether they exist or not within the limitations of the absolute value function. For the meantime, just split the function into its respective parts - one for when the function is greater than 0, and another for when it is less than 0.

$$|\log_e(2x+1)| = 9$$

$$\begin{cases} \log_e(2x+1) = 9 \\ -\log_e(2x+1) = 9 \end{cases}$$

$$\begin{cases} \log_e(2x+1) = 9 \\ \log_e(2x+1) = -9 \end{cases}$$

Every student should be aware of how we define logarithms in terms of indices.
Namely,

$$\begin{aligned} \log_a(b) &= c \\ \text{Therefore, } a^c &= b \end{aligned}$$

Hence,

$$\begin{cases} 2x+1 = e^9, \\ 2x+1 = e^{-9}, \end{cases}$$

$$\begin{cases} x = \frac{e^9-1}{2}, \\ x = \frac{e^{-9}-1}{2}, \end{cases}$$

Always be careful with functions for which you know the domain is restricted for some reason or another. In all these cases, it is worthwhile to check your answers to ensure the solution does not lie outside the required domain. In the case of a logarithmic function, everything inside the bracket must be greater than 0 due to the fact that $e^x > 0$ - refer to the function to check this, and the fact that no matter what index you think of, the value will never be negative.

So our restricted values are

$$\begin{aligned} 2x+1 &> 0 \\ x &> -\frac{1}{2} \end{aligned}$$

The first solution is clearly greater than $-\frac{1}{2}$ since both the denominator and numerator are greater than 0 and so must be postiive. The second solution is a bit harder to consider, but you should recognise that the numerator can be written as $-1+e^{-9}$ and thus is greater than -1. Therefore $\dfrac{e^{-9}-1}{2}$ is also greater than $-\frac{1}{2}$.

Question 5

a.

Writing the rule for a composite function is not dissimilar to merely substituting a value into a function, e.g. $f(2)$. Composite functions are commonly viewed as a machine with inputs and outputs.

For this question, you input a domain of values into $f(x)$ which outputs the range, $f(x)$. This range is now inputted into $g(x)$ and outputs a new range, $g(f(x))$. Take the time to read that sentence over and ensure you understand it.

There is one important thing to consider though - take our example of $f(2)$. We are inputting the value of $x = 2$ or in other words, we substituted x for 2. However, we would have needed to take into account whether 2 was in the domain of $f(x)$. If it is not, the function is undefined for that value of x. The same can be said of $g(f(x))$ where $f(x)$ is the argument. If there are values of $f(x)$ for which $g(f(x))$ is not defined, then we cannot use the whle range of $f(x)$. So we must now determine the values of $f(x)$ (i.e. its range) that lies within the domain of $g(x)$. This must occur for $g(f(x))$ to exist.

Hence the following law applies to composite functions,
For $g(f(x))$ to exist,

$$Ran_{f(x)} \subseteq Dom_{g(x)}$$

In this case $Ran_{f(x)} = [1, \infty]$ and $Dom_{g(x)} = [-\infty, \infty]$. Since the range of $f(x)$ is a subset of the domain of $g(x)$ the function, $g(f(x))$ exists for all values of $f(x)$.

Now that we know it exists with no restrictions, we only have to substitute the x in $g(x)$ for $f(x)$

$$\begin{aligned} g(f(x)) &= 3(e^{x^2})^2 + 1 \\ &= 3e^{2x^2} + 1 \end{aligned}$$

You will realise that in the model solutions, there is no consideration of whether or not the composite function exists - that is only because it is not a necessary part of the solution and the one mark available is only for the rule; however, if you find a composite function that doesn't exist, no marks would be rewarded. It is worth very quickly checking as to whether it exists - it is not unheard of for questions tricking students by asking for a composite function that doesn't exist. In such situations, you merely have to state that this is the case.

b.

For questions involving the finding of an inverse relation, there are commonly two methods taken to do so. The one most commonly used by students is simply swapping the x and y values and then solving for y for the new inverse.

However, there is a second one that students should definitely consider since it is seen as technically more mathematically correct. It is the utilisation of the inverse definition in that for a function, $f(x)$, $f((f^{-1}(x)) = x$, then solving for $f^{-1}(x)$. What students will find is that both methods are actually the same, but the latter prevents certain mathematical errors - also, in questions that ask for the "rule of the inverse", students may be penalised for leaving their answer as a $y =$ inverse in terms of x relation. In any case, either method should be fine as a means of finding the inverse.

For the purposes of this question, I have used the first method due to the clutter of terms in the question (and also to appease the majority of students who use this approach). Refer to the detailed solutions of Algebra Test 2 Question 1 for a use of the second method.

Let $y = 3e^{2x^2} + 1$. To find the inverse, we reflect in the $y = x$ line.
Therefore,

$$\begin{aligned} x &= 3e^{2y^2} + 1 \\ e^{2y^2} &= \frac{x-1}{3} \\ y^2 &= \frac{\log_e(\frac{x-1}{3})}{2} \\ y &= \pm\sqrt{\frac{\log_e(\frac{x-1}{3})}{2}} \end{aligned}$$

Thus, $g(f(x))^{-1} = \pm\sqrt{\frac{\log_e(\frac{x-1}{3})}{2}}$

This last line is necessary for the reason stated above - the question asks for the inverse function $g(f(x))^{-1}$ and thus leaving your answer as above as "$y =$" would be incorrect. There is no mention of "$y =$" in the question; it is a construction of the student, and the student must take effort into returning to what the question asks for.

c.

Typically, the easiest way to find the domain of the inverse is to simply find the range of the original function. However, in this case, our original function is not the easiest to assess for its range (and may not be questioned within the course), though students may be able to work it out relatively simply.

In this case, however, we will just focus on the inverse relation itself and determine the maximum domain it can take. Students should take the time to become familiar with all the possible ways to restrict domains. Certain functions will restrict you (most likely with an asymptote in a logarithmic or tangent function), and others can be seen more easily as natural within mathematics (e.g. for $\frac{1}{x}$, $x \neq 0$ or for $\sqrt{x}$, $x \geqslant 0$). In fact, it is good practice for a student to make a table of all the functions they know from the rational functions of x^{-1} to the transcendental functions like $\tan(x)$.

There are several that apply to this question.

Firstly, within the logarithmic bracket, we should know that

$$\begin{aligned} \frac{x-1}{3} &> 0 \\ x &> 1 \end{aligned}$$

Secondly, within the square root function, we should know that

$$\begin{aligned} \frac{\log_e(\frac{x-1}{3})}{2} &\geqslant 0 \\ \log_e(\frac{x-1}{3}) &\geqslant 0 \\ \frac{x-1}{3} &\geqslant e^0 \\ \frac{x-1}{3} &\geqslant 1 \\ x &\geqslant 4 \end{aligned}$$

We need to satisfy both of these restrictions, and thus we need the intersection of the two.

Therefore, the domain of the inverse is $[4, \infty)$

ALGEBRA

TECH-FREE TEST 2

DETAILED SOLUTIONS

Question 1

a.

In order to find the inverse of this function, we will make use of a helpful identity: $f(f^{-1}(x)) = x$. It is essentially the same as the method that involves swapping the x and y, but is often seen as a more valid (rigorous) method of finding the inverse (though for the purposes of this course, both are fine).

But first, some notes on fractional functions. In all questions, students should look for ways in which to simplify so that questions are easier. When it comes to fractional functions, students will find that when they simplify the fraction, it will then be in a recognisable form. In this question - the function is actually just a basic hyperbola.

So there are two questions students should have: 1) How do I know when I can simplify? and 2) How do I simplify the fraction?

Let us address the first one. There's a reason why we don't have a need to simplify the functions in the truncus or the hyperbola form, i.e. $\frac{1}{x}$ and $\frac{1}{x^2}$ - there is no common factor between the numerator or denominator. But what about $\frac{x}{x}$ and $\frac{x^2}{x}$? It's fairly obvious that these can be simplified to 1 and x respectively.

The general rule is that for $\frac{x^p - a}{x^q - b}$, when $p \geq q \geq 0$, the fraction should be simplified. In other words, if the variable term (in this case x) in the numerator has a higher or equal power to the term in the denominator, it is wise to simplify the fraction.

Now to the second question: how does one simplify the fraction? The method that is most straightforward is to use long division - but it is often a tedious process, especially for higher power functions. There is a very simple, intuitive method to tackle these simplifications which is shown below. Explanation is underneath.

$$\begin{aligned} f(x) &= \frac{x-3}{x+1} \\ &= \frac{(x+1)-4}{x+1} \\ &= 1 - \frac{4}{x+1} \end{aligned}$$

The aim is for students to change the numerator so that a portion of it is equal to the denominator. As you can see, the reason we do this is so that when we divide that part of the numerator with the denominator, we get a whole number. But we have to make sure that when we change the numerator it is still the same. Once you write down the expression you require (in this case the "$x + 1$"), you need to offset it with another value.

$$\begin{aligned} x - 3 &= (x+1) + a \\ a &= -4 \end{aligned}$$

From this point, it becomes much easier to find the inverse. If we didn't simplify there, we would have to simplify later anyway in a more awkward position.

As stated above, we are going to use the rule $f(f^{-1}(x)) = x$. Once we have substituted $f^{-1}(x)$ into our function, we simply have to solve for it to find our inverse.

$$\begin{aligned} f(f^{-1}(x)) &= x \\ 1 - \frac{4}{f^{-1}(x)+1} &= x \\ -\frac{4}{f^{-1}(x)+1} &= x - 1 \\ f^{-1}(x) + 1 &= -\frac{4}{x-1} \\ f^{-1}(x) &= -\frac{4}{x-1} - 1 \\ &= \frac{4}{1-x} - 1 \end{aligned}$$

b.

The easiest way to ensure we get the correct domain of the inverse is to realise that in swapping all the x values for y values, the properties also share this "swap". In this case we know the y-values, i.e the range, will become the x-values, i.e. the domain, of the inverse. The range of the original function is only restricted by the horizontal asymptote of $y = 1$. Hence, the range of $f(x)$ is given by $\mathbb{R} \setminus \{1\}$.

Therefore, the domain of the inverse function, $f^{-1}(x)$, is the range of the original function, $x \in \mathbb{R} \setminus \{1\}$.

Question 2

The first thing that should strike students when they try to solve this is the difference in bases. Students who try to solve the equation with the logs in their given form are likely to struggle greatly. However, a useful tactic to attacking a question is to think about the topic being tested and consider what available resources there are. In most cases, there wouldn't be more than 5 things within a topic that can be applied. So what can we do with logs? There are obviously a whole set of log laws available, such as $\log(ab) = \log(a) + \log(b)$ - most of these seem useless to this question. However, what was the alarming thing about this question? It was the differences in bases. There is in fact a rule within the course that changes the base of logarithms. Unfortunately, many students are uncomfortable memorising and/or utilising this rule.

But never fear, it's not actually very complicated. For example, say we want to change the base from "a" to "c" in the following logarithm, then we can write:

$$\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$$

Now let us apply this to our equation. You can change either of the logs in the question, but you'll find that changing the base x to 3 makes the solution a bit easier.

$$\log_x(3) = \frac{\log_3(3)}{\log_3(x)}$$

Once this has been done, the solution becomes a more ordinary solving question.

$$\begin{aligned}
\log_3(x) &= 4\log_x(3) \\
\log_3(x) &= 4\frac{\log_3(3)}{\log_3(x)} \\
(\log_3(x))^2 &= 4, \quad \text{since } \log_3(3) = 1 \\
\log_3(x) = 2 \quad &or \quad \log_3(x) = -2 \\
x = 3^2 \quad &or \quad x = 3^{-2} \\
x = 9 \quad &or \quad x = \frac{1}{9}
\end{aligned}$$

As always, ensure you read the question carefully in order to be rewarded all the marks available. In many cases, a small sentence added to a question will provide the opportunity for an easy mark or two. In this question, we are required to justify the validity of our solution. For a logarithmic function, we just have to ensure that values within the function are greater than 0. Hence the x-values are limited to all real numbers greater than 0. Hence, both solutions are valid.

Question 3

Students are generally not accustomed to finding when an equation does **not** have a solution. It is more common to see a question looking for an intersection or a solution. However, a question asking for when something doesn't have a solution will still require the same sort of methodology. You'll still need to solve an equation. In this case, we want to find where there are no intersections of $f(x)$ and $g(x)$. However, the most efficient way to solve this question is to simply find when there are actual intersections. This is because we'll have a set equation to solve. Once we find the x-values for which there are intersections, we can then say that there are no intersections for every other value of x.

$f(x)$ and $g(x)$ intersect when,

$$\begin{aligned}
f(x) &= g(x) \\
\frac{1}{x} - 3 &= -ax \\
ax + \frac{1}{x} - 3 &= 0
\end{aligned}$$

The expression on the left-hand side is a hidden quadratic. The detailed solutions for Question 3 in Algebra Test 1 has a sidenote at the end that discusses this type of hidden quadratic. If we multiply both sides of the equation by x we will achieve a quadratic.

$$\begin{aligned}
x \times \left[ax + \frac{1}{x} - 3\right] &= x \times 0 \\
ax^2 - 3x + 1 &= 0
\end{aligned}$$

We now have to question when there is no solution to this equation - this will tell us when there is no intersection. Let us continue and try to solve this equation using the quadratic formula.

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(a)(1)}}{2a}
\end{aligned}$$

When does this expression not have a solution? It has to do with the square root sign. You should realise that if what is under the square root sign is less than 0, then x will not exist - i.e. there will be no solution. Does this sound familiar? It should, because under the square root sign is "$b^2 - 4ac$", and $\triangle = b^2 - 4ac$.

Students should understand that this is where our discriminant comes from. Understanding gives context to the rules often associated with it. Namely,

$$\begin{aligned}\text{Two solutions when } \triangle &> 0 \\ \text{One solution when} \triangle &= 0 \\ \text{No real solutions when } \triangle &< 0\end{aligned}$$

There are two solutions because the square root in the quadratic exists and does not equal 0, therefore the "±" accounts for two solutions. There is one solution when the discriminant equals 0 so the "±" does not have any impact. And lastly no solution when the discriminant is less than 0 since the whole expression will be undefined (for real values anyway).

Returning to the question, we will find the discriminant.

$$\begin{aligned}\triangle &= (-3)^2 - 4a(1) \\ & \quad 9 - 4a\end{aligned}$$

There is no solution when $\triangle < 0$.

$$\begin{aligned}\triangle &< 0 \\ 9 - 4a &< 0 \\ a &> \frac{9}{4}\end{aligned}$$

There is no solution when $a > \frac{9}{4}$

Question 4

Refer to Question 3 in the detailed solutions of Algebra Test 1 for notes on addressing these types of "show that" questions. In this case, the right-hand side is self-explanatory and we merely need to show that the left-hand side reaches the result desired.

However, this question is perhaps a little less simplistic in the sense that steps do not easily follow each other logically. Where do we even begin? What approach should we take?

We could try to show that $(a + 3b)^2 \geq 12ab$ though this is perhaps not the best way to go about things.

Sometimes, a completely valid method is to just manipulate expressions in the most obvious ways - a bit of an exploration. It's useful (and still used in higher level mathematics) because you may happen to stumble across something that is recognisable. These kinds of questions require a bit of intuition and exploration - the question doesn't dictate how to find what you're looking for.

$$\begin{aligned}(a + 3b)^2 - 12ab &= a^2 + 6ab + 9b^2 - 12ab \\ &= a^2 - 6ab + 9b^2\end{aligned}$$

This line should hopefully be familiar - it's not necessarily easy to see, but if you tried to factorise this expression, you would see that it is a perfect square.

$$a^2 - 6ab + 9b^2 = (a - 3b)^2$$

It is very clearly stated in the question that all steps must be shown and justified. To leave the last line as it is, or simply state that it is greater than 0 is not sufficient. You must show some understanding of why $(a - 3b)^2$ is greater than 0 for any value of a and b - it is not a given. There are severals ways to go about this:

If we consider the function $y = x^2$, we know that for any x, $y \geq 0$. Alternatively, simply knowing that it is a perfect square is enough indication that it is greater than 0.

In any case, we have now shown that

$$(a+3b)^2 - 12ab = a^2 - 6ab + 9b^2 = (a-3b)^2 \geq 0$$

Therefore, $(a+3b)^2 - 12ab \geq 0$ as required.

Question 5

Refer to Question 4 in the detailed solutions of Algebra Test 1 for notes on how to separate absolute value functions into their appropriate domains.

$$|x-1|^2 - 2|x-3| \geq 3$$

There are two sets of absolute values to contend with here. In most cases you would end up with a big mess of intersecting sets trying to show the domains for which certain solutions exist. You would split one of the absolute values into two domains and then split the other into within those domains - possibly 4 sets of domains to deal with.

However, for this question, you are in luck if you can recognise that $|x-1|^2 = (x-1)^2$.

This result comes from

$$\begin{aligned} |x-1\,|^2 &= \begin{cases} (-(x-1))^2 & x \leq 1 \\ (x-1)^2 & x > 1 \end{cases} \\ &= \begin{cases} (x-1)^2 & x \leq 1 \\ (x-1)^2 & x > 1 \end{cases} \\ &= (x-1)^2,\ x \in \mathbb{R} \end{aligned}$$

Therefore, there is only one absolute value in the equation!

$$(x-1)^2 - 2|x-3| \geq 3$$

We can now proceed to split the function of $|x-3|$ and solve the equation for when either $x-3 \geq 0$ or $x-3 < 0$.

For $x-3 \geq 0$, $|x-3| = x-3$

$$\begin{aligned} (x-1)^2 - 2(x-3) &\geq 3 \\ x^2 - 2x + 1 - 2x + 6 - 3 &\geq 0 \\ x^2 - 4x + 4 &\geq 0 \\ (x-2)^2 &\geq 0 \\ x &\in \mathbb{R} \end{aligned}$$

We need to find the intersection of this result and $x \geq 3$. This is because we've found a solution of x for when $x-3 \geq 0$ and thus the solutions must lie within this domain - we disregard all the solutions outside this given domain. The intersection of $x \in \mathbb{R}$ and $x-3 \geq 0$ is $x \geq 3$.

And now we can do the same for the other domain.

For $x - 3 < 0$, $|x - 3| = -(x - 3)$

$$\begin{aligned}
(x-1)^2 - 2(-(x-3)) &\geq 3 \\
x^2 - 2x + 1 + 2x - 6 - 3 &\geq 0 \\
x^2 - 8 &\geq 0 \\
x^2 &\geq 8 \\
x \geq 2\sqrt{2} \quad \text{and} &\quad x \leq -2\sqrt{2}
\end{aligned}$$

The intersection of this result and $x < 3$ is $x \leq -2\sqrt{2}$ or $2\sqrt{2} \leq x < 3$.

Now after all of this, we have to put everything together. We have two sets of results which have been shown to be correct, we only need to merge them together. The sets we have are, $x \leq -2\sqrt{2}$, $2\sqrt{2} \leq x < 3$ and $x \geq 3$. The last two can be put together on a number line to be $x \geq 2\sqrt{2}$.

Hence, the values of x that satisfy the equation are $x \geq 2\sqrt{2}$ and $x \leq -2\sqrt{2}$.

ALGEBRA

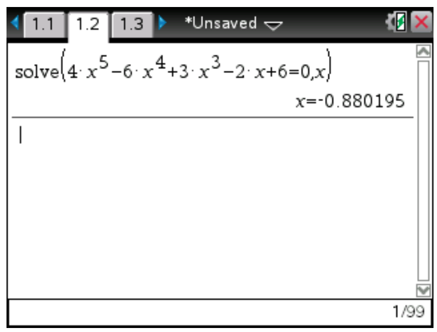
TECH-ACTIVE TEST 1

DETAILED SOLUTIONS

SECTION 1 - Multiple-Choice Questions

Question 1 (E)

The best way to go about this question is to plug this into CAS; this is a quintic equation that we cannot solve by hand. Besides, it is much easier to use CAS to solve equations of any complexity. To solve this equation using CAS, we do the following:



Note that to access the 'solve' function, you press [Menu] [3] [1] for TI-nSpire CX CAS and older models. If you want approximate solutions, like the above, you press [Ctrl] [Enter] after you type the function in.

We end up getting:

$$\begin{aligned} f(x) &= 4x^5 - 6x^4 + 3x^3 - 2x + 6 = 0 \\ x &= -0.88 \end{aligned}$$

Hold up - the domain of the function is specified as $\mathbb{N}$. In other words, x must be a natural number (i.e. a positive integer). Since -0.88 is not an integer, let alone a positive integer, this is an extraneous solution. Thus, there are no solutions to $f(x) = 0$ and so the answer is **E**.

Question 2 (E)

Let us plug each option given into the functional equation:

Option **A:** $f(x) = -2x + 1$. This means that $f(y) = -2y + 1$.

Remember that $f(x) = -2x+1$ simply means that if you 'f' a number (no pun intended), you are simply multiplying by -2 and adding 1. So if you 'f' y, you will get $-2y + 1$.

Substituting this into the functional equation:

$$\begin{aligned}
\text{LHS} &= \frac{f(x)+f(y)}{f(x)-f(y)} \\
&= \frac{-2x+1-2y+1}{-2x+1+2y-1} \\
&= \frac{-2x-2y+2}{-2x+2y} \\
&= \frac{-x-y+1}{-x+y}
\end{aligned}$$

This expression is certainly not equal to RHS $= 1$ for all x and y. Hence, option **A** is incorrect.

Option **B:** $f(x) = 0$

This means that $f(y) = 0$.

Note that the denominator on the LHS is $f(x) - f(y)$. Given this cannot be equal to zero, $f(x) \neq f(y)$. But since $f(x) = f(y) = 0$, we have a contradiction. Hence, $f(x) = 0$ is an incorrect solution.

Option **C:** $f(x) = x^2 \Longrightarrow f(y) = y^2$.

Hence,

$$\begin{aligned}
\text{LHS} &= \frac{f(x)+f(y)}{f(x)-f(y)} \\
&= \frac{x^2+y^2}{x^2-y^2}
\end{aligned}$$

The above expression is certainly not equal to 1 for all x and y. Hence, option **C** is incorrect.

Option **D:** $f(x) = e^{2x} \Longrightarrow f(y) = e^{2y}$.

Hence,

$$\begin{aligned}
\text{LHS} &= \frac{f(x)+f(y)}{f(x)-f(y)} \\
&= \frac{e^{2x}+e^{2y}}{e^{2x}-e^{2y}}
\end{aligned}$$

The above expression is certainly not equal to 1 for all x and y. Hence, option **D** is incorrect.

Since none of the above expressions are correct, option **E** must be the correct answer.

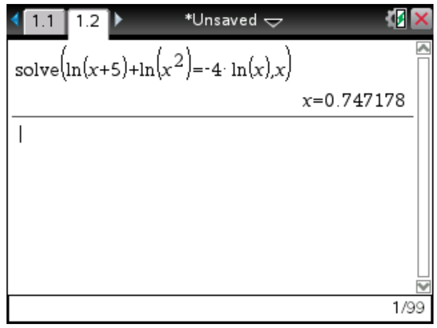
Another way of doing this is by determining conditions for $f(x)$ by manipulating the functional equation:

$$\begin{aligned}
\frac{f(x)+f(y)}{f(x)-f(y)} &= 1 \\
\Longrightarrow f(x)+f(y) &= f(x)-f(y) \\
\Longrightarrow 2f(y) &= 0 \\
\Longrightarrow f(y) &= 0
\end{aligned}$$

Hence, **if** there were a solution, it **must** be $f(x) = 0$. However, since $f(x) \neq f(y)$, yet $f(x) = 0 \Longrightarrow f(x) = f(y) = 0$, we have a contradiction. Thus, $f(x) = 0$ is not a valid answer and hence there is no solution. In other words, option **E** must be the correct answer.

Question 3 (D)

This question is best solved using CAS; you could manipulate it by hand and use CAS to get the answer, but there really is no point. Plugging it into CAS as shown:



Note that to access the 'solve' function, you press [Menu] [3] [1] for TI-nSpire CX CAS and older models. If you want approximate solutions, like the above, you press [Ctrl] [Enter] after you type the function in. We get $x = 0.74$, and hence, the answer is **D**. You could manipulate the expression by hand to get:

$$\begin{aligned}\log_e(x+5) + \log_e(x^2) &= -4\log_e(x) \\ \log_e\left(x^2(x+5)\right) &= \log_e\left(\frac{1}{x^4}\right) \\ x^2(x+5) &= \frac{1}{x^4} \\ x^6(x+5) &= 1\end{aligned}$$

Solving by CAS you can get:

$$x = -5.00\text{, -0.78 and 0.74}$$

However, remember that for $\log_a(\text{stuff})$ to be defined, 'stuff' must be larger than 0. Hence, for $\log_e(x)$ to be defined, x must be larger than 0. Hence, the only valid solution is $x = 0.74$.

Question 4 (E)

Let us have a look at these equations:

$$\begin{aligned}4x - my &= 6 \\ nx + 3y &= 7\end{aligned}$$

For these equations to have an infinite number of solutions, the two equations must be equivalent. In other words, if we multiply the first equation by a factor, we should be able to spit out the second equation. Hence, since the first equation has a '6' on the RHS and the second has a '7', this factor must be $\frac{7}{6}$. Therefore, multiplying the first equation by $\frac{7}{6}$:

$$\frac{14}{3}x - \frac{7}{6}my = 7$$

The above equation and the second equation must be equivalent. Hence:

$$\begin{aligned}-\frac{7}{6}m &= 3 \\ \implies m &= -\frac{18}{7}\end{aligned}$$

Thus, the answer is **E**.

Question 5 (A)

Recall that a domain is basically the set of all values that we can plug into the function for the function to be defined. Now, what conditions need to be satisfied such that $f(x) = \log_e(\sqrt{3} + 2\sin(3x-2))$ can be defined? Recall that for $\log_a(\text{stuff})$ to be defined, 'stuff' must be larger than 0. This means that we know that

$$\begin{aligned} \sqrt{3} + 2\sin(3x-2) &> 0 \\ \sin(3x-2) &> -\frac{\sqrt{3}}{2} \end{aligned}$$

This condition need be satisfied. Now, let us look at each option:

Option **A:** $\left[\frac{2}{3}, \frac{\pi+2}{3}\right)$

If $x \in \left[\frac{2}{3}, \frac{\pi+2}{3}\right)$, this implies that $3x-2 \in [0, \pi)$.

The question is now - given that $3x-2 \in [0,\pi)$, is $\sin(3x-2) > -\frac{\sqrt{3}}{2}$ true for all values within that domain?

As $3x-2$ increases from 0 to $\frac{\pi}{2}$, $\sin(3x-2)$ increases from 0 to 1. As $3x-2$ increases from $\frac{\pi}{2}$ to π, $\sin(3x-2)$ decreases from 1 back to 0.

This means that the range of $\sin(3x-2)$ is [0,1]. Hence, it is true that $\sin(3x-2) > -\frac{\sqrt{3}}{2}$ for all values of $3x-2$ as specified in the domain. Hence, option **A** is the correct answer.

Question 6 (D)

For a function to have an inverse, it must be one-to-one. For a cubic, this would mean that there cannot be any turning points. Hence, there are either no stationary points, or there is only a stationary point of inflection. Now, to determine the values of a for which this is the case, we need to see how the gradient of the tangent to the function behaves over the values of x. Thus, we should differentiate $f(x)$:

$$f'(x) = 6x^2 + 36x + a$$

Remember that $f'(x)$ must not equal 0 for all values of x, or if $f'(x) = 0$ at a particular value, it must be at a stationary point of inflexion (which can be determined by using a sign diagram).

In other words, the discriminant must be less than or equal to 0. Why can the discriminant be equal to zero?

If it were equal to zero, this means that $y = f'(x)$ will have only one x-intercept, and it will be such that $f'(x) > 0$ for all other x. Hence, the gradient of the tangent to $y = f(x)$ would be greater than 0 on both sides of the stationary point. In other words, this would be a stationary point of inflexion.

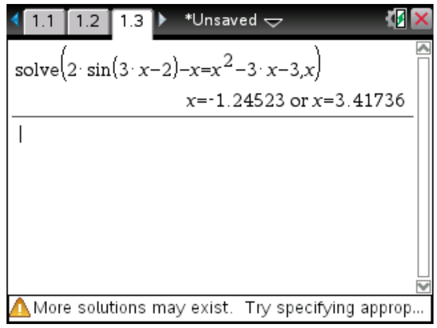
Hence,

$$\begin{aligned} \Delta &= 1296 - 24a \leq 0 \\ a &\geq 54 \end{aligned}$$

Thus, the answer is **D**.

Question 7 (B)

There is no method by which you can solve this equation by hand. The only way to do this question is to use CAS as per the following:



Note that to access the 'solve' function, you press [Menu] [3] [1] for TI-nSpire CX CAS and older models. If you want approximate solutions, like the above, you press [Ctrl] [Enter] after you type the function in.

You get $x = -1.25$ or $x = 3.42$. Hence, **B** is the correct answer.

Question 8 (C)

Seeing as two of the options express y in terms of x, it would be a good idea to do exactly this. We start with:

$$2x = e^y + e^{-y}$$

We need to find y in terms of x. First interesting point is that each term in the equation has an $e^{something}$ term (if we see $2x$ as $2xe^0$); what is more is that the exponents all have the same difference - the exponents are y, 0 and $-y$. Why not make a quadratic out of this? It will be more clear if we multiply both sides by e^y:

$$\begin{aligned} 2xe^y &= e^{2y} + 1 \\ e^{2y} - 2xe^y + 1 &= 0 \\ (e^y)^2 - 2x(e^y) + 1 &= 0 \end{aligned}$$

Now, this is a quadratic with e^y as the term of interest. Solving for e^y,

$$\begin{aligned} e^y &= \frac{(2x) \pm \sqrt{(2x)^2 - 4 \times 1 \times 1}}{2} \\ &= \frac{2x \pm \sqrt{4x^2 - 4}}{2} \\ &= \frac{2x \pm 2\sqrt{x^2 - 1}}{2} \\ &= x \pm \sqrt{x^2 - 1} \end{aligned}$$

Solving for y,

$$y = \log_e\left(x \pm \sqrt{x^2 - 1}\right)$$

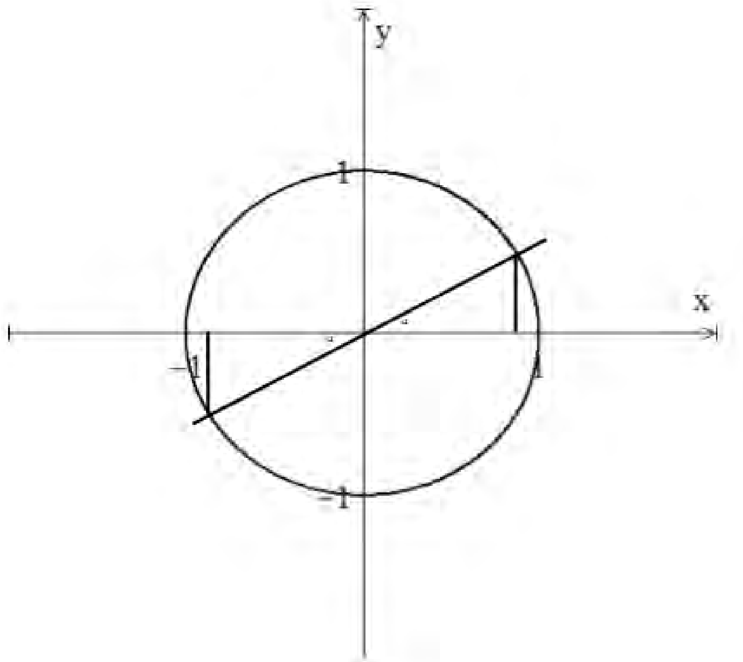
Hold up - that's option **C**. Hence, option **C** is the correct answer.

Question 9 (C)

Let us solve for the general solution:

$$\begin{aligned} 2\tan(2x) &= 1 \\ \tan(2x) &= \frac{1}{2} \end{aligned}$$

Now, what values can $2x$ take? Remember that $\tan(\text{stuff}) = \dfrac{1}{2}$ means that 'stuff' must be the angle subtended by the x-axis and a line with gradient $\dfrac{1}{2}$. Drawing this line on the unit circle:



Note that the angle a is the same on both the first and third quadrants as they are vertically opposite angles.

Hence, to depict a general expression for all angles, we can see that we merely need only add a to a multiple of π, where $a = \tan^{-1}\left(\dfrac{1}{2}\right)$. In other words,

$$\begin{aligned} 2x &= n\pi + \tan^{-1}\left(\frac{1}{2}\right), n \in \mathbb{Z} \\ x &= \frac{1}{2}\left(n\pi + \tan^{-1}\left(\frac{1}{2}\right)\right), n \in \mathbb{Z} \end{aligned}$$

This is equivalent to option **C**.

Question 10 (D)

Let us try out each option in turn:

Option **A:** $f(f(x)) = x$

$$\begin{aligned} f(x) &= 2x^n \\ f(f(x)) &= 2(2x^n)^n \\ &= 2^{n+1}x^{n^2} \end{aligned}$$

Is the following possible?

$$2^{n+1}x^{n^2} \quad = \quad x$$

If yes, then $n^2 = 1 \Longrightarrow n = 1$ (given $n > 0$). If this were the case, then $2^{n+1}x^{n^2} = 4x$. This is not equal to x for all values of x. Hence, option A is false.

Option **B:** $f(xy) = f(x) + f(y)$

$$\begin{aligned} \text{LHS} &= f(xy) \\ &= 2(xy)^n \\ &= 2x^n y^n \\ \text{RHS} &= 2x^n + 2y^n \end{aligned}$$

Can LHS = RHS for all x and y for a particular value of n? No - you can easily substiutue an ordered pair (x, y) such that LHS $\neq$ RHS.

Option **C:** $f(x + y) = f(x) + 2f(y)$

$$\begin{aligned} \text{LHS} &= f(x+y) \\ &= 2(x+y)^n \\ \text{RHS} &= f(x) + 2f(y) \\ & \quad\ 2x^n + 2 \times 2y^n \\ &= 2x^n + 4y^n \end{aligned}$$

This is most definitely not true for any positive values of n for all x, y.

Option **D:**$f(f(f(f(x)))) = 2^{40}x^{81}$

This should be fun.

$$\begin{aligned} f(x) &= 2x^n \\ f(f(x)) &= 2(2x^n)^n \\ &= 2^{n+1}x^{n^2} \\ f(f(f(x))) &= 2(2^{n+1}x^{n^2})^n \\ &= 2^{n^2+n+1}x^{n^3} \\ f(f(f(f(x)))) &= 2(2^{n^2+n+1}x^{n^3})^n \\ &= 2^{n^3+n^2+n+1}x^{n^4} \end{aligned}$$

Can this be possible:

$$2^{n^3+n^2+n+1}x^{n^4} \quad = \quad 2^{40}x^{81}$$

If this were the case, then $n^4 = 81 \Longrightarrow n = 3$. Hence, the only possible value for LHS that includes an x^{81} term is $2^{3^3+3^2+3^1+1}x^{81} = 2^{40}x^{81}$. Hence, option **D** is correct.

Question 11 (E)

This looks suspiciously like a very familiar cubic...

$$\begin{aligned}
\sin^3(x) - 3\sin^2(x) + 3\sin(x) &= 1 \\
\sin^3(x) - 3\sin^2(x) + 3\sin(x) - 1 &= 0 \\
(\sin(x) - 1)^3 &= 0 \\
\sin(x) - 1 &= 0 \\
\sin(x) &= 1
\end{aligned}$$

Now, there are an infinite number of solutions for x. Hence, option **E** is the correct option.

SECTION 2 - Extended-Response Questions

Question 1

Part a.

A point $(x, f(x))$ is a stationary point if $f'(x) = 0$. Hence, to determine the number of stationary points in this function, we need to differentiate the function:

$$f'(x) = 4x^3 - 9x^2 - 12x + p$$

Now, the new question is - what is the minimum number of x-intercepts of the graph of $y = f'(x)$? This is a positive cubic function, where y approaches infinity when x does, and y approaches negative infinity as x does. This means, given this function is continuous, that the graph **must** cross the x-axis at least once. Is it possible to only have one x-intercept in this function? Certainly; increase p to an extremely high value (equivalent to a vertical translation), such that both stationary points of the graph $y = f'(x)$ are above the x-axis. Hence, the answer is **1.**

Part b.

Now, for $f(x)$ to only have one stationary point, the graph $y = f'(x)$ must only have one x-intercept. This means that both turning points must be above the x-axis or both below. Hence, we need to determine the x-coordinates of the turning points of the graph $y = f'(x)$. For clarity's sake, let us define $a(x) = f'(x)$. Hence,

$$a(x) = 4x^3 - 9x^2 - 12x + p$$

Determining the turning points of $y = a(x)$ by taking its derivative,

$$\begin{aligned} a'(x) &= 12x^2 - 18x - 12 = 0 \\ x &= -\frac{1}{2}, 2 \text{ (via CAS)} \end{aligned}$$

Let us take the two cases:

Case 1: both turning points of $y = a(x)$ are above the x-axis; in other words $a(2) > 0$ (as $a(2) < a\left(-\frac{1}{2}\right)$ from the shape of the graph)

$$\begin{aligned} a(2) &> 0 \\ 32 - 36 - 24 + p &> 0 \\ p &> 28 \end{aligned}$$

Case 2: both turning points of $y = a(x)$ are below the x-axis; in other words $a\left(-\frac{1}{2}\right) < 0$ (as $a(2) < a\left(-\frac{1}{2}\right)$ from the shape of the graph)

$$\begin{aligned} a\left(-\frac{1}{2}\right) &< 0 \\ -\frac{1}{2} - \frac{9}{4} + 6 + p &< 0 \\ p &< -\frac{13}{4} \end{aligned}$$

Also note that q can be any value, and its value did not affect the number of stationary points the graph has. Hence, $q \in \mathbb{R}$.

Thus, all ordered pairs (p, q) such that $p \in \left(-\infty, -\frac{13}{4}\right) \cup (28, \infty)$ and $q \in \mathbb{R}$ are valid.

Part c.

Seeing as the gradient of the function is always negative - the function is constantly decreasing, $a < 0$.

The second characteristic of the function is that it has no stationary points. In other words, there exist no values of x such that $f'(x) = 0$. To find some more information, let us differentiate the function:

$$f'(x) = 3ax^2 + 12x - c = 0$$

How do we determine whether a quadratic equation has no solutions? Discriminant - which must be less than 0.

$$\begin{aligned} \Delta &= 144 + 12ac < 0 \\ ac &< -12 \end{aligned}$$

However, the question asked for c to be the subject. Now, note that a is negative. Hence, dividing both sides by a means that the sign changes:

$$c > -\frac{12}{a}$$

There is the relationship. Now, what possible values can c take? Given that a is negative, $-\dfrac{12}{a}$ must be positive. In fact, the minimum value of $-\dfrac{12}{a}$ is when a approaches negative infinity, that being zero. Hence, $c > 0$. Given that this condition is independent of the value of d, we can say that $d \in \mathbb{R}$.

Question 2

Part a.

Show that. Show that. Any 'show that' questions - you have to spell out each step very, very clearly. Note that you have to somehow change the base. However, the question itself gives you a clue. Let us have a look at the LHS and the RHS:

$$\begin{aligned} p(x) &= 5^x \\ q((\log_6(5))(x)) &= 6^{(\log_6(5))(x)} \end{aligned}$$

Hold up. The second expression - have a look at the $6^{\log_6(5)}$ part. This looks like you are performing a function and its inverse simultaneously (exponential and logarithmic functions are inverse). If you perform a function and its inverse on a number...you get that number itself. In fact, $6^{\log_6(5)} = 5$. So, let us feel our way from the LHS to the RHS:

$$\begin{aligned} p(x) &= 5^x \\ &= \left(6^{\log_6(5)}\right)^x \end{aligned}$$

Okay, we've changed our base. Brilliant. Now, how do we get to the RHS? Note the index law that $(a^b)^c = a^{bc}$. We can use it here!

$$\begin{aligned} p(x) &= 5^x \\ &= \left(6^{\log_6(5)}\right)^x \\ &= 6^{(\log_6(5))x} \\ &= q((\log_6(5))x) \end{aligned}$$

Part b.

Okay, let us have a look at $f(x)$ and $g(x)$.

$$f : \mathbb{R} \to \mathbb{R}, f(x) = 2^{x+4}$$

$$g : \mathbb{R} \to \mathbb{R}, g(x) = 2e^{x+3} - 5$$

We need a transformation from $f(x)$ to $g(x)$. Now, the first thing to get out of the way is to change the base. From the last step, changing the base also involves multiplying the exponent by a factor. If we were to do this immediately, then we would get:

$$2^{x+4} \quad \to \quad e^{(\log_e(2))(x+4)}$$

Now, we would have to change the exponent to $x + 3$. Given that if we do transformations this way, we change **only** the 'x' with every transformation. It would be extremely difficult to transform this to $g(x)$.

Let us do it an easier way. Let's make it cleaner by first performing a **translation 4 units in the positive direction of the x-axis:**

$$2^{x+4} \quad \to \quad 2^{x}$$

Now, let us change the base:

$$2^{x} \quad = \quad e^{(\log_e(2))x}$$

We need to get rid of the coefficient of x by performing a **dilation by factor** $\frac{1}{\log_2(e)}$ **from the y-axis:**

$$e^{(\log_e(2)x)} \quad \to \quad e^{x}$$

Now, transforming from here is very simple. Let us perform a **dilation by factor 2 from the x-axis:**

$$e^{x} \quad \to \quad 2e^{x}$$

And now a **translation 3 units in the negative direction of the x-axis:**

$$2e^{x} \quad \to \quad 2e^{x+3}$$

And now a **translation 5 units in the negative direction of the y-axis:**

$$2e^{x} \quad \to \quad 2e^{x+3} - 5$$

Part c.

Deduce a translation. **A translation**. One translation. That's all you are allowed. Let's have a look at the two functions:

$$r : \mathbb{R} \to \mathbb{R}, r(x) = e^{x+4}$$

$$s : \mathbb{R} \to \mathbb{R}, s(x) = 2e^{x-3}$$

At first glance, you seem to require a dilation. But that's not allowed. You have to somehow express $s(x)$ without that coefficient. Consider these index laws:

$$a^{x} \times a^{y} \quad = \quad a^{xy}$$

Hmm. How about if we can express 2 in terms of $e^{something}$? This is certainly possible:

$$2 \quad = \quad e^{\log_e(2)}$$

Back to $s(x)$:

$$\begin{aligned} s(x) &= 2e^{x-3} \\ &= e^{\log_e(2)} \times e^{x-3} \\ &= e^{x-3+\log_e(2)} \end{aligned}$$

Brilliant! We can, by inspection, derive the following transformation:

- **translation $7 - \log_e 2$ units in the positive direction of the x-axis**

Part d.

Let us have a look at the two functions again:

$$a : (0, \infty) \to \mathbb{R}, a(x) = \log_e(x)$$

$$b : S \to \mathbb{R}, b(x) = \log_e\left(\frac{x^6}{3}\right)$$

Given we need at least one translation, we need to express $b(x)$ as something involving a '+' or a '-' term. Now, how do we manipulate $b(x)$? For starters, having an x^6 term makes it impossible to transform from $a(x)$ (at least, within the scope of the course). Hold on, we can use familiar log laws to get past this:

$$\begin{aligned} b(x) &= \log_e(x^6) - \log_e(3) \\ &= 6\log_e(x) - \log_e(3) \end{aligned}$$

Ah, we killed two birds with one stone. Now, we can just, by inspection, determine the transformations:

- **dilation factor of 6 from x-axis**
- **translation $\log_e(3)$ units in the negative direction of the y-axis**

Part e.

What domain is valid here? Remember that domains are transformed just as the function itself is. For instance, if the domain of the original function is [0, 1], a translation of the function 3 units in the positive direction of the x-axis yields a domain of [3, 4]. Vertical translations or dilations from the x axis do not change the domain - you can visualise this. From the above, the two transformations are a vertical translationand a dilation from the x-axis. Hence, the domain of the new function is the same as that of the original - that being $(0, \infty)$.

Question 3

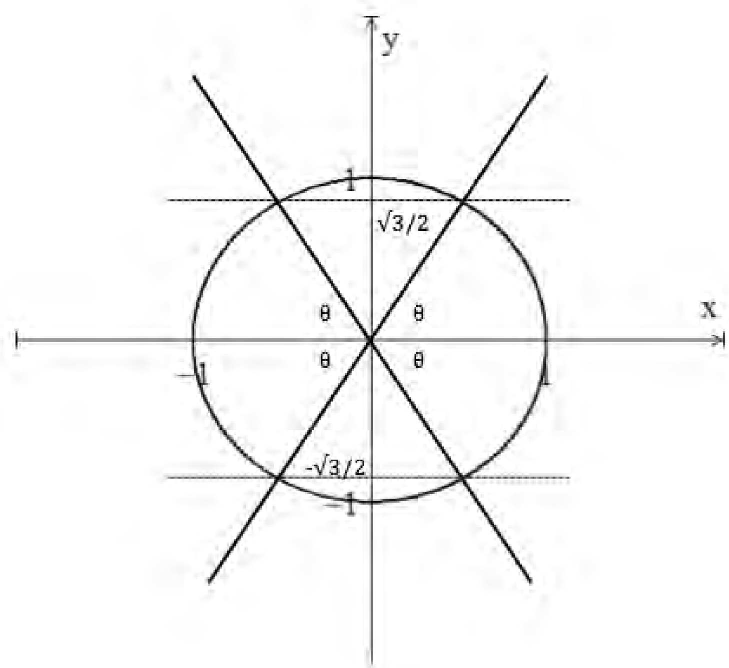
Part a.

The first step here is to get rid of that annoying modulus sign. Remember that if $|a| = b$, $a = \pm b$. Similarly:

$$\begin{aligned} |\sin(\log_e(2x+5))| &= \frac{\sqrt{3}}{2} \\ \sin(\log_e(2x+5)) &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

Now, given this, what can $\log_e(2x+5)$ equal? What the above statement is saying is this: if $\log_e(2x+5)$ were an angle, it would be the angle between the x-axis and a line from the origin to a point on the unit circle such that the y-coordinate of the intersection point is $\frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$.

The best way to do this is to visualise:



From exact values, you should remember that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$. Hence, $\theta = \frac{\pi}{3}$.

How do we express a general solution? Can you see that we can simply take multiples of π and add or subtract $\frac{\pi}{3}$ to get those intersection points? Hence, the next step would be:

$$\begin{aligned} \log_e(2x+5) &= n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z} \\ 2x+5 &= e^{n\pi \pm \frac{\pi}{3}}, n \in \mathbb{Z} \\ x &= \frac{1}{2}\left(e^{n\pi \pm \frac{\pi}{3}} - 5\right), n \in \mathbb{Z} \end{aligned}$$

Part b.

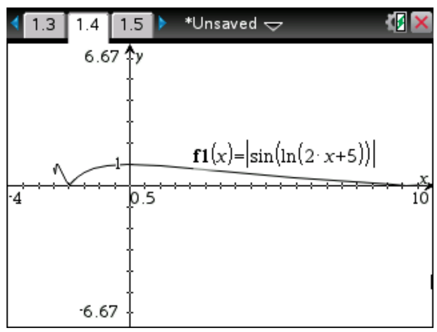
Now, if we have a function $y = \sin(\text{stuff})$, we know that 'stuff' can be any real number. Let us have a look at 'stuff' - $\log_e(2x+5)$.

Recall that $\log_e(\text{stuff})$ is only defined if 'stuff' > 0. Hence, we know that $2x+5 > 0$. Thus, $x > -\dfrac{5}{2}$. In other words, the domain is $\left(-\dfrac{5}{2}, \infty\right)$.

Part c.

This question is extremely tough and requires some degree of complex reasoning.

Before we embark on this question, it is a very good idea to get some idea as to the shape of the graph. To do this, you can use the Graph function and type in the following entry:

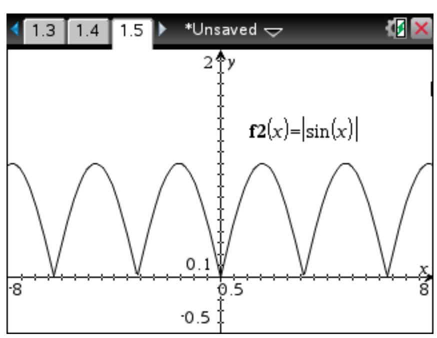


Note that if you want to enter '$|f(x)|$', you type in 'abs($f(x)$)'.

Looking at the graph, it does indeed seem that the stationary points are skewed to the left rather than central. The question is why.

Compare these graphs - $f(x) = |\sin(\log_e(2x+5))|$ (and let us define) $g(x) = |\sin(x)|$.

Using CAS, let us have a look at the graph $y = g(x)$:

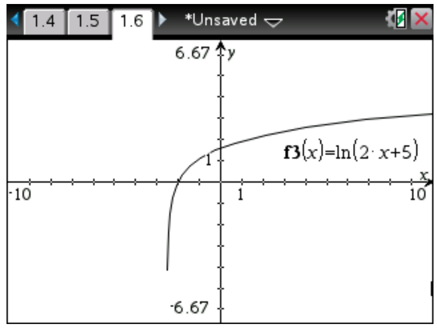


See how the x-coordinates of the stationary points are right in between the x-intercepts?

Now, you have to realise that $y = f(x)$ is really a distorted version of $y = g(x)$. Why do these graphs resemble each other? Because $y = f(x)$ is a composite function of $y = \sin(x)$ and $y = \log_e(2x+5)$. Let $h(x) = \log_e(2x+5)$, so $f(x) = g(h(x))$.

So we are now comparing $y = g(x) = \sin(x)$ and $y = g(h(x)) = \sin(\log_e(2x + 5))$.

Let us graph $y = h(x)$.



Note that the graph of $y = f(x) = g(h(x))$ involves substituting the y-values of the above graph $(h(x))$ into $g(x)$ - to form $g(h(x))$.

The interesting thing to pick up is that for the graph of $y = h(x)$, the gradient of the tangent decreases as x increases.

Now, let's have a look at a segment of $y = h(x)$ - from x_1 to x_2. The relevant coordinates are $(x_1, h(x_1))$ and $(x_2, h(x_2))$. Now, let us have a look at the point where $h(x) = \dfrac{h(x_0) + h(x_1)}{2}$. Why this point? Because we know that $f(x_1) = g(h(x_1)) = 0$ and $f(x_2) = g(h(x_2)) = 0$. Knowing that the x-coordinate of a stationary point for $y = g(x)$ lies between the x-coordinates of the x-intercepts, we can say that there will be a stationary point in between $h(x_1)$ and $h(x_2)$ - in other words, the y-coordinate of the stationary point would be $g\left(\dfrac{h(x_1) + h(x_2)}{2}\right)$. To find the x-coordinate, you have to find when $f(x) = g(h(x)) = g\left(\dfrac{h(x_1) + h(x_2)}{2}\right)$ - in other words, when $h(x)$ $= \dfrac{h(x_1) + h(x_2)}{2}$.

Given the curvature of the above graph (see how it curves down?), you can deduce from the graph that $h(x) = \dfrac{h(x_1) + h(x_2)}{2}$ when $x < \dfrac{x_1 + x_2}{2}$. This means that the stationary point $(x_s,\ f(x_s))$ for the graph between the x-values x_1 and x_2 would be when $x < \dfrac{x_1 + x_2}{2}$, so $x_s < \dfrac{x_1 + x_2}{2}$. This means that the stationary point will be to the left of $\left(\dfrac{x_1 + x_2}{2}, f\left(\dfrac{x_1 + x_2}{2}\right)\right)$.

ALGEBRA

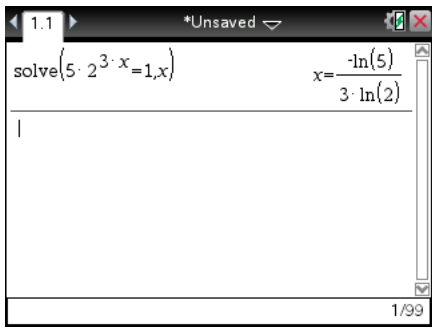
TECH-ACTIVE TEST 2

DETAILED SOLUTIONS

SECTION 1 - Multiple-Choice Questions

Question 1 (E)

We can answer this question either using a CAS or by hand. If we were to use a CAS, we would use the **solve** function, entering the command as follows:



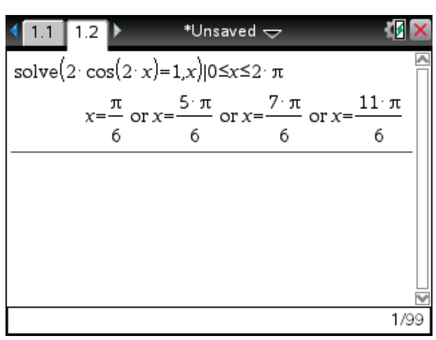
Alternatively, we can solve this question by hand:

$$\begin{aligned} 5 \times 2^{3x} &= 1 \\ 2^{3x} &= \frac{1}{5} \\ 3x &= \log_2\left(\frac{1}{5}\right) \\ x &= \frac{1}{3}\log_2\left(\frac{1}{5}\right) \end{aligned}$$

Both the CAS and the manual method will give us **E** as the correct answer.

Question 2 (D)

Again, like the last question, we can approach this using a CAS or by hand. Before finding the sum of the solutions, which is what we're asked for, we will need to solve the equation first. Using a CAS, we would use the **solve** function, entering the command as follows (remember to include the domain restriction):



Alternatively, if we were to do it manually, we would get:

$$\begin{aligned} 2\cos(2x) &= 1 \\ \cos(2x) &= \frac{1}{2} \\ 2x &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\ x &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

Either method will give us the same four solutions, so what we need to do now is add them:

$$\frac{\pi}{6} + \frac{5\pi}{6} + \frac{7\pi}{6} + \frac{11\pi}{6} = \frac{24\pi}{6} = 4\pi$$

This gives us **D** as the correct answer.

Question 3 (E)

This question requires that we recall our remainder theorem, which would have been covered in Year 11 and revised at the beginning of Year 12. The remainder theorem states that:

If $P(x)$ is divided by $(ax+b)$, the remainder will be $P\left(-\frac{b}{a}\right)$

So we are told that $P(x)$ has a factor of $(x+1)$ and a remainder of 4 when divided by $(x-1)$. We need to also remember that when a polynomial is divided by a factor, its remainder will be zero. Factors divide exactly into a polynomial. Thus, with these two bits of information, we can say that:

$$P(-1) = (-1)^3 + a \times (-1)^2 - b + 6 = 0$$

$$P(1) = 1 + a + b + 6 = 4$$

If we simplify them, we get:

$$a - b + 5 = 0$$

$$a + b + 3 = 0$$

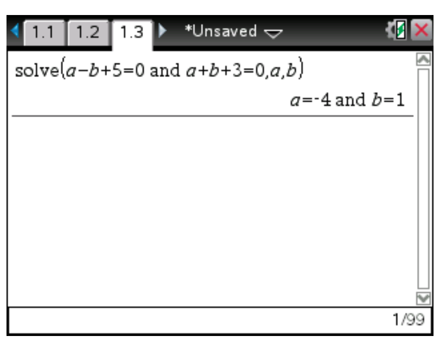
This is a system of two simultaneous equations in two unknowns. We can either solve this with a CAS or by hand, but given that it's already set up, it would be a lot faster to just solve this by hand as opposed to entering both these equations into our CAS. We can see that by adding these two equations, we are able to eliminate the variable b:

$$\begin{aligned} 2a + 8 &= 0 \\ a &= -4 \end{aligned}$$

Now substitute back in to find b:

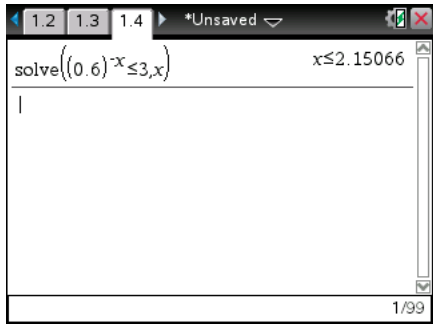
$$\begin{aligned} -4 + b + 3 &= 0 \\ b &= 1 \end{aligned}$$

Hence, **E** is the correct answer. Alternatively, we can use a CAS to solve this system of equations:



Question 4 (C)

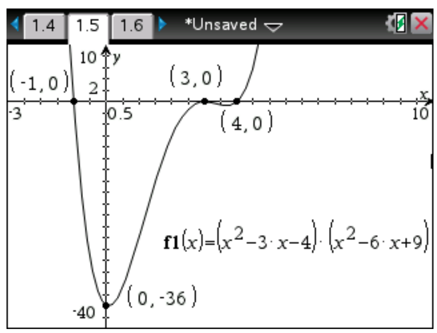
The answer options to this question are all in decimals, so we will need to use a CAS for this question. We can enter the equation into our CAS and use the **solve** function as follows:



This gives us an answer of **C**.

Question 5 (D)

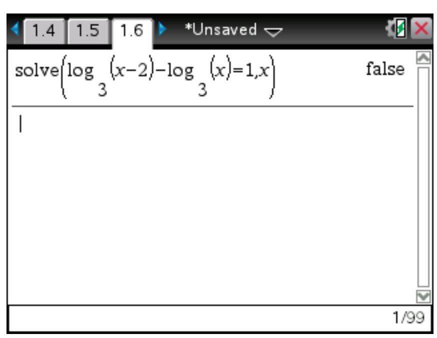
This question can be approached in two ways and is a good example of how a student with CAS experience can have a great advantage over students who prefer to solve all questions by hand. In this question, we are given an equation and asked how many real solutions there are. The easiest way to answer this question will be to use our CAS to sketch a graph. If we sketch the graph of $y = (x^2 - 3x - 4)(x^2 - 6x + 9)$, we will be able to look at how many x-intercepts there are and this will be the number of solutions. We can graph it on the CAS as follows:



Thus, we can see that there are 3 x-intercepts, hence, there are 3 solutions. Alternatively, solving the equation $(x^2 - 3x - 4)(x^2 - 6x + 9) = 0$ on CAS would have also given 3 solutions. In either case, the final answer is **D**.

Question 6 (E)

This is another question which can be solved both by a CAS and by hand. Using a CAS, we can use the **solve** function in order to get our answer. We will get an answer of "false" which tells us that there are no real solutions to this equation, hence, option **E** is the correct answer. We can do this as follows:



Alternatively, we can do this question by hand. Before doing this question, however, we will need to revise our logarithm laws. We should know of the three most basic logarithm laws. These laws work both ways, so they can be used to "expand" or to "simplify" a series of logarithm expressions. For all a, $b > 0$, logarithms satisfy the following properties:

$$\begin{aligned} \log(a) + \log(b) &= \log(ab) \\ \log(a) - \log(b) &= \log\left(\frac{a}{b}\right) \\ a\log(b) &= \log(a^b) \end{aligned}$$

This question asks us to solve:

$$\log_3(x-2) - \log_3(x) = 1$$

Use our second logarithm law:

$$\log_3\left(\frac{x-2}{x}\right) = 1$$

We should know how to interconvert between logarithms and exponentials. They are inverse of one another, thus:

$$\log_a(b) = c \Longleftrightarrow b = a^c$$

Thus, we can also apply the same principle to this question to say that:

$$\begin{aligned} \frac{x-2}{x} &= 3^1 \\ x-2 &= 3x \\ x &= -1 \end{aligned}$$

However, this is outside the domain as there is a $\log_3(x-2)$ in the question, meaning that $x > 2$, so there are no real solutions. This gives us **E** as the correct answer.

Question 7 (B)

This is a binomial expansion question. We should be familiar with the binomial expansion, which is:

$$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + \cdots + {}^nC_{n-1} x^1 y^{n-1} + {}^nC_n x^0 y^n$$

What this expansion tells us is that the coefficients are found using the nCr function, which can be found on both CAS and scientific calculators. The other two terms will be such that their powers will add up to the n. If we look at our expression, we have:

$$(a + x^3)^5$$

This will be easier if we rearrange so that the constant term is second:

$$(x^3 + a)^5$$

We are now asked for the coefficient of the term which contains x^6. We know that this is $(x^3)^2$, so what we can deduce is that if we are going to do a binomial expansion with **descending powers**, then this will be the fourth term, as we will have $(x^3)^5$, $(x^3)^4$, $(x^3)^3$ and then $(x^3)^2$. We now know that since $n = 5$, the power of the a term will have to be $5 - 2 = 3$. Thus, using this information we have, we can say that the term which contains x^6 will be:

$${}^5C_3 \times (x^3)^2 \times a^3 = 10a^3x^6$$

We got the fact that ${}^5C_3 = 10$ from our calculator, although if this was a non-technology assisted test, that could have been found using Pascal's Triangle. We are asked for the coefficient of x^6, which is $10a^3$. The answer is **B**.

Question 8 (E)

This question requires us to remember our **supplementary** and **complementary** angle relationships. We should know our supplementary angle rules for cosine:

$$\begin{aligned}\cos(\pi - x) &= -\cos(x)\\ \cos(\pi + x) &= -\cos(x)\end{aligned}$$

These can be derived from simply realising that shifting the graph of $y = \cos(x)$ by half a period (i.e. π) in either direction is equivalent to reflecting the graph in the x-axis.

The complementary relationships, however, are a little more complex. I've always found that it's easier just to remember the complementary relationships for the first quadrant and then to use them in conjunction with the supplementary relationships in other situations:

$$\begin{aligned}\sin\left(\frac{\pi}{2} - x\right) &= \cos(x)\\ \cos\left(\frac{\pi}{2} - x\right) &= \sin(x)\end{aligned}$$

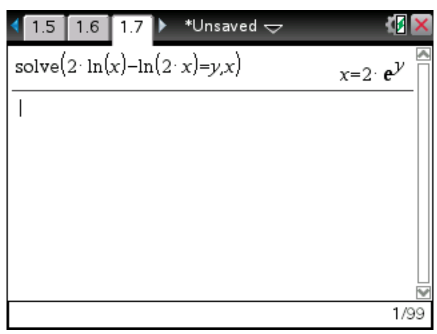
These relationships in particular should be ones you are familiar with. So, using both the supplementary and complementary relationships, we can answer the question:

$$\begin{aligned}\sin\left(\frac{\pi}{2} - x\right) + \cos(x + \pi) &= \cos(x) - \cos(x)\\ &= 0\end{aligned}$$

Note that this is equal to 0 for all x, so the fact that $\cos(x) = 0.7$ didn't affect the answer in this case. The correct answer is **E**.

Question 9 (A)

This is another question which can be solved both by a CAS and by hand. Using a CAS, we can use the **solve** function in order to get our answer. We can do this as follows:



Alternatively, we can do this question by hand. Before doing this question, however, we will need to revise our logarithm laws. We should know of the three most basic logarithm laws. These laws work both ways, so they can be used to "expand" or to "simplify" a series of logarithm expressions. For all a, $b > 0$, logarithms satisfy the following properties:

$$\begin{aligned}\log(a) + \log(b) &= \log(ab)\\ \log(a) - \log(b) &= \log\left(\frac{a}{b}\right)\\ a\log(b) &= \log(a^b)\end{aligned}$$

This question asks us to solve, for x:

$$2\log_e(x) - \log_e(2x) = y$$

We can now apply our third and second logarithm laws to get:

$$\log_e\left(\frac{x^2}{2x}\right) = y$$

We should know how to interconvert between logarithms and exponentials. They are inverse of one another, thus:

$$\log_a(b) = c \Longleftrightarrow b = a^c$$

Thus, we can also apply the same principle to this question to say that:

$$\frac{x^2}{2x} = e^y$$

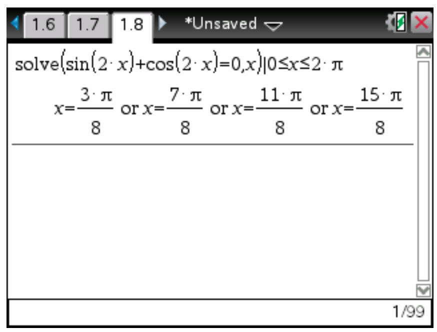
Here, we can take a shortcut. Even though we usually say not to "cancel down" variables, we have a $\log_e(x)$ in the original equation. This implies that $x \neq 0$, because we can't take the logarithm of zero or a negative number. Thus, we can safely cancel down:

$$\begin{aligned} \frac{x}{2} &= e^y \\ x &= 2e^y \end{aligned}$$

Thus, the answer is **A**.

Question 10 (C)

This question requires us to solve $\sin(2x) + \cos(2x) = 0$ over the domain $[0, 2\pi]$. We can either do this with a CAS or by hand. If we wanted to use a CAS, we would use the **solve** function as follows:



We can now also do it by hand, but we need to realise that we can't solve an equation with both sin and cos in it, so we will attempt to convert it to a tan:

$$\begin{aligned} \sin(2x) &= -\cos(2x) \\ \frac{\sin(2x)}{\cos(2x)} &= -1 \\ \tan(2x) &= -1 \end{aligned}$$

We can now solve this equation for x. We should remember that the primary angle required is $\frac{\pi}{4}$ and that tan is negative in the 2nd and 4th quadrants.

$$\begin{aligned} \tan(2x) &= -1 \\ 2x &= \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4} \\ x &= \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8} \end{aligned}$$

Regardless of whether we solved the equation using a CAS or by hand, we would get the same answers. The question is asking for the difference between the largest and smallest solutions, which is:

$$\frac{15\pi}{8} - \frac{3\pi}{8} = \frac{12\pi}{8} = \frac{3\pi}{2}$$

The answer is **C**.

Question 11 (A)

We can begin this question by letting $A = 2^x$, thus, we know that:

$$x = \log_2(A)$$

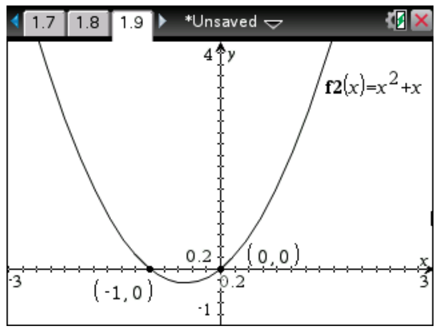
This means that there will be one solution for x when there is only one positive solution for A. Thus, we can now express the original equation as a quadratic in A:

$$A^2 + A + b = 0$$

Let's assume now that $b = 0$ and sketch a graph for:

$$y = x^2 + x$$

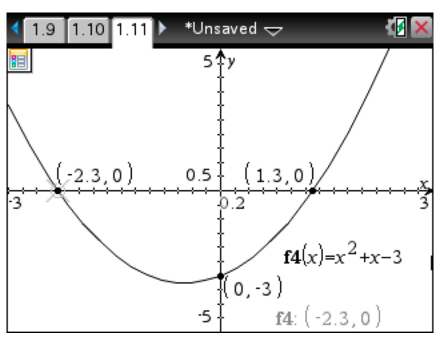
Using a CAS, we will get the following graph:



Here, we can see that we have two solutions for A, -1 and 0 - both of which will not give a real solution for x. Recall that the domain of a log function is $(0, \infty)$. So what we are looking for is how to move this graph such that we will have one positive x-intercept.

The value of b in this question moves the graph up and down. If we move this graph up, we will have two negative solutions, eventually followed by one negative solution, then no negative solutions as the axis is now below the turning point. Thus, we can see that moving the graph up will not give a real solution to the original equation.

The other option, however, which is to move the graph down, will have an effect; no matter how much we move the graph down by, we will end up with one negative and one positive solution. The more we move the graph down, the more negative the negative solution becomes and the more positive the positive solution becomes. Let's say we move it down by 3 units. We will get:



We can see now that we have one negative and one positive solution for A. This means that we will have one real solution for x. Recall that when $b = 0$, we have one negative and one zero solution for A, meaning that there will be no solutions. Thus, we want the interval $b < 0$, because the graph needs to be translated down in order for there to be a solution.

Thus, the answer is **A**.

SECTION 2 - Extended-Response Questions

Question 1

Part a. i.

The best way to answer this question is to first consider $f(x)$. So if we know that $c = 0$, $f(x)$ will be:

$$f(x) = \frac{1}{3}x^3 - x^2 - 3x$$

What we can do from here is see that we can take out a linear factor of x. This will give us:

$$f(x) = \frac{1}{3}x(x^2 - 3x - 9)$$

So now we have a quadratic factor to look at. We know that if we wish to find the number of solutions a quadratic has, we need to use the discriminant, $\Delta = b^2 - 4ac$. So if we find the discriminant of this quadratic factor, we will get:

$$\Delta = (-3)^2 - 4(1)(-9) = 45$$

We should recall several facts regarding the discriminant and using it to find the number of solutions a quadratic function has:

- $\Delta > 0$ - **two** real solutions
- $\Delta = 0$ - **one** real solution
- $\Delta < 0$ - **no** real solutions

We can see that in this example here, $\Delta > 0$, thus, there will be **two** real solutions for the quadratic, and we can see by inspection that neither of them are 0. Combining this with the linear factor of x, which does give a solution of 0, there are **three** distinct real solutions for x. We should also mention that since we are only asked to show that there are three solutions, this is sufficient. There is no need to actually find the solutions. However, it is also possible to solve the equation $f(x) = 0$ and get three solutions, showing that there are three distinct real solutions for x. That is also a valid way to approach this question.

Part a. ii.

This question asks us to show that the turning points of the function $f(x)$ occur at $x = -1$ and $x = 3$. In order to find the x-coordinates of the turning points, we must first find the derivative of the function $f(x)$. Recall that the derivative will give us the gradient of the function at any particular point. At turning points, the gradient is momentarily flat, meaning that the derivative will equal zero at that point. Thus, in order to find the turning points, we will need to calculate the derivative. We should know how to differentiate a polynomial function from Units 1 & 2. A good way to remember it is to "bring down the power and reduce the power by one." Formally:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

So, let's find $f'(x)$, the derivative of $f(x)$:

$$f'(x) = x^2 - 2x - 3 = 0$$

We can now proceed to solve this quadratic in order to get the x co-ordinates where the stationary points occur:

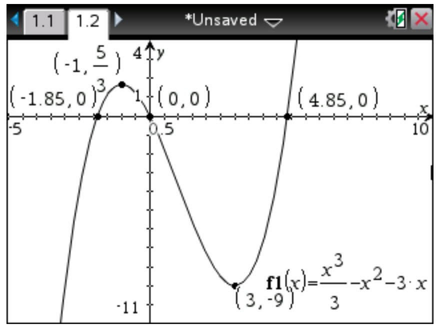
$$f'(x) = x^2 - 2x - 3 = 0$$

$$(x+1)(x-3) = 0$$

$$x = -1 \text{ or } x = 3$$

Part b. i.

The best way to approach this question is to understand the effect that c has on the graph, and to consider the shape and turning points of the graph. First, let's sketch the graph of $f(x)$ on our CAS, assuming that $c = 0$:



We can observe from this sketch that there are three distinct real solutions for this value of c. For the sake of minimising errors later on, note that whatever interval we find for c for $f(x) = 0$ to have three solutions (for part **ii.** of the question), it will need to include $c = 0$.

Let's now discuss the effect that c has on the graph of $f(x)$. It is responsible for the vertical translation, meaning that it moves the graph, as a whole, up and down.

If we look at the graph, there will only be three distinct real solutions when the x-axis passes through between the two turning points. If the graph is moved too low and the x-axis passes through above the two turning points, there will only be one solution. The same thing happens when the graph is moved too high such that the two turning points are above the x-axis.

Thus, we need to realise that the answer to this question has to do with the turning points. We should find the coordinates of the turning points for $c = 0$. From the CAS, we can see that they are:

$$\left(-1, \frac{5}{3}\right) \text{ and } (3, -9)$$

This means two things:

- If the graph is translated **up** by more than 9 units, the x-axis will be below the local minimum point, meaning that there will only be **one** real solution.
- If the graph is translated **down** by more than $\frac{5}{3}$ units, the x-axis will be above the local maximum point, meaning that there will only be **one** real solution.

We should also note that if the graph is translated up by exactly 9 units or down by exactly $\frac{5}{3}$ units, there will be two real solutions because one of the turning points would be lying on the x-axis. The question asks when there is exactly one real solution. This occurs for the two conditions we stated above. Mathematically, we can express this as:

$$c < -\frac{5}{3} \text{ or } c > 9$$

Part b. ii.

This is a continuation of the above question, so we will continue to use the groundwork built up there in order to answer it. We are now asked when there will be three solutions. We remember that when $c = -\dfrac{5}{3}$ or $c = 9$ there will be two solutions, so we don't want to consider these values.

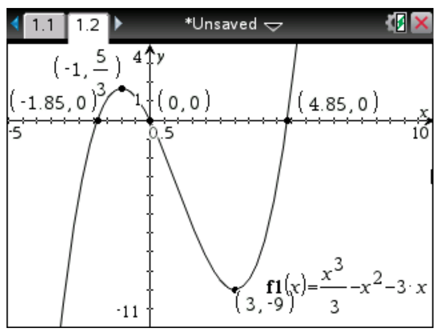
This leaves the interval between $-\dfrac{5}{3}$ and 9 remaining. This makes sense because, if the graph is translated up by less than 9 units or down by less than $\dfrac{5}{3}$ units, the x-axis will cross the graph between the two turning points, meaning three solutions. Thus, the answer is:

$$-\frac{5}{3} < c < 9$$

Part b. iii.

This is a difficult question which follows on from the previous two parts, but at the same time, it also tangents off into a different direction, which can be confusing.

For this question, we are no longer considering whether a particular number of solutions exist, but we are looking at the value of the solutions. First of all, let's think about the consequences of having **two** distinct positive solutions. Revisiting the graph for $c = 0$:



We can see from the graph that there is currently one negative solution, one positive solution and one zero solution. There are a number of things we can do to this graph. Let us list all the things we could possibly do by vertically translating (i.e. by modifying c).

- Translate **down** by more than $\dfrac{5}{3}$ units - i.e. $c < -\dfrac{5}{3}$ - **one positive solution**
- Translate **down** by exactly $\dfrac{5}{3}$ units - i.e. $c = -\dfrac{5}{3}$ - **one positive and one negative solution**
- Translate **down** by less than $\dfrac{5}{3}$ units - i.e. $-\dfrac{5}{3} < c < 0$ - **one positive solution and two negative solutions**
- Translate **up** by more than 9 units - i.e. $c > 9$ - **one negative solution**
- Translate **up** by exactly 9 units - i.e. $c = 9$ - **one positive and one negative solution**
- Translate **up** by less than 9 units - i.e. $0 < c < 9$ - **two positive and one negative solution**

Thus, we can see that we will end up with two distinct positive solutions for last case - translate **up** by less than 9 units. Thus, the answer to this question is:

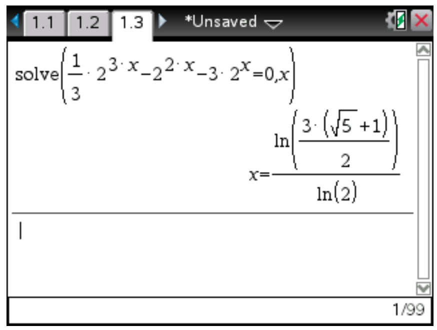
$$0 < c < 9$$

Part c. i.

We should first rearrange the equation so that it becomes easier to solve:

$$\begin{aligned} g(x) &= d \\ \frac{1}{3} \times 2^{3x} - 2^{2x} - 3 \times 2^x + d &= d \\ \frac{1}{3} \times 2^{3x} - 2^{2x} - 3 \times 2^x &= 0 \end{aligned}$$

This question can be completed using a CAS. If we use the solve function, we will get an answer as follows:



If we really wanted to do it by hand, we can. There really is no point in doing it this way when we have access to technology, but anyway: let $A = 2^x$. Then:

$$\frac{1}{3}A^3 - A^2 - 3A = 0$$

$$\frac{1}{3}A(A^2 - 3A - 9) = 0$$

$$A = 0 \text{ or } A = \frac{3 \pm 3\sqrt{5}}{2}$$

Since we need to take the logarithm of A, only $A = \dfrac{3 + 3\sqrt{5}}{2}$ is a possible solution:

$$x = \log_2\left(\frac{3 + 3\sqrt{5}}{2}\right)$$

We should recognise the change of base rule, which makes the CAS answer and our answer equivalent:

$$\log_a(b) = \frac{\log_e(b)}{\log_e(a)}$$

Either of the forms would be acceptable in the exam, though expressing it as $\log_2$ does make the answer look neater.

Part c. ii.

We need to recognise that in order to have two real solutions to this equation, the cubic $\frac{1}{3}A^3 - 2A^2 - 3A = 0$ will need to have two positive solutions for A, because we need to take $\log_2(A)$, which requires A to be a positive number due to the domain of the log function. Thus, the answer will be the same as the answer to **part b. (iii)**, except with d rather than c:

$$0 < d < 9$$

Question 2

Part a.

Let us call these equations 1, 2 and 3:

$$\begin{aligned} x - y - 2z &= -3......(1) \\ tx + y - z &= 3t........(2) \\ x + 3y + tz &= 13.......(3) \end{aligned}$$

Before we do anything else, let's take some time to observe this system. We can see that the easiest variable to eliminate will be y because it is not multiplied by t in any of the three equations, meaning that we won't have to multiply any equations by a t term in order to eliminate. The first two equations can be added in order to eliminate y and we can work with the first and the third to eliminate y. Now, we will make equations (4) and (5) by eliminating y from the system. First, let $(4) = (1) + (2)$ and $(5) = (3) + 3 \times (1)$:

$$\begin{aligned} (t+1)x - 3z &= 3t - 3......(4) \\ 4x + (t-6)z &= 4.............(5) \end{aligned}$$

This is a 2-variable simultaneous equation, which is much easier to work with. Given that we need to find x in terms of t, we need to eliminate z. To do this, we need to multiply equation (4) by a factor such that the coefficient of z is the same as that of equation (5) - that is, $(t-6)$.

So, $3 \times \text{factor} = t - 6$. Hence, $\text{factor} = \dfrac{t-6}{3}$. Thus, we should multiply equation (4) by $\dfrac{t-6}{3}$. Doing so, we get:

$$\frac{1}{3}(t+1)(t-6)x - (t-6)z = \frac{3(t-1)(t-6)}{3} = (t-1)(t-6)........(6)$$

Now, we can eliminate z by adding equations (5) and (6):

$$\left(\frac{1}{3}(t+1)(t-6) + 4\right)x = (t-1)(t-6) + 4$$

This looks a little messy, but do not worry; when you get slightly messy algebra like this, the key is to make sure your handwriting is legible and your steps are neatly set out.

Let us simplify the above equation. Working with fractions is quite nasty, and I don't particularly like the $\dfrac{1}{3}$, so let us get rid of it by multiplying both sides by 3:

$$((t+1)(t-6) + 12)x = 3(t-1)(t-6) + 12$$

This is much easier to work with; let's simplify:

$$\begin{aligned} (t^2 - 5t - 6 + 12)x &= 3(t^2 - 7t + 6) + 12 \\ (t^2 - 5t + 6)x &= 3t^2 - 21t + 18 + 12 \\ &= 3t^2 - 21t + 30 \end{aligned}$$

All we need to do now is to isolate x and we end up with the expression we were asked to show:

$$x = \frac{3t^2 - 21t + 30}{t^2 - 5t + 6}$$

Part b.

This question requires us to follow on from the last question. We realise that when the denominator of the above answer is zero, what will happen is that x will be undefined. This might give us a lead. Quickly finding the values of t such that the denominator is equal to 0:

$$\begin{aligned} t^2-5t+6 &= 0 \\ (t-2)(t-3) &= 0 \\ t &= 2,3 \end{aligned}$$

Now, we have to ask ourselves this question - what happens when $t=2$ or $t=3$? After all, when we solve simultaneous equations (or do anything, for that matter), we should never be dividing by 0. Suppose $t=2$ or $t=3$. We have to find - where does our working to part **a.** break down?

Let's look at the steps to part **a.** backwards:

$$\begin{aligned} (t^2-5t-6+12)x &= 3(t^2-7t+6)+12 \\ (t^2-5t+6)x &= 3t^2-21t+18+12 \\ &= 3t^2-21t+30 \end{aligned}$$

If $t^2-5t+6=0$, this means that for both steps the coefficient of x is 0. What!? Why does this happen?

Well, think about it. We eliminated z to get the above equation. If the coefficient of x is also 0 after this elimination of z, that means that we accidentally eliminated x **and** z at the same time. Let's look at equations (5) and (6), the equations we used to eliminate z:

$$\begin{aligned} 4x+(t-6)z &= 4 \text{.........................(5)} \\ \frac{1}{3}(t+1)(t-6)x-(t-6)z &= (t-1)(t-6) \text{........(6)} \end{aligned}$$

Given we added (5) and (6) together to eliminate z, **and** we accidentally eliminated x too, it follows that the coefficient of x in equation (6) must be -4. So we get:

$$\begin{aligned} 4x+(t-6)z &= 4 \text{.........................(5)} \\ -4x-(t-6)z &= (t-1)(t-6) \text{........(7)} \end{aligned}$$

Now, the coefficients of x and z here are in the same 'proportions.' In other words, to get the LHS of one equation, you need only multiply the LHS of the other equation by a factor. Doing so with equation (7), multiplying it by -1:

$$\begin{aligned} 4x+(t-6)z &= 4 \text{...........................(5)} \\ 4x+(t-6)z &= -(t-1)(t-6) \text{........(8)} \end{aligned}$$

Now **this** is when you think alarm bells. If you have a set of simultaneous equations where the LHS is the same in both equations, you are basically saying one of two things:

- If the RHS of both equations (remember they are constants!) are the same, you are effectively saying that only equation (5) need be true since equations (5) and (8) would be the same equation, in which case infnite pairs of (x,z) would be solutions.
- If the RHS of both equations are different, then the two equations cannot have any solutions for (x,z).

Note that the above set of simultaneous equations are true if $t=2$ or $t=3$ in that the coefficient of x in equation (8) is 4 only if $t=2$ or $t=3$.

So, when $t=2$ or $t=3$, you can see that there would either be infinite or no solutions.

Part c.

We need to find a way to determine whether there will be inifinite solutions, or whether there will be no solutions. Let's go back to equations (5) and (8) in part **b.**:

$$\begin{aligned} 4x + (t-6)z &= 4 \text{............................}(5) \\ 4x + (t-6)z &= -(t-1)(t-6) \text{........}(8) \end{aligned}$$

For there to be no solutions, as we said in part **b.**, the RHS of equations (5) and (8) need to be different. Let's see the values of t for which they are the same by solving the following equation:

$$\begin{aligned} -(t-1)(t-6) &= 4 \\ t^2 - 7t + 6 &= -4 \\ t^2 - 7t + 10 &= 0 \\ (t-5)(t-2) &= 0 \\ t &= 2, 5 \end{aligned}$$

Given that equation (8) is only true if $t = 2$ or 3, the only valid solution is $t = 2$. Hence, $t = 3$ would yield different RHSs for equations (5) and (8), meaning there are no real solutions to the equation. To demonstrate, let's substitute $t = 3$ into the equations:

$$\begin{aligned} 4x - 3z &= 4 \text{............................}(5) \\ 4x - 3z &= 6 \text{............................}(8) \end{aligned}$$

Clearly, 4 cannot equal 6. So the answer is $t = 3$.

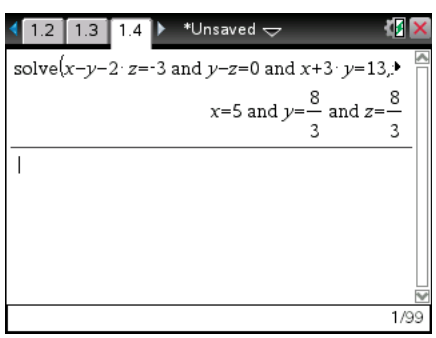
Part d.

$$\begin{aligned} x - y - 2z &= -3 \\ tx + y - z &= 3t \\ x + 3y + tz &= 13 \end{aligned}$$

If $t = 0$,

$$\begin{aligned} x - y - 2z &= -3 \\ y - z &= 0 \\ x + 3y &= 13 \end{aligned}$$

Since the question does not specify that we have to use algebra, we can use any method we like. There is no reason to solve this by hand as it is tedious and complex. We can use the **solve** function on our CAS, entering it in the following syntax in order to get the correct answer.



$$\begin{aligned} x &= 5 \\ y &= \frac{8}{3} \\ z &= \frac{8}{3} \end{aligned}$$

CALCULUS

TECH-FREE TEST 1

DETAILED SOLUTIONS

Question 1

As always, we should consider the tools available to us to answer questions - in this question, we're finding the derivative of a function. Usually, a question will require the use of a differentiation rule (quotient/product/chain). However, $\tan(x)$ as a function doesn't present itself as open to using a particular rule. And even though it's a given that the derivative of $\tan(x)$ is $\sec^2(x)$, that will not be sufficient for a "show that" question. Whenever you see this phrase, you must take care in showing all steps follow on from each other in a simple and logical way.

Just a you are expected to recognise identities and formulas in simplifying trignometric expressions, a basic one students are expected to remember is $\tan(x) = \frac{\sin(x)}{\cos(x)}$. while obvious, this identity is very handy for many questions involving simplication. In this case, it is used to incorporate the "quotient rule" for differentiation.
Note: Some students abstain from the use of this rule, and will use the product rule opting for the denominator to be a product to the power of -1 (in this case $\tan(x) = \sin(x)(\cos(x))^{-1}$). This is acceptable, however, many would claim this to be a more difficult method.

The quotient rule states that for $f(x) = \frac{g(x)}{h(x)}$, $f'(x) = \frac{g'(x)h(x)-h'(x)g(x)}{(h(x))^2}$
Hence,

$$f'(x) = \frac{\cos(x) \times \cos(x) - \sin(x) \times - \sin(x)}{(\cos(x))^2}$$

In order to simplify this expression, the use of another identity is required. The pythagorean identity states that $\sin^2(x) + \cos^2(x) = 1$. This is a very significant identity that students should memorise.
As a result,

$$\begin{aligned} \frac{\cos(x) \times \cos(x) - \sin(x) \times - \sin(x)}{(\cos(x))^2} &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \\ &= \sec^2(x) \text{ as required} \end{aligned}$$

Question 2

a.
This question is typical of the first question in a VCAA Exam 1, i.e. A relatively simple "find the derivative/anti-derivative". Looking ahead to part b), you will see that this is a standard "integration by recognition" question.

As mentioned in question 1, it is important that students realise that the use of a differentiation rule is almost always required. In this case students should immediately recognise the form $f(x)g(x)$ and hence use the product rule.

The product rule states that if $f(x) = g(x)h(x)$ then $f'(x) = g(x)h'(x) + g'(x)h(x)$.

It is also worth noting a very handy thing to remember with regards to differentiating logarithmic function:. $\frac{d}{dx}\log(f(x)) = \frac{f'(x)}{f(x)}$. This result is easily shown using the chain rule, but remembering it in that form will often save students a lot of time.

Hence,

$$\begin{aligned}\frac{d}{dx}(x\log_e(x^2)) &= x\left(\frac{2x}{x^2}\right)+1(\log_e(x^2))\\ &= 2+\log_e(x^2)\end{aligned}$$

b.

When we consider integration and antidifferentiation, there are commonly two schools of thought. 1) The integral of a function gives the area between the function and the coordinate axes, and 2) If we take the derivative of a function, the anti-derivative of the gradient function is the original function. The subtle difference between these two is insignificant within the scope of the Mathematical Methods course, but it's always worth keeping in mind since students can be tested on either.

This question does not concern itself with areas so let us consider the 2nd way of looking at antidifferentiation. Since we know the derivative of the original function, we also know that its anti-derivative is in fact the original function.

Therefore,

$$\int \left(2+\log_e(x^2)\right)dx = x\log_e(x^2)+c_1$$

However, note the "c_1". This was not in the original function, was it? This constant is important because it signifies the fact that the derivative $\frac{d}{dx}(c_1)=0$, so that when we derive we still end up with $2+\log_e(x^2)$. It's not in the original function because when you derive, you only have one solution curve (the derivative). When you integrate, you get a family of solution curves, represented by the $+c$ (or c_1 in this case).

When simplifying $\int 2+\log_e(x^2)dx$, we make use of an important rule that the intergral of sums is equal to the sum of integrals. It's the same principle that states that $\frac{d}{dx}(f(x)+g(x))=\frac{d}{dx}(f(x))+\frac{d}{dx}(g(x))$, e.g. $\frac{d}{dx}(x^2+x)=\frac{d}{dx}(x^2)+\frac{d}{dx}(x)=2x+1$.

$$\begin{aligned}\int \left(2+\log_e(x^2)\right)dx &= \int 2dx+\int \log_e(x^2)dx\\ \therefore \int \log_e(x^2)dx &= -2x+x\log_e(x^2)+c_2\end{aligned}$$

Again, notice the use of "c_2", which while potentially a different value to "c_1", has no effect in the resulting anti-derivative since c can be any constant value.

Question 3

This question is a slightly twisted take on a standard linear approximation question. However, the difference is that instead of finding the approximate value, we are instead looking at the approximate change.

If you refer to the model solutions, you will note two sets of solution - the choice is yours as to whether you wish x to be the radius or the surface area of the sphere. The only difference is whether you look for "h" as the change of radius (if x is used for radius) OR you look for "$hf'(x)$" (if x is your surface area, and hence $f(x)$ is your radius). Either method results in the same answer. You can also use the approximation $\delta y=\delta x\frac{dy}{dx}$. This method is not explored here, as VCAA tends to use the $f(x+h)$ notation.

For the purposes of this solution, I will run through the first method in the model solutions.

If we consider the rule used in linear approximation, $f(x+h) = f(x) + hf'(x)$, it should be clear that we need to define 3 things: $f(x)$, x, and h. $f'(x)$ is found after defining $f(x)$. Defining these three things should be the first thing a student does when seeing a linear approximation question.

In most questions, defining the function requires the most thought and is dependent on the question. Read carefully to determine what you wish to do - in this case we know of radius and surface area, so we may use either.

Let $f(r) = SA = 4\pi r^2$. You should know from your mensuration formula sheets that the surface area of a sphere is $4\pi r^2$. In this case, the x we have above is interchangeable with the r which defines the radius, meaning for this question we will have $f(r+h) = f(r) + hf'(r)$. Substituting into our equation, we see that we have $f(r) = 64\pi$ and $f(r+h) = 65\pi$

The only thing missing is $f'(r)$, we can find its derivative using the function defined above. Therefore, $f'(r) = 8\pi r$

In this question finding r, and hence h, is perhaps more difficult (or tedious rather). When looking to define a suitable r, we wish for it to be simple enough that when substituted into the function, $f(r)$ can be easily found. The whole point behind linear approximation is that for the x-value (r-value) given, it is not easy to substitute and calculate an exact value. In this question, finding the radius of a sphere with surface area 65π is not necessarily easy without the use of surds. Luckily, in this question, we are given the values of $f(x)$ to begin with so we can work backwards to find the x value required.

$$\begin{aligned} 64\pi &= 4\pi r^2 \\ r^2 &= 16 \\ r &= 4 \end{aligned}$$

Now this question requires us to find the change in radius, h. If we substitute the value defined above into the rule for linear approximation, calculating h should be very simple.

$$\begin{aligned} h &= \frac{f(r+h) - f(r)}{f'(r)} \\ &= \frac{65\pi - 64\pi}{8\pi \times 4} \\ &= \frac{1}{32} \end{aligned}$$

Hence almost all questions for linear approximation can be broken down into a few steps.

1. From reading the question, determine what term from the linear approximation rule is required to answer the question.
2. Define "$f(x)$" (or other relevant function) and all other terms apart from the one required.
3. Substitute the values into the linear approximation rule and solve for the term required.

Side note: It is noteworthy that h is a relatively small value. It needs to be small in order for the approximation to be close to the actual result. In fact, the exact change in radius is 0.0311 whereas $\frac{1}{32} = 0.0313$. You might question why h needs to be as small as possible. Well consider where the linear approximation rule comes from - it's the first principles of finding the derivative that states:

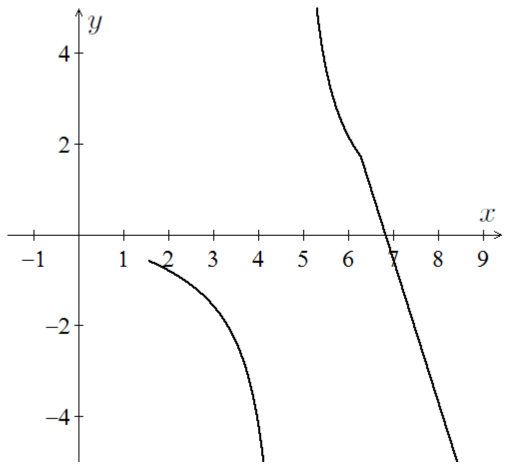
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

When rearranged for $f(x+h)$, we get the linear approximation rule. However, the first principles rule gives an exact derivative, whereas we want an approximation. The difference is in the "$\lim_{h\to 0}$". It is this requirement that h be as close to 0 as possible that gives it its accuracy and the closer h is to 0, the more accurate the result will be.

Question 4

The phrase "strictly decreasing" is one that often confuses students. A set that is strictly decreasing is usually defined as, $[a, b]$ such that $f(a) > f(b)$ and $b \geq a$. This means that endpoints are included - the INCORRECT definition is one that involves the derivative being less than 0, because that would exclude the endpoints.

A rough drawing will suffice for this question considering we are only interested in seeing the behaviour of the graph



We can see that the $\tan(-\frac{x}{3})$ function and linear function are always decreasing.

However, there is an asymptote that interrupts the set for the definition given above. To find the asymptote, we can consider where the asymptote is for $\tan(x)$ is, $x = \frac{\pi}{2}$. To get $\tan(-\frac{x}{3})$ there is a dilation of 3 from the y-axis. Hence, the aymptote for the required domain lies at $x = \frac{3\pi}{2}$.

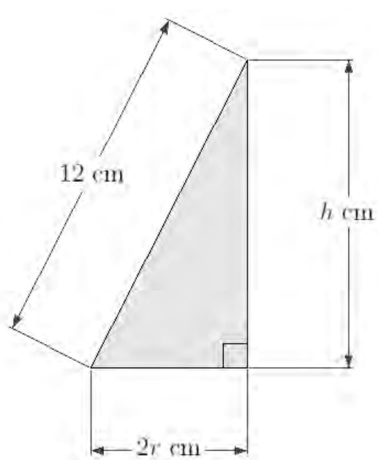
Therefore, there are two sets for us to consider, and we just have to pick the larger set. The two sets: $[0, \frac{3\pi}{2})$ and $(\frac{3\pi}{2}, 6\pi]$

Hence, the largest subset for which $f(x)$ is strictly decreasing is $(\frac{3\pi}{2}, 6\pi]$

Question 5

a.

Students should recognise a right angled triangle that is formed by cutting the cylinder from corner to corner. It will have a hypotenuse length of 12cm due to the diameter of the sphere.



Using the Pythagorean theorem, we can get the following expression that has both r and h.

$$\begin{aligned}(2r)^2 + h^2 &= 12^2 \\ 4r^2 &= 144 - h^2 \\ r &= \pm\frac{\sqrt{144 - h^2}}{2} \\ r &= \frac{\sqrt{144 - h^2}}{2}, \; r \geq 0\end{aligned}$$

Remember that this question has a realistic application and that means that students must be aware of restrictions in values - in this case, radius is a distance within a sphere and thus must be positive or 0. Please ensure that the answer you end up with is the same as the one required in the answer. Also, this "show that" question allows students to continue in later parts without claiming the marks for that part - don't give up on the whole question just because you can't get the first part!

b.

The volume of a cylinder is given as base×height (or you can just refer to formula sheet). Knowing the basic volumes and areas provided in the formula sheet can shave valuable time in an exam setting.

$$V = \pi r^2 h$$

We can then substitute the result of **part a.** into this equation given.

$$\begin{aligned}V &= \pi h\left(\frac{144 - h^2}{4}\right) \\ &= 36\pi h - \frac{\pi h^3}{4}\end{aligned}$$

c.

When tackling methods questions, thanks to the limited scope of the course, you can rely on certain words as a trigger. There are some very common and obvious ones, and in this question the word "maximum" should pop out to students. The first thing you should think about is finding a derivative and letting it equal to 0. When we consider what this property of $\frac{dy}{dx} = 0$ amounts to, students should be aware that it will either be a turning point (local maximum or minimum) or a stationary point of inflection. For now, we will simply find when $\frac{dV}{dh} = 0$ and worry about whether it is a maximum or not at a later point.

Hence,

$$\frac{dV}{dh} = 36\pi - \frac{3\pi h^2}{4}$$

Let $\frac{dV}{dh} = 0$

$$\begin{aligned}\frac{dV}{dh} &= 0 \\ 36\pi - \frac{3\pi h^2}{4} &= 0 \\ \frac{3\pi h^2}{4} &= 36\pi \\ h^2 &= 48 \\ h &= 4\sqrt{3}\end{aligned}$$

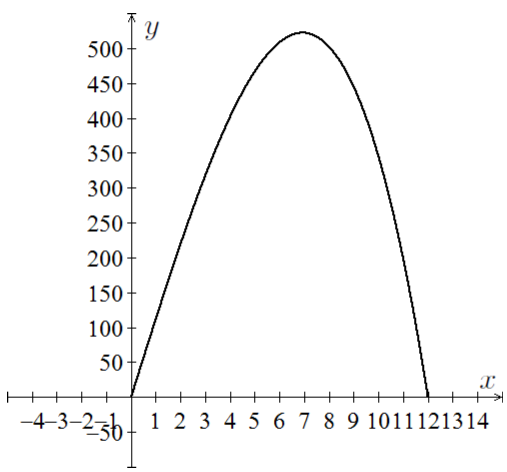
For those who struggle with simplifying their surds, just always think about the square values you know - more often than not, VCAA will not test anything above 13^2 so you will always be able to find a square number, e.g. 4 in $\sqrt{4 \times 4 \times 3}$, that can be simplified and taken out of the surd as shown in the result of h.

Remember, we still have to question whether this value of h results in a maximum volume or not - it could be a minimum or a stationary point of inflection in the volume function. Firstly, there is only one value, and hence, it is likely that this value is correct. In fact, you will probably get away with continuing this question without proving this is the maximum. Secondly, we can find out exactly with the use of a sign table or a second derivative (however, since second derivatives are not in the Methods course, it will not be covered here - that said, it is in Specialist Mathematics and can be used within Methods).

The table below shows how to set up a sign table - essentially, we pick two values on either side of $4\sqrt{3}$ that will hence show us the behaviour of the function around that value.

h	0	$4\sqrt{3}$	7
$\frac{dV}{dh}$	36π	0	$-\frac{3}{4}\pi$
+/-	/	−	\

The lines in the "+/-" row signify what the graph is doing at that point - if $\frac{dy}{dx}$ is positive, then the slope is positive. In this sense, we can gauge what occurs around $h = 4\sqrt{3}$. What the slopes in the last row show is a maximum point (akin to a $y = -x^2$ graph). Hence, we are assured that at $h = 4\sqrt{3}$, V is a maximum. In fact if we graph the volume function (though not required to answer the question), we can see this clearer.



One last important note about considering whether a point is a maximum or not! The Mathematical Methods course specifically makes a distinction between a "local maximum" and an "absolute maximum". It's always important to consider the end points of a function that has a practical element to it - the practicality of a question will commonly restrict the domain. In this instance, the volume and height of a cylinder cannot be less than zero. However, this is not a concern for this particular question as the graph indicates - the maximum is well and truly the turning point local maximum. All we have to do now is substitute the value of h to get the value of the maximum volume.

$$\begin{aligned} V &= 36\pi(4\sqrt{3}) - \frac{\pi(4\sqrt{3})^3}{4} \\ &= 96\sqrt{3}\pi \end{aligned}$$

Hence, the maximum volume of the cylinder is $96\sqrt{3}\pi$ cubic units
As always, don't forget to include the units in your final answer.

CALCULUS

TECH-FREE TEST 2

DETAILED SOLUTIONS

Question 1

a.

This is a typical Question 1 in a VCAA exam 1 with a slight twist. Students are expected to be able to find the anti-derivative of many functions and also evaluate definite integrals. This question is only irregular in that one of the terminals is not a value but a variable, t. You should not be alarmed by this, but rather continue to evaluate the definite integral as one would usually.

$$\int_0^t \left(e^{-x} + x\right) dx = \left[-e^{-x} + \frac{x^2}{2}\right]_0^t$$

Be sure to take care with finding the anti-derivative - there is often one mark involved in doing so, particularly if the question makes mention of the phrase "using calculus". The actual step that involves the "use of calculus" is not calculating the definite integral, but rather the finding of the anti-derivative while doing so. From this stage, we only have to take care in substituting the values in the terminals correctly.

$$\begin{aligned} \left[-e^{-x} + \frac{x^2}{2}\right]_0^t &= (-e^{-t} + \frac{t^2}{2}) - (-1 + 0) \\ &= -e^{-t} + \frac{t^2}{2} + 1 \end{aligned}$$

Again, students should not be uncomfortable in this result - as explained in the next question, it is just a function with a certain purpose, but not a particularly tricky concept to understand at all.

b.

Let us first address what the definite integral tells us (ignoring the variable t). At a simple level, you only have to consider what the integral has meant in the more typical calculation of $\int_a^b f(x)\, dx$. This definite integral calculates the **net signed area** of the region between the function and the x-axis for the domain (a, b).

What do I mean by net signed area? This is different to just the area between the function and the x-axis, because it takes into account the fact that an area might be negative. While not having much application in the Mathematical Methods course, signed areas can have various physical applications within integral calculus, and you are thus expected to know that they occur. If you recall the left/right-endpoint approximations, they relied on the area of rectangles with a height defined by the function - if the function is negative at a particular point, then the area of that rectangle will be negative.

So if you add together the positive areas with the negative areas, you will end up with less than the "total area" because the negative areas have effectively been subtracted from the positive. If we wanted the total area, we would have to ensure that $f(x)$ is positive for all values without compromising the value itself. This is done by finding the absolute value of the function and evaluating $\displaystyle\int_a^b |f(x)| dx$. In the context of the rectangle approximations, the physical height has not changed its value, so the area of negative regions will simply be positive now. It is important to recognise this distinction between the "signed area" and "total area".

The only issue now is what to make of the variable "t". To a large extent, we don't have to be concerned. Instead of finding the area from $x = 0$ to say $x = 9$, it is now to some value of t. It can be any value - it doesn't even need to be greater than 0. So essentially, our answer is a function, e.g. $g(t)$, in terms of t and we can let this value of t be whatever we desire, and the value of the function is equal to the signed area of the region between the function and the x-axis from 0 to the value of the t input.

Thus, the answer is a function in terms of t, that represents the net signed area under the curve $y = e^{-x} + x$ between $x = 0$ and $x = t$

Question 2

a.

The normal is generally defined as the line that passes through a point on a curve and is perpendicular to the tangent at that point. For the purposes of this course, students need to at least be aware of how to find the gradient of the normal to a curve with respect to the gradient of the tangent at the same point.

The rule given for this is Gradient of normal$= -\frac{1}{m}$, where "m" is the gradient of the tangent.
This rule comes from the result that two lines with gradients m_1, and m_2, are perpendicular if, and only if, $m_1 m_2 = -1$. For those interested, this definition can be derived from some moderately interesting geometry, though this is beyond your requirements for this course.

In any case, we know that we only need the gradient of the line itself to find the gradient of the normal.

$$\begin{aligned}\text{Gradient of line} &= 2 \\ \text{Gradient of normal} &= -\frac{1}{2}\end{aligned}$$

b.

When defining a straight line, most students will recognise that they need to find its gradient and y-intercept according to $y = mx + c$, where m is the gradient, and c is the y-intercept. We already have the gradient of the normal. We only need to find the "c", and this is usually done by substituting one point into the straight-line equation and solving for c. This is a perfectly viable method, but there is perhaps a (very) slightly more efficient method of finding the straight line - for the lazy (or as some wish to be called, the efficiency seekers) amongst you, this will save several lines.

You may remember that the gradient of a line is given by $m = \dfrac{\text{rise}}{\text{run}} = \dfrac{y_2 - y_1}{x_2 - x_1}$. If we state y and x to be the variables of the equation, this result can be rearragned to achieve $y - y_1 = m(x - x_1)$ and if you substitute any point (x_1, y_1) then the result is the straight line you desire. There is no need to specifically solve for c and no need to rewrite the equation after doing so. If we were asked for a specific line, not a "general" normal, we would substitute a specific point, and not a general one as required in the question.

Hence, we substitute the general point (x_a, y_a) into general form of a line with gradient $-\frac{1}{2}$

$$\begin{aligned}y - y_a &= -\frac{1}{2}(x - x_a) \\ y &= -\frac{1}{2}x + \frac{1}{2}x_a + y_a\end{aligned}$$

c.

Students should think two things when they see the phrase "shortest distance". 1) "Shortest" should make students think about finding the minimum of a function - This method will be covered later. And 2) The line that represents the shortest distance between a point and line forms a right angle with the line. This is not the easiest thing to prove by most students (and is too involved to include here), but for those interested, a geometric proof can be found elsewhere.

Back on task, the shortest distance can be found by a line that connects the point to the original line at a right angle, i.e. the normal found above. Students have to understand that questions are often **designed** in a way that guides you through step-by-step. Especially if the question contains "hence", you know that answers found in previous parts are likely to be useful in answering following parts - in fact, you must use them to derive your answer or you will lose marks. The only time you can use an alternate method in a "hence" question is if the words "or otherwise" follow shortly after.

If we are to know the equation of the normal (i.e. a straight line), we must know at least one point. We can substitute point A into the general normal found in **part b.** to find the equation of the normal in terms of $x_a$and y_a.

$$\begin{aligned} 2 &= -\frac{1}{2}(-4) + \frac{1}{2}x_a + y_a \\ y_a &= -\frac{x_a}{2} \end{aligned}$$

Now if we were to find the intersection between this normal and the original line, we would find the point on the line that is closest to point A. We can represent any point on the normal as $\left(x_a, -\frac{x_a}{2}\right)$ i.e. If we substitute x_a into the normal, the y-value would be $-\frac{x_a}{2}$. If we substitute this "any point" into the original line (effectively doing simultaneous equations and finding the intersection), we can get the coordinates of the point at which the right angle is formed.

$$\begin{aligned} -\frac{x_a}{2} &= 2x_a - 3 \\ \frac{5x_a}{2} &= 3 \\ x_a &= \frac{6}{5} \\ \text{since } y_a &= -\frac{x_a}{2} \\ y_a &= -\frac{3}{5} \end{aligned}$$

Hence the closest point on the line to point A is $(2, 1)$. From here, we only have to apply our distance formula.

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This rule is not a difficult one to remember - if you were to forget it during the exam, go back to basics, and you will realise that on a co-ordinate axes, it's as simple as applying the pythagorean theorem and finding the distance of the hypotenuse.

$$\begin{aligned} \text{Distance} &= \sqrt{(\frac{6}{5} - (-4))^2 + (\frac{-3}{5} - 2)^2} \\ &= \sqrt{\frac{676}{25} + \frac{169}{25}} \\ &= \frac{13\sqrt{5}}{5} \text{ units} \end{aligned}$$

The shortest distance between the line and point A is $\sqrt{37}$ units. Please remember to include "units" after your answer to ensure you don't lose any marks. The vague term is required only because an actual unit has not been given.

As mentioned in the Model Solutions and above, there is an alternative solution. We can define a function that describes the distance between the general point on $y = 2x - 3$ and point A, then finding the minimum of this function. The general point on $y = 2x - 3$ is simply $(x_b, 2x_b - 3)$. The distance between this point and point A can be found by using the distance formula again.

$$\begin{aligned} \text{Distance} &= \sqrt{(-4 - x_b)^2 + (2 - (2x_b - 3)^2} \\ &= \sqrt{5x_b^2 - 12x_b + 41} \end{aligned}$$

Now that we have a function that describes distance, we only have to find the minimum of this function to find the shortest distance. As usual, "minimum" should be a trigger words for students to automatically launch into finding the derivative of this function and solving for when the derivative is equal to 0. Now while all Mathematical Methods students would be expected to be able to find the derivative of the above function by hand, there is a useful note that will make it a lot easier.

Students should realise that for a set of points all equal to or above 0 (as this square root function must be since it has not be translated down at all), if all the values were to be squared, the order of the squared values will be the same as the original order prior to being squared. In other words, squaring the smallest number will result in the smallest squared number and squaring the largest number will result in the largest squared number - take the time to think about this if it doesn't immediately click.

What does that mean for this question? If we find the x_b value that gives us the minimum Distance^2, we will actually find the x_b value for the minimum of the Distance.

$$\begin{aligned} \text{Distance}^2 &= 5x_b^2 - 12x_b + 41 \\ \frac{d}{dx_b}(\text{Distance}^2) &= 10x_b - 12 \end{aligned}$$

Now we can let this derivative equal to 0 in order to find the stationary point (which will be the minimum in this case).

$$\begin{aligned} \frac{d}{dx_b}(\text{Distance}^2) &= 0 \\ 10x_b - 12 &= 0 \\ x_b &= \frac{6}{5} \end{aligned}$$

Now that we have a value for x_b that corresponds to the minimum distance, we only have to substitute that back into the distance formula. Be careful! Do not substitute the value into the squared distance rule.

$$\begin{aligned} \text{when } x_b = \frac{6}{5}, \text{ Distance} &= \sqrt{5x_b^2 - 12x_b + 41} \\ &= \frac{13\sqrt{5}}{5} \text{ units} \end{aligned}$$

One of the issues with this method is that it doesn't provide the actual point of intersection between the line and its normal - the previous method found it. Fortunately, you are not necessarily expected to find this point since the question does not ask for it, but make sure of this if you decide to use this distance function method in the future.

Question 3

Refer to Question 3 in the Detailed Solutions of Calculus Test 1. This question is in fact more standard and straightforward than the one found in the other calculus test. As mentioned in those detailed solutions, you only need to refer to the rule for linear approximation, and then find appropriate values for x and h, and lastly define a function.

$$f(x+h) \approx f(x) + hf'(x)$$

In this question, since we wish to find an approximation of $\frac{1}{\sqrt[3]{28}}$, $f(x+h) = \frac{1}{\sqrt[3]{28}}$ since $f(x+h)$ is what we find the approximation for in the above rule.

Defining the function is usually the most difficult part of these question - in this question it isn't so difficult. It is clear that the only function that seems present is $f(x) = \frac{1}{\sqrt[3]{x}}$. For this function, our $x + h = 28$.

Don't forget that we need to find the derivative to make use of linear approximation (hence its application within calculus.

$$f'(x) = -\frac{1}{3}x^{-\frac{4}{3}}$$

In terms of defining appropriate values for x and h, we are obviously interested in a value of x that will make finding $f(x)$ easy, but we also want it to be one such that our h value (for $x + h = 28$) is relatively small. x will be easy to find if it is a perfect cube so that when the cube-root is taken, it will be an integer (or an easily calculable) result. The closest perfect cube to 28 is 27. Hence, let $x = 27$, and $h = 1$.

Now we only have to substitute all the small parts we've found above into our rule to find the approximation, $f(x+h)$

$$\begin{aligned} f(x+h) &\approx \frac{1}{\sqrt[3]{27}} + 1(-\frac{1}{3})(27^{-\frac{4}{3}}) \\ &= \frac{1}{3} - \frac{1}{243} \\ &= \frac{80}{243} \end{aligned}$$

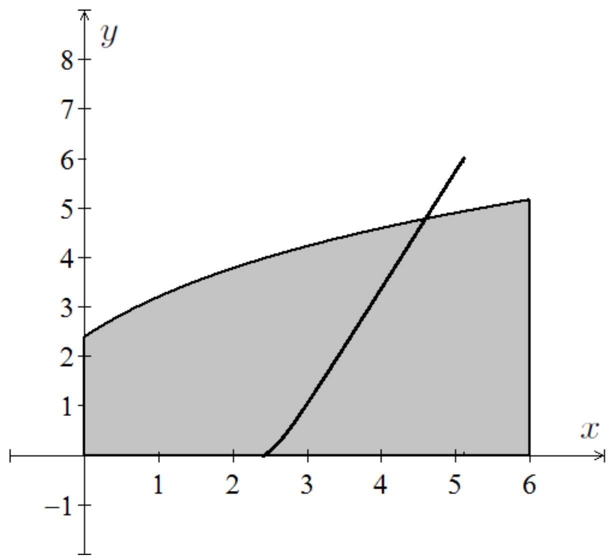
For those interested, the value we found has a decimal of 0.3292 whereas the true value of $\frac{1}{\sqrt[3]{28}}$ is in fact 0.3293 so it is within 4 decimals places of a accuracy as an approximation. This is always a good check to ensure your answer is correct (though probably not possible in this tech-free test - hence the need to approximate).

Question 4

This question teaches an important lesson in the use of an inverse in order to find an area. It arises from the fact that there are many functions that we cannot find the anti-derivative of by hand (using techniques from VCE mathematics) and hence using an integral for that function to find the area is not a possibility for us.

a.

When shading in the area, you only have to read the question very carefully and ensure that the area is touched by ALL of the bounds mentioned in the question. Fortunately, the area in this question is fairly obvious, but don't get complacent - please check to avoid losing marks and setting yourself up for error in later questions.



For notes regarding how to find inverses and what they signify, refer to the Detailed Solutions of Question 1 Algebra Tech-Free Test 2.

$$\begin{aligned} f(f^{-1}(x)) &= x \\ 2\log_e(f^{-1}(x)+2)+1 &= x \\ \log_e(f^{-1}(x)+2) &= \frac{x-1}{2} \\ f^{-1}(x)+2 &= e^{\frac{x-1}{2}} \\ f^{-1}(x) &= e^{\frac{x-1}{2}} - 2,\ x \in \mathbb{R} \end{aligned}$$

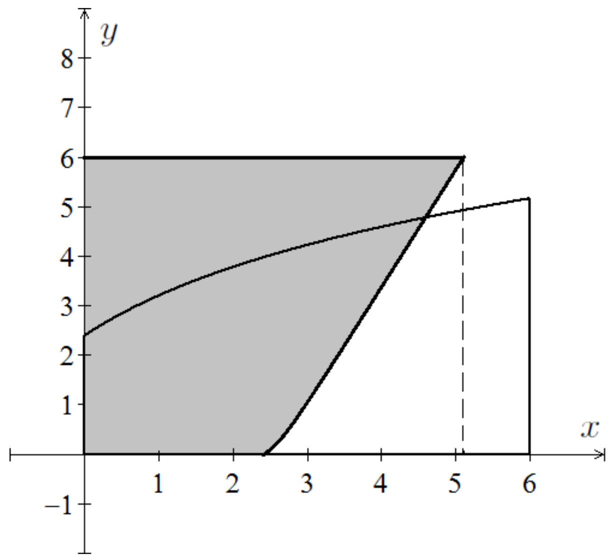
A correct sketch of this function is now required, you are only required to sketch the relevant portion that is akin to what has been provided. In essence, the point of this sketch is to mirror the field (along the $y = x$ line) in order to make it easier to find the area itself.

The shape of the exponential should be a given to all students, and so we only need to consider the axial intercepts. We know what the x-int will be based on the y-int of the original function which is $2\log_2(2)+1$ and there is no y-intercept, however it is worth noting in the $y = 6$ line since we had $x = 6$ in the original. This $y = 6$ gives an x value of $2\log_e(8)+1$ in the inverse.

b.

As stated above, it should be clear from this question why we were required to find the inverse. In this course, we are not expected to be able to antidifferentiate logarithmic functions, however, we must be able to antidifferentiate the exponential function. Now we can evaluate an integral in order to find the area of the field, but not without some more thought.

From the diagram, the area between the y-axis and the inverse from 0 to $y = 6$ is the same as the area required from the original function. However, we know how to find the area between the curve and the x-axis not the y-axis. What can we do? Construct a rectangle with the height from 0 to $y = 6$ and the base from $x = 0$ to the corresponding x-value for when $y = 6$. Now consider the area under the exponential from where it cuts the x-axis to the corresponding x-value for when $y = 6$. If you subtract this area from the rectangle, what area do you end up with? That's right, the area you desire as shown below.



Now to find those necessary x-values that make up the interval for the area under the exponential.

$f^{-1}(x)$ cuts the x-axis when $f(x)$ cuts the y-axis, i.e. $2\log_e(0+2)+1 = 2\log_e(2)+1$
We now need to find the x-value for when $f^{-1}(x) = 6$

$$\begin{aligned} 6 &= e^{\frac{x-1}{2}} - 2 \\ x &= 2\log_e(8)+1 \end{aligned}$$

Hence, we can find the area underneath the inverse between the x-intercept and $2\log_e(8)+1$

$$\begin{aligned} \int_{2\log_e(2)+1}^{2\log_e(8)+1} \left(e^{\frac{x-1}{2}} - 2\right) dx &= \left[2e^{\frac{x-1}{2}} - 2x\right]_{2\log_e(2)+1}^{2\log_e(8)+1} \\ &= \left[2e^{\frac{2\log_e(8)+1-1}{2}} - 2(2\log_e(8)+1)\right] - \left[2e^{\frac{2\log_e(2)+1-1}{2}} - 2(2\log_e(2)+1)\right] \\ &= 2e^{\log_e(8)} - 4\log_e(8) - 2 - (2e^{\log_e(2)} - 4\log_e(2) - 2) \\ &= 2(8) - 4\log_e(2^3) - 2 - 2(2) + 4\log_e(2) + 2) \\ &= 16 - 12\log_e(2) - 2 - 4 + 4\log_e(2) + 2) \\ & \quad 12 - 8\log_e(2) \end{aligned}$$

Students should take note of the log laws being used in the above calculations, especially $e^{\log_e a} = a$ and $\log_e(a^b) = b \times \log_e(a)$. Knowing these rules will make calculations significantly easier.

If we subtract the area found above from the square formed by the axes, $y = 6$ and $x = 2\log_e(8) + 1$ we get the required area.

$$\begin{aligned}
\text{Area of field} &= \text{Area of rectangle - Area under the exponential} \\
&= 6 \times (2\log_e(8) + 1) - (12 - 8\log_e(2)) \\
&= 44\log_e(2) - 6
\end{aligned}$$

Hence the area of Farmer Bebbington's field is $44\log_e(2) - 6$ square units

Take note of the above method - this question is a particular type that students should be aware of and the method of finding the inverse, then subtracting an area from a rectangle is one that can be always be applied when you can't find the area by an integral because the anti-derivative of the function cannot be easily calculated.

CALCULUS

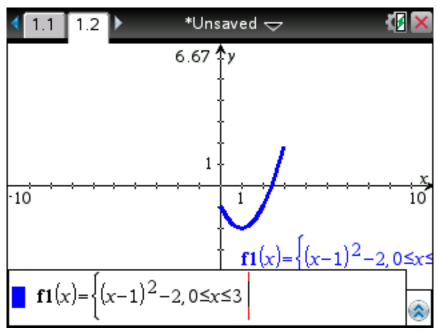
TECH-ACTIVE TEST 1

DETAILED SOLUTIONS

SECTION 1 - Multiple Choice Questions

Question 1 (B)

The question asks for the maximum and minimum value of the function, now these could be at either turning points or at endpoints. Completing the square using "[Menu] [3] [5]", so that we can get it into turning point form 'completeSquare($x^2 - 2x - 1$, x)' gives $y = (x-1)^2 - 2$. You should always plot the graph so that you don't blindly plug in a turning point or endpoint incorrectly as a maximum or minimum.



As shown on the calculator, the minimum will be at the turning point, that is at $(1, -2)$, so our minimum value will be -2. The maximum will occur at the right end point, substituting the value of x in, $f(3) = 2$. So the maximum value is 2. So the answer is **B**.

Question 2 (C)

Since we are given the derivative function, and want to find the original function, we need to integrate, but this will give a family of antiderivatives, so we must **add the constant C**, and solve for it, to find the unique anti-derivative, which will be the equation of the curve that passes through the point $(0, 2)$.

$$\int \left(\frac{1}{2}e^{3x} - \sin\left(\frac{1}{2}x\right)\right) dx = \frac{1}{6}e^{3x} + 2\cos\left(\frac{1}{2}x\right) + C$$

Substituting in the point $(0, 2)$ and solving for C

$$\begin{aligned} \frac{1}{6}e^{3(0)} + 2\cos(\frac{1}{2}(0)) + C &= 2 \\ C &= -\frac{1}{6} \\ \therefore f(x) &= \frac{1}{6}e^{3x} + 2\cos(\frac{1}{2}x) - \frac{1}{6} \end{aligned}$$

That leaves option **C** as the answer.

Question 3 (B)

The average value of a function over the interval $[a, b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)dx$$

If the test is calculator allowed, **always plot the curve** so that you can get an idea of the situation, and so that you don't make any silly errors.

The quickest way to obtain the answer here is to use the integral function on the calculator, "[⇑]+[−]" and use abs for our modulus/absoulte value function.

If we were to do this by hand, we would need to split the modulus function, up, then integrate it. The modulus makes whatever is inside it positive. So when the original function is negative, it is flipped in the x-axis, and stays the same elsewhere. So our function is

$$f(x) = \begin{cases} x^2-4 & x < -2 \\ 4-x^2 & -2 \leq x \leq 2 \\ x^2-4 & x > 2 \end{cases}$$

So our average value of the function will be

$$\begin{aligned} y_{avg} &= \frac{1}{3-(-3)}\left(\int_{-3}^{-2}\left(x^2-4\right)dx + \int_{-2}^{2}\left(4-x^2\right)dx + \int_{2}^{3}\left(x^2-4\right)dx\right) \\ &= \frac{1}{6}\left(\frac{7}{3}+\frac{32}{3}+\frac{7}{3}\right) \\ &= \frac{23}{9} \end{aligned}$$

We could also note that since the function is an even function, we can find the average value over the intervale $[0, 3]$, which will result in the same average value over the interval $[-3, 3]$.

$$\begin{aligned} y_{avg} &= \frac{1}{3-0}\int_0^2\left(4-x^2\right)dx + \int_2^3\left(x^2-4\right)dx \\ &= \frac{1}{3}\left(\frac{16}{3}+\frac{7}{3}\right) \\ &= \frac{23}{9} \end{aligned}$$

So our answer would be option **B**.

Question 4 (A)

We are given the graph of a function and asked to pick the derivative function. Since our original function appears to be a cubic, the derivative function should look like a parabola, so that knocks out options C and E as they are a quartic and a cubic respectively. As stationary points have a gradient of zero, when we derive a function, the

derivative will pass through the x-axis at the same value of x as the turning point of the original funciton. There appears to be a local maximum near $x = -1$, but for the quesiton we can approximate a turning point just left of $x = -1$, so there will be an x intercept just left of $x = -1$ for the graph of the derivative. Since the gradient is negative to the left of this turning point and positive to the right, the graph of the derivative should be negative to the left of this intercept and positive to the right of this intercept. The gradient is also zero at $x = \frac{1}{2}$, so the curve of the derivative passes through the x-axis at this value of x. Finally the gradient is positive for $x > \frac{1}{2}$, so the derivative is above the x-axis for these values of x.

The only option that satisfies all of this is option **A**.

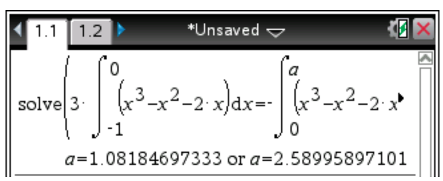
Question 5 (A)

Since we need to equate the two areas, we first need to find expressions for the two areas. We also note that according to the diagram, $0 < a \leq 2$. So that rules out options D and E.

We equate the areas under the curve by the following integrals and their terminals, then solve for a. Note that since the region B is below the x-axis, to get the area we have a negative in front of the integral.

$$3\int_{-1}^{0} (x(x-2)(x+1))\,dx = -\int_{0}^{a} (x(x-2)(x+1))\,dx$$
$$3\int_{-1}^{0} \left(x^3 - x^2 - 2x\right)dx = -\int_{0}^{a} \left(x^3 - x^2 - 2x\right)dx$$

Solving using the calculator, "[Menu] [3] [1]" for solve and "[⇑ shift] [+]" for the integral, we obtain 2 solutions.



Since the value of a has to be between 0 and 2 we reject the second solution, leaving $a = 1.0818$ as the answer, so option **A** is correct.

Question 6 (D)

We are asked to find the nature of the stationary points of the function from the sign of the derivative at particular intervals, that is the whether the gradient is positive or negative. The best way to approach this problem is to draw out a gradient table.

x	-2	-1	0	$\frac{5}{2}$	3
$f'(x)$	$+ve$	0	$+ve$	0	$-ve$
Shape	/	—	/	—	\

Stationary points of inflections have a gradient that is the same sign on both sides of the stationary point, where the gradient is 0. Local maximums go from a positive gradient, to 0, then to negative as you go from left to right. Local minimums go from a negative gradient, to 0, then to positive as you go from left to right.

As the gradient before $x = -1$ is positive, then 0 at $x = 1$ and positive to the right, there is a stationary point of inflection at $x = -1$. At $x = \frac{5}{2}$, the gradient to the left is positive, then zero at $x = \frac{5}{2}$ and negative to the right, so we have a local maximum at $x = \frac{5}{2}$. So option **D** is correct.

Question 7 (C)

Since we are asked for the derivative of the product of two functions, we will need to apply the product rule,

$$\text{If } y = u(x)v(x)$$

$$\text{then}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

So our two functions are $u(x) = e^{f(x)}$ and $v(x) = \cos(f(x))$, using an application of the chain rule to find the individual derivatives.
Using the chain rule, if $y = e^w$ then $\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dx}$. There are a lot of variables here because we are using a chain rule within the product rule. You might consider this to be "calculus-ception". It is therefore a bit confusing, so take the time to read carefully and fully understand what's going on here.

$$u'(x) = e^{f(x)} f'(x)$$
$$v'(x) = -\sin(f(x)) f'(x)$$

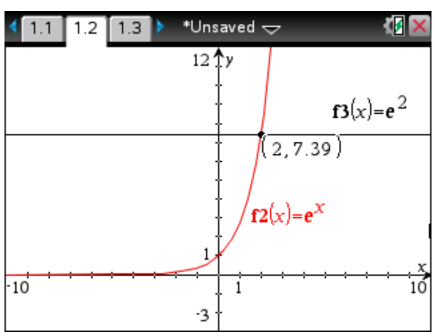
Using the product rule,

$$\begin{aligned} \frac{dy}{dx} &= e^{f(x)} \times -\sin(f(x)) f'(x) + \cos(f(x)) \times e^{f(x)} f'(x) \\ &= e^{f(x)} f'(x) (\cos(f(x)) - \sin(f(x))) \end{aligned}$$

So that makes option **C** the correct answer.

Question 8 (E)

We have two functions $f(x) = e^2$ and $g(x) = e^x$, and we want the area between them. So firstly we need to find the terminals of integration, so a quick sketch on the calculator and solving for the intersection gives an intersection at $x = 2$.



So our terminals are from 0 to 2 with $y = e^2$ being the upper function. For the area we want we need to integrate the upper function take the lower function. Using "[⇑ shift] [+]" to obtain an integral on the calculator we obtain.

$$\begin{aligned} \int_0^2 \left(e^2 - e^x\right) dx &= \left[e^2 x - e^x\right]_0^2 \\ &= 2e^2 - e^2 - \left(0 - e^0\right) \\ &= e^2 + 1 \end{aligned}$$

Hence the answer is **E**.

Question 9 (E)

Two lines that are perpendicular have gradients that multiply to -1, that is $m_1 m_2 = -1$. So we need the gradient of the tangent. Differentiating the function,

$$\begin{aligned} y &= x^2 - 4x + 4 \\ \frac{dy}{dx} &= 2x - 4 \end{aligned}$$

If we make m_1 the gradient of the tangent and m_2 to be perpendicular to the gradient of the tangent, substituting in the point to find the gradient

$$\begin{aligned} m_1 &= 2(3) - 4 \\ &= 2 \\ m_2 &= -\frac{1}{m_1} \\ &= -\frac{1}{2} \end{aligned}$$

The only answer with this gradient is option **E**.

Question 10 (E)

To find the angle between the two tangents, we need to find the gradient of the tangents, then the angle will be the difference between the two angles that those tangents make with the positive direction of the x-axis. So the angle we are looking for is α where $\alpha = \tan^{-1}(m_1) - \tan^{-1}(m_2)$.

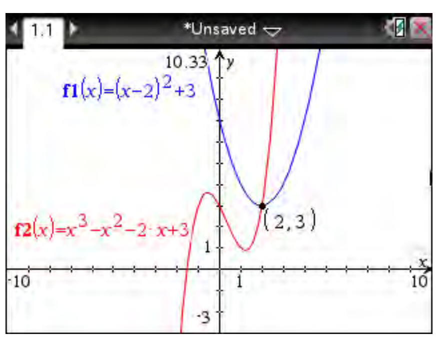
Firstly we need to know where the two functions intersect. Solving $f(x) = g(x)$ using the solve function on the calculator, "[Menu] [3] [1]" gives $x = 2$. So Finding the gradient of the tangents, we need to derive each function,

$$\begin{aligned} f'(x) &= 3x^2 - 2x - 2 \\ g'(x) &= 2(x - 2) \end{aligned}$$

Substituting in $x = 2$ gives $f'(2) = 6$ and $g'(2) = 0$. So the acute angle between the tangents is

$$\begin{aligned} \alpha &= \tan^{-1}(2) - \tan^{-1}(0) \\ &= 80.54^o \end{aligned}$$

(Make sure your calculator is in degrees for this part).



This makes **E** the correct answer.

Question 11 (A)

A stationary point of a function will occur when the gradient, and thus derivative is zero. Since the given function is a cubic, it can have either 0 stationary points, 2 tunring points or 1 stationary point of inflection.

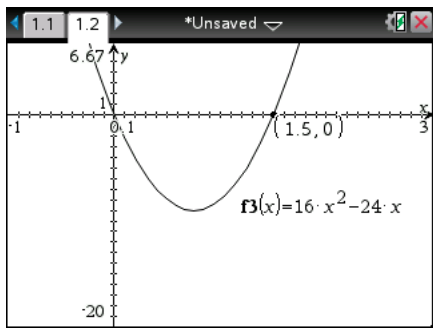
This implies that there will be two turning points when there is two unique solutions to $f'(x) = 0$

$$f'(x) \quad = \quad 3ax^2 + 4ax + 2 = 0$$

If the discriminant $\left(\triangle = b^2 - 4ac\right)$ is greater than 0, then we have two unique solutions.

$$\begin{aligned} \triangle &= (4a)^2 - 4\,(3a)\,(2) \\ &= 16a^2 - 24a \\ 8a\,(2a-3) &> 0 \end{aligned}$$

We can see that this has intercepts at $(0, 0)$ and $(\frac{3}{2}, 0)$. We should aslo note that if $a = 0$, then the cubic reduces to a linear function and would not satisfy the question. Drawing it out on the calculator gives



So the graph is greater than 0 for $R \setminus \left[0, \frac{3}{2}\right]$. This means the function has two turning points when $a \in R \setminus \left[0, \frac{3}{2}\right]$. Hence option **A** is correct.

SECTION 2 - Extended Response Questions

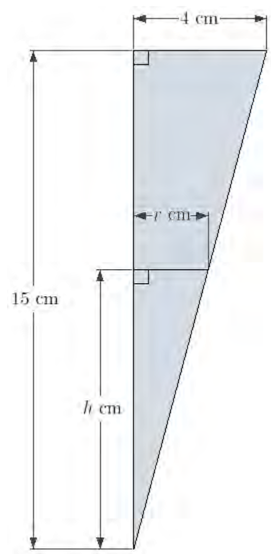
Question 1

Before we start answering the parts to the question, take a look at what information we're giving and the situation. We have

- an **inverted** cone that is being filled with ice-cream
- radius of 4 cm
- heigth of 15 cm
- h is the height of the **ice-cream**
- r is the radius of the surface of the **ice-cream**

Part a.

We are asked to find r in terms of h, so we need to find some relationship between the two. If we view a cross section of the ice-cream cone, then we have two similar triangles. They are similar because all three angles inside both are the same.



So the ratio of the side lengths is

$$\begin{aligned} \frac{4}{r} &= \frac{15}{h} \\ r &= \frac{4h}{15} \end{aligned}$$

Part b.

The voulme of a cone is given by

$$V = \frac{1}{3}\pi r^2 h$$

Since we have r in terms of h, we can substitute this into the formula above to find V in terms of h

$$\begin{aligned} V &= \frac{1}{3}\pi \left(\frac{4h}{15}\right)^2 h \\ &= \frac{16\pi}{675} h^3 \text{ cm}^3 \end{aligned}$$

Part c.

Now since we are looking for the height h, when the cone is *"half filled"*, we need to find the total volume of the cone and half it. Since the maximum height is 15 cm, we substitute that into our expression for h.

$$\begin{aligned} V_{max} &= \frac{16\pi}{675}(15)^3 \\ &= 80\pi \text{ cm}^3 \end{aligned}$$

Half the volume and equating it to our expression for V

$$\begin{aligned} \frac{16\pi}{675}h^3 &= 40\pi \\ h^3 &= 1687.5 \\ h &= 11.91 \text{ cm} \end{aligned}$$

Part d.

Now we are filling the cone, and we know that when $h = 5$cm that the *"the volume is increasing at* $20\text{cm}^3/\text{s}$". That is

$$\frac{dV}{dt} = 20 \text{ cm}^3/\text{s}$$

Now what we are actually looking for is *"the rate at which the height is increasing when* $h = 5$ cm", that is $\dfrac{dh}{dt}$.
Using the chain rule

$$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$$

Now we already have $\frac{dV}{dt}$, but we need to differentiate $V = \dfrac{16\pi}{675}h^3$,

$$\begin{aligned} \frac{dV}{dh} &= 3 \times \frac{16\pi}{675}h^2 \\ &= \frac{16}{226}\pi h^2 \end{aligned}$$

Make sure you flip the $\frac{dV}{dh}$ when you substitute it into $\frac{dh}{dt}$.
When $h = 5$ cm

$$\begin{aligned} \frac{dh}{dt} &= \frac{225}{16\pi \times 5^2} \times 20 \\ &= 3.58\, cm/s \end{aligned}$$

Part e.

We are told that "ice-cream is added at a rate of $20t\ cm^3/s$", that is

$$\frac{dV}{dt} = 20t\ \mathrm{cm^3/s}$$

If we integrate with respect to t, we will get a family of antiderivatives, adding the constant of integration, $+C$, we can then use the initial condition of $V = 0$ when $t = 0$ to find C, as the cone is initially empty.

$$\begin{aligned}
\frac{dV}{dt} &= 20t \\
\int \frac{dV}{dt} dt &= \int 20t dt \\
V(t) &= 10t^2 + C \\
\text{When } V = 0\,\mathrm{cm^3}, t = 0\ s & \\
\therefore C &= 0 \\
V(t) &= 10t^2
\end{aligned}$$

Substituting in the maximum volume and solving for t

$$\begin{aligned}
V = 80\pi & \\
10t^2 &= 80\pi \\
t^2 &= 8\pi\ s \\
t &= \pm 2\sqrt{2\pi}\ s
\end{aligned}$$

But $t > 0$

$$\therefore t = 2\sqrt{2\pi}\ s$$

So it takes $2\sqrt{2\pi}$ seconds to fill the cone with ice-cream**Question 2**

Part a.

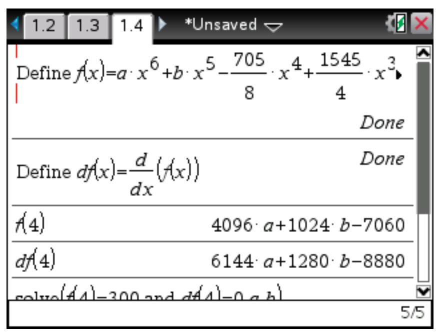
We have 2 unknowns to find, so that means we need 2 equations. The first piece of information we have is the point itself, so we can substitute that into the function to get our first equation.

$$\begin{aligned} f(4) &= 300 \\ 4096a + 1024b - 7060 &= 300...[1] \end{aligned}$$

Now we also have a second piece of information, that is the minor peak occurs when $x = 4\,km$, that is we have a turning point here. The gradient at a turning point is 0, so that is the derivative will be 0 , when $x = 4\,km$. This allows us to find our second equation.

$$\begin{aligned} f'(4) &= 0 \\ 6ax^5 + 5bx^4 - \frac{705}{2}x^3 + \frac{4635}{4}x^2 - 1215x &= 0...[2] \end{aligned}$$

The quickest way to do this on the calculator is to use a small shortcut that can so that we don't have to continually type out long equations. That is we can define the function using "[Menu] [1] [1]", we then type in $f(x) = ...$ Next then refer to the function using $f\,(<x$ value $>)$. I.e. We can type $f(3)$ without having to type out the entire equation again. If we were to take the derivative of the function, we would still end up with quite a messy function, but we can define a derivative $df(x)$, in terms of $f(x)$. To do this we again use "[Menu] [1] [1]" to bring up define, then the shortcut keys "[⇑ shift] [−]", and place x in the lower box that appears, as we are differentiating with respect to x. To solve the equation we can then use "[Menu] [3] [1]" to bring up the solve function, then type *"solve(*$f(4) = 300$ *and* $df(4) = 0, a, b$*)"*. Your screen should look something like the screenshot below.



Solving the two equations gives

$$\therefore a = -\frac{5}{16}, \qquad b = \frac{135}{16}$$

Part bi.

The intercepts of a curve with the x and y axis will occur when $y = 0$ and $x = 0$ respectively.

$$\begin{aligned} x = 0, \quad f(0) = &\ 500 \\ &\ (0, 500) \\ y = 0, \quad &\ 0 = -\frac{5}{16}x^6 + \frac{135}{16}x^5 - \frac{705}{8}x^4 + \frac{1545}{4}x^3 - \frac{1215}{2}x^2 + 500 \\ &\ x = -0.73,\ 2,\ 5 \end{aligned}$$

But our function is only defined for $x \in [0, 5]$, so we reject the $x = -0.73$ solution.

$$\begin{aligned} \therefore x &= 2,\ 5 \\ & \quad (2, 0)\,,\ (5, 0) \end{aligned}$$

Part bii.

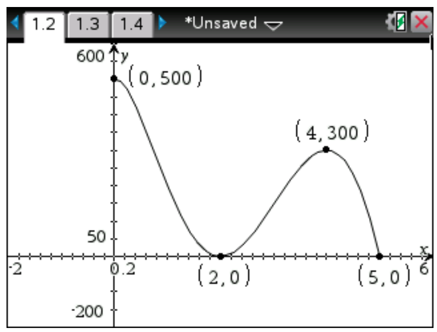
Stationary points have a gradient of 0, so that means our derivative, $f'(x) = 0$. To solve use "[Menu] [3] [1]", then place the equation in, putting a comma and the variable we want to solve for before closing the bracket. *"solve(df(x) = 0, x)"* which gives $x = 0,\ 2,\ 4$. (which is inside our domain, make sure to check this)
Now we need to find the actual points, so substituing in those values of x we get

$$\begin{aligned} f(0) &= 500 \\ f(2) &= 0 \\ f(4) &= 300 \end{aligned}$$

So our stationary points are at $(0, 500)\,,\ (2, 0)\,,$ and $(4, 300)$

Part biii.

Before we sketch anything, its always best to note the minimum and maximum domain and range of the curve. Here our domain is $0 \le x \le 5$ and our range is $0 \le y \le 500$. You should also sketch it on the calculator to get a general idea of the shape of the curve. Not all curves will show up on the right scale right away on the calculator, you could use "[Menu] [4] [A]" to get an automatically fitted window, but it doesn't always produce the best results. If needed set it manually using "[Menu] [4] [1]", and for each option set a little bit more than what the domain or range is.
e.g. $XMin:\ -2$, $XMax:\ 6$, $YMin:\ -200$ and $YMax:\ 600$



When you draw the curve, make sure that the gradient is 0 at $x = 0,\ 2,\ ,4$, that is even though $(0, 500)$ is an endpoint, it is still a stationary point, and so the curve has to flatten out at this point.

Part c.

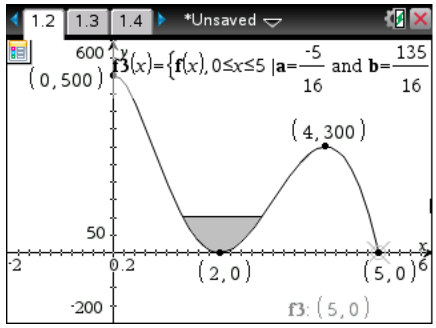
The best way to approach this question is to add to the graph from the previous question. Since the water fills up between the two peaks to a height of 100 m, we can draw a horizontal line of $y = 100$ between the two peaks. Since we need the area between this line and the function, we need to find the intersection of the two.

$$\begin{aligned} -\frac{5}{16}x^6 + \frac{135}{16}x^5 - \frac{705}{8}x^4 + \frac{1545}{4}x^3 - \frac{1215}{2}x^2 + 500 &= 100 \\ x &= -0.665,\ 1.300,\ 2.795,\ 4.831 \end{aligned}$$

But the intersections we are interested in are for $0 < x < 4$, so we can reject the other solutions.

$$\therefore x = 1.300,\ 2.795$$

Now we are looking for the area that is between the two functions, since $y = 100$ is greater than $f(x)$ for the region we are interested in, we will integrate the top function minus the lower function, with the terminals being the intersection of the two curves. This is give the shaded region that is in the graph below.



Now since the y axis is in metres and the x axis is in kilometres, the area that we would get would be in $km \times m$. There are two ways of looking at this.

The first is since the question asks for the cross-sectional area in m^3, we can multiply the resulting area by 1000, to convert into m^2.

$$1000 \int_{1.300}^{2.795} \left(100 - \left(-\frac{5}{16}x^6 + \frac{135}{16}x^5 - \frac{705}{8}x^4 + \frac{1545}{4}x^3 - \frac{1215}{2}x^2 + 500\right)\right) dx \quad = \quad 97,882.752\, m^2$$

The second is through a dilation from the y axis by a factor of 1000. This changes the x values of the function. The mapping for this dilation would be

$$(x, y) \quad \rightarrow \quad (1000x, y)$$

So solving for x' and y'

$$\begin{aligned} x' &= 1000x \\ x &= \frac{1}{1000}x' \\ y' &= y \end{aligned}$$

This also means the terminals of the integral would change, so our integral becomes

$$\int_{1300}^{2795} \left(100 - f\left(\frac{x}{1000}\right)\right) dx \quad = \quad 97,882.752\, m^2$$

This is only the cross-sectional area, the mountain range is also $1000\, m$ long, so we need to multiply by 1000 to get the **volume** of water in the temporary dam.

$$\begin{aligned} \text{Volume} &= 97,882.752 \times 1000 \\ &= 97,883,000\, m^3 \end{aligned}$$

Part d.

The angle a line or a tangent makes with the positive direction of the x axis is given by $\tan(\theta) = m$ where m is the gradient of the line. So to find the angle Nina hits the water at, we need two things, the point at which she hits the water and the gradient of the curve at that point. We already know this point to be $(2.795, 100)$ from the previous question.

To find the gradient, we need the derivative of the function.

$$\begin{aligned} x &= 2.795 \\ f'(2.795) &= 214.343\, m/km \end{aligned}$$

Since the x axis is in kilometres and the y axis is in metres, the gradient will be m/km, so we need to convert it to m/m by dividing by 1000.

$$\begin{aligned} &= \frac{214.343}{1000}\, m/m \\ &= 0.214 \end{aligned}$$

Since the question asks for the answer in **degrees**, be careful of what **mode** your calculator is in.

Equating this gradient to $\tan(\theta)$

$$\begin{aligned} \tan(\theta) &= 0.214 \\ \theta &= \tan^{-1}(0.214) \\ &= 12.08^o \end{aligned}$$

$\therefore$ Nina hits the water at 12.08^o

Part e.

For the temporary dam to overflow the water must fill to a height above that of the smaller peak, that is the water needs to reach $y = 300$ m. So our terminals are going to change, but other than that its similar to the previous question. We need to set up our integral for the cross-sectional area, account for the different units, then mutiply by the length of the mountain range.

Solving for the intersection between the curve and $y = 300$.

$$\begin{aligned} -\frac{5}{16}x^6 + \frac{135}{16}x^5 - \frac{705}{8}x^4 + \frac{1545}{4}x^3 - \frac{1215}{2}x^2 + 500 &= 300 \\ x &= -0.493,\ 0.738,\ 4 \end{aligned}$$

But for out intersection $0 < x < 4$

So $x = 0.738$, and our upper terminal will still be $x = 4$.

Then using the same method as the previous question

$$\begin{aligned} 1000\int_{0.738}^{4}\left(300 - \left(-\frac{5}{16}x^6 + \frac{135}{16}x^5 - \frac{705}{8}x^4 + \frac{1545}{4}x^3 - \frac{1215}{2}x^2 + 500\right)\right) dx &= 550641.708\, m^2 \\ \text{Volume} &= 550641.708 \times 1000 \\ &= 550{,}642{,}000\, m^3 \end{aligned}$$

CALCULUS

TECH-ACTIVE TEST 2

DETAILED SOLUTIONS

SECTION A - Multiple-choice questions

Question 1 (B)

To find the derivative of two multiplied functions, $g(x) = \log_e(f(3x+1))$ and $h(x) = x^2$ we need to use the product rule. We are going to need to use the chain rule twice, since we have $f(3x+1)$ inside the log and the log itself

$$\frac{d}{dx}(f(3x+1)) = 3f'(3x+1)$$

Then to look at the log

$$\frac{d}{dx}(\log_e(f(x))) = \frac{f'(x)}{f(x)}$$

So by the chain rule twice we have

$$\begin{aligned} g'(x) &= \frac{3f'(3x+1)}{f(3x+1)} \\ h'(x) &= 2x \end{aligned}$$

By the product rule,

$$\begin{aligned} y' &= g(x)h'(x) + h(x)g'(x) \\ \frac{dy}{dx} &= 2x\log_e(f(3x+1)) + 3x^2\frac{f'(3x+1)}{f(3x+1)} \end{aligned}$$

So option **B** is our answer.

Question 2 (B)

As the two tangents are parallel, their gradients are equal. If they weren't equal then they would meet at some point and wouldn't be parallel. The gradient of the tangents at a particular point are found by finding the derivative and substituting in that point.

$$\begin{aligned} f'(4) &= g'(4) \\ \frac{-2}{4-2} &= -2(4-a) \\ \therefore a &= \frac{7}{2} \end{aligned}$$

So **B** is the correct answer.

Question 3 (B)

The approximation is giving by the area of the shaded rectangles. Each rectangle will have an area of length $\times$ height where the length is 1 unit and the height is given by the value of the function at that value of x.

$$\begin{aligned}
\text{Area} &= ((1)\times f(1)) + (1\times f(2)) + (1\times f(3)) \\
&= 2\log_e(1+1) + 2\log_e(2+1) + 2\log_e(3+1) \\
&= \log_e\left(2^2\times 3^2\times 4^2\right) \\
&= \log_e(576)
\end{aligned}$$

If we evaluate this on the calculator, then we get $2\log_e(24)$. We need to convert this into an equivalent expression, so using our log rules, that the coefficient out the from of the log can be brought into the log as a power since

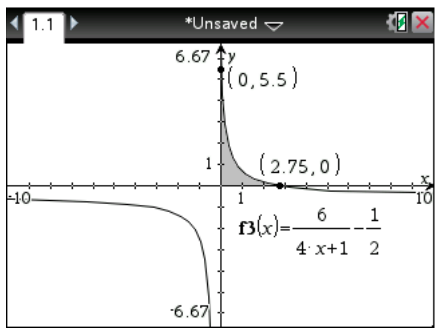
$$\begin{aligned}
2\log_e(24) &= \log_e(24) + \log_e(24) \\
&= \log_e(24\times 24) \\
&= \log_e\left(24^2\right) \\
&= \log_e(576)
\end{aligned}$$

i.e. $a\log_e(x) = \log_e(x^a)$

So the required answer is **B.**

Question 4 (A)

Firstly draw the graph on the calculator to get an idea of the situation.



So we are looking for the shaded area, as this is bound by the two axis and the curve. To find the terminals of the integral, we need to solve for the x intercept of the function, $f(x)=0$, which gives $\left(\frac{11}{4},0\right)$. Then integrating that using "[⇑ shift] + [+]", then clear the terminals using "[del]" and integrate the function.

$$\int\left(\frac{6}{4x+1}-\frac{1}{2}\right)dx = \frac{3\ln(|4x+1|)}{2} - \frac{1}{2}x + C$$

So evaluating the definite integral will mean we will need to evaluate

$$\int_0^{\frac{11}{4}}\left(\frac{6}{4x+1}-\frac{1}{2}\right)dx = \left[\frac{3\ln(|4x+1|)}{2} - \frac{1}{2}x\right]_0^{\frac{11}{4}}$$

Thus, the required answer is **A.**

Question 5 (E)

A function will have only one stationary point when there is <u>only</u> one solution to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 3ax^2 + 4ax - b \\ &= 0 \end{aligned}$$

Using the discriminant, $\triangle = b^2 - 4ac$ (for $y = ax^2 + bx + c$), there is one solution when $\triangle = 0$.

$$\begin{aligned} (4a)^2 - 4(3a)(-b) &= 0 \\ 4a(4a + 3b) &= 0 \\ a = 0 \text{ or } a &= -\frac{3b}{4} \end{aligned}$$

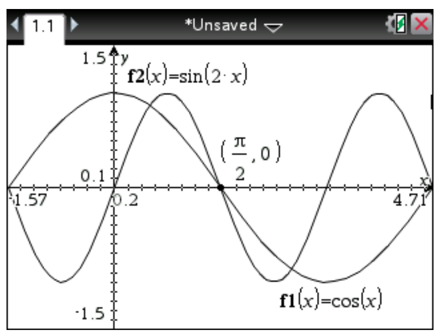
But $f(x)$ won't be a cubic if $a = 0$, which as a is a multiple of b means that $b \neq 0$.

$$\therefore a = -\frac{3b}{4} \text{ and } b \neq 0$$

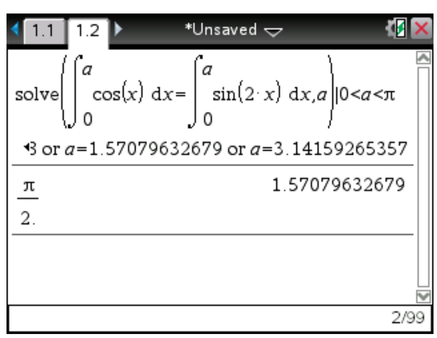
So **E** is the correct answer.

Question 6 (D)

Again, firstly graph the situation out so that we don't make any mistakes in calculating the correct area.

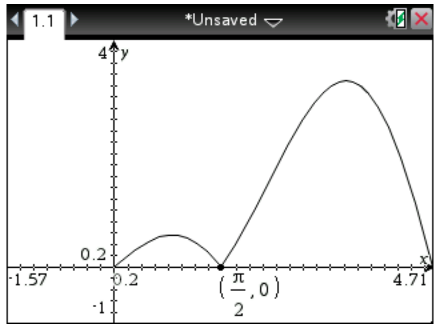


So as we can see that since the area is bettwen both axis and the curves can be anywhere from 0 up to $\frac{\pi}{2}$, that means our value of a which is our upper bound of the region, has to be $0 < a < \frac{\pi}{2}$. Integrating the area under the two curves using "[⇑ shift]+[+]", starting at $x = 0$ to $x = a$, letting them equal each other and solving for a with the domain restriction by adding "$|0 < a < \pi$" to the end of the solve function.



Now since the first and last solution are outside our domain and the approximate value of the middle solution is $\frac{\pi}{2}$, that makes **D** our answer.

Question 7 (C)



Firstly we note that 4 out of 5 of the options are integrals that are broken up and do not involve the modulus function and the option that does isn't calculating the average value. So we need to break the integral up into parts, depending on whether $x\cos(x)$ is above or below the x-axis. Using trace on the graph page, "[Menu] [5] [1]" we find that the $x$intercept of $f(x) = x\cos(x)$ is at $\left(\frac{\pi}{2}, 0\right)$. So that is where we need to split the integral. Now if we look at the original function (that is without the modulus), then $f(x) > 0$ for $0 < x < \frac{\pi}{2}$ so the integral will be positive. For $\frac{\pi}{2} < x < \frac{3\pi}{2}$, $f(x) < 0$, so the integral will be negative. Now the average value is given by $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$, we need to put the $\dfrac{1}{b-a}$ out the front of the total area, that is in front of the two integrals added together. Since the second region of the intergral is below the x-axis for $f(x)$, but the equivalent area is above the x axis for $|f(x)|$, we need to put a negative in front of the integral. That is our average value is given by

$$y_{avg} = \frac{2}{3\pi}\left(\int_0^{\frac{\pi}{2}} (x\cos(x))\,dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (x\cos(x))\,dx\right)$$

An alternate more time-consuming way to approach this question if you get really stuck is to evaluate the integral for the average value with the modulus on the calculator, that is

$$\begin{aligned} y_{avg} &= \frac{1}{\frac{3\pi}{2} - 0}\int_0^{\frac{3\pi}{2}} |x\cos(x)|dx \\ &= 1.4545 \end{aligned}$$

Now evaulating each of the options the only one that gives the same answer is

$$\frac{2}{3\pi}\left(\int_0^{\frac{\pi}{2}} (x\cos(x))\,dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (x\cos(x))\,dx\right)$$

That is option **C** is the correct answer.

Question 8 (C)

This is a common question involving the properties of definite integrals that students should be familar with. Since we know that $\int_0^{10} f(x)dx = 2a$, we can expand our expression out and substitute it in, then simplify

$$\begin{aligned}
\int_0^{10} (6f(x) - 2)\,dx &= \int 6f(x)dx - \int_0^{10} (2)\,dx \\
&= 6\int_0^{10} f(x)dx - \int_0^{10} (2)\,dx \\
&= 6\,(2a) - [2x]_0^{10} \\
&= 12a - (20 - 0) \\
&= 12a - 20
\end{aligned}$$

Question 9 (E)

The key to this question is getting the inequalities right in the domains for the derivative. When we have a hybrid function the derivative will only be defined at the end points of the domain of the functions inside the hybrid, when the graph at these end points is "smooth". That is the gradient of both the functions at that end point is equal. If we check at $x = 2$, for the left function $f'(2) = 2\,(2-1) = 2$, for the right function $f'(2) = 2$. So the gradient is smooth at $x = 2$, so we can include $x = 2$ in the domain. Checking at $x = 4$, for the left function $f'(4) = 2$, for the right function, $f'(4) = 0$, so the gradient is not smooth at $x = 4$ and so $x = 4$ is not included in the domain. So we have

$$f'(x) = \begin{cases} 2\,(x-1) & x \le 2 \\ 2 & 2 < x < 4 \\ 0 & x > 4 \end{cases}$$

Question 10 (A)

- If the gradient goes from "positive to zero to negative", then we have a local maximum.
- If the gradient goes from "negative to zero to positive" then we have a local minimum.
- If the gradient is the same on both sides of the point at which the gradient is zero, then we have a stationary point of inflection.

Since the gradient to the left of $x = -1$ is positive, and to the right negative, and 0 at $x = -1$, this point is a local maximum. $-ve \to 0 \to +ve$

Since the gradient to the left and right of $x = 1$ is negative, and 0 at $x = 1$, there is a stationary point of infleciton at $x = 1$. $-ve \to 0 \to -ve$. (Stationary Points of inflection may also have $+ve \to 0 \to +ve$)

Since the gradient to the left of $x = 3$ is negative, and to the right positive, and 0 at $x = 3$, this point is a local minimum. $+ve \to 0 \to -ve$

The best way to approach the question, is to draw up a gradient table and draw it out, so that we can visualise each stationary point. We pick the points that we know have a gradient of 0, that is $x = -1,\ 1,\ 3$. Then we can pick any point between the values to test, it is best to pick a whole number as it is easier to calculate, but any number between the two points will do.

x	-2	-1	0	1	2	3	4
$f'(x)$	$+ve$	0	$-ve$	0	$-ve$	0	$+ve$
Shape	/	—	\	—	\	—	/
		Local Maximum		Stationary Point of Inflection		Local Minimum	

So we have 1 stationary point of inflection, 1 local maxmimum and 1 local minimum, so the correct answer is **A**.

Question 11 (C)

Integrating $f'(x)$ will give us the general anti-derivative, so we need to add the $+C$ and solve for it given the point that lies on the curve.

$$\begin{aligned} f(x) &= \int 4x^3 + \frac{1}{x^2} - \frac{1}{x} dx \\ &= x^4 - \frac{1}{x} - \ln|x| + C \\ f(1) &= 2 \\ 1 - 1 - 0 + C &= 2 \\ \therefore C &= 2 \\ \therefore f(x) &= x^4 - \frac{1}{x} - \ln|x| + 2 \end{aligned}$$

So **C** is the answer.

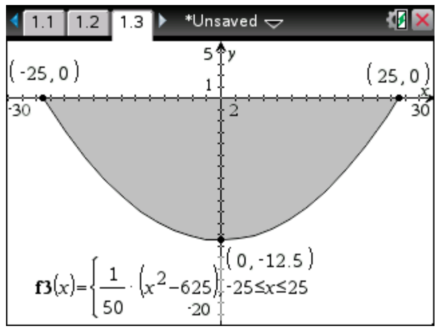
SECTION B - Extended response questions

Question 1

a. Firstly plot the function out so we can see what we are dealing with. Since we have a restricted domain we can add "|lower $x \leq x \leq$upper x" to the end of the function.

i.e. In the graph bar type

$$"f(x) = \frac{1}{50}\left(x^2 - 625\right) \mid -25 \leq x \leq 25"$$



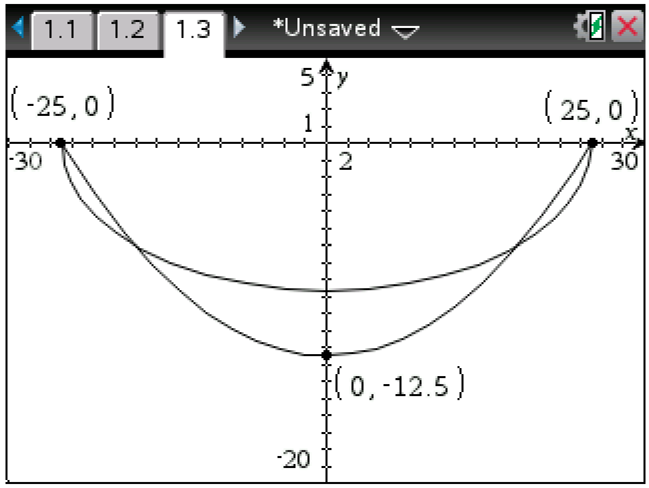
The cross-sectional area of the river is represented by the shaded region. Since this area is below the x-axis we need to place a negative in front of the integral. Our terminals will be the x-intercepts of the function.

$$\begin{aligned}
\text{Area} &= -\int_{-25}^{25} \frac{1}{50}\left(x^2 - 625\right) dx \\
&= -\frac{1}{50}\left[\frac{1}{3}x^3 - 625x\right]_{-25}^{25} \\
&= \frac{1250}{3}\ m^2
\end{aligned}$$

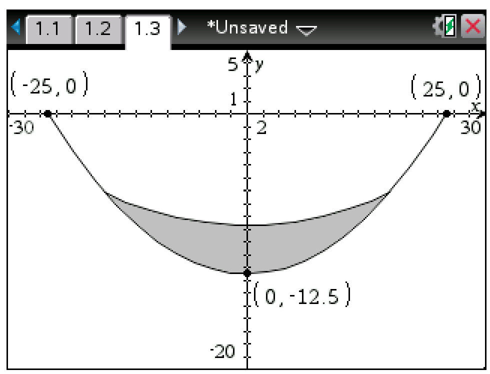
Since the function is an even functions, that is $f(-x) = f(x)$, we could just evaluate the region for $x > 0$ and double it.

$$\begin{aligned}
\text{Area} &= -2\int_{0}^{25} \frac{1}{50}\left(x^2 - 625\right) dx \\
&= \frac{1250}{3}\ m^2
\end{aligned}$$

b. Plotting the two functions on the calculator to give us an idea of the situation we obtain:



Looking at the curves, the sediment will be in the cross-sectional area that is between the lower parts of the two curves, that is $s(x)$ needs to be restricted to be within the intersection of $f(x)$ and $s(x)$. So effectively we have the following situation, and we need to find the intersections of the two curves, that is not on the x-axis.



Finding the intersection gives $x = \pm 25,\ \pm 5\sqrt{13}$, so the domain we want is $\left[-5\sqrt{13}, 5\sqrt{13}\right]$.
$\therefore a = 5\sqrt{13}$

c. The cross-sectional area will be given by the integral of the upper function minus the lower function, since the integral for the lower function will give a larger negative area, when this is subtracted from the area for the top function, it will result in a positive region that is the area of the shaded region in the figure in the previos question. Since we are finding the area from the extremes of the restricted function, the terminals of the integral will be the lowest and highest values in our domain for $s(x)$, that is $-5\sqrt{13}$ and $5\sqrt{13}$. Make note that we need to keep the negative in $s(x)$ and then minus $f(x)$. Evaluating on the calculator, using "[⇑ shift]+[+]"

$$\begin{aligned}
\text{Area}_{\text{sediment}} &= \int_{-5\sqrt{13}}^{5\sqrt{13}} (s(x) - f(x))\,dx \\
&= \int_{-5\sqrt{13}}^{5\sqrt{13}} \left(-\sqrt{\frac{3}{25}\left(625 - x^2\right)} - \frac{1}{50}\left(x^2 - 625\right)\right) dx \\
&= 90\,m^2
\end{aligned}$$

d. We have two unknowns to find, so we need two equations, the first comes from the point itself. Substituting in $g\left(\sqrt{6}-3\right)=\frac{13}{2}-\sqrt{6}$ gives equation [1].

$$\begin{aligned} g\left(\sqrt{6}-3\right) &= \frac{13}{2}-\sqrt{6} \\ \frac{-3}{a+\sqrt{6}-3}+b\left(3-\sqrt{6}\right)+\sqrt{6} &= \frac{3}{2}....[1] \end{aligned}$$

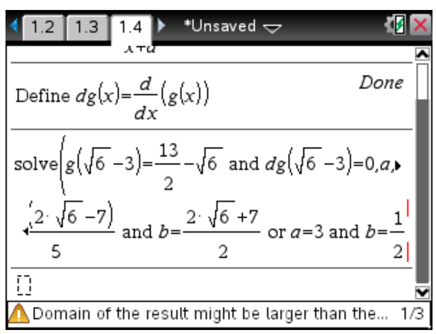
The second equation comes from the fact that there is a turning point at $\left(\sqrt{6}-3,\frac{13}{2}-\sqrt{6}\right)$. As the gradient at a stationary point is 0, our derivative at $x=\sqrt{6}-3$ is 0. That is $g'\left(\sqrt{6}-3\right)=0$. Substituing $x=\sqrt{6}-3$ into $g'(x)=\dfrac{3}{(x+a)^2}-b$ gives equation [2].

$$\begin{aligned} g'(x) &= 0 \\ \frac{3}{(x+a)^2}-b &= 0 \\ x &= \sqrt{6}-3 \\ \frac{3}{\left(\sqrt{6}-3+a\right)^2} &= b....[2] \end{aligned}$$

Solving those two equations gives 2 sets of solutions, $a=1.261$ and $b=5.950$ or $a=3$ and $b=\frac{1}{2}$. But $a>2$ and $b<5$,

$$\therefore a=3 \text{ and } b=\frac{1}{2}$$

Possibly the quickest way of doing this on the calculator would be to define $g(x)$ using "[Menu] [1] [1]", then define the derivative using "[⇑ shift] + [-]". Then when using solve, instead of typing out the long equation and subsituting in the values, you can just use $g\left(\sqrt{6}-3\right)=\frac{13}{2}-\sqrt{6}$ and $dg\left(\sqrt{6}-3\right)=0$



e. There is two ways to approach this question, the first by minimising the distance, and the second by using a line that is perpendicular to the line.

In general the distance between two points is given by

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

This can be show by using Pythagoras' Theorem. Now as one of our points is at the origin, that is $(0,0)$, we obtain $d=\sqrt{x^2+y^2}$ Substituing in $y=\frac{1}{3}x+2$ gives an expression for the distance

$$\begin{aligned} d(x) &= \sqrt{x^2+\left(\frac{1}{3}x+2\right)^2} \\ &= \frac{1}{3}\sqrt{2\left(5x^2+6x+18\right)} \end{aligned}$$

To find the minimum distance, we need to find the stationary point(s) of $d(x)$, so finding the derivative and letting it equal 0

$$\begin{aligned} d'(x) &= \frac{\sqrt{2}\,(5x+3)}{3\sqrt{5x^2+3x+9}} \\ \frac{\sqrt{2}\,(5x+3)}{3\sqrt{5x^2+3x+9}} &= 0 \\ 5x+3 &= 0 \\ x &= -\frac{3}{5} \end{aligned}$$

results in $x = -\frac{3}{5}$. Subsitituing that into the equation for the canal gives

$$\begin{aligned} g\left(-\frac{3}{5}\right) &= -\frac{3}{-\frac{3}{5}+3} - \frac{1}{2}\left(-\frac{3}{5}\right) + 5 \\ &= \frac{9}{5} \end{aligned}$$

$y = \frac{9}{5}$, which means our point on the canal is $\left(-\frac{3}{5}, \frac{9}{5}\right)$ and substituing into $d(x)$ gives

$$\begin{aligned} d_{min} &= \frac{1}{3}\sqrt{2\left(5\left(-\frac{3}{5}\right)^2 + 6\left(-\frac{3}{5}\right) + 18\right)} \\ &= \frac{3\sqrt{10}}{5}\ m \end{aligned}$$

The minimum distance between the Nuclear Power Plant and the canal is $\frac{3\sqrt{10}}{5}$ metres and this occurs at the point $\left(-\frac{3}{5}, \frac{9}{5}\right)$ on the canal.

An alternative solution to the problem is to use the fact that the line of minimum distance is perpendicular to the function at the point of interest and passes through that point. So we need to find the gradient at the point

$$\begin{aligned} h(x) &= \frac{1}{3}x + 2 \\ h'(x) &= \frac{1}{3} \end{aligned}$$

Now since the line of minimum distance is perpendicular to the function, we can use the fact that the gradients of two perpendicular lines multiply to -1. That is

$$\begin{aligned} m_1 m_2 &= -1 \\ m_1 &= \frac{1}{3} \\ m_2 &= -3 \end{aligned}$$

The line will be of the form

$$y - y_1 = m\,(x - x_1)$$

Now since this line passes through the origin, $(0,0)$, the line will be given by

$$\begin{aligned} y - 0 &= -3\,(x-0) \\ y &= -3x \end{aligned}$$

So we then need to find the intersection of this line and the function, as that will be the point at which the distance between the origin and the function is a minimum.

$$\begin{aligned} \frac{1}{3}x + 2 &= -3x \\ \frac{10}{3}x &= -2 \\ \therefore x &= -\frac{3}{5} \end{aligned}$$

Then finding the y value

$$\begin{aligned} y &= \frac{1}{3}\left(-\frac{3}{5}\right)+2 \\ &= \frac{9}{5} \end{aligned}$$

So then to find the minimum distance we need the formula for the distance between two points

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Substituting in the point

$$\begin{aligned} d_{min} &= \sqrt{\left(-\frac{3}{5}-0\right)^2+\left(\frac{9}{5}-0\right)^2} \\ &= \frac{3\sqrt{10}}{5}\,m \end{aligned}$$

The minimum distance between the Nuclear Power Plant and the canal is $\frac{3\sqrt{10}}{5}$ metres and this occurs at the point $\left(-\frac{3}{5}, \frac{9}{5}\right)$ on the canal.

f. The area under $V'(x)$ represents the water added/taken out of the reactor. Integrating $V'(x)$ gives a general anti-derivative, so we need to add the constant of integreation, $+C$ and find it using the initial conditions $V = 20$ and $t = 0$.

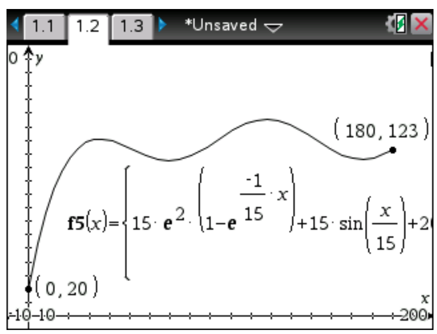
$$\begin{aligned}
V(t) &= \int \left(e^{-\frac{1}{15}(t-30)} + \cos\left(\frac{1}{15}t\right)\right) dt \\
&= -15e^{-\frac{1}{15}(t-30)} + 15\sin\left(\frac{1}{15}t\right) + C \\
V(0) &= 20 \\
20 &= -15e^2 + C \\
\therefore C &= 15e^2 + 20 \\
\therefore V(t) &= 15e^2\left(1 - e^{-\frac{1}{15}t}\right) + 15\sin\left(\frac{1}{15}t\right) + 20
\end{aligned}$$

As the quesiton asks for the function, it is best to write it out in the form

$$V : [0, 180] \to R,\ V(t) = 15e^2\left(1 - e^{-\frac{1}{15}t}\right) + 15\sin\left(\frac{1}{15}t\right) + 20$$

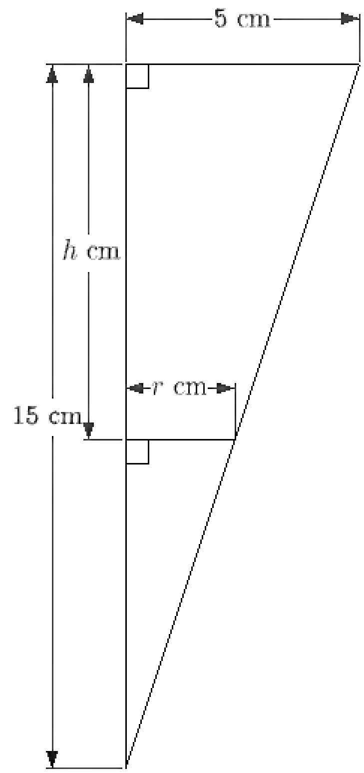
Although the rule and domain should be ok aswell

Checking that the graph makes sense by graphing it we find that $V(x) > 0$ for $0 \le x \le 180$. That is the volume of water has to be greater than 0.



Question 2

ai. Drawing the situation out we obtain two triangles, and as they share the same angle, and have the right angles algined, they are similar. So we can use ratios of the side lengths.



$$\frac{5}{r} = \frac{15}{15-h}$$
$$r = 5 - \frac{1}{3}h$$

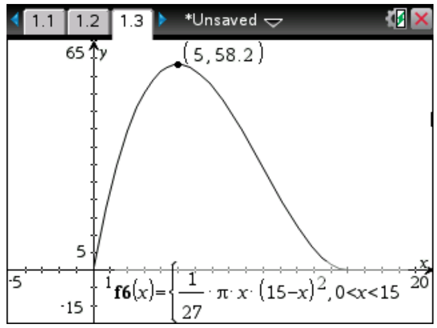
aii. The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$ where r is the radius and h is the heigth of the cone. Now that we have r in terms of h we can substitute that into the formula for the volume of a cone, obtaining

$$\begin{aligned} V(h) &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi\left(5 - \frac{1}{3}h\right)^2 h \\ &= \frac{1}{27}\pi h (15-h)^2 \end{aligned}$$

Now since the quesiton asks for the function, we need to find the domain, $h > 0$ because lengths cannot be negative and $h < 15$ because the cone has to fit inside the larger cone. So we get $0 < h < 15$, that is no inclusive since if $h = 0$ or $h = 15$ then the cone will not exist. Writing it out as function notation gives

$$V : (0, 15) \to R,\ V(h) = \frac{1}{27}\pi h (15-h)^2$$

b. The maximum volume will occur at either a stationary point (if it is a maximum stationary point) or at an endpoint. The best way to find this out is to graph it.



So as we can see the maximum occurs at a maximum stationary point, so that the gradient is 0 at that point. So finding the derivative, letting it equal 0

$$
\begin{aligned}
\frac{dV}{dh} &= \frac{\pi}{9}(h-15)(h-5) \\
\frac{dV}{dh} &= 0 \\
\frac{\pi}{9}(h-15)(h-5) &= 0 \\
h = 5 &\quad \text{or} \quad h = 15
\end{aligned}
$$

But our domain for h is $0 < h < 15$ so we reject the $h = 15\,cm$ solution as the smaller cone would have a radius of 0, giving a volume of 0. So this doesn't correspond to the stationary point we're looking for. Finding $V(5)$ for the maximum volume

$$
\begin{aligned}
V(5) &= \frac{1}{27}\pi(5)(15-5)^2 \\
&= 58.18\,cm^3
\end{aligned}
$$

So the maximum volume of the smaller cone is $55.18\,cm^3$ and occurs when $h = 5\,cm$.

c. The graph will be that of the figure in the previous part of the question above. The key points to label is the stationary point at $(5, 58.2)$ and the two endpoints of $(0,0)$ and $(15,0)$ should be **open circles** since the values of $x = 0$ and $x = 15$ are not included in the domain, so the endpoints here are **not** included. We should also show that the slope of the curve at $x = 15$ is 0.

Question 3

a. Firstly we need to find the stationary points of the function, so we need to find the derivative, equate it to 0 and solve for x.

$$\begin{aligned} f'(x) &= 2a\cos(ax) \\ -2a\cos(ax) &= 0 \\ \cos(ax) &= 0 \\ ax &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2} \\ x &= \frac{\pi}{2a}, \frac{3\pi}{2a}, \frac{5\pi}{2a}, \frac{7\pi}{2a}, \frac{9\pi}{2a}, \frac{11\pi}{2a} \end{aligned}$$

Since we want the first 5 stationary points, we list the first 6 solutions. Now we want the 5^{th} solution to be before $x = 2\pi$, as we don't include the endpoint since we have cuvred brackets in the domain. So that means we can solve

$$\begin{aligned} \frac{9\pi}{2a} &< 2\pi \\ a &> \frac{9}{4} \end{aligned}$$

Likewise we want the 6^{th} solution to be on or after 2π. We can include 2π here as we don't want the 6^{th} solution to be in our domain given. So we can solve

$$\begin{aligned} \frac{11\pi}{2a} &\geq 2\pi \\ a &\leq \frac{11}{4} \end{aligned}$$

So we are left with

$$\frac{9}{4} < a \leq \frac{11}{4}$$

b. To find the stationary points of the graph, we need to find when the derivative equals 0. Finding the derivative and equating to 0 gives

$$\begin{aligned} \frac{dy}{dx} &= 3ax^2 + 4ax + 3a \\ &= a\left(3x^2 + 4x + 3\right) \\ a\left(3x^2 + 4x + 3\right) &= 0 \end{aligned}$$

Now the solutions we have are $a = 0$ and $3x^2 + 4x + 3 = 0$ but as $a \neq 0$ then we can only solve the second equation. To find the number of solutions to this equation we use the discriminant for a quadratic, for $y = ax^2 + bx + c$ is $\triangle = b^2 - 4ac$. If $\triangle > 0$ we have 2 unique solutions, $\triangle = 0$ then 1 unique solution and $\triangle < 0$ we have no real solutions.

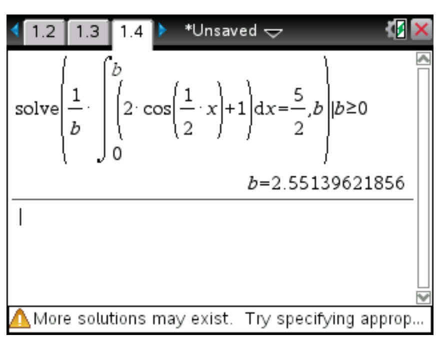
$$\begin{aligned} \triangle &= (4)^2 - 4(3)(3) \\ &= -20 \\ \therefore \triangle &< 0 \end{aligned}$$

In out case $\triangle = -20$ so there are no solutions to $\frac{dy}{dx} = 0$ and thus there are no stationary points for the graph for any value of a. So the only possible value of n is 0.

c. The average value of a function $f(x)$ over the interval $[a, b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$. So solving

$$\begin{aligned} \frac{1}{b}\int_0^b \left(2\cos\left(\frac{1}{2}x\right) + 1\right) dx &= \frac{5}{2} \\ \frac{1}{b}\left[4\sin\left(\frac{1}{2}x\right) + x\right]_0^b &= \frac{5}{2} \\ \frac{1}{b}\left(4\sin\left(\frac{1}{2}b\right) + b\right) &= \frac{5}{2} \\ b &= 2.55 \end{aligned}$$

The quickest way to solve this is to use the solve function "[Menu] [3] [1]", then the integral function [⇑ shift]+[+] with a domain restriction $|b \geq 0$



So the required value of b is 2.55.

PROBABILITY

TECH-FREE TEST 1

DETAILED SOLUTIONS

Question 1

This is a standard question that tests a student's understanding of basic probability definitions. Students should immediately think about what is involved in probability distributions.

The obvious ones are split into two portions
1) Measures of centre/location - mean, median, mode.
2) Measures of spread - variance and standard deviation.

This question doesn't actually require you to know any of the above, but students should ensure they know these definitions clearly and how to calculate them. This question in fact requires a much simpler rule and one that is often taken for granted by students: The sum of probabilities must equal 1.

It can be represented as a summation (not tested in this course).

$$\sum_x p(x) = 1$$

But the point is that this rule provides an easy equation in many questions that can be used to solve an unknown. Typically, the question that requires this rule is one that asks you to find an unknown value used in the distribution. Without knowing the mean, median, variance etc. which change from question to question, this is the method to use because you know the sum of probabilities **always** equals 1.

$$\begin{aligned} 2p + p^2 + \frac{p^2}{2} + \frac{p+2}{3} + p^2 &= 1 \\ \frac{5}{2}p^2 + \frac{7p}{3} - \frac{1}{3} &= 0 \\ 15p^2 + 14p - 2 &= 0 \end{aligned}$$

Using the quadratic formula,

$$\begin{aligned} p &= \frac{-14 \pm \sqrt{14^2 - 4(15)(-2)}}{2(15)} \\ &= \frac{-14 \pm \sqrt{316}}{30} \\ &= \frac{-14 \pm 2\sqrt{79}}{30} \\ &= \frac{-7 \pm \sqrt{79}}{15} \\ &= \frac{-7 + \sqrt{79}}{15} \text{ since } p \geq 0 \end{aligned}$$

Be sure to check your solutions because there are a lot of restrictions involved with probaility. Above, you can understand that it's impossible for a probability to be less than 0 - what would it mean to have the probability of drawing a heart in a deck of cards as -0.5?

Question 2

This question requires students to have a strong understanding of the Binomial distribution and the various calculations required.

a.

A lot of students neglect to define a distribution when answering probability questions. while it is possible to get through a question without doing so, it is recommended since it helps clarify solutions - this is beneficial to both you and the marker trying to scrawl through your answers and understand what you have written. Get familiar with the short-hand notation, particularly for defining Binomial and Normal distributions.

For Binomial distributions, we use $\mathrm{Bi}(n, p)$, where n represents the number of trials and p represents the probability of a defined "success". For the Normal distribution, we use $\mathrm{N}(\mu, \sigma^2)$ where μ is the mean of the distribution and σ is its standard deviation. Then like defining a function, we assign a letter, e.g. X, but instead of using an equal sign, we use "~" to denote that the variable X is approximated by this certain distribution. You may also see an equal sign with a "d" above it which also denotes that a variable has been assigned a distribution.

For this question, we are told that each question has 5 options, so if we were to guess "at random" we would have a one in five (i.e. 0.2) chance of getting the answer correct (if it were actually possible to guess at random). If there are 20 questions, we would effectively be repeating this random guessing with a "probability of success 0.2" 20 times.

Hence, we let $X \sim \mathrm{Bi}(20, 0.2)$

The question asks for the standard deviation - students should immediately recognise that the square root of the variance is the standard deviation and should aim to find the variance first.

The tricky thing about the Binomial distribution is that finding measures such as mean and variance are found in a way not similar to the other distributions in the course (however, they come from the same definition).

For the Binomial distribution, variance is found by

$$\mathrm{Var}(X) = np(1-p)$$

The useful thing about defining the distribution comes in now, where n and p are clearly given in the distribution and can be easily substituted into the rule.

$$\begin{aligned} \mathrm{Var}(X) &= 20(0.2)(0.8) \\ & \quad 3.2 \end{aligned}$$

Now we just have to find the square root of this in order to find the standard deviation.

$$\begin{aligned} \sigma &= \sqrt{\mathrm{Var}(X)} \\ &= \sqrt{\frac{16}{5}} \\ &= \frac{4\sqrt{5}}{5} \end{aligned}$$

b.

In probability, errors made by students are often attributed to students misinterpreting or not being able to interpret the question in such a way that allows them to enter calculations - reading questions carefully and breaking down complicated phrases is the key to removing these errors.

To "get 95% or more" in this test, he must score 19 or more. Binomial distributions are never given as a continuous function as per their purpose of discrete trials - there is no 1.2 trial, only integers. Therefore, we really only need to find the probabilities of when he scores 19 or 20 (it is impossible to score between this in the multiple choice test).

We can thus write our required probability as such:

$$\Pr(X \geq 19) = \Pr(19) + \Pr(20)$$

How do we find probabilities in the Binomial distribution by hand? A rule exists based on the idea of the binomial theorem (hence the name of the distribution).

Consider the following. Expand $(a+b)^2$

$$(a+b)^2 = a^2 + 2ab + b^2$$

This expansion is technically a binomial expansion because it involves two terms a and b which have been put to a certain power. The coefficients of "$(1)a^2 + 2ab + (1)b^2$" are what makes this expansion interesting. In terms of combinations, we can write out the combinations of a and b being drawn twice as: a,a; a,b; b,a and b,b. Note how "a,a" and "b,b" have only one combination because reversing the order does not make a difference. Now look back to the coefficients. There is "1" in front of the $a \times a$ combination (and similarly for the b^2) and "2" in front of the "$a \times b$". In other words, the coefficients are the number of different combinations a, b can take if they are drawn from twice. The binomial theorem allows us to find that coefficient and hence the probability of specific values in the Binomial distribution.

The "a,b" in the Binomial distribution is actually the probabilities of success and failure where p is the probability of success and $1-p$ is the probability of failure.

Probabilities can be found as such:

$$\Pr(X = x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}, \text{ where } n! = n \times (n-1) \times (n-2) \cdots \times 1$$

The similarities of this result with when you expand $(a+b)^x$ should be fairly clear. If $x = 3$ then you would expect the term with the variable that contains a^2 to be

$$\begin{aligned}
\frac{3!}{2!(3-2)!}a^2b^{3-1} &= \frac{3 \times 2 \times 1}{2 \times 1 \times 1}a^2b \\
&= 3a^2b
\end{aligned}$$

And this clearly matches the expansion$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Hence back to the question,

$$\begin{aligned}
\Pr(19) + \Pr(20) &= \frac{20!}{19!1!}0.2^{19}\,(0.8)^1 + \frac{20!}{20!0!}0.2^{20}\,(0.8)^0 \\
&= 20(0.2)^{19}(0.8) + 0.2^{20} \\
&= 20 \times \frac{4}{5} \times 0.2^{19} + 0.2^{20} \\
&= 16 \times 0.2^{19} + 0.2 \times 0.2^{19} \\
&= (16.2)0.2^{19}
\end{aligned}$$

Question 3

One of the most common confusions in the probability course is the distinction between "mutually exclusive events" and "independent events".

To put it simply:

1) Mutually exclusive events indicates that there are no results shared between two events. e.g. Say event A contains the results $\{1, 2, 3, 4\}$ and event B contains $\{5, 6, 7, 8\}$, then the events are said to be mutually exclusive.

It is commonly associated with a change in the addition rule from $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ to $\Pr(A \cup B) = \Pr(A) + \Pr(B)$, the intersection removed because there is none between the two events.

2) If two events are Independent, then the probability of one does not affect the other. Importantly, $\Pr(A \mid B) = \Pr(A)$ instead of $\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$. Though this is the rule that most easily represents the meaning of independent events, the more commonly used one by students is $\Pr(A \cap B) = \Pr(A)\Pr(B)$ where the intersection is merely the multiple of the probability of the two events.

Knowing these rules, the following questions become relatively simple - it is up to students to understand both concepts and not get confused as to which one is which.

a.

Conditional probability tells us that $\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

If A and B are mutually exclusive, then they share no common outcome, therefore $\Pr(A \cap B) = 0$

$$\begin{aligned}\Pr(A \mid B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\ &= 0\end{aligned}$$

Some of the information in the question is given as a 'red herring' - hopefully this does not distract you in the exam proper.

b.

As stated above, $\Pr(C|D) = \Pr(C)$ since C and D are independent events. Again utilising $\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$,

$$\begin{aligned}\Pr(C|D) &= \frac{\Pr(C \cap D)}{\Pr(D)} \\ \Pr(C) &= \frac{\Pr(C \cap D)}{\Pr(D)} \\ \Pr(C \cap D) &= \Pr(C)\Pr(D) \text{ as required}\end{aligned}$$

Question 4

a.

This first part of the question should be reminiscent of Question 1 in this test. Again we just want to find an unknown in the distribution so we will use the same rule:

$$\sum_x p(x) = 1$$

However, this rule must be adapted for use in a continuous probability distribution where the values of p are now continuous - we can't just add every value because there is an infinite amount of values within a certain interval. It should be understood that the area under a probability curve between a certain interval is the probability of that event between the given interval.

Hence, for a probability density function, $\int_{-\infty}^{\infty} f(x)dx = 1$.

Now that we have an equation, we can just solve for k. The only thing to be careful of is the hybrid function - the rule still applies though. If you add the probability of the functions for where they exist, it will still add to 1 as shown below. Understanding of definite integrals and how to evaluate them in a probability context is expected - for more detailed integral solutions, refer to the detailed solutions of the Calculus Tests.

$$\begin{aligned}
\int_0^2 kx\,dx + \int_2^4 5k(x-2)\,dx &= 1 \\
\left[\frac{kx^2}{2}\right]_0^2 + \left[\frac{5kx^2}{2} - 10kx\right]_2^4 &= 1 \\
(\frac{4k}{2} - 0) + [(40k - 40k) - (10k - 20k)] &= 1 \\
2k + [-(-10)] &= 1 \\
2k + 10k &= 1 \\
12k &= 1 \\
k &= \frac{1}{12}
\end{aligned}$$

b.

For finding the mean we should be aware that $E(X) = \sum_x x\Pr(x)$. In other words, we find the sum of x-values multiplied with each of their probabilities. Just like above in **part a.** the mean of a continuous PDF has an analogous rule, which is $\mu = \int_{-\infty}^{\infty} xf(x)dx$.

Hence,

$$\begin{aligned}
\mu &= \frac{1}{12}\int_0^2 x^2\,dx + \int_2^4 5x(x-2)\,dx \\
&= \frac{1}{12}\left(\left[\frac{x^3}{3}\right]_0^2 + \left[\frac{5x^3}{3} - 5x^2\right]_2^4\right) \\
&= \frac{1}{12}\left[(\frac{8}{3} - 0) + \left[(\frac{320}{3} - 80) - (\frac{40}{3} - 20)\right]\right] \\
&= \frac{1}{12}(\frac{8}{3} + (\frac{280}{3} - 60) \\
&= \frac{1}{12}(\frac{8}{3} + \frac{100}{3}) \\
&= 3
\end{aligned}$$

c.

This question will likely trip up students only because it is slightly uncommon, but it only takes one step for it to become a standard question (and essentially a simple one, disregarding the tedious calculations involved).

It is dealing with the modulus sign that will confuse students - but mathematics notation is somewhat consistent and universal, unless it is obvious, the use of modulus signs should be no different in probability as it is anywhere else. Expand the modulus bracket and it becomes simple.

$\mid X - \mu \mid < 1$ can be split into the following

$$\begin{aligned} +(X-\mu) &< 1 \\ \text{and } -(X-\mu) &< 1 \\ \text{i.e. (X-}\mu\text{)} &> -1 \end{aligned}$$

Now we can rearrange for X

$$\begin{aligned} X &< \mu+1 \\ &\text{and} \\ X &> \mu-1 \end{aligned}$$

You should realise that there is a value less than X and a value greater than X, hence we can merge the expressions together to form

$$\mu - 1 < X < \mu + 1$$

Hence,

$$\begin{aligned} \Pr(|X-\mu\,|<1) &= \Pr(\mu-1<X<\mu+1) \\ &= \Pr(3-1<X<3+1) \\ &= \Pr(2<X<4) \end{aligned}$$

Now it is a straightforward question that wants us to find the probability for an interval. For a continuous probability distribution, we just have to find the area under the curve between this interval. The lowest bound is at $X = 2$ hence we only have to consider the latter portion of the distribution function.

$$\begin{aligned} \Pr(2<X<4) &= \int_2^4 5k(x-2)\,dx \\ &= \left[\frac{5kx^2}{2} - 10kx\right]_2^4 \\ &= k\left[(40-40)-(10-20)\right] \\ &= k\left[0-(-10)\right] \\ &= 10k \\ &= \frac{10}{12} \\ &= \frac{5}{6} \end{aligned}$$

PROBABILITY

TECH-FREE TEST 2

DETAILED SOLUTIONS

It is worth noting that this test is one that hones in on a student's ability to read questions carefully, break them down, interpret and finally launch into the mathematics learnt in the course. One of the things students struggle with most in probability is interpreting the given information in a question then knowing how to start their solution. while it is important for students to maintain a totality of the question and all the information given, you can almost always pull apart a question into smaller bits that will help you to understand what is going on in the question. More often than not, the mathematics and calculations involved in probability are relatively easy compared to the rest of the course - it is the interpretation that makes them difficult.

Question 1

This is a classic question on Markov chains that asks students what the probability of an event occuring is at a given time in the future. Students should immediately pick up on the fact that the question requires an understanding of Markov chains based on the wording that alludes to an occurence at one time, and the probability that it will occur (or not occur) the next time. For the purposes of this course, you will only ever have to deal with two events potentially occuring at the same time - in this case, Mr. Li either drives his car or rides his bike, and we are told the probabilities of which he will drive/ride dependent on what he did the previous day.

a.

So for this part, we want the probability that he drives the first day, then drives the second day AND the third day. It is possible to find the probability of each result occuring for each day individually, but then how do we put the probabilities together? Students who are accustomed to drawing tree diagrams will understand that you simply multiply along each event in the tree. However, let's consider what doing that actually means. Between consectutive (different) days, the events are independent of each other and for independent events, students should know that $\Pr(A \cap B) = \Pr(A)\Pr(B)$. This is slightly complicated in this question because the probability of driving depends on the day before, but if we calculate that seperately from the total probability asked for in this question, we can still use the multiplication rule. Essentially, students only have to multiply probabilities for each consecutive day according to what the probability is on each day.

Firstly, we should define some variables to make the solution less complicated. Let C represent the event that Mr Li drives in his car, and B is the event that he rides his bike. We have been told in the question that Mr. Li definitely drives on the first day - it is certain, and hence its probability is 1. The probability of driving on the second day is given by the fact that he drove on the first day, and the information given tells us that this probability is 0.3. As on the second day, the probability he drives on the third day is also 0.3. Now we just have to multiply these results together.

$$\begin{aligned}\Pr(C, C, C) &= 1 \times 0.3 \times 0.3 \\ &= 0.09\end{aligned}$$

Even though this is a tech-free test, you should be expected to deal with simple decimals like this - at the very least, you are expected to multiply fractions together and you can simply write 0.3 as $\frac{3}{10}$.

b.

With most probability questions, the first thing students should look to do is define two things: 1) The transition matrix and 2) The initial state matrix (if given). It is also important to consider which state is required (i.e. the matrix we need to find that contains the probability asked for in the question).

Let us do the last part first. We want the probability on day 3, so we wish to find the matrix S_3 which can be defined by $S_3 = T^2 S_1$. Where does this expression come from?

It is actually the same as what we see in tree diagrams but with the conditional probability described in terms of matrices. Essentially, we multiply the probability of Time1 with the conditional probability (represented by the transition matrix) for Time2.

We can write this as $S_2 = T \times S_1$.

What about finding Time3? We multiply the probability of Time2 by the conditional probability to produce $S_3 = T \times S_2$. But we also have an expression for S_2 so we can rewrite Time3 to be $S_3 = T \times T \times S_1 = T^2 S_1$.

If you were to continue along this path you would notice a pattern - $S_n = T^{n-1} \times S_1$ or $S_n = T^n \times S_0$. This is an incredibly useful result because it means that we don't have to find all the probabilities between Time1 and Time1000 to find the probabilities at Time1000 - this is not the case if we just use a tree diagram. Hence for this question, we want $S_3 = T^2 S_1$.

Now let us examine the transition matrix. The general look of a transition matrix is $T = \begin{array}{cc} & \begin{array}{cc} A_0 & B_0 \end{array} \\ \begin{array}{c} A_1 \\ B_1 \end{array} & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$.
Note: the letters "A and B" are technically not a part of the matrix itself, however they have a specific purpose and one that should be utilised all the time whe just defining the matrix. We should read the matrix as "a is the probability of A_1 given A_0." and similarly for the other probabilities. Always read the probability across rows. Using the notation in part a) We will use C and B again. From the information given, we can write the following transition matrix $T = \begin{array}{cc} & \begin{array}{cc} C & B \end{array} \\ \begin{array}{c} C \\ B \end{array} & \begin{bmatrix} 0.3 & b \\ c & 0.6 \end{bmatrix} \end{array}$. But what about "$b$" and "$c$"? Well consider the event that he drove the previous day. He MUST either drives or he rides his bike - i.e. the probability of him driving or riding is 1 because there are no other possibilities. Hence if the probability he drives after driving is 0.3, then the probability he rides his bike is $1 - 0.3 = 0.7$. Thus the transition matrix is $T = \begin{array}{cc} & \begin{array}{cc} C & B \end{array} \\ \begin{array}{c} C \\ B \end{array} & \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix} \end{array}$.

The last thing to define is the initial state matrix. Take care when defining the initial state matrix - it will often be "S_0" because the information given states that an event has happened prior to when the question begins.

However, on occassion, you will instead define "S_1" because the information states the first event that occurs for when the question begins. In this question we are told what happens on Day 1 of the tournament in question, not what happens before the tournament. This is very important because using the incorrect initial state matrix notation will cause all future probabilities calculated to be incorrect as you find the probability of the wrong day (e.g. Day 4 instead of Day 3).

Hence we let $S_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ since we know he drives his car on the first day.

Ensure that the rows in this matrix match the ones you defined in the transition matrix - otherwise future calculations will be incorrect. This can be easily helped by writing the letters next to the matrices such as the C and B in the transition matrix. It does not matter which way around the letters are as long as the probabilities within are consistent with the order.

With all of that settled, we only have to use matrix arithmetic to find the probability.

$$\begin{aligned}
S_3 &= T^2 S_1 \\
&= \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.3 \times 0.3 + 0.4 \times 0.7 & 0.3 \times 0.4 + 0.4 \times 0.6 \\ 0.7 \times 0.3 + 0.6 \times 0.7 & 0.7 \times 0.4 + 0.6 \times 0.6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.09 + 0.28 & 0.12 + 0.24 \\ 0.21 + 0.42 & 0.28 + 0.36 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.37 & 0.36 \\ 0.63 & 0.64 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.37 \\ 0.63 \end{bmatrix}
\end{aligned}$$

Again, the importance of using the letters above help to clarify which probability is which since we know that the second row is associated with B, the event that Mr Li rides his bike. Hence, the probability that he rides his bike on day 3 is 0.63.

Question 2

a.

This question that benefits greatly from students clearing their minds and considering each portion individually. The only question we need to ask in order to answer this question is what is the probability of landing it in each particular region? Since a dart is equally likely to land at any point on the board, it does not matter what the region is, it only matters how large the region is, because the larger the region is the greater the chance of landing it there. In other words, we want to find the area of each region - its propotion to the entire dartboard will give us the probability of landing it there (assuming every dart lands on the board with an equal probability of landing on any given point).

The only trick to this question is that students need to take care that they don't just find the areas of each circle and leave it at that - they need to subtract the circles within to get the true region (though there is no circle within the innermost circle). We are finding the area of an annulus. Area of the circle is given as πr^2 - students should know this readily.

$$\begin{aligned}
\text{Total Area of Dartboard} &= \pi \times 30^2 \\
&= 900\pi \text{ square units}
\end{aligned}$$

$$\begin{aligned}
\text{Area of inner circle} &= \pi \times 3^2 \\
&= 9\pi \text{ square units} \\
\text{Portion of whole board} &= \frac{9\pi}{900\pi} \\
&= 0.01
\end{aligned}$$

$$\begin{aligned}
\text{Area of outer region} &= \text{Area of Board-Circle of radius 15 cm} \\
&= 900\pi - \pi \times 15^2 \\
&= 675\pi \text{ square units} \\
\text{Portion of whole board} &= \frac{675\pi}{900\pi} \\
&= 0.75
\end{aligned}$$

$$\begin{aligned}
\text{Area of middle region} &= \text{Area of board - Area of outer region - Area of inner circle} \\
&= 900\pi - 675\pi - 9\pi \\
&= 216\pi \text{ square units} \\
\text{Portion of whole board} &= \frac{216\pi}{900\pi} \\
&= 0.24
\end{aligned}$$

Now that we have the proportional areas of each region on the board, we can define our probability distribution accordingly. We will write the distribution as an x-value, the score of one throw, depending on the region it lands in.

Hence, the probability distribution is:

x	10	25	100
$\Pr(X = x)$	0.75	0.24	0.01

, X is the score from one throw.

b.

This question will generally be solved in one of two ways. A probability distribution was previously defined in **part a.** that descibed the probabilities of a score off one throw, students may wish to define another probability distribution that describes the scores for two throws - a tedious method. However, writing out a completely new distribution is unecessary and those who understand the idea of independent events will find this question straightforward. It should be realised that the first throw does not affect the probability of the second throw and hence the probabilities can be multiplied between the first and second throw according to $\Pr(A \cap B) = \Pr(A)\Pr(B)$ for A and B as independent events.

As said above, there's no need to define a whole new probability distribution - you would be finding uncessary probabilities. Let us only look at all the possibilities that results in a score of more than 50 points. The possible throws to score more than 50 points are: 10,100; 100,10; 25,100; 100,25; 100,100. It may look silly to write out all the possibilities in this way, but in reality, it makes the question easier to see all the possibilities laid out in front of you - suddenly the question seems less daunting now that you know what you need to find.

Finding the probabilities of the combination is easier enough according to the rule of multiplication for independent events, but how do we account for the different probabilities between combinations? They are all possibilities of occuring, so each of them simply has to be added together. An easier example is a die - the probability of each side is $\frac{1}{6}$ but you know that if you want to find the probability of a 1 or 2 occuring, it becomes $\frac{2}{6} = \Pr(1) + \Pr(2)$ where $\Pr(1) = \Pr(2) = \frac{1}{6}$. It's just an addition we have to deal with.

$$\begin{aligned}
\Pr(Score > 50) &= 0.75 \times 0.01 + 0.01 \times 0.75 + 0.24 \times 0.01 + 0.01 \times 0.24 + 0.01 \times 0.01 \\
&= 2 \times 0.0075 + 2 \times 0.0024 + 0.0001 \\
&= 0.015 + 0.0048 + 0.0001 \\
&= 0.0199
\end{aligned}$$

Question 3

a.

As you should in most probability questions involving the Normal or Binomial distribution, it is worth defining a variable that describes the distribution. For more information, please refer to **Question 2 a.** in the Detailed Solutions of Probability Tech-Free Test 1.

For this question, let $X \sim \mathrm{N}(75, 3^2)$

From this point we have to think about how we can find $\Pr(X > 81)$ since we want to find when Mr. Li jumps **over** 81 cm. There is an advantage to the astute student in Tech-Free probability, becuase normal probabilities are almost always transcendentals (which are irrational numbers and therefore impossible to use in tech-free). Therefore, we are going to make use of very easy identities and symmetries to solve these questions. What do we know for sure about normal distributions that doesn't require a calculator? The standard Normal distribution which is denoted by $Z \sim \mathrm{N}(0, 1)$. The only way to make use of this standard normal is to standardise the values of X in the defined normal distribution above. This is done by finding $Z = \frac{x-\mu}{\sigma}$.

For $X = 81$, $Z = \frac{81-75}{3} = 2$.

But what exactly does this achieve? It allows us to understand how many standard deviations away a score is from the mean. Because on the standard normal distribution which has a standard deviation of 1, a z-score of 2 is obviously 2 standard deviations away from the mean.

So let's write $\Pr(X > 81)$ in a different form.

$$\Pr(X > 81) = \Pr(X > \mu + 2\sigma)$$

But without a calculator, how are we supposed to know what this probability is? Read the question carefully - there is a big clue in the word "approximate" which is amplified by the fact that this is a tech-free test.

Students need to know the 68-95-99.7% approximation rule (as most textbooks call it). These are **approximations**, and are not acceptable answers unless the question specifies the need for an approximation.

It states the following:

1) For the values of X which are 1 standard deviation away from the mean, the probability is 68%

2) For the values of X which are 2 standard deviations away from the mean, the probability is 95%

3) For the values of X which are 3 standard deviations away from the mean, the probability is 99.7%

But be careful! Standard deviations away from the mean goes both ways, e.g. $\mu \pm \sigma$. The probability we require is only for $X > \mu + 2\sigma$. Since the normal distribution is symmetrical about the mean, it is not difficult to find this probability.

Consider, $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) = 0.95$.

Then everything outside of that will be $1 - 0.95 = 0.05$.

But we only want the one tail outside of the 0.95 region, therefore,

$$\begin{aligned}\Pr(X > \mu + 2\sigma) &= \frac{1}{2}(1 - 0.95) \\ &= 0.025\end{aligned}$$

b.

This question requires students to make use of the normal distribution in order to define a binomial distribution. This may seem tricky at first, but once the distribution is defined, it can once again just be considered a binomial question. Students should take note of this type of question - the combination of these two distributions is a common occurrence within the Mathematical Methods probability course so it is worthwhile spending the short time to really understand what is being done.

What do we need to define a binomial distribution? The number of trials and the probability of success. There are 7 days in the tournament, and hence 7 vertical jump "trials" Mr. Li will undertake.

The question wants the probability of his jumping over 75 cm on 5 days, so let us define the "success" as when he jumps over 75 cm. Now we just want the probability of jumping over 75 cm once, according to the normal distribution defined above.

$$\begin{aligned}\Pr(X > 75) &= \Pr(X > \mu) \\ &= 0.5\end{aligned}$$

This probability should make sense since it is the very definition of the median - the probability of values greater than the median is 0.5 (and hence the probability of values less than the mean is also 0.5 because the normal distribution is symmetrical, and therefore the median coincides with the mean, which is not necessarily the case in every probability distribution).

Now we can define our Binomial distribution - be sure to use a different pronumeral to the one you have used in previous parts of a question to avoid confusion.

Let $Y \sim \text{Bi}(7, 0.5)$

No order is required for this question - it just says "on 5 days" as opposed to in 5 consecutive days or something similar. We want to find the probability that he will be successful on **any** 5 days. Hence, the need to account for the different combinations of days he could jump over 75 cm on 5 days. Refer to **Question 2 b.** in the Detailed Solutions of Probability Tech-Free Test 1 for more information on how to find probabilities using the Binomial Theorem.

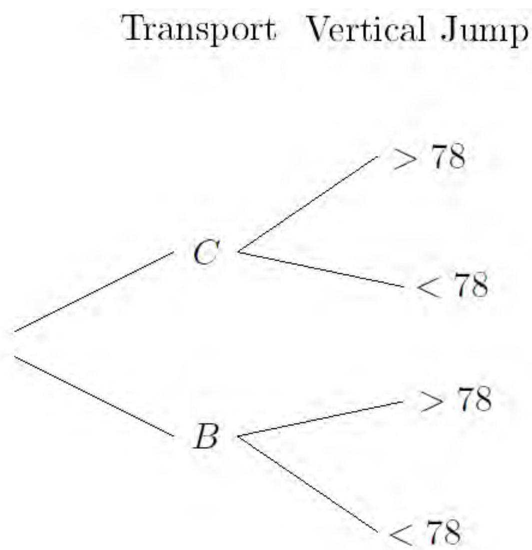
$$\begin{aligned}\Pr(Y = 5) &= \frac{7!}{5!2!}0.5^5\,(0.5)^2 \\ &= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1 \times 2 \times 1}(0.5)^5(0.5)^2 \\ &= \frac{7 \times 6}{2 \times 1}(0.5)^5(1 - 0.5)^2 \\ &= 21(0.5)^5(1 - 0.5)^2\end{aligned}$$

Be careful to take note of when questions ask for a specific form. Particularly in Tech-Free exams/questions, you can expect a specific form to be required due to the practicality of working out complicated calculations - VCAA wants to test the concepts within the course rather than waste your time and test you rigorously on calculations.

Question 4

The purpose of this question is to accumulate all the information given in this question - the whole test has been slowly guiding you towards this one. It is up to the student to be able to break apart these long-winded questions packed with information. There are three portions to this question - whether he rides his bike or drives his car, whether or not he jumps over 78 cm in the vertical jump, and lastly, whether he wins the event or not. Each of these three portions accumulate on top of each other to give us a final probability. We'll have to break it up because it's too much information to put together for the meantime.

The information can be split into the following tree diagram. These are very useful just as a diagram to illustrate all the information given and help students with a visual aid - it cannot be overstated how useful it is for most students to be able to see information as an image as opposed to words.



Let's start from the beginning - the first thing that occurs is Mr. Li travels to the stadium. This is the first fork in the tree diagram - he will either ride his bike or drive his car. So all we need to do for now is find the probabilities involved in day 2.

Probabilities of whether he takes the car or bike on day 2 (using the result from **Question 1 b.**),

$$\begin{aligned} S_2 &= TS_1 \\ &= \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} \end{aligned}$$

Now don't get muddled by the fact that there are already two possible paths. Take it in your stride to just look at one path at a time.

First, when he drives his car. The next part is whether he jumps over 78 cm or not. This is simply given by the normal distribution in **Question 3**. The method of finding the probability below is also outlined in **Question 3**

$$\begin{aligned} \Pr(X > 78) &= \Pr(X > \mu + \sigma) \\ &= \frac{1}{2}(1 - 0.68) \\ &= 0.16 \end{aligned}$$

Lastly, the chances of him winning are predicated on his jumping over 78 cm, the question has no consideration of what happens when he doesn't jump over 78 cm because the probability of winning is 0%, so when multiplied, the probability is simply 0. Hence, no matter what, we will multiply our probability of him jumping over 78 cm when he drives his car by 1, since he always wins when he jumps over 78 cm.

The probabilities of the three portions will simply be multiplied together as per the tree diagram.

Therefore, for the case that he drives his car, the probability of winning is:

$$\begin{aligned}\Pr(Win_{car}) &= 0.3 \times \Pr(X > 78) \times 1 \\ &= 0.3 \times \frac{1}{2}(1 - 0.68) \\ &= 0.048\end{aligned}$$

We've already covered half of the tree diagram! And the second half will be solved much the same way, however we have to read the question to understand that if he rides his bike, the distribution of his vertical jump changes (probably because he's more tired from the ride).

We already have the probability of when he rides his bike, so let's look at the vertical jump now. We have to adjust the distribution according to the information given.

Hence, let $A \sim \mathrm{N}(74, 2^2)$, the distribution that describes his vertical jump after he rides his bike. And again after this, we just have to multiply by 1, the probability of him winning when he jumps over 78 cm.

Hence, the probability of winning when he rides his bike is:

$$\begin{aligned}\Pr(Win_{bike}) &= 0.7 \times \Pr(A > 78) \times 1 \\ &= 0.7 \times \Pr(A > \mu + 2\sigma) \\ &= 0.7 \times \frac{1}{2}(1 - 0.95) \\ &= 0.0175\end{aligned}$$

Now that we have the probabilities of both paths found, we just have to add them together to find the total probability.

Therefore, the probability of winning is:

$$\begin{aligned}\Pr(Win) &= \Pr(Win_{car}) + \Pr(Win_{bike}) \\ &= 0.048 + 0.0175 \\ &= 0.0655 \\ &\sim 0.066\end{aligned}$$

Again, be sure to leave your answer to the correct number of decimal places - in this case 3.

PROBABILITY
TECH-ACTIVE TEST 1
DETAILED SOLUTIONS

SECTION 1 - Multiple Choice Questions

Question 1 (E)

The key properties of a probability distribution are that

$$0 \leq p(x) \leq 1$$
$$\sum_x p(x) = 1$$

That is none of the individual properties can be negative and all the sum of the probabilities must add to 1. Adding the probabilities results in

$$\begin{aligned} \frac{1}{2}k + k^2 + 2k^2 + 3k^2 + \frac{1}{2}k &= 1 \\ 6k^2 + k - 1 &= 0 \\ (3k-1)(2k+1) &= 0 \end{aligned}$$

Using the null factor law, the first bracket could be 0, or the second bracket could be 0 or both brackets could equal 0.

$$\begin{aligned} 3k - 1 = 0 \quad &\text{or} \quad 2k + 1 = 0 \\ k = \frac{1}{3} \quad &\text{or} \quad k = -\frac{1}{2} \end{aligned}$$

But since our individual probabilities have to be greater than 0, we have $\frac{1}{2}k > 0$ and $k^2 > 0$. i.e. $k > 0$ so we reject the negative solution.

$$\begin{aligned} k > 0 & \\ \therefore k &= \frac{1}{3} \end{aligned}$$

So the required option is **E**.

Question 2 (B)

Firstly we note that we have a fair die, and the event that we are interested in, which we will call X, is obtaining a 4, and the number of trials we run is 3. So we can either "obtain a 4" or we can "not obtain a 4". So that is we have either a success or a failure, as soon as you narrow down the options to this. We also note that each trial or 'run' will be independent of one another, that is one trial does not have an effect on the other. So we should be able to represent the situation as a Binomial distribution.

$$X \sim \text{Bi}\left(3, \frac{1}{6}\right)$$

Now since we are looking at the *"probabilitiy of obtaining at least* 1 *occurrence of a* 4*"* then we are looking for $\Pr(X \geq 1)$. So we recall the formula for the probability of obtaining an event in a binomial distribution

$$\Pr(X = x) = \frac{n!}{x!\,(n-x)!}(p)^x (1-p)^{n-x}$$

So an equivalent way of expressing $\Pr(X \geq 1)$ is

$$\Pr(X \geq 1) = \Pr(X=1) + \Pr(X=2) + \Pr(X=3)$$

But we note that there is no options that are a combination of three single binomial probabilities, so we need to look at the question from another angle. If we look at the event that we are not interested in, then we may be able to find an equivalent expression

$$\begin{aligned}\Pr(X \geq 1) &= 1 - \Pr(X=0) \\ &= 1 - \frac{3!}{0!3!}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3\end{aligned}$$

Which we note is option **B**.

Going through the other options we see what they represent

A.	$\Pr(X=2) + \Pr(X=3) = \Pr(X \geq 2)$
C.	$\Pr(X=0)$
D.	$\Pr(X=3)$
E.	Not part of the binomial distribution of $X \sim \text{Bi}\left(3, \frac{1}{6}\right)$

Question 3 (A)

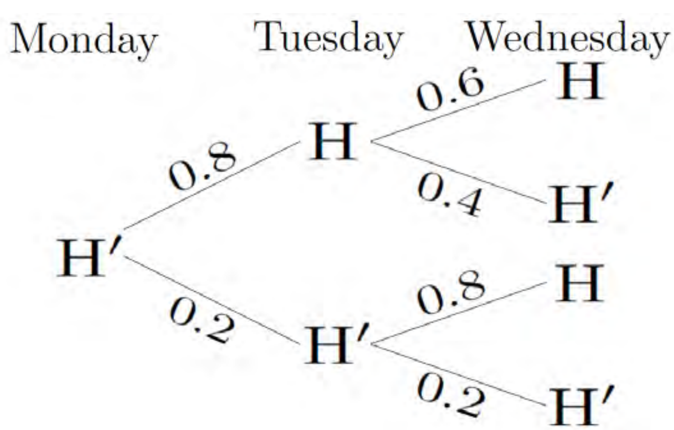
One of the keys to mastering probability on the methods course is to work out what type of probability methods needs to be used in each question, students may know the theory behind each section but struggle to pick out when to apply what.

There is two ways to approach this question, one is to use a tree diagram which is useful when there are a small number of trials, the second is to use markov chains, which come in handy when we have a large number of trials.

Looking at using a tree diagram, if we denote H as being the event he does do his homework on any day then the first method, then we get

$$\begin{aligned}\Pr(\mathrm{H}|\mathrm{H}) &= 0.6 \\ \Pr(\mathrm{H}'|\mathrm{H}') &= 0.2\end{aligned}$$

So drawing the tree diagram out, we note that we 3 columns of events since we have the event occurring on Monday, Tuesday and Wednesday.



We calculate the values of 0.4 and 0.8 as the probabilities of all the events on an arm has to add to 1. Now we are looking for the probability of him not doing his homework on Wednesday, which will be H′ on Wednesday, that is

$$\begin{aligned}\Pr(\text{no homework on wednesday}) &= (0.8 \times 0.4) + (0.2 \times 0.2) \\ &= 0.32 + 0.04 \\ &= 0.36\end{aligned}$$

So the probability that Andrew does not do his homework on Wednesday is 0.36. Now the second method using

Markov chains. Looking through the question we note that Andrew can either "do his homework" or he can "not do his homework". So lets denote A to be the event that he does do his homework on the previous day, and denote B to be the event that he does do his homework on the current day.

Looking at the question we need to pick out the key bits of information we are supplied with. We are told that *"The probability that Andrew does homework on a day given that he did homework on the previous day is* 0.6". So this tells us that whether Andrew does his homework or not is dependent on what he does the day before. This is evident by the word "given", which leads us to conditional probability. So we have

$$\Pr(B|A) \quad = \quad 0.6$$

We also know that *"the probability he doesn't do homework on a day given that he didn't do homework on the previous day is* 0.2*"*. That is

$$\Pr(B'|A') \quad = \quad 0.2$$

So putting that into a transition matrix we get

$$\mathrm{T} \quad = \quad \begin{bmatrix} 0.6 & \\ & 0.2 \end{bmatrix}$$

We note that the columns in a transition matrix have to add to 1.

$$\mathrm{T} \quad = \quad \begin{bmatrix} 0.6 & 0.8 \\ 0.4 & 0.2 \end{bmatrix}$$

Now there are two forms to a markov chain,

$$S_n \quad = \quad T \times S_{n-1} = T^n \times S_0$$

Since we have an initial state that counts as our first day, we have the second form, and since on Monday he doesn't do his homework, our initial condition is

$$S_0 \quad = \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We have 2 days after Monday, that is Tuesday and Wednesday, so $n = 2$.

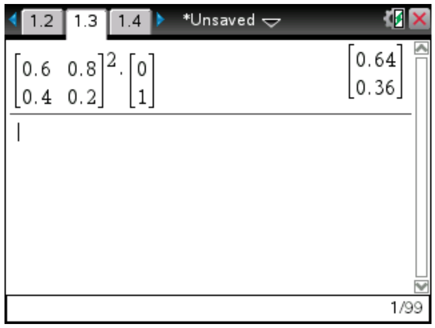
$$\begin{aligned} S_2 &= T^2 \times S_0 \\ &= \begin{bmatrix} 0.6 & 0.8 \\ 0.4 & 0.2 \end{bmatrix}^2 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.64 \\ 0.36 \end{bmatrix} \end{aligned}$$

So on Wednesday

$$\begin{bmatrix} \Pr(\mathrm{H}) \\ \Pr(\mathrm{H}') \end{bmatrix} \quad = \quad \begin{bmatrix} 0.64 \\ 0.36 \end{bmatrix}$$

$\therefore \Pr(\mathrm{H}') = 0.36$

To evaluate this on the calculator we need to open the matrix window, this can be down by clicking the "$\left[|\square| \begin{cases} \square \\ \square \end{cases} \right]$" button and selecting the appropriate matrix in the second row. If we need bigger than a 2×2 then select the 3×3 matrix and enter in the dimensions.



So the probability that Andrew does not do his homework on Wednesday is 0.36.

That makes our answer would be option **A**.

Question 4 (C)

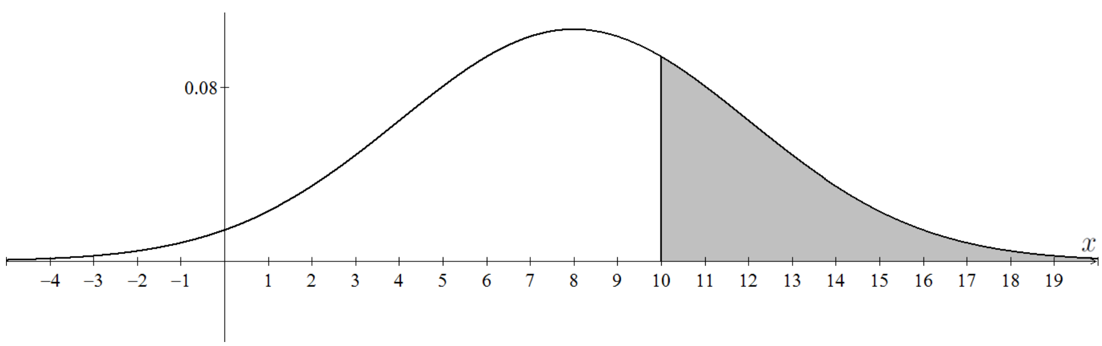
Firstly we are given a normal distribution X, which we know to have mean $\mu = 8$ and variance $\sigma^2 = 16$. Since the variance of a distribution is the square of the standard deviation, the standard deviation of X is

$$\begin{aligned}\sigma^2 &= 16\\ \sigma &= \pm\sqrt{16} \text{ but } \sigma \geq 0\\ \therefore \sigma &= 4\end{aligned}$$

So

$$X \sim N(8, 16)$$

We are looking for $\Pr(X > 10)$ and note that all of the options we have to choose from involve the standard normal distribution, Z.



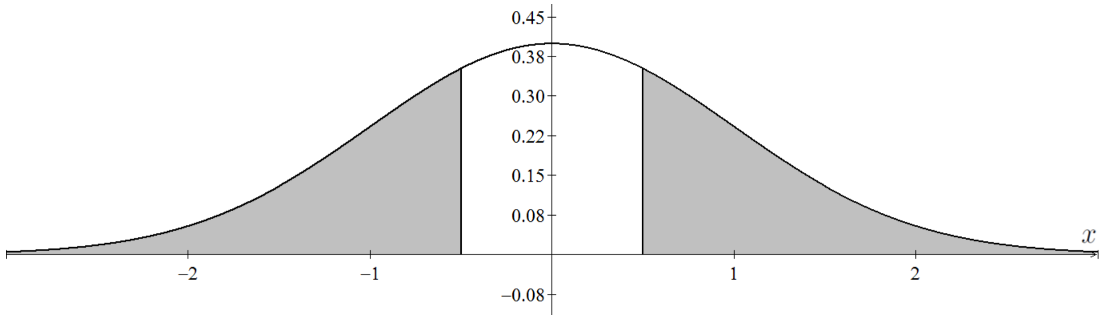
So we need to convert our value of X into a Z-score. The Z-score is given by

$$\begin{aligned}z &= \frac{x-\mu}{\sigma}\\ &= \frac{10-8}{4}\\ &= \frac{1}{2}\end{aligned}$$

That means that

$$\Pr(X > 10) = \Pr\left(Z > \frac{1}{2}\right)$$

But this isn't any of our options, so we still need to find an equivalent expression. Now one of the properties of the standard normal distribution is that it is symmetrical about $x = 0$.



$$\Pr\left(Z > \frac{1}{2}\right) = \Pr\left(Z < -\frac{1}{2}\right)$$

So option **C** is the correct answer.

Question 5 (B)

We are asked to find the median and variance of the distribution. To find the median we add each probability from the lowest value of x to the highest until we reach or exceed 0.5. If we exceed 0.5 then that value of x for when the sum exceeds 0.5 is the median. If we reach 0.5 exactly, then we sum the current value of x and the next value of x and divide by 2.

$$0.05 + 0.05 + 0.2 + 0.2 = 0.5$$

So since the sum reaches 0.5 at $x = 1$ the median m will be give by

$$\begin{aligned} m &= \frac{1+2}{2} \\ &= 1.5 \end{aligned}$$

So already the only options left are B and C. Now the variance of the distribution will be given by

$$\operatorname{Var}(X) = \mathrm{E}\left[(X-\mu)^2\right] = \mathrm{E}\left(X^2\right) - \left[\mathrm{E}(X)\right]^2$$

So we can calculate the variance in two ways, both require the mean μ, so lets calculate that first by summing the product x and $\Pr(X = x)$

$$\begin{aligned} \mu &= \mathrm{E}(X) \\ &= (-2 \times 0.05) + (-1 \times 0.05) + (0 \times 0.20) + (1 \times 0.2) + (2 \times 0.25) + (3 \times 0.25) \\ \therefore \mu &= 1.3 \end{aligned}$$

So lets use the first method, we find the expected value of the square of each of the x values minus the mean.

$$\begin{aligned} \operatorname{Var}(X) &= (-2-1.3)^2(0.05) + (-1-1.3)^2(0.05) + (0-1.3)^2(0.20) + (1-1.3)^2(0.20) + (2-1.3)^2(0.25) + (3-1.3)^2(0.25) \\ &= 2.01 \end{aligned}$$

Using the second method, we need to find $\mathrm{E}\left(X^2\right)$

$$\begin{aligned} \mathrm{E}\left(X^2\right) &= (-2)^2(0.05) + (-1)^2(0.05) + 0^2(0.20) + 1^2(0.20) + 2^2(0.25) + 3^2(0.25) \\ &= 3.7 \end{aligned}$$

So the variance of X is given by

$$\begin{aligned} \operatorname{Var}(X) &= 3.7 - 1.3^2 \\ &= 2.01 \end{aligned}$$

So option **B** is the correct answer.

Question 6 (D)

The first piece of information to note is the type of probability distribution we're dealing with, here we have a binomial distribution. Now we are given the variance and expected value of the distribution.

$$\begin{aligned} \mathrm{E}(X) &= 3 \\ \sigma(X) &= \frac{3}{2} \end{aligned}$$

Now looking at the options we have, we are looking for the values of n and p of the distribution, that is the number of trials and the probability of success on a trial. We also know the following for a binomial distribution

$$\begin{aligned} \mathrm{E}(X) &= np \\ \mathrm{Var}(X) &= np(1-p) \end{aligned}$$

So we need the variance of X, which is easy to obtain as the variance of a distribution is the square of the standard deviation of the distribution. We can also represent the variance by σ^2 and the standard deviation by σ.

$$\begin{aligned} \sigma^2 &= \left(\frac{3}{2}\right)^2 \\ &= \frac{9}{4} \end{aligned}$$

So we have two simultaneous equations

$$\begin{aligned} np &= 3....[1] \\ np(1-p) &= \frac{9}{4}....[2] \end{aligned}$$

Substituting [1] into [2]

$$\begin{aligned} 3(1-p) &= \frac{9}{4} \\ 1-p &= \frac{3}{4} \\ \therefore p &= \frac{1}{4} \end{aligned}$$

Substituting $p = \frac{1}{4}$ into [1]

$$\begin{aligned} n\left(\frac{1}{4}\right) &= 3 \\ \therefore n &= 12 \end{aligned}$$

Now we can represent the binomial distribution by $X \sim \mathrm{Bi}(n, p)$. So our distribution is given by $X \sim \mathrm{Bi}\left(12, \frac{1}{4}\right)$. So option **D** is correct.

Question 7 (A)

The type of probability that we are now dealing with here is continuous probability. The expected value or mean of a continuous probability density function is given by

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} (xf(x))\, dx$$

We need to integrate across the entire domain for which the function isn't 0.

$$\mathrm{E}(X) = \int_{0}^{e-1} \left(\frac{x}{x+1}\right) dx$$

Although the quickest way to do this on the calculator, it is good to know how to do it by hand as well. So we can use a little trick to expand the fraction out, we try to make the top look like the bottom by adding a number, then subtracting it.

$$\begin{aligned} \frac{x}{x+1} &= \frac{x+1-1}{x+1} \\ &= \frac{x+1}{x+1} - \frac{1}{x+1} \\ &= 1 - \frac{1}{x+1} \end{aligned}$$

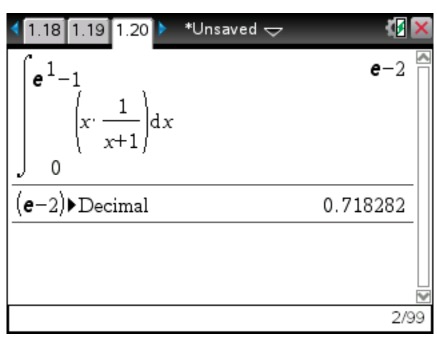
Now we can integrate it by hand

$$\begin{aligned} \mathrm{E}(X) &= \int_0^{e-1} \left(1 - \frac{1}{x+1}\right) dx \\ &= \left[x - \log_e(|x+1|)\right]_0^{e-1} \\ &= e - 1 - \log_e(|e-1+1|) - (0 - \log_e(|0+1|)) \\ &= e - 2 \\ &= 0.7183 \end{aligned}$$

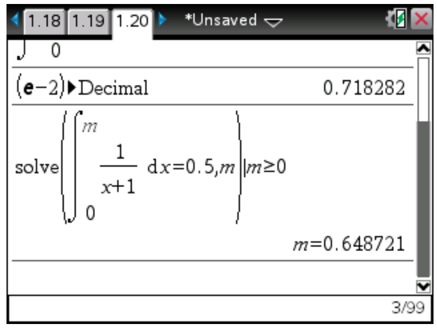
The median can be found by integrating the function to an upper terminal of m, and solving for when that integral equals 0.5.

$$\begin{aligned} \int_0^{m} \left(\frac{1}{x+1}\right) dx &= \frac{1}{2} \\ \left[\log_e(|x+1|)\right]_0^{m} &= \frac{1}{2} \\ \log_e(m+1) - \log_e(1) &= \frac{1}{2} \\ \log_e(m+1) &= \frac{1}{2} \\ m+1 &= e^{\frac{1}{2}} \\ m &= e^{\frac{1}{2}} - 1 \\ \therefore m &= 0.6487 \end{aligned}$$

To find the answers on the calculator we make use of the integral function, "[⇑shift]+[+]". Then to get the decimal answer we can use "[Menu] [2] [1]".



For the median, we again need to use the integral function, but also the solve function, "[Menu] [3] [1]" but also add a domain restriction using "$|m > 0$", the vertical bar can be found using "[ctrl]+[=]" which is under the control button.



So that makes option **A** the correct answer.

Question 8 (B)

This questions asks for the *"steady state probability that Australia wins a game"* if *"Sri Lanka wins the first game"*. If we keep running trials over and over then the probability of a particular event should settle down to a certain value, which we call the steady state probability. There are two ways to go about finding the answer, one is to use the formula for the steady state equation, the other is to raise the transition matrix to a large power, usually 50 or 100, until raising it to any larger power results in the same probabilities.

So firstly we need to find the transition matrix. If we let the event that Australia wins the previous game be event A and the event that they win the current game to be event B, then the transition matrix will be given by

$$T = \begin{bmatrix} \Pr(B|A) & \Pr(B|A') \\ \Pr(B'|A) & \Pr(B'|A') \end{bmatrix}$$

So since if Australia won the previous game, that is event A, the probability of them winning the next game, that is event B is $\frac{1}{2}$, we have $\Pr(B|A) = \frac{1}{2}$. If Sri Lanka wins the previous game, that is event A', then the probability of winning the next game, that is event B', is $\frac{1}{3}$, we have $\Pr(B'|A') = \frac{1}{3}$. Now we also know that the columns of a transition matrix must add to 1. So our transition matrix is

$$T = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

Now if we have the transition matrix

$$T = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$$

Then the steady state probability of each event is

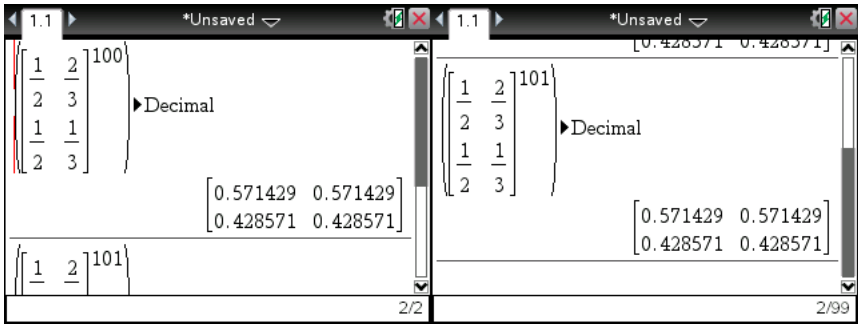
$$\begin{aligned} \Pr(B_n) &= \frac{b}{a+b} \\ \Pr(B_n') &= \frac{a}{a+b} \end{aligned}$$

So since we want the event that Australia wins, we are looking for $\Pr(B_n)$, so

$$\begin{aligned} \Pr(B_n) &= \frac{\frac{2}{3}}{\frac{1}{2}+\frac{2}{3}} \\ &= \frac{2}{3} \div \frac{3+4}{6} \\ &= \frac{4}{7} \end{aligned}$$

So the steady state probability of Australia winning a game is $\frac{4}{7}$.

Now using the second method, we take the transition matrix and raise it to a larger and larger power until the probabilities don't change. It's best to start off with say a power of 100 and try from there.



The top row represents event B and the bottom row represents event B', so we need to look at the top row. So the steady state probability of Australia winning a game is $0.571429 \approx \frac{4}{7}$.
Hence the answer is **B**.

Question 9 (E)

Firstly we note that we don't have all the probabilities for the distribution, and since they have to add to 1, we can find the value of a.

$$\begin{aligned} 0.1 + 0.1 + a + 0.2 + 0.15 + 0.1 + 0.05 &= 1 \\ \therefore a &= 0.3 \end{aligned}$$

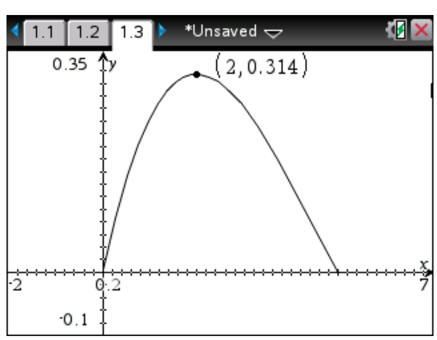
Now since we are looking for 'probability that X is greater than 2, given that it is equal to or greater than 0', we are going to have to use conditional probability, that is

$$\begin{aligned} \Pr(X \geq 2 | X \geq 0) &= \frac{\Pr(X \geq 2) \cap \Pr(X \geq 0)}{\Pr(X \geq 0)} \\ &= \frac{\Pr(X \geq 2)}{\Pr(X \geq 0)} \\ &= \frac{0.15 + 0.1 + 0.05}{0.3 + 0.2 + 0.15 + 0.1 + 0.05} \\ &= \frac{0.3}{0.8} \\ &= \frac{3}{8} \end{aligned}$$

So the correct option is **E**.

Question 10 (A)

The mode of a continuous distribution occurs when $f(M) \geq f(x)$ for all other values of x. So firstly the best task to do is to graph the function on the calculator.



So from the shape of the graph our maximum will be be at the turning point, so to find the turning point we need to find the derivative of the function and let it equal zero.

$$\begin{aligned}
f'(x) &= \frac{12(x-2)(3x-20)}{1375} \\
\frac{12(x-2)(3x-20)}{1375} &= 0 \\
x-2=0 \quad &\text{or} \quad 3x-20=0 \\
x=2 \quad &\text{or} \quad x=\frac{20}{3} \\
\text{But } 0 \le x \le 5 & \\
\therefore x=2 &
\end{aligned}$$

A second approach would be to plot the graph and use the trace function, that is "[Menu] [5] [1]", using the clickpad to move left or right until the word minimum appears, which we then note down the value of x.

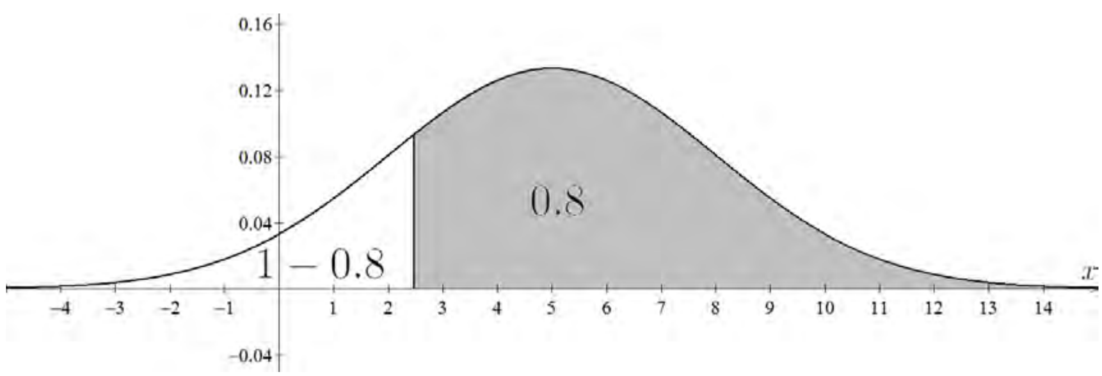
So the mode of the distribution is $x = 2$. This makes **A** the correct answer.

Question 11 (C)

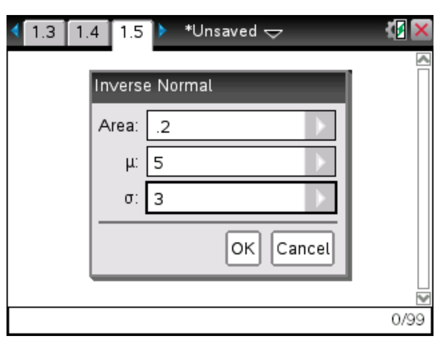
We are given a normal distribution X that has $\mu = 5$ and $\sigma = 3$. This can be represented by $X \sim \mathrm{N}(\mu, \sigma^2)$. So we have

$$X \sim \mathrm{N}(5, 9)$$

Now we are told $\Pr(X \ge a) = 0.8$



So that leave the remaining area as $1 - 0.8 = 0.2$. The reason we do this is because we need to use the inverse normal function on the calculator, and that takes the area from $-\infty$. So using the invnorm function, "[Menu] [5] [5] [3]"



We obtain $c = 2.4751$. Hence option **C** is correct.

SECTION 2 - Extended Response Questions

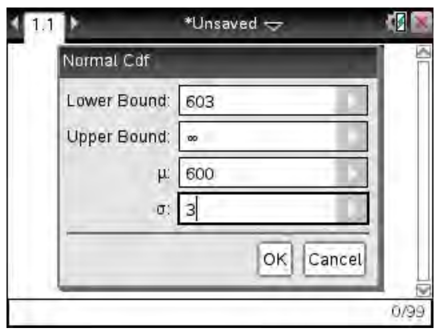
Question 1
Part ai.
We need to find $\Pr(X \geq 603)$ **correct to 4 decimal places**, that is since X follows a normal distribution, it will follow

$$X \sim \mathrm{N}\left(\mu, \sigma^2\right)$$

Our mean is 600 ml and our standard deviation is 3 ml, so

$$\begin{aligned} \mu &= 600 \\ \sigma &= 3 \\ X &\sim \mathrm{N}(600, 9) \end{aligned}$$

To find out probability we need to use the normal cumulative distribution function, "[Menu] [5] [5] [2]", with a lower bound of 603 and an upper bound of ∞. ∞ can be obtained from "[ctrl] + [π]" then selecting ∞ from the menu that appears.



$$\therefore \Pr(X \geq 603) = 0.1587$$

Part aii.
Now we are dealing with Y, which follows a continuous distribution. The probability over an interval will be given by the area under the function for that interval. So we need to integrate the function given from $y = 597$ to $y = 606$.

$$\begin{aligned} \Pr(597 \leq Y \leq 606) &= \int_{597}^{606} \left(\frac{3y^2 - 3600y + 1080064}{2048}\right) dy \\ &= \left[\frac{1}{2048}\left(y^3 - 1800y^2 + 1080064y\right)\right]_{597}^{606} \\ &= 0.3999 \end{aligned}$$

Part b.
The mean value of a continuous probability distribution will be given by

$$\mathrm{E}(Y) = \int_{-\infty}^{\infty} y f(y)\, dy$$

But since the function is 0 for certain parts, we only need to integrate for the intervals where it is not 0. That is

$$\begin{aligned} \mathrm{E}(Y) &= \int_{592}^{608} \left(y \times \frac{3y^2 - 3600y + 1080064}{2048}\right) dy \\ &= \left[\frac{y^2}{8192}\left(3y^2 - 4800y + 2160128\right)\right]_{592}^{608} \\ &= 600\,\mathrm{ml} \end{aligned}$$

The expected amount of liquid in a lemon flavoured soft drink is 600 ml.

Part c.

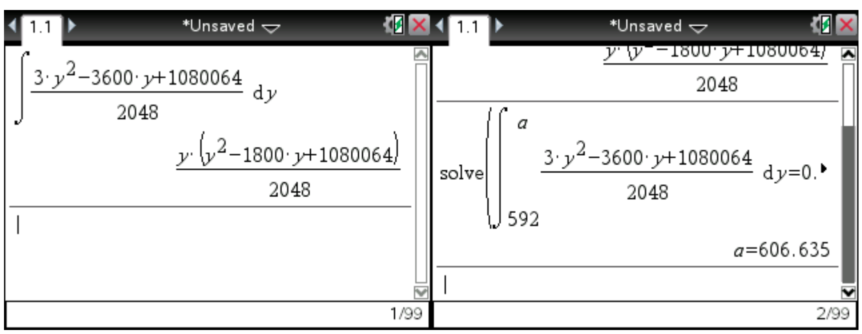
We are asked to find the value of a for which $\Pr(Y \leq a) = 0.85$. So that is the value of Y for which the probability of obtaining less than the value is 0.85. Also note that our answer must be **to 2 decimal places.** Whenever you start a question always note the amount of decimal places the answer should be to, as it is easy to lose marks for not having the answer to the right number of decimal places.

Again since this is a continuous distribution, to find the probability over an interval, we need to integrate the function. Since we are looking for the values less than a, we will need to integrate from the lowest value of $f(y)$ that isn't 0 up to a.

$$\begin{aligned}
\Pr(Y \leq a) &= 0.85 \\
\int_{592}^{a} \left(\frac{3y^2 - 3600y + 1080064}{2048} \right) dy &= 0.85 \\
\left[\frac{1}{2048} \left(y^3 - 1800y^2 + 1080064y \right) \right]_{592}^{a} &= 0.85 \\
a &= 606.64
\end{aligned}$$

It should be noted to be careful on the rounding, the value of a that the caclulator gives here is actually 606.635336116, which we round up since the third digit is ≥ 5. But in the case where we have $a = 606.649$, take care to round it down, instead of taking 606.65 and rounding up, which would be incorrect in that situation.

The quickest way to get the answer would be to evaluate this on the calculator using the solve, "[Menu] [3] [1]", and integral "[⇑shift]+[+]" functions. To get the definite integral, you can use the clear button to remove the terminals, then to get the answer, solve using the terminals.



Part d.

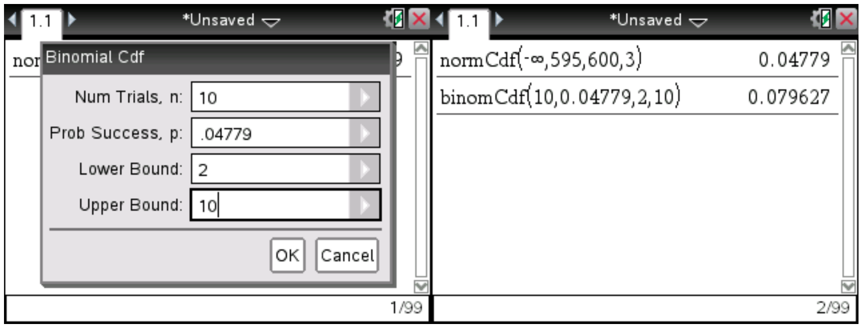
This question mixes using two different types of probability distributions. Firstly the orange soft drink still follows a normal distribution, and we find the probability of $X \leq 595$.

$$\begin{aligned}
X &\sim \mathrm{N}(600, 9) \\
\Pr(X \leq 595) &= 0.04779
\end{aligned}$$

This can be evaluated using the normal cumulative distribution function, "[Menu] [5] [5] [2]". Then this outcome becomes our "event" for either a success or failure, and we have 10 chances of picking it. That is the event that "the bottle selected contains less than 595 ml of orange soft drink", so lets denote that event A. Then A follows a binomial distribution

$$A \sim \mathrm{Bi}(10, 0.04779)$$

So then we are looking for $\Pr(A \geq 2)$, which can be evaluated using the binomial cumulative distribution function, "[Menu] [5] [5] [E]", since we want for more than one value of A. If we wanted just a single value of A then we could use the binomial distribution function, "[Menu] [5] [5] [D]". Our number of trials is 10, $p = 0.04779$ with a lower bound and upper bound of 2 and 10 respectively.



$$\therefore \Pr(A \geq 2) \quad = \quad 0.0796$$

Part e.

Firstly we are dealing with conditional probability, that is

$$\Pr(Y \leq A | Y \leq B) \quad = \quad \frac{\Pr(Y \leq A \cap Y \leq B)}{\Pr(Y \leq B)}$$

So we have

$$\begin{aligned}
\Pr(Y \leq 595 | Y \leq 600) &= \frac{\Pr(Y \leq 595 \cap Y \leq 600)}{\Pr(Y \leq 600)} \\
&= \frac{\Pr(Y \leq 595)}{\Pr(Y \leq 600)}
\end{aligned}$$

So we need the values of $\Pr(Y \leq 595)$ and $\Pr(Y \leq 600)$, which we can get using the same methods above for the continuous probability distribution.

$$\begin{aligned}
\Pr(Y \leq 595) &= \int_{592}^{595} \left(\frac{3y^2 - 3600y + 1080064}{2048} \right) dy \\
&= \left[\frac{1}{2048} \left(y^3 - 1800y^2 + 1080064y \right) \right]_{592}^{595} \\
&= 0.282715 \\
\Pr(Y \leq 600) &= \int_{592}^{600} \left(\frac{3y^2 - 3600y + 1080064}{2048} \right) dy \\
&= \left[\frac{1}{2048} \left(y^3 - 1800y^2 + 1080064y \right) \right]_{592}^{600} \\
&= 0.5000
\end{aligned}$$

At this point we could also take advantage of the fact that since the parabola is symmetrical about the mean of 600, and the area under it has to be 1, that the area from $Y = 592$ to $Y = 600$ would have to be $\frac{1}{2}$.

So now putting that into our conditional probability formula

$$\begin{aligned}
\Pr(Y \leq 595 | Y \leq 600) &= \frac{\Pr(Y \leq 595)}{\Pr(Y \leq 600)} \\
&= \frac{0.282715}{0.5} \\
&= 0.5654
\end{aligned}$$

Part f.

We are looking for the probability that a bottle picked will contain Orange soft drink **given** that it contains less than 595 ml of soft drink. So if we denote the event of getting an Orange falvoured bottle of soft drink to be O, and getting a lemon flavoured bottle of soft drink to be L, and the amount of liquid in a bottle to be V, then we are looking for $\Pr(O|V \leq 595)$. By conditional probability, this will be

$$\Pr(O|V \leq 595) = \frac{\Pr(O \cap V \leq 595)}{\Pr(V \leq 595)}$$

So we need to find the probability that we select a bottle that contains less than 595 ml of soft drink and the probability that any bottle selected will contain less than 595 ml of soft drink. We already know from previous questions that

$$\begin{aligned} \Pr(X \leq 595) &= 0.04779 \\ \Pr(Y \leq 595) &= 0.282715 \end{aligned}$$

We also know that we have 20 bottles of X and 10 bottles of Y and so the total number of bottles we have is 30. That is we have a $\frac{20}{30}$ chance of picking an bottle containing Orange soft drink. So this means that the probability of obtaining a bottle that contains less than 595 ml of soft drink is

$$\begin{aligned} \Pr(V \leq 595) &= \left(\frac{20}{30} \times 0.04779\right) + \left(\frac{10}{30} \times 0.282715\right) \\ &= 0.12610 \end{aligned}$$

Now the probability of selecting a bottle containing Orange soft drink and contains less than 595 ml of soft drink is

$$\begin{aligned} \Pr(O \cap V \leq 595) &= \frac{20}{30} \times 0.04779 \\ &= 0.03186 \end{aligned}$$

So we have

$$\begin{aligned} \Pr(O|V \leq 595) &= \frac{0.03186}{0.12610} \\ &= 0.2527 \end{aligned}$$

Question 2

Part a.

Firstly lets go through the information in the question and pick our the key parts and ideas. If we let A be the event that Will scores on his previous shot and B be the event that he scores on the current shot. Now since *"The probability that Will scores the goal given that he scored on his previous attempt is* 0.7", we have

$$\Pr(B|A) = 0.7$$

Now we also have know that *"while the probability that he scores given that he did not score on his previous attempt is* 0.45."That is

$$\Pr(B|A') = 0.45$$

Now we also know that the columns in a transition matrix have to add to 1. So our transition matrix is

$$\begin{aligned} T &= \begin{bmatrix} \Pr(B|A) & \Pr(B|A') \\ \Pr(B'|A) & \Pr(B'|A') \end{bmatrix} \\ &= \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix} \end{aligned}$$

Now if Will scores on his first attempt, we have our initial state being

$$S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since we want the probability of scoring on the fifth attempt, there are 4 attempts after the first attempt, so we will need to use the transition matrix to the 4^{th} power.

$$\begin{aligned} S_n &= \mathrm{T}^n \times S_0 \\ S_4 &= \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix}^4 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.6016 \\ 0.3984 \end{bmatrix} \end{aligned}$$

Now since we are looking for the "failure" to the original event, we take the probability from the bottom row, if we were looking for the "success" of the event, then we would take the probability from the top row. So the probability that Will does not score on his fifth attempt is 0.3984.

Part b.

If we let S be the event that Will scores a goal then we are looking for $\Pr(SSS|S')$, as our first outcome which has already occured was S'. Since we want a success on all three outcomes instead of just at a certain point, it is better to use a single pathway approach instead of a markov chain approach. So that is

$$\begin{aligned} \Pr(SSS|S') &= \Pr(\mathrm{S}|\mathrm{S}') \times \Pr(\mathrm{S}|\mathrm{S}) \times \Pr(\mathrm{S}|\mathrm{S}) \\ &= 0.45 \times 0.7 \times 0.7 \\ &= 0.2205 \end{aligned}$$

Part c.

If we have the transition matrix

$$\mathrm{T} = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$$

Then the steady state probability of a success will be given by

$$\Pr(\text{"Success"}) = \frac{b}{a+b}$$

and of failure

$$\Pr(\text{"Failure"}) = \frac{a}{a+b}$$

So in our transition matrix we have, $a = 0.30$ and $b = 0.45$. So we have

$$\begin{aligned} \Pr(S_\infty) &= \frac{0.45}{0.30+0.45} \\ &= \frac{3}{5} \end{aligned}$$

So the long term probability that Will scores a goal is $\frac{3}{5}$.

Part d.

There is two ways to approach this question, with the first being quicker and less tedious than the second.

The first being by matrix methods, where we can get the expected value from

$$\mathrm{E}\left(\begin{bmatrix} X_0 \\ X_1 \end{bmatrix}\right) = T^1 S_0 + T^2 S_0 + T^3 S_0$$

Now since he is successful on his previous attempt we have $S_0 = \begin{bmatrix}1\\0\end{bmatrix}$, and our transition matrix from the previous question.

$$\begin{aligned}
\mathrm{E}\left(\begin{bmatrix} X_0 \\ X_1 \end{bmatrix}\right) &= \begin{bmatrix}0.70 & 0.45\\0.30 & 0.55\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0.70 & 0.45\\0.30 & 0.55\end{bmatrix}^2\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0.70 & 0.45\\0.30 & 0.55\end{bmatrix}^3\begin{bmatrix}1\\0\end{bmatrix} \\
&= \begin{bmatrix}1.93125\\1.06875\end{bmatrix}
\end{aligned}$$

Now since we are looking for the expected number that he scores, that is for the expected value of a success, we take the top row, so

$$\mathrm{E}(X) = 1.93$$

The second method is a lot more tedious. Firstly, from 3 shots, Will can either score 0, 1, 2 or 3 goals. So to find the expected value, we can draw up a discrete probability distribution, and find the corresponding probabilities for each number of goals using markov chains. To make it a bit clearer we will represent the event that he does not score as event F. We will also make x the number of goals that he scores.

There is only one way of scoring zero goals, and that is $\Pr(FFF)$, which we can find from

$$\begin{aligned}
\Pr(FFF) &= \Pr(F|S) \times \Pr(F|F) \times \Pr(F|F) \\
&= 0.30 \times 0.55 \times 0.55 \\
&= 0.09075 \\
\therefore \Pr(X = 0) &= 0.09075
\end{aligned}$$

There are three ways of scoring 1 goal, that is $\Pr(FFS)$, $\Pr(FSF)$ and $\Pr(SFF)$.

$$\begin{aligned}
\Pr(FFS) &= \Pr(F|S) \times \Pr(F|F) \times \Pr(S|F) \\
&= 0.30 \times 0.55 \times 0.45 \\
&= 0.07425 \\
\Pr(FSF) &= \Pr(F|S) \times \Pr(S|F) \times \Pr(F|S) \\
&= 0.30 \times 0.45 \times 0.30 \\
&= 0.0405 \\
\Pr(SFF) &= \Pr(S|S) \times \Pr(F|S) \times \Pr(F|F) \\
&= 0.70 \times 0.30 \times 0.55 \\
&= 0.1155 \\
\therefore \Pr(X = 1) &= 0.07425 + 0.0405 + 0.1155 \\
&= 0.23025
\end{aligned}$$

There are three ways of scoring 2 goals, that is $\Pr(SSF)$, $\Pr(SFS)$ and $\Pr(FSS)$.

$$\begin{aligned}
\Pr(SSF) &= \Pr(S|S) \times \Pr(S|S) \times \Pr(F|S) \\
&= 0.70 \times 0.70 \times 0.30 \\
&= 0.1470 \\
\Pr(SFS) &= \Pr(S|S) \times \Pr(F|S) \times \Pr(S|F) \\
&= 0.70 \times 0.30 \times 0.45 \\
&= 0.0945 \\
\Pr(FSS) &= \Pr(F|S) \times \Pr(S|F) \times \Pr(S|S) \\
&= 0.30 \times 0.45 \times 0.70 \\
&= 0.0945 \\
\therefore \Pr(X = 2) &= 0.1470 + 0.0945 + 0.0945 \\
&= 0.3360
\end{aligned}$$

There is only one way of scoring all 3 goals, that is $\Pr(SSS)$.

$$\begin{aligned}
\Pr(SSS) &= \Pr(S|S) \times \Pr(S|S) \times \Pr(S|S) \\
&= 0.70 \times 0.70 \times 0.70 \\
&= 0.3430 \\
\therefore \Pr(X=3) &= 0.3430
\end{aligned}$$

So we have the discrete distribution below

x	0	1	2	3
$\Pr(X=x)$	0.09075	0.23025	0.3360	0.3430

Just to make sure we haven't made a mistake somewhere, we check that the probabilities in the distribution add to 1.

$$0.09075 + 0.23025 + 0.3360 + 0.3430 = 1$$

Now to find the expected number of goals scored, we add the product of the x values and their associated probabilities.

$$\begin{aligned}
\mathrm{E}(X) &= (0 \times 0.09075) + (1 \times 0.23025) + (2 \times 0.3360) + (3 \times 0.3430) \\
&= 1.93
\end{aligned}$$

So the expected number of goals scored will be 1.93 goals

Part e.

Firstly we can either have Michael blocking a shot, or letting a shot through, so that is we can either have a success or a failure. We also know that the probability of blocking a shot is the same on all shots, that is the outcome of one shot doesn't affect the outcome of another. So from these two points we can tell that we need to use a binomial distribution. So lets let Y be the event that Michael blocks a goal. The probability of blocking a goal is 0.4 and the number of trials is 5, so we have

$$Y \sim \mathrm{Bi}(5, 0.4)$$

So now we need to find *"the probability that Michael blocks* 2 *out of* 5 *shots, given that he blocks at least* 1 *shot."* Since we have one event "given" that another has already occurred, we are going to have to use conditional probability. That is

$$\begin{aligned}
\Pr(Y=2|Y \geq 1) &= \frac{\Pr(Y=2 \cap Y \geq 1)}{\Pr(Y \geq 1)} \\
&= \frac{\Pr(Y=2)}{\Pr(Y \geq 1)}
\end{aligned}$$

Now to find these individual probabilities we need to first use the Binomial Probability Distribution function, that is "[Menu] [5] [5] [D]". For $\Pr(Y=2)$, our number of trials is 5, our probability of success is 0.4 and our "X-value" (which is our Y-value) will be 2. For $Pr(Y \geq 1)$, we will need to use the Binomial Cumulative Distribution function - [Menu] [5] [5] [E]. Our lower bound will be 1 and our upper bound will be the total number of trials, which is 5. That gives

$$\begin{aligned}
\Pr(Y=2|Y \geq 1) &= \frac{0.34560}{0.92224} \\
&= 0.3747
\end{aligned}$$

Part f.

We have the probability of *"at least* 2 *shots to be greater than* 0.7", that is we have

$$\Pr(X \geq 2) \quad > \quad 0.7$$

Now we are looking for the *"total number of shots that he should attempt to block"*. So since we don't have an upper bound for the number of shots, we need to look at the situation in reverse, that is we need to look at

$$\begin{aligned} 1 - \Pr(X < 2) &> 0.7 \\ \Pr(X < 2) &< 0.3 \end{aligned}$$

The sign flips because we are dividing by a negative number. Now we can work with what we have, that is we break up the $X < 2$ into the relevant situations.

$$\Pr(X = 0) + \Pr(X = 1) \quad < \quad 0.3$$

Now our binomial theorem for $Y \sim \mathrm{Bi}(n, p)$ is

$$\Pr(X = x) \quad = \quad \frac{n!}{x!\,(n-x)!}(p)^x (1-p)^{n-x}$$

So we have

$$\frac{n!}{0!\,(n-0)!}(0.4)^0 (0.6)^{n-0} + \frac{n!}{1!\,(n-1)!}(0.4)^1 (0.6)^{n-1} < 0.3$$

Now if we look at the first term

$$\begin{aligned} \frac{n!}{0!\,(n-0)!} &= \frac{n!}{n!} \\ &= 1 \end{aligned}$$

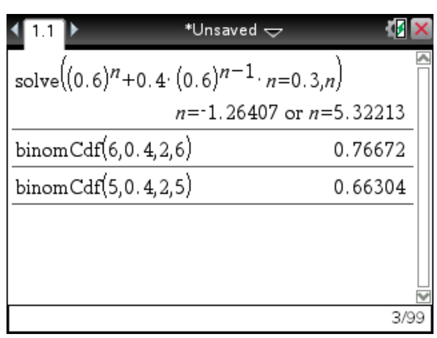
Looking at the first term of the second use of the binomial theorem

$$\begin{aligned} \frac{n!}{1!\,(n-1)!} &= \frac{n \times (n-1) \times (n-2) \times (n-3) \times \ldots.}{(n-1) \times (n-2) \times (n-3) \times \ldots.} \\ &= n \end{aligned}$$

So we get to

$$0.6^n + 0.4 \times 0.6^{n-1} n \quad < \quad 0.3$$

Which we can solve on the calculator using the solve function, "[Menu] [3] [1]", but we need to replace the "$<$" sign with an "$=$" sign otherwise we won't get an actual value. This gives $n = -1.26407$ and $n = 5.32213$. Since we can't have a negative value of n we reject the first solution but we also have to have a whole number value of n. So we need to round the second solution since it asks for the *"minimum total number of shots"*?. If we round down then we don't get a probability that is greater than 0.7, but we do it we round up. This can be checked using "[Menu] [5] [5] [E]".



At least 6 shots are required so that the probability of blocking at least 2 shots is grater than 0.7.

PROBABILITY

TECH-ACTIVE TEST 2

DETAILED SOLUTIONS

SECTION 1 - Multiple Choice Questions

Question 1 (C)

Calculating $\mathrm{E}\left(X^2\right)$ is similar to calculating $\mathrm{E}(X)$ except that we square the x values when we calculate it.

$$\begin{aligned}\mathrm{E}\left(X^2\right) &= (-3)^2(0.2)+(-2)^2(0.1)+(-1)^2(0.2)+\left(0^2\times 0.2\right)+\left(1^2\times 0.05\right)+\left(2^2\times 0.05\right)+\left(3^2\times 0.2\right)\\ &= 4.45\end{aligned}$$

Now we also have that

$$\begin{aligned}\operatorname{Var}(2X+3) &= 2^2\operatorname{Var}(X)\\ &= 4\operatorname{Var}(X)\end{aligned}$$

So we need to find the variance of the distribution, which is given by $\operatorname{Var}(X)=\mathrm{E}\left(X^2\right)-[\mathrm{E}(X)]^2$. We already have $\mathrm{E}\left(X^2\right)$, so we just need to find the expected value of X, $\mathrm{E}(X)$. So to find the expected value, we sum the product of x and the associated probabilities.

$$\begin{aligned}\mathrm{E}(X) &= (-3\times 0.2)+(-2\times 0.1)+(-1\times 0.2)+(0\times 0.2)+(1\times 0.05)+(2\times 0.05)+(3\times 0.2)\\ &= -0.25\end{aligned}$$

So we can now calculate $\operatorname{Var}(2X+3)$

$$\begin{aligned}\operatorname{Var}(2X+3) &= 4\left(4.45-(-0.25)^2\right)\\ &= 17.55\end{aligned}$$

So the correct answer is option **C**.

Question 2 (A)

The variance of a continuous probability distribution is given by

$$\begin{aligned}\operatorname{Var}(X) &= \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx\\ &= \mathrm{E}\left(X^2\right)-\mu^2\\ &= \int_{-\infty}^{\infty}\left(x^2 f(x)\right)dx-\left(\int_{-\infty}^{\infty}(xf(x))\,dx\right)^2\end{aligned}$$

Working on $\mathrm{E}\left(X^2\right)$ first

$$\begin{aligned}\mathrm{E}\left(X^2\right) &= \int_{-\infty}^{\infty}\left(x^2 f(x)\right)dx\\ &= \int_0^6\left(x^2\times\frac{1}{72}\left(x^2-6x+18\right)\right)dx\\ &= 12.60\end{aligned}$$

Then working on μ

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} (xf(x))\,dx \\ &= \int_0^6 \left(x \times \frac{1}{72}\left(x^2-6x+18\right)\right)dx \\ &= 3.00\end{aligned}$$

So $\text{Var}(X)$ is

$$\begin{aligned}\text{Var}(X) &= 12.60 - 3.00^2 \\ &= 3.60\end{aligned}$$

So option **A** is the correct answer.

Question 3 (A)

The first piece of information to note is that Sam can either *"win or lose"* the game, that is we have either a success or a failure, and that one game doesn't affect the other since the probability is the same regardless of each game. This leads us to applying a binomial distribution.

Lets denote the event that Sam wins a game to be event X, then the probability p, of success is 0.3 and the number of trials is n. $X \sim \text{Bi}(n, 0.3)$.

Now what we're actually looking for is *"the minimum number of games required"* given that $\Pr(X \geq 2) > 0.90$

If we tried to solve this on the calculator using the binomial function it wouldn't work, so what we need to do is expand out for each value of X, but since at the moment that is unbounded, we need to look at the problem in reverse. That is

$$\begin{aligned}\Pr(X<2) &< 0.10 \\ \Pr(\text{X}=0)+\Pr(X=1) &< 0.10 \\ \frac{n!}{0!\,(n-0)!}(0.3)^0(0.7)^n + \frac{n!}{1!\,(n-1)!}(0.3)^1(0.7)^{n-1} &< 0.10\end{aligned}$$

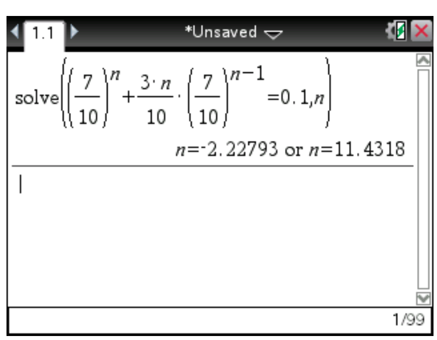
Now if we look at the term $\frac{n!}{0!\,(n-0)!}$, we can see that we have $\frac{n!}{n!}=1$, leaving $0!=1$. So $\frac{n!}{0!\,(n-0)!}=1$. So the first three terms become 0.7^n. If we look at the $\frac{n!}{1!\,(n-1)!}$ term, we see that we have

$$\frac{n\times(n-1)\times(n-2)\times(n-3)\times\ldots}{(n-1)\times(n-2)\times(n-3)\times\ldots} = n$$

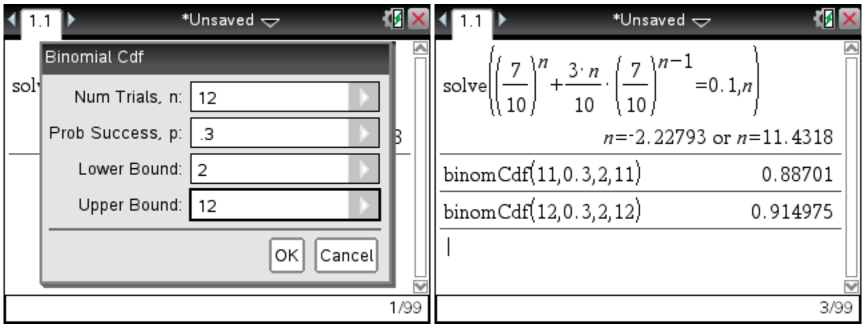
So the second set of terms become $0.3n \times 0.7^{n-1}$. So we now have

$$0.7^n + 0.3n\times 0.7^{n-1} < 0.1$$

Now we solve this on the calculator using the solve function, "[Menu] [3] [1]", but we need to replace the "$<$" with an "$=$" sign otherwise we won't get a value for n.



Now the restrictions on n is since it is the number of trials, it has to be greater than zero and has to be a whole number. So we can reject the first solution and need to round the second solution. Now we need to round this solution to a whole number, but if we round down to $n = 11$, then the $\Pr(X \geq 2) < 0.90$. So if we round up to $n = 12$ our conditions will be satisfied. We can check this using the binomial cumulative function, "[Menu] [5] [5] [E]".



That makes our answer would be option **A**.

Question 4 (E)

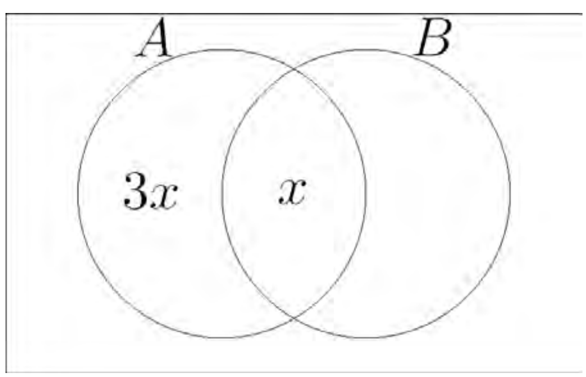
We note that events A and events B are independent. For two events to be independent of one another, they must not have an effect on each other, that is

$$\Pr(A|B) = \Pr(A)$$

Now we also know our rule for conditional probability

$$\begin{aligned}
\Pr(A|B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\
\Pr(A) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\
\Pr(A \cap B) &= \Pr(A)\Pr(B)
\end{aligned}$$

The last line being the formula we need to manipulate when it comes to independent probability. Now we also know that $\Pr(A \cap B') = 3\Pr(A \cap B)$, so to make it easier let $\Pr(A \cap B) = x$. Then we have the following diagram.



$$\begin{aligned}
\Pr(A \cap B) &= \Pr(A)\Pr(B) \\
x &= (x + 3x)\Pr(B) \\
\Pr(B) &= \frac{x}{4x} \\
\therefore \Pr(B) &= \frac{1}{4}
\end{aligned}$$

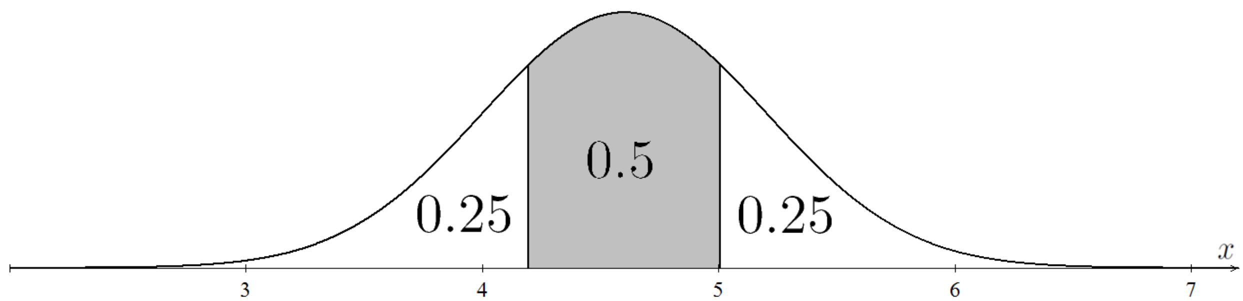
That makes **E** the correct answer.

Question 5 (D)

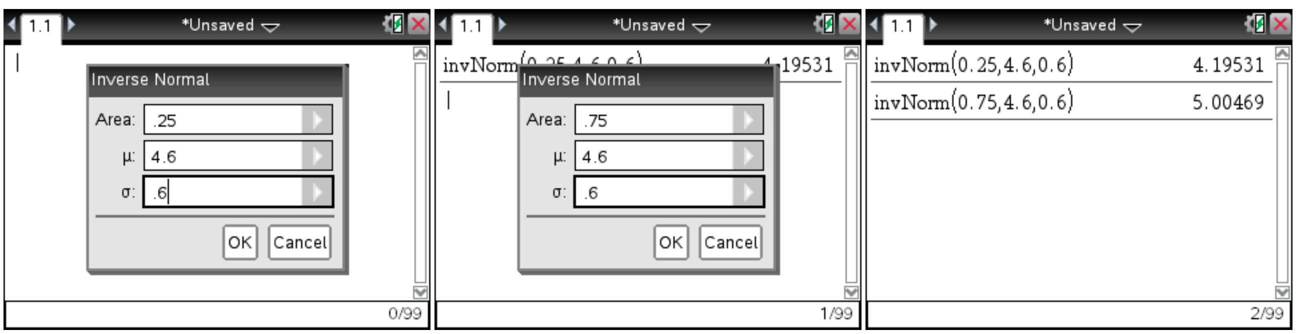
We first note that we are dealing with a normal distribution of $\mu = 4.6$ and $\sigma = 0.6$. So if X is the time taken to complete a pit stop then

$$X \sim \mathrm{N}(4.6, 0.36)$$

Now we want the "*middle* 50% *of pitstops*". So that means the probability that the time is less than the lowest time of this 50% will be $\left(\frac{100-50}{2}\right) = 0.25$. This is represented in the diagram below.



So to find the lower value, we use the inverse normal function on the calculator, "[Menu] [5] [5] [3]", with the area being 0.25. For the upper value, the area will be 0.75.



This means that time taken for the middle 50% of pit stops will be between 4.20 seconds and 5.00 seconds. So option **D** is the correct answer.

Question 6 (B)

The two key features that a continuous probability density functions has to have is

$$f(x) \geq 0 \text{ for all } x$$
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

So the area under the function has to equal 1, so we can integrate the function from the lowest value to the highest value for which it is not 0, and equate this to 1, solving for k.

$$\begin{aligned}
\int_0^{\pi} k^2 (\cos(x) + 1)\,dx &= 1 \\
k^2 [\sin(x) + x]_0^{\pi} &= 1 \\
k^2 (\sin(\pi) + \pi - \sin(0) - 0) &= 1 \\
\pi k^2 &= 1 \\
k &= \pm \frac{1}{\sqrt{\pi}}
\end{aligned}$$

Because we have k^2 and not k in the probability function, we don't need to care about negative answers as they will always be positive once squared. Thus, $k = -\frac{1}{\sqrt{\pi}}, \frac{1}{\sqrt{\pi}}$ are both valid solutions. Hence, option **D** is correct.

Possibly the quickest way to achieve this on the calculator is to use the solve function, "[Menu] [3] [1]" and the integral function, "[⇑shift] + [+]".

Question 7 (C)

Before we start, we note that since the probabilities in the distribution have to add to 1, the value of $a+b$ must be 0.50, so that means we can cross off options D and E.

We have 2 unknowns so we need 2 equations to be able to find them. The first comes from the expected value of the distribution, which is the sum of the product of the x values and their probabilities.

$$\begin{aligned}(-4 \times a) + (-2 \times 0.3) + (0 \times b) + (2 \times 0.1) + (4 \times 0.1) &= -1.0 \\ -4a &= -1.0 \\ \therefore a &= 0.25\end{aligned}$$

The second comes from that fact that the sum of the probabilities in the distribution must equal 1.

$$\begin{aligned}a + 0.3 + b + 0.1 + 0.1 &= 1 \\ b &= 0.5 - a \\ b &= 0.5 - 0.25 \\ \therefore b &= 0.25\end{aligned}$$

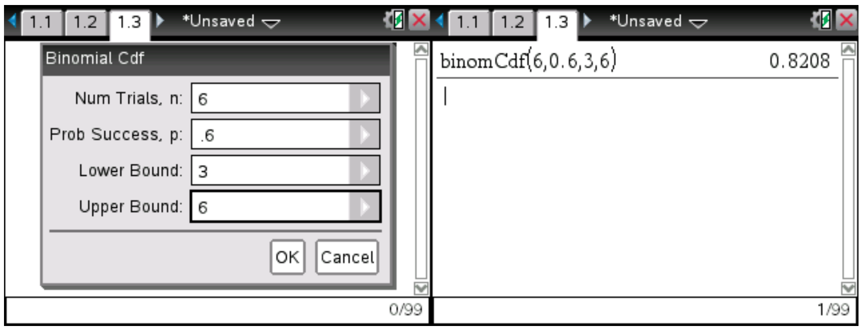
Thus option **C** is the correct answer.

Question 8 (D)

Firstly since Matt can *"either catch or drop the ball"*, we have either a success or failure, and since each ball is independent of the other, we will have to use a binomial distribution. We also have $n = 0.6$ and $p = 6$. If we denote X to be the event that Matt catches a ball then

$$X \sim \text{Bi}(6, 0.6)$$

Now we want to find *"Then the probability that out of* 6 *balls, he catches at least* 3 *of them"*, that is we want $\Pr(X \geq 3)$. Since we want more than just once value of X, we will need to use the cumulative binomial distribution function on the calculator, that is "[Menu] [5] [5] [E]".



So $\Pr(X \geq 3) = 0.8208$. So the required answer is **D**.

Question 9 (B)

Since we have an event that is conditional on the event before, and we are going to run it over a couple of days, we can use markov chains. If we let event A be the event that Rahad attends lectures on the previous day and event B be the event that he attends lectures on the current day then our transition matrix will be given by

$$T = \begin{bmatrix} \Pr(B|A) & \Pr(B|A') \\ \\ \Pr(B'|A) & \Pr(B'|A') \end{bmatrix}$$

Now since *"he probability that Rahad attends lectures given that he attended the day before is* 0.75*"*, we have $\Pr(B|A) = 0.75$ and since *"the probability that he attends lectures given that he didn't the previous day is* 0.5", we have $\Pr(B|A') = 0.5$. We also know that the columns of the transition matrix must add to one, which gives us the other two probabilities.

$$T = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}$$

Now since *"Rahad attends lectures on Monday"*,

$$S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Now we count Monday as the initial condition, so Tuesday will be S_1, Wednesday will be S_2 and so on making Friday S_4. To evaluate this on the calculator we need to open the matrix window, this can be down by clicking the "$\begin{bmatrix} |\square| & \begin{cases} \square \\ \square \end{cases} \end{bmatrix}$" button and selecting the appropriate matrix in the second row. If we need bigger than a 2×2 then select the 3×3 matrix and enter in the dimensions.

$$\begin{aligned} S_4 &= T^4 \times S_0 \\ &= \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}^4 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.6680 \\ 0.3320 \end{bmatrix} \end{aligned}$$

Now the top row represents the probability that he does do homework on Friday, while the bottom represents the probability that he doesn't do homework on Friday. So The probability that he does do work on Friday is 0.6680. So option **B** is the required answer.

Question 10 (E)

The probability that X is greater than 1 will be given by the area under $f(x)$ from $x = 1$ to $x = 2$. We cannot evaluate this integral to 4 decinmal places by hand in Methods, so we turn to the calculator, using the integral function, "[⇑ shift] [+]". Not that we must use euler's "e" and not the letter e. Euler's "e" can be obtained from the "e^x" button on the left hand side of the calculator.

$$\int_1^2 \left(\frac{1}{e^2-3}\left(e^{2-x}-1\right)\right)dx = 0.1637$$

This makes **E** the correct answer.

Question 11 (D)

Firstly we are given a normal distribution for X, which we know to have mean $\mu = 5$ and variance $\sigma^2 = 4$. Since the variance of a distribution is the square of the standard deviation, the standard deviation of X is

$$\begin{aligned} \sigma^2 &= 4 \\ \sigma &= \pm\sqrt{4} \text{ but } \sigma \geq 0 \\ \therefore \sigma &= 2 \end{aligned}$$

So

$$X \sim N(5,4)$$

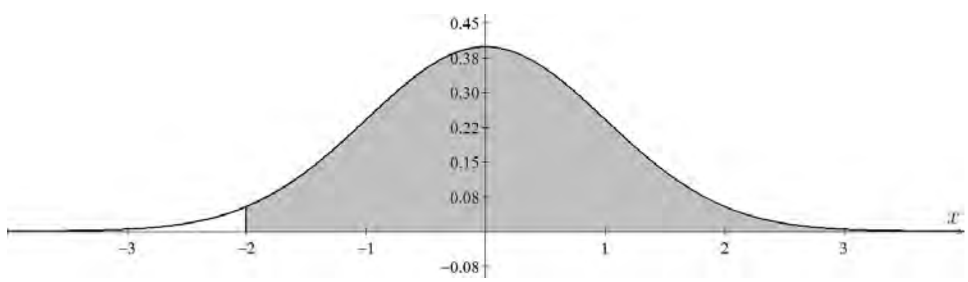
We are looking for $\Pr(X > 1)$ and note that all of the options we have to choose from involve the standard normal distribution, Z. So we need to convert our value of X into a Z-score. The Z-score is given by

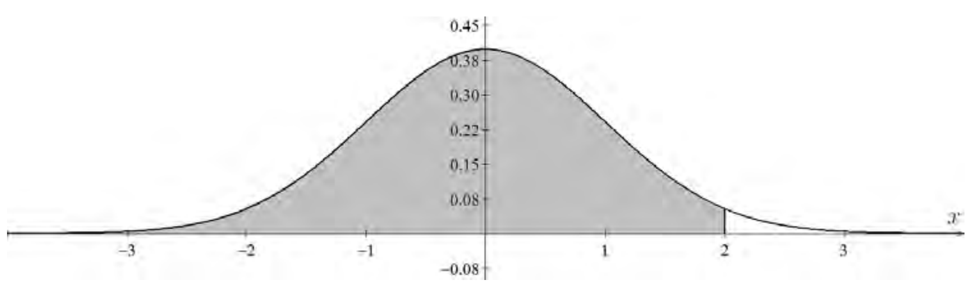
$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ &= \frac{1-5}{2} \\ &= -2 \end{aligned}$$

That means that

$$\Pr(X > 1) = \Pr(Z > -2)$$

But that isn't any of the options, so we need to look for an equivalent expression. As the distribution is symmetric, the area greater than -2 will be the same as the area less than 2. We can see this if we do a quick sketch of the standard normal distribution.





$$\therefore \Pr(Z > -2) = \Pr(Z < 2)$$

Hence option **D** is correct.

SECTION 2 - Extended Response Questions

Question 1

Part a.

As stated previously, one of the more common problems students encounter when facing probability questions is determining what type of probability they need to apply to the situation. He we have a scenario where there is either a success or a failure, i.e. *"he makes it onto the train'* or *'he misses the train completely"*. This "success" or "failure" idea is parlty a giveaway towards the use of a binomial distribution, as long as one event doesn't have an effect on the other, that is each trial is independent of the other trials.

Now we don't know the actual value of p, as that is what we are asked to find, but we do know the value of n, as Dennis goes to university *"5 days a week, for a five week period"*. So we have $n = 5 \times 5 = 25$ days. That is

$$X \sim \mathrm{Bi}(25, p)$$

We know that

$$\begin{aligned}
\Pr(X > 22) &= 4\Pr(X = 25) \\
\Pr(X = 23) + \Pr(X = 24) + \Pr(X = 25) &= 4\Pr(X = 25) \\
\Pr(X = 23) + \Pr(X = 24) &= 3\Pr(X = 25)
\end{aligned}$$

We know that for any value of x that the probability of that event will be given by

$$\Pr(X = x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

So that brings us to

$$\begin{aligned}
\frac{25!}{23!2!} p^{23} (1-p)^2 + \frac{25!}{24!1!} p^{24} (1-p)^1 &= 3 \times \frac{25!}{25!0!} p^{25} (1-p)^0 \\
300p^{23} (1-p)^2 + 25p^{24} (1-p) &= 3p^{25} \\
p^{23} \left(300\left(1 - 2p + p^2\right) + 25p - 25p^2 - 3p^2\right) &= 0 \\
p^{23} \left(272p^2 - 575p + 300\right) &= 0 \\
p^{23} (16p - 15)(17p - 20) &= 0 \\
p = 0 \quad \text{or} \quad 16p - 15 = 0 &\quad \text{or} \quad 17p - 20 = 0 \\
p = 0 \quad \text{or} \quad p = \frac{15}{16} &\quad \text{or} \quad p = \frac{20}{17}
\end{aligned}$$

But $0 < p \leq 1$ as $p > 0$ is given in the question, so we reject the $p = 0$ and $p = \frac{20}{17}$ solutions.

$$\therefore p = \frac{15}{16}$$

Part bi.

The expected value and variance of a binomial distribution is given by

$$\begin{aligned}
\mathrm{E}(X) &= np \\
\mathrm{Var}(X) &= np(1-p)
\end{aligned}$$

Now we know the number of trials to be 25, and $\mathrm{E}(X) = 23$. Solving for p

$$\begin{aligned}
23 &= 25p \\
\therefore p &= \frac{23}{25}
\end{aligned}$$

Then we can substitute this in to find the variance of the distribution

$$\begin{aligned}\mathrm{Var}(X) &= 25 \times \frac{23}{25}\left(1 - \frac{23}{25}\right) \\ &= \frac{46}{25} \\ &= 1.84\end{aligned}$$

Part bii.

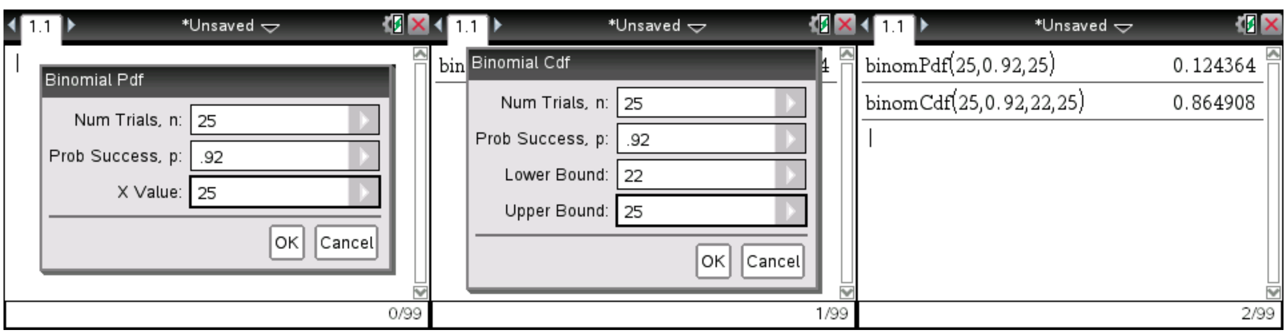
We are asked to find μ and σ of the distribution for Y. We don't know anything else about Y except for its relationships to the distribution for X. That is

$$\begin{aligned}\Pr(X = 25) &= \Pr(Y \le 34.2) \\ \Pr(X \ge 22) &= \Pr(Y \ge 34.5)\end{aligned}$$

So that means that we need to find the two probabilities for $\Pr(X = 25)$ and $\Pr(X \ge 22)$. Now we can represent our two distributions by

$$\begin{aligned}X &\sim \mathrm{Bi}(25, 0.92) \\ Y &\sim \mathrm{N}\left(\mu, \sigma^2\right)\end{aligned}$$

Using the binomial probability distribution function, "[Menu] [5] [5] [D]" will give us our value at one particular value of X. But for $\Pr(X \ge 22)$ we will use the binomial cumulative distribution function, "[Menu] [5] [5] [E]", as we want the probability for more than one value of X.



$$\begin{aligned}\Pr(X = 25) &= 0.124364 \\ \Pr(X \ge 22) &= 0.864908\end{aligned}$$

So that means we now have some information about the Y distribution. That is

$$\begin{aligned}\Pr(Y \le 34.2) &= 0.124364 \\ \Pr(Y \ge 34.5) &= 0.864908\end{aligned}$$

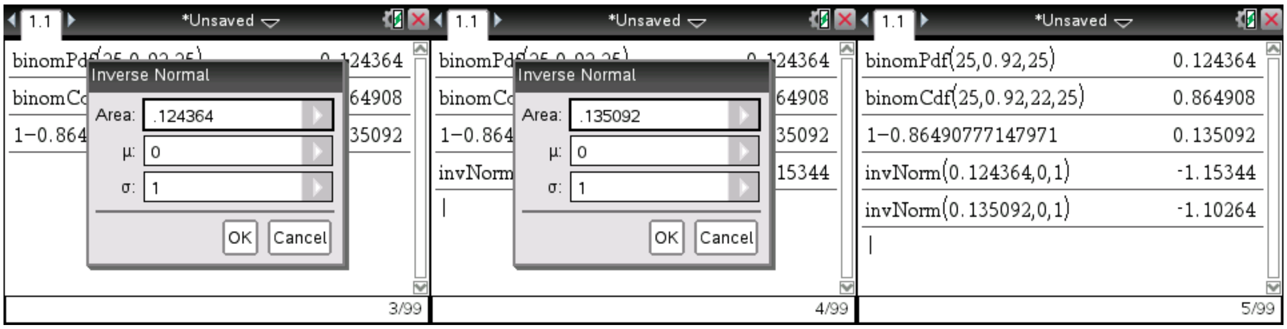
Now since we have the probability, that is the area under the normal distribution curve, we need to transform the distribution into a standard normal distribution, that is with $\mu_n = 0$ and $\sigma_n = 1$. To do this we convert out Y value to a Z score, which effectively tells us how many standard deviations that value is from the mean. That is

$$\begin{aligned}Z &= \frac{x - \mu}{\sigma} \\ \Pr\left(Z \le \frac{34.2 - \mu}{\sigma}\right) &= 0.124364....[1] \\ \Pr\left(Z \ge \frac{34.5 - \mu}{\sigma}\right) &= 0.864908....[2]\end{aligned}$$

Now we can use the inverse normal function, "[Menu] [5] [5] [3]" to find the values of Z. But since this inverse normal function measures the lower end from$-\infty$, we need to have the area that is less than or equal to Z, that is the second relationship we have becomes

$$\begin{aligned} 1-\Pr\left(Z \leq \frac{34.5-\mu}{\sigma}\right) &= 0.864908 \\ \Pr\left(Z \leq \frac{34.5-\mu}{\sigma}\right) &= 1-0.864908 \\ \Pr\left(Z \leq \frac{34.5-\mu}{\sigma}\right) &= 0.135092....[3] \end{aligned}$$

So from [1] and [3] we have



That is we now know that

$$\begin{aligned} \frac{34.2-\mu}{\sigma} &= -1.15344....[4] \\ \frac{34.5-\mu}{\sigma} &= -1.10264....[5] \end{aligned}$$

Solving them simultaneously or using the solve function, "[Menu] [3] [1]" on the calculator gives

$$\begin{aligned} \mu &= 34.2+1.15344\sigma \quad \text{(from [4])} \\ \text{Substitute into [5]} & \\ 34.5-34.2-1.15344\sigma &= -1.10264\sigma \\ \sigma &= 5.905512 \\ \mu &= 34.2+(1.15344\times 5.905512) \\ &= 41.011654 \end{aligned}$$

Rounding to 4 decimal places we have

$$\begin{aligned} \therefore \mu &= 41.0117 \\ \therefore \sigma &= 5.9055 \end{aligned}$$

Part biii.

Since we have the event that *"the train journey takes less than 30 minutes given that the train runs on time*", we are going to be dealing with conditional probability, that is if we have event A given that event B has already occurred, then

$$\Pr(A|B) = \frac{\Pr(A\cap B)}{\Pr(B)}$$

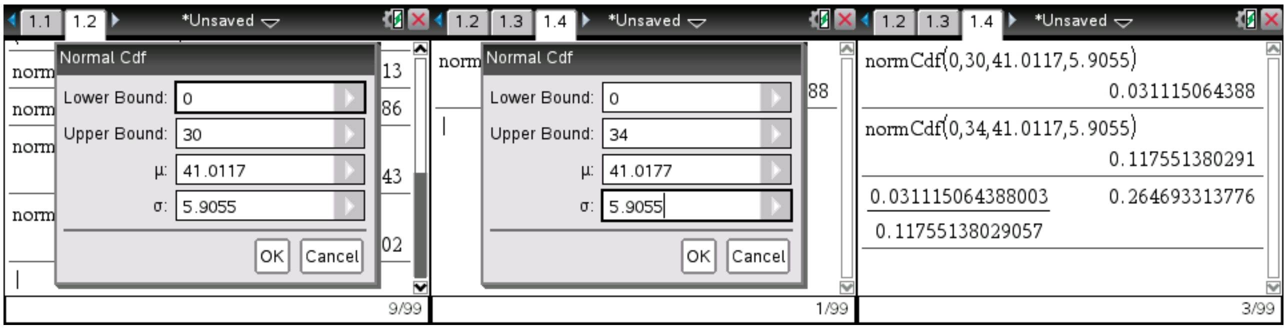
So here event B will be the event that the train runs on time, that is the journey takes at most 34 minutes, and we want the probability that the journey takes less than 30 minutes, given that the train is on time. The time Y in minutes still follows the normal distribution

$$Y \sim \mathrm{N}(41.0117, 34.8751)$$

Note that the second term is variance, not standard deviation. So we are looking for

$$\begin{aligned}\Pr(Y<30|Y<34) &= \frac{\Pr(Y<30\cap Y<34)}{\Pr(Y<34)}\\ &= \frac{\Pr(Y<30)}{\Pr(Y<34)}\end{aligned}$$

Now we can get the those two values by using the cumulative normal distribution function, "[Menu] [5] [5] [2]".



$$\begin{aligned}\Pr(Y<30) &= 0.03112\\ \Pr(Y<34) &= 0.11755\\ \Pr(Y<30|Y<34) &= \frac{0.03112}{0.11755}\\ &= 0.2647\end{aligned}$$

So the probability that the journey takes less than 30 minutes, given that the train runs on time is 0.2647.

Question 2

Part a.

Firstly we need to determine what type of probability to apply to the situation at hand. We know that the probability of the event occurring is dependent on the outcome of the event previous to it, and that we are also given the *"long run probability"*, that is the steady state probability. So we can apply markov chains to the situation. Whenever we have a question that involves markov chains, there will most likely be a mark for the transition matrix, so that is what we always try to start off with. So if we make event B be the event that Jake goes to an event currently, and event A be the event that Jake goes to the previous event then our transition matrix is

$$T = \begin{bmatrix}\Pr(B|A) & \Pr(B|A')\\ \Pr(B'|A) & \Pr(B'|A')\end{bmatrix}$$

Now since we know that *"The probability that Jake turns up to an event given that he turned up to the last one is p"*, we have $\Pr(B|A)=p$. Also as *"the probability that he turns up to an event given that he didn't turn up to the previous event is $p+\dfrac{3}{20}$"*, we have $\Pr(B|A')=p+\dfrac{3}{20}$. That is

$$T = \begin{bmatrix}p & p+\frac{3}{20}\\ & \end{bmatrix}$$

We also know that the columns of a transition matrix must add to 1, that is we have

$$T = \begin{bmatrix}p & p+\frac{3}{20}\\ 1-p & \frac{17}{20}-p\end{bmatrix}$$

Now if we have the transition matrix $\begin{bmatrix}1-a & b\\ a & 1-b\end{bmatrix}$, the steady state solution will be given by

$$\begin{bmatrix}\Pr(\text{"Success"})\\ \Pr(\text{"Failure"})\end{bmatrix} = \begin{bmatrix}\frac{b}{a+b}\\ \frac{a}{a+b}\end{bmatrix}$$

Now we know that *"the long run probability of Jake showing up to an event is* $\frac{7}{23}$", and we also know that the probability of a success plus the probability of a failure in the long term must add to 1. So we have

$$\begin{aligned}
\begin{bmatrix} \frac{b}{a+b} \\ \frac{a}{a+b} \end{bmatrix} &= \begin{bmatrix} \frac{7}{23} \\ \frac{16}{23} \end{bmatrix} \\
\begin{bmatrix} \frac{p+\frac{3}{20}}{1-p+p+\frac{3}{20}} \\ \frac{1-p}{1-p+p+\frac{3}{20}} \end{bmatrix} &= \begin{bmatrix} \frac{7}{23} \\ \frac{16}{23} \end{bmatrix}
\end{aligned}$$

Since we only have to find one unknown, we only need one equation, so lets use the top equation, the one for the probability of a success in the long run.

$$\begin{aligned}
\frac{p+\frac{3}{20}}{1-p+p+\frac{3}{20}} &= \frac{7}{23} \\
p+\frac{3}{20} &= \frac{7}{23}\times\frac{23}{20} \\
p &= \frac{7-3}{20} \\
\therefore p &= \frac{1}{5} \text{ as required}
\end{aligned}$$

Which is what we are required to show. You will get a mark for showing the last line if, and only if you have legitimate steps leading to that result.

We could have done the same method with the second row and the long run probability for a failure. Either working is fine.

$$\begin{aligned}
\frac{1-p}{1-p+p+\frac{3}{20}} &= \frac{16}{23} \\
1-p &= \frac{16}{23}\times\frac{23}{20} \\
p &= \frac{20-16}{20} \\
\therefore p &= \frac{1}{5} \text{ as required}
\end{aligned}$$

Part b.

The 15^{th} event that Jake goes to will be the 14^{th} event, after the first. That is if the first event is S_0, then as he doesn't go to this event, we have a failure, and no success, and as the top row is our success and the bottom row is our failure we have

$$S_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now with the value of p from the previous question, we have

$$\begin{aligned}
T &= \begin{bmatrix} \frac{1}{5} & \frac{1}{5}+\frac{3}{20} \\ 1-\frac{1}{5} & \frac{17}{20}-\frac{1}{5} \end{bmatrix} \\
\therefore T &= \begin{bmatrix} \frac{1}{5} & \frac{7}{20} \\ \frac{4}{5} & \frac{13}{20} \end{bmatrix}
\end{aligned}$$

Now applying the markov chain we obtain

$$\begin{aligned}
S_{14} &= T^{14}\times S_0 \\
&= \begin{bmatrix} \frac{1}{5} & \frac{7}{20} \\ \frac{4}{5} & \frac{13}{20} \end{bmatrix}^{14} \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0.304348 \\ 0.695652 \end{bmatrix}
\end{aligned}$$

Now as the top row is associated with the success to our event we have the top row represents the probability of Jake attending the 15^{th} event given that he did not attend the first event, while the bottom row represents the probability of Jake not attending the 15^{th} event given that he did not attend the first event.

So the probability that Jake attends the 15^{th} event given that he didn't attend the first is 0.3043, correct to 4 decimal places.

Part c.

The 4^{th} event will be the third event after the first, that is after the initial state. Since we are looking for whether he attends the first event we are looking for the initial state, S_0. So we need our markov chain with the same transition matrix as before, except this time our value of n is 3.

$$S_3 \quad = \quad T^3 \times S_0$$

There are three ways to approach this problem from here, the first by using matrix inverses, the second letting the caclulator do all the work, and the thrid is to expand out both sides to arrive at two simultaneous equations, which then can be solved.

Using the matrix inverse methods, since these are matrices, to get S_0 on it's own we need to multiple both sides by the inverse of T^3. That is

$$\begin{aligned}
\left(T^3\right)^{-1} \times S_3 &= \left(T^3\right)^{-1} \times T^3 \times S_0 \\
\left(T^3\right)^{-1} S_3 &= I \times S_0 \\
S_0 &= \left(T^3\right)^{-1} S_3
\end{aligned}$$

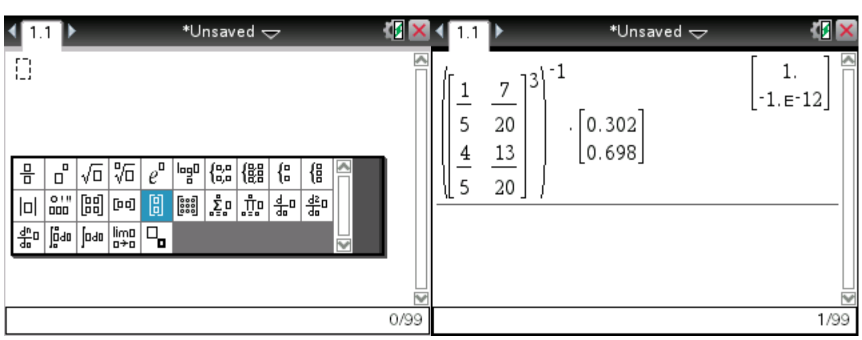
So we need the matrix S_3, we already know that the bottom entry corresponds to Jake not attending the 4^{th} event, which is 0.6980, and the sum of the probabilities in this column matrix has to sum to 1. So that makes the top entry 0.3020 and our matrix becomes

$$S_3 \quad = \quad \begin{bmatrix} 0.3020 \\ 0.6980 \end{bmatrix}$$

Solving through gives

$$\begin{aligned}
S_0 &= \left(\begin{bmatrix} \frac{1}{5} & \frac{7}{20} \\ \frac{4}{5} & \frac{13}{20} \end{bmatrix}^3\right)^{-1} \begin{bmatrix} 0.3020 \\ 0.6980 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1.\text{E} - 12 \end{bmatrix}
\end{aligned}$$

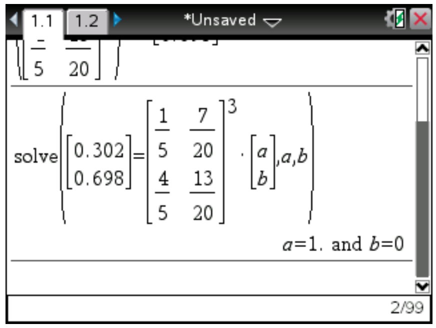
This can be done on the calculator using the matrix pop up window, which is on the right hand side of the keyboard, that is by clicking the $\begin{bmatrix} |\square| \begin{cases} \square \\ \square \end{cases} \end{bmatrix}$ button and selecting the appropriate matrix. The inverse of the matrix can be found by raising it to the power of -1.



The reason we get the small error is due to the values being given in the question being rounded to 4 decimal places, rounding to the nearest whole number, and remembering that the two probabilities have to add to 1 gives

$$S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

You could also use the solve function "[Menu] [3] [1]" to solve for S_3, as below.



The final method invovles expanding out both sides, and simultanouly solving the resulting equations.

$$\begin{aligned} \begin{bmatrix} 0.3020 \\ 0.6980 \end{bmatrix} &= \begin{bmatrix} \frac{1}{5} & \frac{7}{20} \\ \frac{4}{5} & \frac{13}{20} \end{bmatrix}^3 \times \begin{bmatrix} a \\ b \end{bmatrix} \\ \begin{bmatrix} 0.3020 \\ 0.6980 \end{bmatrix} &= \frac{1}{8000} \begin{bmatrix} 2416a + 2443b \\ 5584a + 5557b \end{bmatrix} \end{aligned}$$

Which gives the two equations

$$\begin{aligned} 2416 &= 2416a + 2443b....[1] \\ 5584 &= 5584a + 5557b....[2] \end{aligned}$$

Solving simultaneously using the calculator or by hand gives $a = 1$ and $b = 0$, thus

$$S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Now since the top entry is associated with our success, that is that he goes to the event, is 1, that is this event occurred. If the bottom entry was a 1 and the top entry was a 0 then he would not have attended the first event.

So Jake did attend the first event.

Question 3

Part a.

We are asked to find 2 unknowns, so we need 2 equations to do that. One will come from the fact that the probabilities in the distribution must add to 1 and the other from the standard deviation value. Since we have discrete values of x we are dealing with a discrete probability distribution. The sum of the probabilities must be 1. That is we have

$$\begin{aligned} 0.3 + a + 0.2 + 0.05 + b &= 1 \\ a + b &= 0.45....[1] \end{aligned}$$

The standard deviation, σ and variance, σ^2 of a discrete probability distribution is given by

$$\begin{aligned} \sigma(X) &= \sqrt{\text{Var}(X)} \\ \text{Var}(X) &= \text{E}\left(X^2\right) - [\text{E}(X)]^2 \end{aligned}$$

So we need to first find $\text{E}(X)$, that is our mean, which is the sum of the product of the x values and their associated probabilities. That is

$$\begin{aligned} \text{E}(X) &= (0 \times 0.3) + (1 \times a) + (2 \times 0.2) + (3 \times 0.05) + (4 \times b) \\ &= a + 4b + 0.55 \end{aligned}$$

But we need $\text{E}\left(X^2\right)$, which comes from the sum of the product of x^2 and the associated probabilities, that is

$$\begin{aligned} \text{E}\left(X^2\right) &= \left(0^2 \times 0.3\right) + \left(1^2 \times a\right) + \left(2^2 \times 0.2\right) + \left(3^2 \times 0.05\right) + \left(4^2 \times b\right) \\ &= a + 16b + 1.25 \end{aligned}$$

So putting that back into our formula for variance

$$\begin{aligned} (1.06184)^2 &= a + 16b + 1.25 - (a + 4b + 0.55)^2 \\ 0.1800 &= -a^2 - 16b^2 - a(8b + 0.1) + 11.6b....[2] \end{aligned}$$

Now we can solve equation [1] and [2] on the calculator using the solve function, "[Menu] [3] [1]" or we can solve them by hand, which is the longer option. This gives

$$a = -0.5 \text{ and } b = 0.95 \quad \text{or} \quad a = 0.4 \text{ and } b = 0.05$$

But since a and b are probabilities, they must be between or equal to 0 and 1. That is $0 \leq a \leq 1$ and $0 \leq b \leq 1$. So we reject the first solution.

$$\begin{aligned} \therefore a &= 0.4 \\ \therefore b &= 0.05 \end{aligned}$$

Part b.

Since we "*Assume the customer buys at least* 1 *item*", we will have a reduced sample space and will need to adjust the probabilities. Now since Y is the cost of items to the customer, we need to find out the values of Y that we can obtain. So if the customer buys 1 item, the value of Y will be \$2.00, if they buy 2 items, the value of Y will be \$4.00. So that is if

$$\begin{aligned} X = 1, &\quad Y = 2.00 \\ X = 2, &\quad Y = 4.00 \\ X = 3, &\quad Y = 4.50 \\ X = 4, &\quad Y = 6.00 \end{aligned}$$

Now since we know that at least on Item was bought, we need to adjust the probabilities

$$\Pr(Y = y | X \geq 1) = \frac{\Pr(Y = y \cap X \geq 1)}{\Pr(X \geq 1)}$$

So that will give us each of the probabilities for the corresponding value of Y.

$$\begin{aligned}
\Pr(Y=2.00|X\geq 1) &= \frac{0.4}{0.4+0.2+0.05+0.05} \\
&= \frac{4}{7} \\
\Pr(Y=4.00|X\geq 1) &= \frac{0.2}{0.4+0.2+0.05+0.05} \\
&= \frac{2}{7} \\
\Pr(Y=4.50|X\geq 1) &= \frac{0.05}{0.4+0.2+0.05+0.05} \\
&= \frac{1}{14} \\
\Pr(Y=6.00|X\geq 1) &= \frac{0.05}{0.4+0.2+0.05+0.05} \\
&= \frac{1}{14}
\end{aligned}$$

So that gives us our discrete probability table

y	2	4	4.5	6
$\Pr(Y=y)$	$\frac{4}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	$\frac{1}{14}$

Part c.

Now since part c requires values that were found in part a and b, if you were unable to complete part a and/or b, then making up some values and using those would get you method marks only, but no answer marks. But if you really are desperate then those method marks are still valuable.

The expected price will be given by the sum of the product of y and the corresponding probabilities. That is

$$\begin{aligned}
\mathrm{E}(Y) &= \left(2\times\frac{4}{7}\right)+\left(4\times\frac{2}{7}\right)+\left(4.5\times\frac{1}{14}\right)+\left(6\times\frac{1}{14}\right) \\
&= \$3.04
\end{aligned}$$

To find the median, we sum the probabilities until we either hit, or exceed 0.5. If we hit 0.5 exactly, then we take the average of that value and the next value of y. If we exceed 0.5 then the median is the value of y that we were on when we exceeded 0.5.

$$\begin{aligned}
\frac{4}{7} &\approx 0.571429 \\
\frac{4}{7} &> 0.5 \\
\therefore m &= \$2.00
\end{aligned}$$

Now since our first value is already exceeds 0.5, our median is 2.00, correct to 2 decimal places.

SET 1 EXAM 1
DETAILED SOLUTIONS

Question 1
Part a.
This question is a fairly simple problem to start off with, it requires the use of the chain rule, that is if we have

$$\begin{aligned} h(x) &= f(g(x)) \\ \text{Then } h'(x) &= f'(g(x))\, g'(x) \end{aligned}$$

In the alternate notation that is

$$\begin{aligned} \text{If } y &= y(u) \\ \text{Then } \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \end{aligned}$$

So in our case we have

$$\begin{aligned} \frac{d}{dx}(2\cos(2x+3)) &= -2\sin(2x+3) \times 2 \\ &= -4\sin(2x+3) \end{aligned}$$

A way to remember this is that we always need to multiple by the "derivative of the inside". That is in our case the function that is "inside" $y = 2\cos(u)$ is $u = 2x+3$, so we need to multiply y' by the derivative of $2x+3$.

Part b.
This problem is a little ticker than the previous, as it requires an application of the product rule. If we have

$$\begin{aligned} f(x) &= g(x)\, h(x) \\ \text{Then } f'(x) &= g(x)\, h'(x) + h(x)\, g'(x) \end{aligned}$$

In alternate notation that is

$$\begin{aligned} \text{If } y &= u \times v \\ \text{Then } \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ y' &= uv' + vu' \end{aligned}$$

So in our case we can look at the two functions as being $u = 3x^3$ and $v = \log_e(2x) - 1$. That is we have

$$\begin{aligned} \frac{du}{dx} &= 9x^2 \\ \frac{dv}{dx} &= \frac{1}{x} \end{aligned}$$

Note that the derivative of $\log_e(2x)$ is $\frac{1}{x}$ **not** $\frac{2}{x}$. This is a common mistake that students make, and the reasoning to why the derivative is the first term is given below, by an application of the chain rule.

$$\begin{aligned} \text{If } y &= \log_e(ax) \\ \text{and } u &= ax \\ y' &= \frac{d}{du}(\log_e(u)) \times \frac{d}{dx}(ax) \\ &= \frac{1}{u} \times a \\ &= \frac{a}{ax} \\ &= \frac{1}{x} \end{aligned}$$

Now back to the problem at hand. We now have

$$\begin{aligned}
f'(x) &= uv' + vu' \\
&= \left(3x^3 \times \frac{1}{x}\right) + \left((\log_e(2x) - 1) \times 9x^2\right) \\
&= 3x^2 + 9x^2(\log_e(2x) - 1) \\
&= 3x^2(1 + 3\log_e(2x) - 3) \\
&= 3x^2(3\log_e(2x) - 2)
\end{aligned}$$

Note that is is not necessary to factorise the expression, and if you factorise the expression incorrectly, you may not gain the mark. You can obtain the same answer by substituting $x = 2$ into $3x^2 + 9x^2(\log_e(2x) - 1)$. Now we need to actually find $f'(2)$, so we have

$$\begin{aligned}
f'(2) &= 3(2)^2(3\log_e(2 \times 2) - 2) \\
&= 12(3\log_e(4) - 2) \\
&= 72\log_e(2) - 24
\end{aligned}$$

Part c.

This is an integration by recognition question, so we are asked to use our result from **part b**, that is

$$\begin{aligned}
f'(x) &= 3x^2(3\log_e(2x) - 2) \\
&= 9x^2\log_e(2x) - 6x^2
\end{aligned}$$

Now since we know that the original function is

$$\begin{aligned}
f(x) &= 3x^3(\log_e(2x) - 1) \\
&= 3x^3\log_e(2x) - 3x^3
\end{aligned}$$

We can manipulate the fact that

$$f(x) = \int f'(x)\,dx + C$$

That is

$$3x^3\log_e(2x) - 3x^3 = \int \left(9x^2\log_e(2x) - 6x^2\right) dx + C_1$$

Now we are actually looking for *"an anti-derivative of* $x^2\log_e(2x)$*"*. So we look for the expression $x^2\log_e(2x)$ somewhere inside the integral, which we do have. So we can split the integral up into two parts, separating what we want to solve for, $\int \left(x^2\log_e(2x)\right) dx$. That is we have

$$\begin{aligned}
3x^3\log_e(2x) - 3x^3 &= 9\int \left(x^2\log_e(2x)\right) dx - 6\int \left(x^2\right) dx + C_1 \\
9\int \left(x^2\log_e(2x)\right) dx &= 3x^3\log_e(2x) - 3x^3 + 6\int \left(x^2\right) dx + C_1 \\
9\int \left(x^2\log_e(2x)\right) dx &= 3x^3\log_e(2x) - 3x^3 + 2x^3 + C_2 \\
\int \left(x^2\log_e(2x)\right) dx &= \frac{1}{9}\left(3x^3\log_e(2x) - x^3\right) + \frac{1}{9}C_2 \\
\therefore \int \left(x^2\log_e(2x)\right) dx &= \frac{1}{9}x^3(3\log_e(2x) - 1) + C
\end{aligned}$$

Since we are asked for *"an anti-derivative"* the $+C$ is not necessary, but a mark will be given whether its there or not.

Question 2

Part a.

We are asked to find an anti-derivative, that is any anti-derivative of y. So now we first need to recognize that we have a hyperbola, and so the anti-derivative will contain a log function. Our general case for the antiderivatives of a rectangular hyperbola is given below

$$\int \left(\frac{1}{ax+b}\right) dx \quad = \quad \frac{1}{a}\log_e\left(|ax+b|\right), \text{ for } a \neq 0$$

So that is when we get to a situation like we have, we take what is inside the denominator of the fraction (as long as it is a linear function), and the modulus of this term inside a natural log function, but we need to remember to divide by the coefficient in front of the x. The reason being, if we were to differentiate the resulting anti-derivative, it must go back to what it was originally. Differentiating $\log_e\left(|ax+b|\right)$ gives

$$\frac{d}{dx}\left(\log_e\left(|ax+b|\right)\right) \quad = \quad \frac{a}{ax+b}$$

Which we note that by the chain rule, we have a as the numerator, not 1. So to get back to what we had originally, we need to divide by a, that is divide by the coefficient in front of the x. You can think of this as "reversing the chain rule". So we have

$$\int \left(\frac{1}{2x-1}+4\right) dx \quad = \quad \frac{1}{2}\log_e\left(|2x-1|\right)+4x+C$$

Where C is any real number, but not necessary. As the question asks for **an anti-derivative**, the C is not required, but so that you don't forget it when it is required, get used to putting it on the end anyway, the answer is still correct.

There are two key points that students go wrong with a question like this, they either forget to divide by the coefficient in front of the x or forget to that the log needs to have a modulus function inside it.

Part b.

Since we are integrating the same expression as from the previous question, we can use the result from **part a**. That is we have

$$\begin{aligned}\int_1^3 \left(\frac{1}{2x-1}+4\right) dx &= \left[\frac{1}{2}\log_e\left(|2x-1|\right)+4x\right]_1^3 \\ &= \frac{1}{2}\log_e\left(|6-1|\right)+12-\left(\frac{1}{2}\log_e\left(|2-1|\right)+4\right) \\ &= \frac{1}{2}\log_e\left(5\right)+8\end{aligned}$$

We can then can apply our log laws to this expression because the form that we are given is different to the form we currently have. We need to get the coefficient of the log to be 1, so we can apply the log law

$$a\log_e\left(x\right) \quad = \quad \log_e\left(x^a\right)$$

This is due to

$$\begin{aligned}2\log_e\left(x\right) &= \log_e\left(x\right)+\log_e\left(x\right) \\ &= \log_e\left(x\times x\right) \\ &= \log_e\left(x^2\right)\end{aligned}$$

As an exercise, you can extend this for higher powers of a to see that this is true for all $a \in \mathbb{N}$ such that $a\log_e(x) = \log_e(x^a)$.

Thus, our expression becomes

$$\begin{aligned}
\int_1^3 \left(\frac{1}{2x-1}+4\right)dx &= \log_e(5)^{\frac{1}{2}}+8 \\
&= \log_e\sqrt{5}+8 \\
&= \log_e(a)+b \\
\therefore a &= \sqrt{5} \\
\therefore b &= 8
\end{aligned}$$

Part c.

We need to find the axis intercepts and the asymptotes of $y = \dfrac{1}{2x-1}+4$. The x intercepts will occur when $y=0$ and the y intercepts will occur when $x=0$. So we have

$$\begin{aligned}
x = 0 & \\
y &= \frac{1}{-1}+4 \\
\therefore y &= 3 \\
& (0,3) \\
y = 0 & \\
0 &= \frac{1}{2x-1}+4 \\
\frac{1}{2x-1} &= -4 \\
-8x+4 &= 1 \\
8x &= 3 \\
\therefore x &= \frac{3}{8} \\
& \left(\frac{3}{8},0\right)
\end{aligned}$$

So we have x and y intercepts at $\left(\frac{3}{8},0\right)$ and $(0,3)$ respectively.
Now to find the asymptotes. Asymptotes occur when the curve approaches a certain line and keeps getting closer and closer to this line but never touches it. In our case, these lines will be horizontal and vertical lines. Now we know that the hyperbola $y=\dfrac{1}{x}$ has asymptotes at $x=0$ and $y=0$. Our curve is just a transformation of the previous curve. So if we try to make our function look like $\dfrac{1}{x}$ we get

$$\begin{aligned}
y' &= \frac{1}{2x'-1}+4 \\
y'-4 &= \frac{1}{2x'-1} \\
y &= y'-4 \\
y' &= y+4 \\
x &= 2x'-1 \\
x' &= \frac{x+1}{2} \\
(x,y) &\to \left(\frac{1}{2}(x+1), y+4\right)
\end{aligned}$$

That is we've translated the curve of $y = \dfrac{1}{x}$ 1 unit in the positive direction of the x axis, 4 units in the positive direction of the y axis and then dilated the curve by a factor of $\dfrac{1}{2}$ from the y axis. This means our horizontal asymptote at $y = 0$ will be translated to $y = 4$ and our vertical asymptote at $x = 0$ will be translated and then dilated, that is

$$\begin{aligned} x = 0 \quad &\rightarrow \quad x = 1 \\ x = 1 \quad &\rightarrow \quad x = \frac{1}{2} \end{aligned}$$

So our new asymptotes are now at $x = \dfrac{1}{2}$ and $y = 4$.

And alternate way to understand where the asymptotes will be is to look at the expression $y = \dfrac{1}{2x-1} + 4$. If we were to choose a large value of x, that keeps going off to ∞, then what would happen to y? As we make x really large, the denominator of the fraction becomes large, which makes the fraction itself become increasingly smaller and smaller. The fraction approaches 0, which means that y approaches 4. That is

$$\begin{aligned} x &\rightarrow \infty \\ 2x - 1 &\rightarrow \infty \\ \frac{1}{2x-1} &\rightarrow 0 \\ y &\rightarrow 4 \end{aligned}$$

The same situation goes for the vertical asymptote, so lets rearrange for x

$$\begin{aligned} \frac{1}{2x-1} &= y - 4 \\ 2x - 1 &= \frac{1}{y-4} \\ x &= \frac{1}{2(y-4)} + \frac{1}{2} \end{aligned}$$

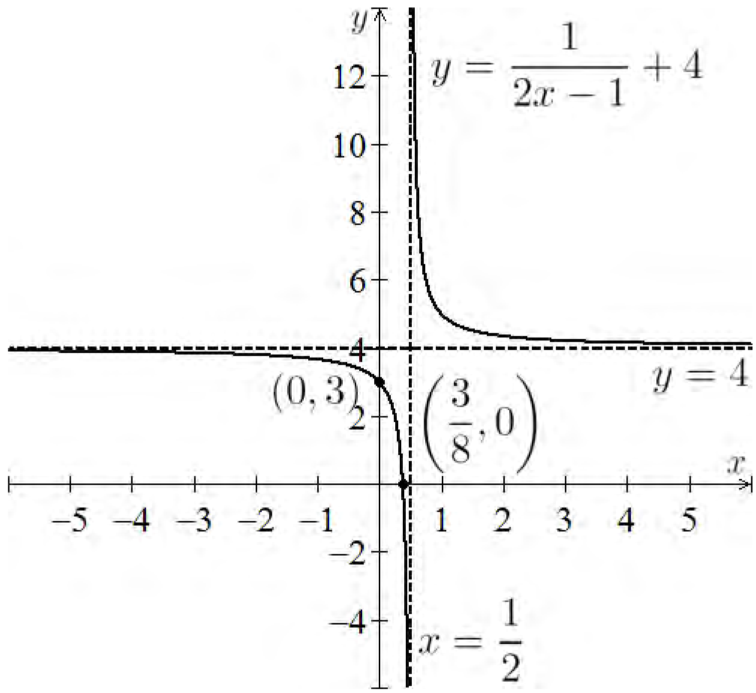
Now if we make y approach ∞, we get

$$\begin{aligned} y &\rightarrow \infty \\ y - 4 &\rightarrow \infty \\ \frac{1}{2(y-4)} &\rightarrow 0 \\ x &\rightarrow \frac{1}{2} \end{aligned}$$

So we have asymptotes at $x = \dfrac{1}{2}$ and $y = 4$

Over time you will be able to see what the asymptotes will be by looking at the equation for the curve, where the vertical asymptote will come from the denominator of the fraction and the horizontal asymptote will come from the other terms that are not the fraction containing the x.

Now to drawing the graph out, we note that the **asymptotes** must be **dotted lines** and **labeled with their equation**. We also need to **labeled our axis intercepts**.



Question 3

Now we need to solve the equation for x, so we will need to move everything over to the one side so that we have a 0 on the other side, then factorise and apply the null factor law. But in this case there is a little trick to get us started, that is

$$\begin{aligned} \text{Let } a &= 3^x \\ 2a^2 - 48a - 162 &= 0 \end{aligned}$$

Now all three terms have a factor of 2 in common, so we can take that out first

$$2\left(a^2 - 24a - 81\right) = 0$$

Now to factorise this quadratic into the form $2\,(a+b)\,(a+c)$, we think to ourselves, "what two numbers multiply to -81 and add to -24?" Now because the two numbers have to multiply to a negative number, we know that one of our numbers has to be negative and the other has to be positive. So we then go through the factors of -81. Since 3^4 is 81 if we divide by 3 we will get 27. So then we can check if a combination of these two numbers (positive or negative) will result in -24. So

$$\begin{aligned} -3 \times 27 &= -81 \\ -3 + 27 &= 24 \end{aligned}$$

So -3 and 27 won't work.

$$\begin{aligned} 3 \times -27 &= -81 \\ 27 - 3 &= -24 \end{aligned}$$

So that means that 3 and -27 will work. That is we know have

$$2\,(a+3)\,(a-27) = 0$$

We can now apply the null factor law, that is if we have a combination of brackets giving 0, then either the first bracket equals 0, or the second bracket equals 0 or both brackets equal 0. That is

$$\begin{aligned} a + 3 = 0 \quad &\text{or} \quad a - 27 = 0 \\ a = -3 \quad &\text{or} \quad a = 27 \end{aligned}$$

Now we need to go back and look at what a was in the first place, that is $a = 3^x$. That is

$$3^x = -3 \quad \text{or} \quad 3^x = 27$$

But since we are raising positive three to any number, we can never get a negative number, so that means there is no solution to the first part. So we then have

$$\begin{aligned} 3^x &= 27 \\ 3^x &= 3^3 \\ \therefore x &= 3 \end{aligned}$$

So our only solution is $x = 3$.

Question 4

Before you ever start finding the inverse of a function, **always check that the original function is a one-to-one function**, that is if it is not then we would **need to restrict the function**. In our case f is already a one-to-one function since it is a exponential function.

To find the inverse of a function, we need to swap x and y, and then resolve for the new y. That is

$$\begin{aligned} &\text{For inverse, swap } x \text{and } y \\ x &= 2e^{-3y} - 2 \\ \frac{x+2}{2} &= e^{-3y} \end{aligned}$$

Now in the next step, we need to convert our exponential expression into a log expression. They way to remember how to do this is *"log of the base of the answer gives the power"*. That is our base is e, our "answer" is $\frac{x+2}{2}$ and our power is $-3y$. So we now have

$$\begin{aligned} \log_e\left(\frac{x+2}{2}\right) &= -3y \\ \therefore y &= -\frac{1}{3}\log_e\left(\frac{x+2}{2}\right) \\ \therefore f^{-1}(x) &= -\frac{1}{3}\log_e\left(\frac{x+2}{2}\right) \end{aligned}$$

We now have our rule but we need to find out domain. The domain of the inverse will be the range of the original and the range of the inverse will be the domain of the original. That is

$$\begin{aligned} \text{Ran } f^{-1} &= \text{Dom } f \\ \text{Dom } f^{-1} &= \text{Ran } f \end{aligned}$$

Now we note that our original function is restricted anyway, so the endpoints in this case will give the maximum and minimum of the function since there are no turning points. So we need to find the endpoints

$$\begin{aligned} f(-1) &= 2e^3 - 2 \\ f\left(\frac{4}{3}\log_e(2)\right) &= 2e^{-3\times\frac{4}{3}\log_e(2)} - 2 \\ &= 2e^{\log_e\left(\frac{1}{2^4}\right)} - 2 \end{aligned}$$

Now here is where the tricky part comes in, we need to simplify the $e^{\log_e}$ expression. Now if we look at a general case

$$e^{\log(x)} = a$$

If we then take the natural logarithm of both sides we get

$$\log_e\left(e^{\log(x)}\right) = \log_e(a)$$

Now we can bring the power down to the front using the power log law. That is

$$\begin{aligned}
\log_e(x)\log_e(e) &= \log_e(a) \\
\text{as } \log_e e &= 1 \\
\log_e(x) &= \log_e(a) \\
x &= a \\
\therefore e^{\log_e(x)} &= x
\end{aligned}$$

Now back to the question, we now have

$$\begin{aligned}
f\left(\frac{4}{3}\log_e(2)\right) &= 2e^{\log_e\left(\frac{1}{2^4}\right)} - 2 \\
&= \frac{2^1}{2^4} - 2 \\
&= \frac{1}{8} - 2 \\
&= \frac{1-16}{8} \\
&= -\frac{15}{8}
\end{aligned}$$

Now we also have to note that in the domain given, $\left(-1, \frac{4}{3}\log_e(2)\right]$ the endpoint at $x = -1$ is not included as we have a circular bracket while the endpoint at $x = \frac{4}{3}\log_e(2)$ is included as we have a square bracket. As f is a decreasing function the lower bound and upper bound of the range of f will be swapped compared to the domain of f, as $2e^3 - 2 > -\frac{15}{8}$. We can easily tell this as $e > 1$, we have $2\left(e^3 - 1\right)$ which will always be positive, and thus greater than $-\frac{15}{8}$. So our range of f is

$$\begin{aligned}
\text{Ran } f &= \left[-\frac{15}{8}, 2e^3 - 2\right) \\
\text{Dom } f^{-1} &= \text{Ran } f \\
\therefore \text{Dom } f^{-1} &= \left[-\frac{15}{8}, 2e^3 - 2\right)
\end{aligned}$$

Put together, we now have:

$$f^{-1} : \left[-\frac{15}{8}, 2e^3 - 2\right) \to R, \quad f^{-1}(x) = -\frac{1}{3}\log_e\left(\frac{x+2}{2}\right)$$

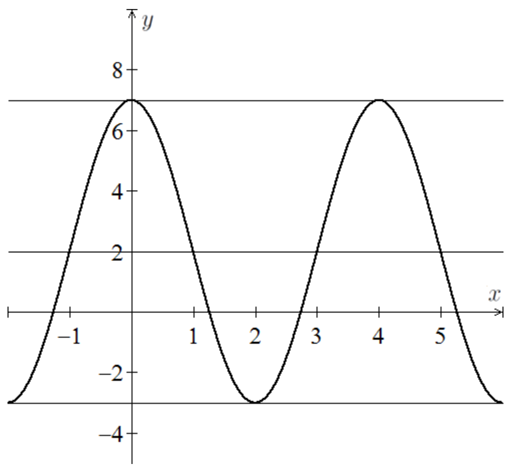
Question 5

Part a.

We have a cosine wave, which has from the standard cosine wave been translated 4 units in the positive direction of the x axis, dilated by factor of $\frac{2}{\pi}$ from the y axis and a factor of 5 from the x axis, then translated 2 units in the positive direction of the y axis. That means that the amplitude of our new curve is 5 and the mean position of the curve is 2. So from the mean position the curve goes a maximum of 2 units up or down. That is our range will be

$$\begin{aligned}\text{Ran } f &= [2-5, 2+5] \\ \text{Ran } f &= [-3, 7]\end{aligned}$$

This is depicted in the plot below.



Now if we had the function

$$g(x) = \cos(nx)$$

Then the period of the function would be given by

$$\text{Period} = \frac{2\pi}{n}$$

So in our case we have

$$\begin{aligned}\text{Period} &= 2\pi \div \frac{\pi}{2} \\ &= \frac{2\pi \times 2}{\pi} \\ &= 4\end{aligned}$$

Part b.

When we solve an equation like this, we will at some point end up with a set of solutions for $2x - \frac{\pi}{2}$, and we will need to ignore some of these solutions as they won't be in the domain given, that is in $[0, 2\pi]$. What we can do is work out a domain for $2x - \frac{\pi}{2}$. That is we will try and turn x below into $2x - \frac{\pi}{2}$

$$\begin{aligned}0 &\le x \le & 2\pi \\ 0 &\le 2x \le & 4\pi \\ 0 - \frac{\pi}{2} &\le 2x - \tfrac{\pi}{2} \le & 4\pi - \frac{\pi}{2} \\ -\frac{\pi}{2} &\le 2x - \tfrac{\pi}{2} \le & \frac{7\pi}{2}\end{aligned}$$

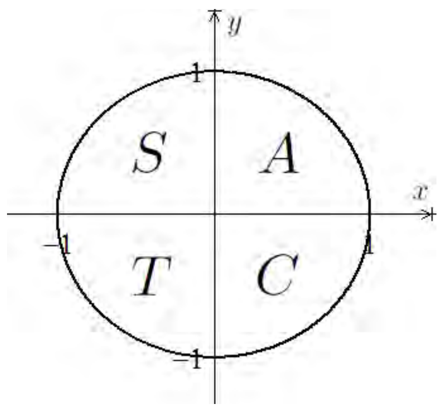
Now to actually solving the equation, firstly we want to get the $\sin\left(2x - \frac{\pi}{2}\right)$ on its own.

$$\begin{aligned} 2\sin\left(2x - \frac{\pi}{2}\right) + 3 &= 2 \\ \sin\left(2x - \frac{\pi}{2}\right) &= -\frac{1}{2} \end{aligned}$$

Now the next step is to find $\sin^{-1}\left(-\frac{1}{2}\right)$. So we need to recall our exact values table, which you should have committed to memory. So here we have

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

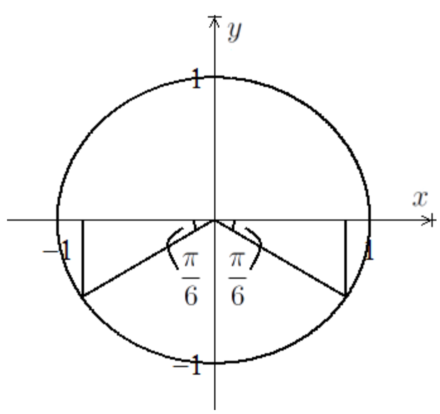
Now from this solution, we need to look at our unit circle, and work out which other angles will give the same value. CAST is a way to remember which quadrants have which trigonometric functions as positive. That is



Where

C	–	Cosine only positive
A	–	All positive
S	–	Sine only positive
T	–	Tangent only positive

Now the base angle is in the fourth quadrant, so since we are dealing with sine, we need another quadrant where sine is negative, that will be in the third quadrant as shown below.



That is our second angle will be

$$-\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

Now we have the two angles $-\frac{\pi}{6}$ and $-\frac{5\pi}{6}$, we can add multiples of 2π to these angles to get the angles in the next cycle of the curve, and we need to remember that our solutions here we want to be within $-\frac{\pi}{2} \leq 2x - \frac{\pi}{2} \leq \frac{7\pi}{2}$. So we have

$$\begin{aligned}
\sin\left(2x - \frac{\pi}{2}\right) &= -\frac{1}{2} \\
2x - \frac{\pi}{2} &= -\frac{5\pi}{6}, -\frac{\pi}{6}, -\frac{5\pi}{6} + 2\pi, -\frac{\pi}{6} + 2\pi, -\frac{5\pi}{6} + 4\pi, -\frac{\pi}{6} + 4\pi \\
&= -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}
\end{aligned}$$

Now we note that $-\frac{5\pi}{6}$ and $\frac{23\pi}{6}$ are outside our allowable domain that we found, so we can cross those off, and continue working with the other solutions.

$$\begin{aligned}
2x - \frac{\pi}{2} &= -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6} \\
2x &= -\frac{\pi}{6} + \frac{3\pi}{6}, \frac{7\pi}{6} + \frac{3\pi}{6}, \frac{11\pi}{6} + \frac{3\pi}{6}, \frac{19\pi}{6} + \frac{3\pi}{6} \\
2x &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{22\pi}{3} \\
\therefore x &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}
\end{aligned}$$

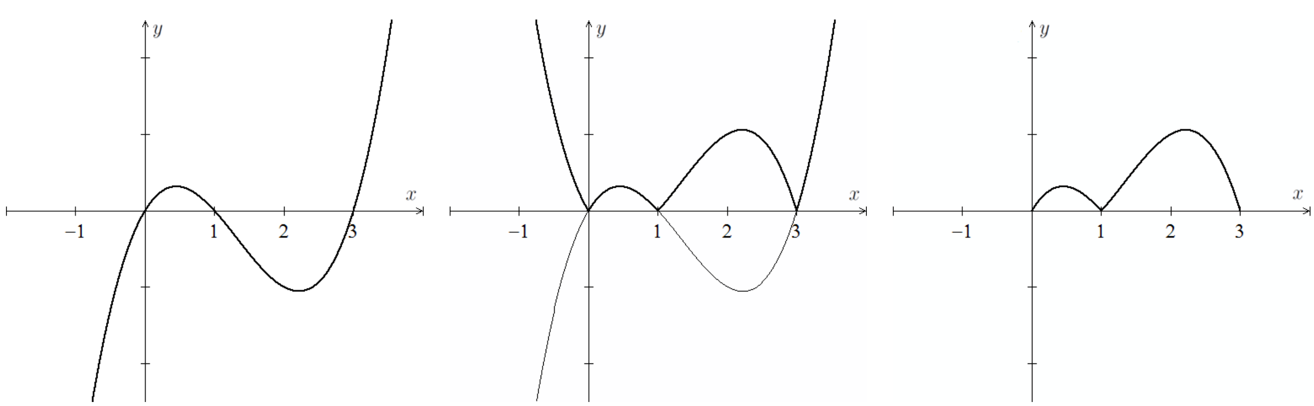
Also note that all of our solutions are within $[0, 2\pi]$

Question 6

Part a.

There are two key features to a probability density function, the first is that the curve has to be equal to or greater than zero where it is defined, as we cannot have negative probabilities. The second is the area under the curve representing the probability has to be equal to one. We use the latter fact to find the value of k.

Now since we have a modulus function, we will need to split the integral up into two parts, and we will also need to expand the function. We don't know which part is positive and which part is negative unless we make a small sketch of $x(x-1)(x-3)$ and then turn that into $|x(x-1)(x-3)|$. We will have intercepts at $x = 0$, 1, 3 and since the coefficient of x^3 will be positive if we expanded it out, the curve will start from the bottom left and continue out through the top right. That is we have something similar to the situation below. We then apply the modulus and flip all the negative parts of the graph in the x axis. Finally we get rid of the sections that our curve is not defined for.



We now have

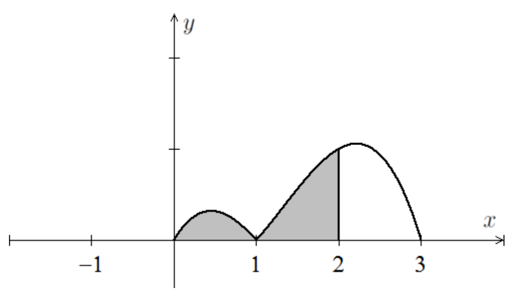
$$f(x) = \begin{cases} kx(x-1)(x-3) & 0 \leq x \leq 1 \\ -kx(x-1)(x-3) & 1 \leq x \leq 3 \end{cases}$$

Now don't forger the negative inside the second integral as we have the negative section of the curve for $1 \leq x \leq 3$.

$$\begin{aligned}
\int_0^1 (kx(x-1)(x-3))\,dx + \int_1^3 (-kx(x-1)(x-3))\,dx &= 1 \\
k\int_0^1 \left(x^3-4x^2+3x\right)dx - k\int_1^3 \left(x^3-4x^2+3x\right)dx &= 1 \\
k\left[\frac{1}{4}x^4-\frac{4}{3}x^3+\frac{3}{2}x^2\right]_0^1 - k\left[\frac{1}{4}x^4-\frac{4}{3}x^3+\frac{3}{2}x^2\right]_1^3 &= 1 \\
k\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}-0\right) - k\left(\frac{81}{4}-\frac{108}{3}+\frac{27}{2}-\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}\right)\right) &= 1 \\
k\left(2\left(\frac{3-16+18}{12}\right)-\left(\frac{81+54}{4}-36\right)\right) &= 1 \\
k\left(\frac{5}{12}-\frac{135-144}{4}\right) &= 1 \\
k\left(\frac{5}{6}+\frac{9}{4}\right) &= 1 \\
\frac{37}{12}k &= 1 \\
\therefore k &= \frac{12}{37}
\end{aligned}$$

Part b.

Since we have a continuous probability distribution, to find $\Pr(X \leq 2)$ we will need to find the area under the curve for $X \leq 2$, that is we will need to integrate the function. But as the function is split into the two parts at $x = 1$, we will again need to split our integral into two parts. The area that we need to integrate is given below



Now since we have already integrated the function from the previous question we can use the result of this integration.

$$\begin{aligned}
\Pr(X \leq 2) &= \int_0^1 \left(\frac{12}{37}x(x-1)(x-3)\right)dx - \int_1^2 \left(\frac{12}{37}x(x-1)(x-3)\right)dx \\
&= \frac{12}{37}\left(\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}-0\right)-\left(\frac{1}{4}(16)-\frac{4}{3}(8)+\frac{3}{2}(4)-\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}\right)\right)\right) \\
&= \frac{12}{37}\left(\left(\frac{3-16+18}{12}\right)-\left(4-\frac{32}{3}+6-\frac{3-16+18}{12}\right)\right) \\
&= \frac{12}{37}\left(\frac{5}{6}-\frac{30-32}{3}\right) \\
&= \frac{12}{37}\left(\frac{5}{6}+\frac{4}{6}\right) \\
&= \frac{12}{37}\left(\frac{9}{6}\right) \\
\therefore \Pr(X \leq 2) &= \frac{18}{37}
\end{aligned}$$

Part c.

This next part makes use of conditional probability, and also makes use of a result from **part b**. We know that

$$\Pr(X \le 2) = \frac{18}{37}$$

But we need $\Pr(X \ge 2)$. Now we know that

$$\begin{aligned}
\Pr(X \le 2) + \Pr(X \ge 2) &= 1 \\
\Pr(X \ge 2) &= 1 - \Pr(X \le 2) \\
&= 1 - \frac{18}{37} \\
&= \frac{19}{37}
\end{aligned}$$

Now we will also need $\Pr(X \ge 1)$. Again since we already integrated the same function in the previous part of the question, we can use that result of the integral. That is

$$\begin{aligned}
\Pr(X \ge 1) &= -\int_1^3 \left(\frac{12}{37}\left(x^3 - 4x^2 + 3\right) x\right) dx \\
&= -\frac{12}{37}\left(\frac{81}{4} - 36 + \frac{27}{2} - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right)\right) \\
&= -\frac{12}{37}\left(\frac{81+54}{4} - 36 - \left(\frac{3-16+18}{12}\right)\right) \\
&= -\frac{12}{37}\left(\frac{135}{4} - \frac{5}{12} - 36\right) \\
&= \frac{12}{37}\left(\frac{100}{3} - 36\right) \\
&= \frac{12}{37}\left(\frac{100-108}{3}\right) \\
&= \frac{32}{37}
\end{aligned}$$

As an alternate method, we can note that from **part a** we could have found that the ratio of the areas of $\Pr(0 \le X \le 1) : \Pr(1 \le X \le 3)$ was $\dfrac{5}{12} : \dfrac{8}{3}$. So that would mean that

$$\begin{aligned}
\Pr(1 \le X \le 3) &= \frac{8}{3} \div \left(\frac{5}{12} + \frac{8}{3}\right) \\
&= \frac{8}{3} \div \left(\frac{37}{12}\right) \\
&= \frac{8}{3} \times \frac{12}{37} \\
&= \frac{32}{37}
\end{aligned}$$

Next we need to apply conditional probability, that is

$$\begin{aligned}
\Pr(X \ge 2 | X \ge 1) &= \frac{\Pr(X \ge 2 \cap X \ge 1)}{\Pr(X \ge 1)} \\
&= \frac{\Pr(X \ge 2)}{\Pr(X \ge 1)} \\
&= \frac{19}{37} \div \frac{32}{37} \\
&= \frac{19}{37} \times \frac{37}{32} \\
&= \frac{19}{32}
\end{aligned}$$

Question 7

There is two ways to approach this question, the first using by rearranging the equation and looking at the gradient and y intercepts, and the second my using matrices and determinants.

Using the first method, we will rearrange the equations into $y = mx + c$ form so that we can compare the gradient and y intercepts.

$$\begin{aligned} mx + y &= m - 2 \\ \therefore y &= -mx + (m-2) \\ 6x + (m-1)\,y &= 12 \\ \therefore y &= -\frac{6}{m-1}x + \frac{12}{m-1} \end{aligned}$$

Now for two linear equations to have no solutions if **they are parallel** and **are not that same line**. That is

- For them to be parallel the **gradients have to be the same**
- For them to not be the same line **the y intercepts cannot be the same**

So equating the gradients gives

$$\begin{aligned} -m &= -\frac{6}{m-1} \\ m^2 - m - 6 &= 0 \\ (m-3)(m+2) &= 0 \\ m - 3 = 0 \quad &\text{or} \quad m + 2 = 0 \\ \therefore m &= 3,\ -2 \end{aligned}$$

Now they y intercepts are **not** equal, that is

$$\begin{aligned} m - 2 &\neq \frac{12}{m-1} \\ (m-2)(m-1) &\neq 12 \\ m^2 - 3m + 2 &\neq 12 \\ m^2 - 3m - 10 &\neq 0 \\ (m-5)(m+2) &\neq 0 \\ m - 5 \neq 0 \quad &\text{or} \quad m + 2 \neq 0 \\ \therefore m &\neq 5,\ -2 \end{aligned}$$

Now the values of m that satisfy both sets of solutions we have. Since $m = -2$ but $m \neq -2$, that will not be a solution. So we are left with $m = 3$ as our only solution.

$$\therefore m = 3$$

So $m = 3$ is the only value of m for which the two simultaneous equations have no solutions.

Now for the second method which makes use of matrices. If we write the equations out in matrix form we will have

$$\begin{bmatrix} m & 1 \\ 6 & m-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

Now if the determinant of the square matrix is zero, then we either have either no solutions or infinite solutions. This means that we can let the determinate equal zero and solve for the values of m, then plug these values back into our original equations and see if they result in the same lines or different lines. Now if we have the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then the determinate will be

$$\det(A) = ad - bc$$

That is we multiply the two diagonals, and minus the positive sloping diagonal. Thus we have

$$\begin{aligned}
m(m-1) - (1 \times 6) &= 0 \\
m^2 - m - 6 &= 0 \\
(m-3)(m+2) &= 0 \\
m - 3 = 0 \quad \text{or} &\quad m + 2 = 0 \\
\therefore m &= -2,\ 3
\end{aligned}$$

Now these are the values of m for **both** no solutions and infinite solutions, so we know substitute these back in to find which values give two different lines. Substituting in $m = -2$

$$\begin{aligned}
-2x + y &= -4 \\
6x - 3y &= 12
\end{aligned}$$

Now the two lines appear different, but one is really a multiple of the other, that is dividing the second by -3 gives

$$\begin{aligned}
\frac{6}{-3}x + \frac{-3}{-3}y &= \frac{12}{-3} \\
-2x + y &= -4
\end{aligned}$$

Which is the same as the first equation, so we know that $m = -2$ gives infinite solutions, but we don't want that, so we disregard $m = -2$. Substituting in $m = 3$ gives

$$\begin{aligned}
3x + y &= 3....[1] \\
6x + 2y &= 12....[2]
\end{aligned}$$

Rearranging into $y = mx + c$ form gives

$$\begin{aligned}
[1]\ y &= 3 - 3x \\
[2]\ y &= 6 - 3x
\end{aligned}$$

Which will be two lines that are **parallel** but **not the same**, which means there will be no solutions for the two lines.

$$\therefore m = 3$$

Question 8

We have a cylinder with a fixed thickness of 3 mm and the radius of the cylinder increases as the volume increase. Now the volume of a cylinder is given by

$$\begin{aligned} V &= \pi r^2 h \\ &= 3\pi r^2 \end{aligned}$$

Now we know that the oil is flowing in at $5\pi\, mm^3/s$, that is

$$\frac{dV}{dt} = 5\pi\, mm^3/s$$

We are asked to find the rate that the radius is increasing, that is $\frac{dr}{dt}$. So we will need to apply the chain rule to our rates situation. Since we have $\frac{dV}{dt}$ there is only one other rate that will multiply with this to give $\frac{dr}{dt}$, that is

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

So we will need $\frac{dr}{dV}$ which we can get from differentiating the expression for V in terms of r with respect to r. That is

$$\begin{aligned} \frac{dV}{dr} &= 6\pi r \\ \frac{dr}{dV} &= \frac{1}{6\pi r} \end{aligned}$$

We need to remember to flip this for $\frac{dr}{dV}$. Now substituting in the value of r, $10\, mm$.

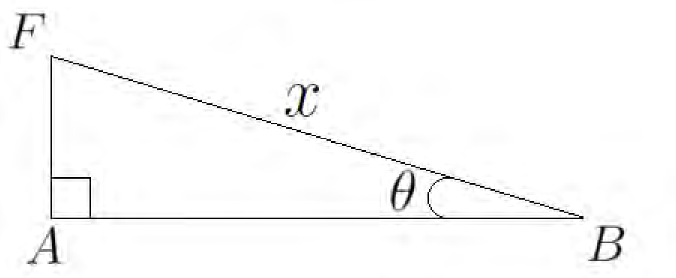
$$\begin{aligned} \frac{dr}{dt} &= \frac{dr}{dV} \times \frac{dV}{dt} \\ & \frac{1}{6\pi \times 10} \times 5\pi \\ &= \frac{1}{12}\, mm/s \end{aligned}$$

That is the radius of the puddle is increasing at $\frac{1}{12}\, mm/s$.

Question 9

Part a.

We need to find AB and AF in terms of x and θ, so we look for a shape that relates the four of these. So we will look at $\triangle ABF$.



Now we can use our simple trigonometric ratios to find the relation ship between them, remember SOH CAH TOA. So we have

$$\begin{aligned}
\cos(\theta) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\
\cos(\theta) &= \frac{AB}{x} \\
\therefore AB &= x\cos(\theta) \\
\sin(\theta) &= \frac{\text{opposite}}{\text{hypotenuse}} \\
\sin(\theta) &= \frac{AF}{x} \\
\therefore AF &= x\sin(\theta)
\end{aligned}$$

Part b.

Next we need to find the volume of the structure in terms of x and θ, so we can split the structure up into three sections, a rectangular prism and two triangular prisms. We then add the volumes of each of the individual sections. So for the rectangular prism we have

$$\begin{aligned}
V_R &= l \times w \times h \\
&= (x)(x)(x\sin(\theta)) \\
&= x^3 \sin\theta
\end{aligned}$$

For the two triangular prisms we have

$$\begin{aligned}
2V_T &= 2 \times \frac{1}{2} bh \times l \\
&= (x\cos(\theta))(x\sin(\theta))(x) \\
&= x^3 \cos(\theta)\sin(\theta)
\end{aligned}$$

Adding those two sections together we obtain

$$\begin{aligned}
V &= V_R + 2V_T \\
&= x^3 \sin(\theta) + x^3 \cos(\theta)\sin(\theta)
\end{aligned}$$

Part c.

Now we are asked to find the derivative of V with respect to θ, but we have V in terms of x and θ. To do this we need to view x as a constant, and differentiate with respect to θ **only**. Since we have the $\cos(\theta)\sin(\theta)$ term we could make use of a little trick that will make our differentiation easier, so that we won't have to apply the chain rule, but in this situation it turns out that we have to re-expand it in the next question, so its best not to apply the trick. We know that from the double angle formula for sine we have

$$\begin{aligned} \sin(2x) &= 2\sin(x)\cos(x) \\ \text{Thus } \sin(x)\cos(x) &= \frac{1}{2}\sin(2x) \end{aligned}$$

That is our expression for V becomes

$$V = x^3\sin(\theta) + \frac{1}{2}x^3\sin(2\theta)$$

If we were to do this without making use of this trick, we would have to use the product rule as below

$$\begin{aligned} \frac{dV}{d\theta} &= x^3\cos(\theta) + x^3(\cos(\theta)\times\cos(\theta) + \sin(\theta)\times -\sin(\theta)) \\ &= x^3\cos(\theta) + x^3\left(\cos^2(\theta) - \sin^2(\theta)\right) \end{aligned}$$

Which as $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$, is equivalent to the expression that we will obtain using the little trick, which is below.

$$\begin{aligned} \frac{dV}{d\theta} &= x^3\cos(\theta) + \frac{1}{2}x^3\cos(2\theta)\times 2 \\ &= x^3\cos(\theta) + x^3\cos(2\theta) \\ &= x^3(\cos(\theta) + \cos(2\theta)) \end{aligned}$$

Now we are also asked to find the value of θ for which $\frac{dV}{d\theta} = 0$. So we now need to solve for x. To do this we will need to make all the θ's inside the cosines to be the same, so we need to either convert back using an alternate form of the first identity $\cos(2\theta) = 2\cos^2(\theta) - 1$, or use the Pythagorean identity $\cos^2(\theta) + \sin^2(\theta) = 1$. That is from the first method we have

$$\begin{aligned} \frac{dV}{d\theta} &= x^3(\cos(\theta) + \cos(2\theta)) \\ &= x^3\left(2\cos^2(\theta) + \cos(\theta) - 1\right) \end{aligned}$$

For the second method we will have

$$\begin{aligned} \frac{dV}{d\theta} &= x^3\cos(\theta) + x^3\left(\cos^2(\theta) - \sin^2(\theta)\right) \\ &= x^3\left(\cos^2(\theta) + \cos(\theta) - \left(1 - \cos^2(\theta)\right)\right) \\ &= x^3\left(2\cos^2(\theta) + \cos(\theta) - 1\right) \end{aligned}$$

Equating to zero

$$\begin{aligned} x^3\left(2\cos^2(\theta) + \cos(\theta) - 1\right) &= 0 \\ x^3 = 0 \quad &\text{or} \quad 2\cos^2(\theta) + \cos(\theta) - 1 = 0 \end{aligned}$$

But if $x = 0$ then the structure ceases to exist, so we can ignore that solution, we then need to factorise the second expression. We can view it just like any other quadratic, but instead of having x, we have $\cos(\theta)$ in it's place. Now since the coefficient of $\cos^2(\theta)$ is 2, we will have factors that have $2\cos(\theta)$ and $\cos(\theta)$ inside of them.

$$(2\cos(\theta) \quad\;)(\cos(\theta) \quad\;) = 0$$

We then need to pick two numbers, that when we sum the product of the coefficients of the first and last terms with the product of the coefficients of the second and third terms, we will get 1 and the product of these two numbers will be -1. So in our case -1 and 1 respectively will fit. That is we have

$$(2\cos(\theta) - 1)(\cos(\theta) + 1) \quad = \quad 0$$

Then we apply the null factor law, that is

$$\begin{aligned} 2\cos(\theta) - 1 = 0 \quad &\text{or} \quad \cos(\theta) + 1 = 0 \\ \cos(\theta) = \frac{1}{2} \quad &\text{or} \quad \cos(\theta) = -1 \end{aligned}$$

But for the situation to make sense, the angle has to be between 0 and $\frac{\pi}{2}$. That is

$$0 < \quad \theta \quad < \frac{\pi}{2}$$

This means that there will be no solutions to the second equation, while there will only be one solution in the first quadrant for the first equation. Now we recall our basic trigonometric angles, that is

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{1}{2}\right) \\ &= \frac{\pi}{3} \end{aligned}$$

So $\theta = \frac{\pi}{3}$ when $\frac{dV}{d\theta} = 0$.

Part d.

Finally we are asked to find the maximum volume, but this time we are given a value for x, 5 m. Now the maximum volume will occur at the turning point of V vs θ, that is when $\frac{dV}{d\theta} = 0$ which we already found in **part c**, that is $\theta = \frac{\pi}{3}$. So this last question is just a matter of substituting in the values of $x = 5$ and $\theta = \frac{\pi}{3}$ into our expression for the volume of the structure.

$$\begin{aligned} V &= x^3\sin(\theta) + \frac{1}{2}x^3\cos(\theta)\sin(\theta) \\ x = 5\,, \theta = \frac{\pi}{3} & \\ V_{max} &= (5)^3\sin\left(\frac{\pi}{3}\right) + \frac{1}{2}(5)^3\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) \\ &= \frac{125\sqrt{3}}{2} + 125 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} \\ &= \frac{375\sqrt{3}}{4}\, m^3 \end{aligned}$$

So our maximum volume of the structure when $x = 5$ is $\frac{375\sqrt{3}}{4}\, m^3$.

SET 1 EXAM 2

DETAILED SOLUTIONS

SECTION 1 - Multiple Choice Questions

Question 1 (C)

There are three ways to approach this question, one by looking at the gradient and intercepts of the two equations, the second by using matrices and the third by substituting in the multiple choice options.

When we are asked to find "the values of m for which .....", there are three questions that we can be asked, that is whether the lines "have one solution", "have no solutions" or "have infinite solutions", with each case having a different way to go about finding the answer.

For two **straight** lines to have only one solution, they must not be parallel, otherwise they would either never meet or be the same line. So this means that **the gradient of the two lines must be different.**

For two **straight** lines to have no solutions, they must be parallel, but cannot be the same line. So that means that they must **have the same gradient** but to be different lines they must **have different y-intercepts.**

For two **straight** lines to have infinite solutions they have to be the same line, that is they have to have the **same gradients** and the **same y-intercepts**. This is the situation that we are asked in the question.

So by the first method we rearrange the equations into $y = mx + c$ form, and look at the m and c values.

$$\begin{aligned}
(m+2)\,x + 3y &= (m+3) \\
y &= \frac{m+3}{3} - \frac{m+2}{3}x....[1] \\
4x + (2m-1)\,y &= 5 \\
y &= \frac{5}{2m-1} - \frac{4}{2m-1}x....[2]
\end{aligned}$$

Now since the **gradients have to be the same**, we equate the gradients.

$$\begin{aligned}
-\frac{m+2}{3} &= -\frac{4}{2m-1} \\
(2m-1)\,(m+2) &= 12 \\
2m^2 + 4m - m - 2 &= 12 \\
2m^2 + 3m - 14 &= 0 \\
(2m+7)\,(m-2) &= 0 \\
2m+7=0 \quad &\text{or} \quad m-2=0 \\
m = -\frac{7}{2} \quad &\text{or} \quad m=2
\end{aligned}$$

So now we look at the y-intercepts; they need to be **the same** to have infinite solutions, that is

$$\begin{aligned}
\frac{m+3}{3} &= \frac{5}{2m-1} \\
(m+3)(2m-1) &= 15 \\
2m^2 - m + 6m - 3 &= 15 \\
2m^2 + 5m - 18 &= 0 \\
(2m+9)(m-2) &= 0 \\
2m+9=0 \quad &\text{or} \quad m-2=0 \\
m = -\frac{9}{2} \quad &\text{or} \quad m=2
\end{aligned}$$

So since we need both to be satisfied we need to take the intersection of the two sets, that is $\left\{-\frac{7}{2}, 2\right\} \cap \left\{-\frac{9}{2}, 2\right\} = \{2\}$.

$$\therefore m = 2$$

The second method is to write the two equations in the form of a matrix equation, in the form of $AX = B$ where the coefficients fill up the matrices. So we have

$$\begin{aligned}
AX &= B \\
\begin{bmatrix} m+2 & 3 \\ 4 & 2m-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} m+3 \\ 5 \end{bmatrix}
\end{aligned}$$

Now if the determinate of the matrix A is 0, then there is either no solutions or infinite solutions to the set of linear equations. Now the determinate of a matrix will be

$$\begin{aligned}
\text{If } A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
\det(A) &= ad - bc
\end{aligned}$$

So our determinate is

$$\begin{aligned}
(3 \times 4) - (m+2)(2m-1) &= 0 \\
12 - \left(2m^2 - m + 4m - 2\right) &= 0 \\
-2m^2 - 3m + 14 &= 0 \\
-(2m+7)(m-2) &= 0 \\
2m+7=0 \quad &\text{or} \quad m-2=0 \\
m = -\frac{7}{2} \quad &\text{or} \quad m=2
\end{aligned}$$

Now we substitute these values back in to check whether the equations end up being exactly the same.

$$\begin{aligned}
\text{For } m &= -\frac{7}{2} \\
\left(-\frac{7}{2}+2\right)x + 3y &= \left(-\frac{7}{2}+3\right) \\
-\frac{3}{2}x + 3y &= -\frac{1}{2} \text{....[1]} \\
4x + \left(2 \times -\frac{7}{2} - 1\right)y &= 5 \\
4x - 8y &= 5\text{....[2]}
\end{aligned}$$

Those two equations are not the same so that means that $m = -\frac{7}{2}$ does not give infinite solutions.

So now we try again

$$\begin{aligned}
\text{For } m = 2 & \\
(2+2)\,x + 3y &= (2+3) \\
4x + 3y &= 5....[1] \\
4x + (2 \times 2 - 1)\,y &= 5 \\
4x + 3y &= 5....[2] \\
[1] = [2] &
\end{aligned}$$

Now as both equations are exactly the same, we have an infinite number of solutions, so the value of m that we are looking for is $m = 2$. We should note that if we get one equation as a multiple of the other, then the two lines are still the same, for example $2x + y = 1$ and $4x + 2y = 2$ are the same lines.

The third method is the least mathematical, in the sense you are guessing with the values given, i.e. we try each option and see if we get the same line. For example option A, trying $m = 2$

$$\begin{aligned}
m = 2 & \\
(2+2)\,x + 3y &= 2 + 3 \\
4x + 3y &= 5 \\
4x + (4-1)\,y &= 5 \\
4x + 3y &= 5
\end{aligned}$$

So since the two lines are the same, $m = 2$ is an answer, so lets try $m = -\dfrac{7}{2}$.

$$\begin{aligned}
\left(-\frac{7}{2} + 2\right) x + 3y &= -\frac{7}{2} + 3 \\
-\frac{3}{2}x + 3y &= -\frac{1}{2} \\
-3x + 6y &= -1 \\
y &= \frac{-1 + 3x}{6} \\
-14x + (-7-1)\,y &= 5 \\
-14x - 8y &= 5 \\
y &= \frac{5 + 14x}{8}
\end{aligned}$$

Now we can see that the two lines are different, so $m = -\dfrac{7}{2}$ is not the answer. Going through the options, $m = 2$ is the only one that fits.
So the correct answer is option **C**.

Question 2 (E)

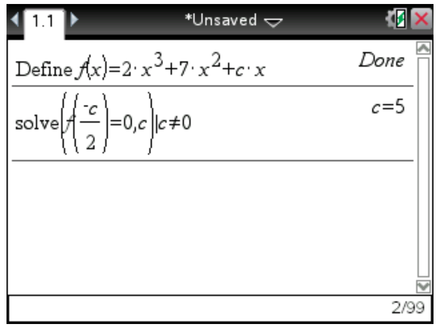
For this question we turn to the factor theorem. If $ax + b$ is a factor of $f(x)$, then $f\left(\dfrac{-b}{a}\right) = 0$. So in our case for $2x + c$ to be a factor of $2x^3 + 7x^2 + cx$, we need

$$\begin{aligned}
f\left(-\frac{c}{2}\right) &= 0 \\
2\left(-\frac{c}{2}\right)^3 + 7\left(-\frac{c}{2}\right)^2 + c\left(-\frac{c}{2}\right) &= 0 \\
-\frac{1}{4}c^3 + \frac{7}{4}c^2 - \frac{1}{2}c^2 &= 0 \\
c^3 - 5c^2 &= 0 \\
c^2\,(c-5) &= 0
\end{aligned}$$

So then using the null factor law we obtain

$$\begin{aligned} c^2 = 0 \quad &\text{or} \quad c - 5 = 0 \\ c = 0 \quad &\text{or} \quad c = 5 \end{aligned}$$

But $c \in R\backslash\{0\}$ so we reject the first solution, so $\therefore c = 5$. We could also solve this on the calculator, which in this situation would be quicker, by defining the function using "[Menu] [1] [1]" and then using the solve function, "[Menu] [3] [1]". Adding the '$|c \neq 0$' at the end of the solve will ignore the $c = 0$ solution.



That makes **E** the correct answer.

Question 3 (A)

We are asked to find the **normal** to a curve at a particular point. The normal is perpendicular to that the tangent, so we can find the derivative at the point.

$$\begin{aligned} f'(x) &= 3x^2 + 10x + 2 \\ f'(-1) &= 3(-1)^2 + 10(-1) + 2 \\ &= -5 \\ \therefore m_1 &= -5 \end{aligned}$$

If two lines are perpendicular, the their gradients will multiply to -1. That is

$$\begin{aligned} m_1 m_2 &= -1 \\ m_2 &= -\frac{1}{m_1} \\ \therefore m_2 &= \frac{1}{5} \end{aligned}$$

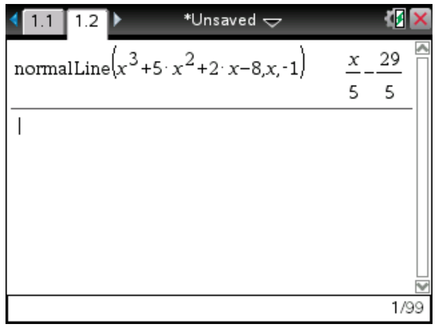
Now we know that the gradient of the line that is normal to the curve is $\frac{1}{5}$ and that the line passes through the line at $x = -1$, so we need to find the point so that we will be able to find the equation of the line.

$$\begin{aligned} f(-1) &= (-1)^3 + 5(-1)^2 + 2(-1) - 8 \\ &= -6 \end{aligned}$$

So we have the point $(-1, -6)$. Now we need to solve for c

$$\begin{aligned} y &= mx + c \\ -6 &= \frac{1}{5} \times -1 + c \\ c &= \frac{1}{5} - 6 \\ \therefore c &= -\frac{29}{5} \\ y &= \frac{1}{5}x - \frac{29}{5} \\ \therefore y &= \frac{1}{5}(x - 29) \end{aligned}$$

We could also find this on the calculator, making use of the normal line function, that is "[Menu] [4] [A]".

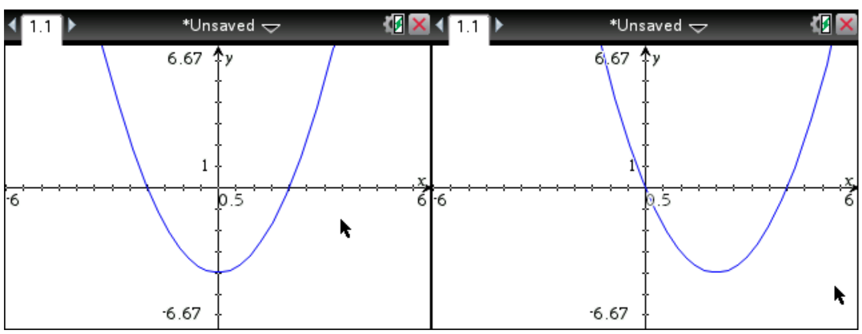


Hence option **A** is correct.

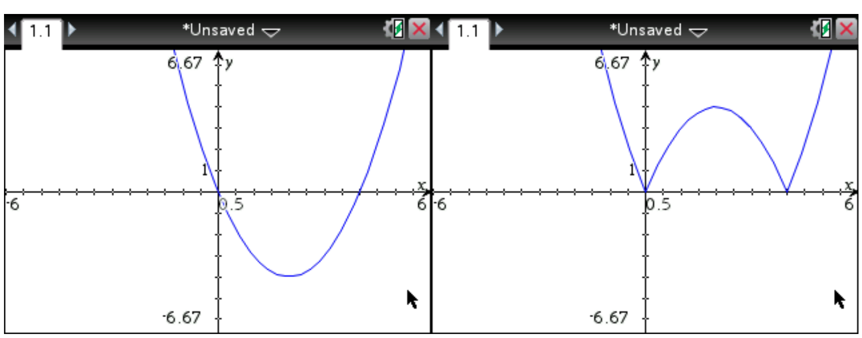
Question 4 (E)

This question asks us to find the graph of a transformed composite function. That is we are given $f(x)$ and asked to find $g(f(x-2))$. Now we cannot work out the equation of $f(x)$ from the graph as we don't have enough information given. So what we need to do is apply the transformations one by one, doing a little sketch each time.

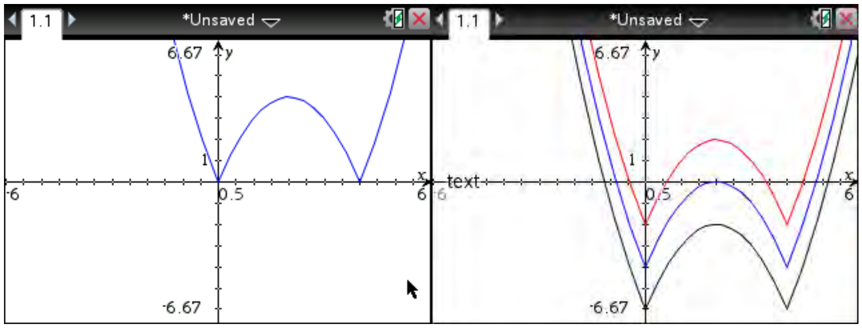
Firstly we start off with a parabola, that is $f(x)$, noting that it has a minimum value. Now if we were to draw $f(x-2)$, we would be translating the parabola 2 units to the right, remembering for translations along the x-axis, negative means right and positive means left. Now the turning point of the original parabola is on the y-axis, that is at $x=0$, we have shifted it to $x=2$.



Next we will be "putting" $f(x-2)$ into $g(x)$, as we have $g(x)=|x|-4$, we will first only focus on the $|x|$, that is we are looking at $|f(x-2)|$. Now the modulus function will make everything that is negative into a positive with the same value. In other words everything that is below the x-axis will be flipped in the x-axis, as if we were flipping the curve in a mirror along $y=0$. So in our case this only affects the part of the curve for which $f(x-2)<0$. We have the curve before and after applying the modulus below.



Now we have the final transformation that we need to apply, that is the translation of 4 units in the negative direction of the y-axis. So basically shifting the curve down 4 units but since we don't know the y value at the turning point, we do not know whether the turning point will end up above, on or below the x-axis.



But there is only one option which the translation could end up resulting in, that is option E which has the turning point on the x-axis. We also note that it could not be option A because there is no horizontal shift of the curve. Options B and C have not had the modulus applied and option D has had the modulus applied in the wrong order.

So option **E** is the correct answer.

Question 5 (D)

We are asked to find $g'(x)$, that is the derivative of $g(x)$ in terms $f(x)$. We can write the expression in another form

$$g(x) = \left(1 - [f(x)]^2\right)^{-\frac{1}{2}}$$

Now we need to apply the chain rule, twice, once for the square root and once for the $f(x)$. So that is if we have $y = u(v(x))$, then the derivative will be

$$\frac{dy}{dx} = \frac{du}{dv} \times \frac{dv}{dx}$$

So for the $[f(x)]^2$ expression, our derivative will be

$$2[f(x)]^1 \times f'(x)$$

So now if we go back to the full expression, our derivative will be

$$\begin{aligned} g'(x) &= -\frac{1}{2}\left(1 - [f(x)]^2\right)^{-\frac{3}{2}} (-2f(x) f'(x)) \\ &= \frac{f(x) f'(x)}{\left(1 - [f(x)]^2\right)^{\frac{3}{2}}} \end{aligned}$$

Hence option **D** is correct.

Question 6 (A)

Firstly we note that the answer must be in the form

$$\begin{bmatrix} a & b & c \\ e & f & g \\ i & j & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ h \\ l \end{bmatrix}$$

This is due to the fact that if we had

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} a & b & c \\ e & f & g \\ i & j & k \end{bmatrix} = \begin{bmatrix} d \\ h \\ l \end{bmatrix}$$

then the matrix on the left hand side could not be defined, as the orders of the matrices don't match, that is we have a 3×1 and a 3×3. The number of columns of the first matrix (in this case 1) must match the number of rows of the second matrix (in this case 3). That rules out options B and D. If we look at the format of the other options, they make sense, since we multiply the top row by the first column and so on, i.e.

$$\begin{aligned}
\begin{bmatrix} a & b & c \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} d \\ \\ \end{bmatrix} \\
ax + by + cz &= d \\
\begin{bmatrix} & & \\ e & f & g \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} \\ h \\ \end{bmatrix} \\
ex + fy + gz &= h \\
\begin{bmatrix} & & \\ & & \\ i & j & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} \\ \\ l \end{bmatrix} \\
ix + jy + jz &= l
\end{aligned}$$

Firstly we should write the equations out with the coefficients aligned up, so that we don't make any mistakes putting the wrong coefficients in the wrong spots. That is we have

$$\begin{aligned}
x + 3y - 2z &= 1 \\
-x + 0y + 4z &= 2 \\
x + 8y + 0z &= 4
\end{aligned}$$

Which gives us

$$\begin{bmatrix} 1 & 3 & -2 \\ -1 & 0 & 4 \\ 1 & 8 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Now we note that none of the options are what we have, but we can move the order of the equations around, that is we can swap rows of the first and last matrix. Doing so will let us arrive at

$$\begin{bmatrix} 1 & 8 & 0 \\ 1 & 3 & -2 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Which is the same as option A.

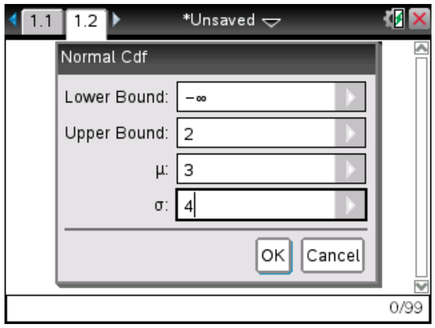
A second approach to the question could have been to manually evaluate the left hand side matrix of each of the options and see if it matched up with the equations given, depending on which answer is the correct one, this could be quicker or take a lot longer. In this case it is probably the quicker method.

Question 7 (A)

Firstly we know the mean of X is 3, that is $\mu = 3$ and a standard deviation of 4, that is $\sigma = 4$. That means we can represent X by

$$X \sim \mathrm{N}(3, 16)$$

Now the probability will be the area under the normal distribution curve from $-\infty$ until 2, so we use the normal cumulative distribution function on the calculator, that is "[Menu] [5] [5] [2]".

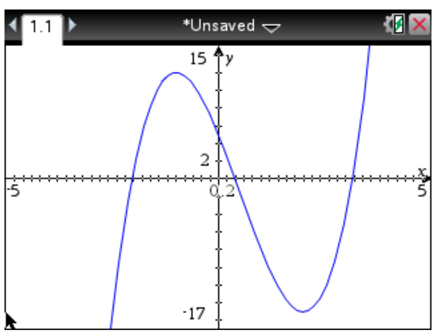


$$\therefore \Pr(X \leq 2) = 0.4013$$

Hence option **A** is correct.

Question 8 (D)

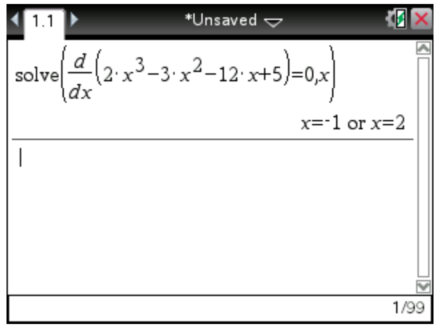
A function will have an inverse function provided that the original function is a one-to-one function, that is if we draw a horizontal line anywhere on the graph, it will only intersect the curve once. So lets draw out the original graph



Now since the domain we are give is $[a, \infty)$, we know it must end going towards positive ∞, that is we are looking at the curve from the rightmost turning point onwards to the right. So we need to find this turning point, we can do this by using the trace function on the graphing window, "[Menu] [5] [1]" and then slide along until we get to the turning point, or we can use calculus to find it.

$$\begin{aligned} f'(x) &= 6x^2 - 6x - 12 \\ 6x^2 - 6x - 12 &= 0 \\ 6(x-2)(x+1) &= 0 \\ x - 2 = 0 \quad &\text{or} \quad x + 1 = 0 \\ x = 2 \quad &\text{or} \quad x = -1 \end{aligned}$$

We could also evaluate this on the calculator using the solve, "[Menu] [3] [1]" and derivative, "[⇑ shift] [−]" functions.



So the point we are looking for is $x=-2$. So the function f will have an inverse function provided that the domain is $[-2,\infty)$, that is $a=-2$.

That makes our answer would be option **D**.

Question 9 (B)

This is quite a simple multiple choice question that requires a bit of manipulation and log rules. We start off by substituting $4x-1$ everywhere we see a x, that is we have

$$\begin{aligned}
f(4x-1) &= \log_e(y) \\
\frac{1}{2}\log_e\left(2(4x-1)^2+1\right) &= \log_e(y) \\
\frac{1}{2}\log_e\left(2\left(16x^2-8x+1\right)+1\right) &= \log_e(y) \\
\frac{1}{2}\log_e\left(32x^2-16x+3\right) &= \log_e(y) \\
\log_e\left(\sqrt{32x^2-16x+3}\right) &= \log_e(y) \\
\therefore y &= \sqrt{32x^2-16x+3}
\end{aligned}$$

The main log rule that students have trouble with is the coefficient out the front of the log can become the power of what is inside the log, that is $a\log_e(x) \iff \log_e(x^a)$.

Thus option **B** is the correct answer.

Question 10 (A)

The period of the function $f(x)=a\cos(nx+b)+c$ is $\dfrac{2\pi}{n}$, that is our period is

$$\begin{aligned}
\text{Period} &= 2\pi \div \frac{\pi}{2} \\
&= 2\pi \times \frac{2}{\pi} \\
&= 4
\end{aligned}$$

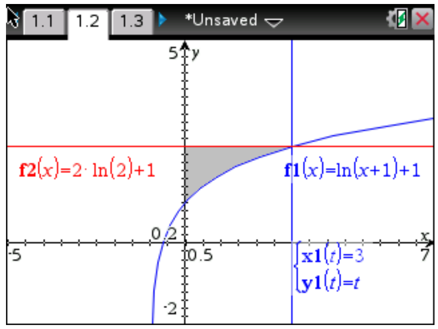
The amplitude will be the value of a, that is the amplitude is 4. The range of the function will be the vertical shift, plus or minus the amplitude, that is

$$\begin{aligned}
\text{Range} &= [1-4, 1+4] \\
&= [-3, 5]
\end{aligned}$$

So our answer is 4, 4, $[-3,5]$, which is option **A**.

Question 11 (A)

We are asked to find the area between a curve, the y-axis and the line $f(x) = \log_e(x+1) + 1$. If we integrate the function, we will get the area between the curve and the x-axis. I.e. We are looking for the grey region below



Now there are two ways to approach this, the first by subtracting the area under the curve from a rectangle formed, and the second by finding x in terms of y and integrating with respect to the y-axis.

For the first method we can find the area of the rectangle formed and take away the area that is between the curve and the x-axis. That is we can take away the area that we integrate. But first we need to know the area of the rectangle, so we need the x value when $y = 2\log_e(2) + 1$.

$$\begin{aligned} \log_e(x+1) + 1 &= 2\log_e(2) + 1 \\ \log_e(x+1) &= \log_e(4) \\ x + 1 &= 4 \\ \therefore x &= 3 \end{aligned}$$

So our area of the rectangle will be

$$\begin{aligned} \text{Area} &= (2\log_e(2) + 1)(3) - \int_0^3 \log_e((x+1) + 1)\, dx \\ &= 6\log_e(2) + 3 - (8\log_e(2)) \\ &= 3 - 2\log_e(2) \end{aligned}$$

We cannot do this integral by hand in methods, so we evaluate on the calculator using the integral function, that is using "[⇑ shift] [−]".

For the second method we first x in terms of y, that is

$$\begin{aligned} y &= \log_e(x+1) + 1 \\ y - 1 &= \log_e(x+1) \\ x + 1 &= e^{y-1} \\ x &= e^{y-1} - 1 \end{aligned}$$

Now if we look back at the graph above we see that if we integrate along the y-axis, we will get the area of the shaded region, so we need to find our terminals, which will now correspond to y values instead of x values since we are integrating with respect to y. So finding the y-intercept

$$\begin{aligned} f(0) &= \log_e(0+1) + 1 \\ &= 1 \end{aligned}$$

Then the upper terminal will be the y value that we're already given, that is $2\log_e(2) + 1$.

So now we will integrate from $y = 1$ to $y = \log_e(4) + 1$

$$\begin{aligned}\text{Area} &= \int_1^{\log_e(4)+1} \left(e^{y-1} - 1\right) dy \\ &= \left[e^{y-1} - y\right]_1^{\log_e(4)-1} \\ &= e^{\log_e(4)+1-1} - \log_e(4) - 1 - \left(e^{1-1} - 1\right) \\ &= 3 - \log_e(4) \\ &= 3 - 2\log_e(2)\end{aligned}$$

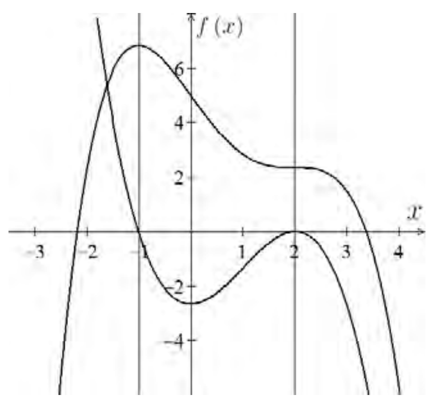
Hence option **A** is correct.

Question 12 (B)

The questions asks us to find the derivative graph of $f(x)$. Before we event start, since $f(x)$ looks like a quartic function, we know that the derivative must lot like some kind of cubic function, that rules out options A, D and E. The derivative, $f'(x)$ will be zero when $f(x)$ has a stationary point. From inspection $f(x)$ has turning points at $x = -1$ and $x = 2$. That is $f'(x)$ will have x-intercepts at $x = -1$ and $x = 2$. Both options B and C satisfy this, so we will need to look at the sign of the gradient between stationary points. So lets draw up a little gradient table, the values of x between the stationary points can be any value.

x	-2	-1	0	2	3
$f'(x)$	$+ve$	0	$-ve$	0	$-ve$
Shape	╱	—	╲	—	╲

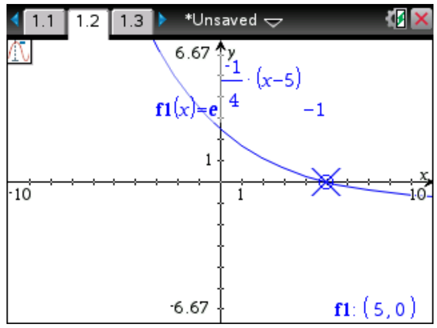
Now from the table we can see that for $x < -2$, $f'(x) > 0$, that is $f'(x)$ is above the x-axis. For $-1 < x < 0$ and $x > 2$, $f'(x) < 0$, that is $f'(x)$ is below the x-axis. The only option that satisfies this is option B. When doing these types of questions, it helps to draw over the original graph, so that you can visually see the gradient changes, like so



Hence option **B** is correct.

Question 13 (B)

We are looking for the value of k that will make the function a continuous probability distribution. The two key features of a continuous probability distribution are that $f(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)\, dx = 1$. That is the area under the curve has to equal 1 and the function can never be negative, as we cannot have negative probabilities. Now if we go to integrate the function, since we have $x \geq 0$ our lower terminal will be 0, but we don't know what our upper terminal will be. But we do know that the domain for the curve is the intersection of $f(x) \geq 0$ and $x \geq 0$. So if we draw out the function without the k dilation we will get the following



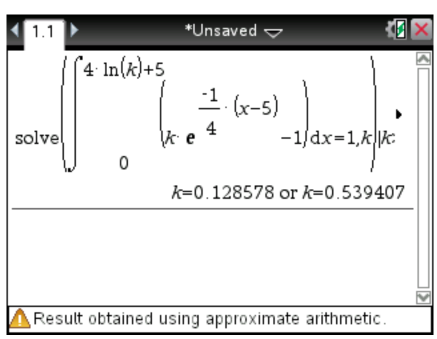
So we can see that the upper limit to the domain will be from $f(x) \geq 0$. So to find the upper terminal, we need to find the x-intercept of our function, that is we let $y = 0$ and solve for x.

$$\begin{aligned}
ke^{-\frac{1}{4}(x-5)} - 1 &= 0 \\
e^{-\frac{1}{4}(x-5)} &= \frac{1}{k} \\
\log_e\left(\frac{1}{k}\right) &= -\frac{1}{4}(x-5) \\
-\log_e(k) &= -\frac{1}{4}(x-5) \\
x - 5 &= 4\log_e(k) \\
\therefore x &= 4\log_e(k) + 5
\end{aligned}$$

So we have an x-intercept at $(4\log_e(k) + 5, 0)$. So that will be our upper terminal when we integrate the area to be 1. So we now have

$$\begin{aligned}
\int_0^{4\log_e(k)+5} \left(ke^{-\frac{1}{4}(x-5)} - 1\right) dx &= 1 \\
\left[-4ke^{-\frac{1}{4}(x-5)} - x\right]_0^{4\log_e(k)+5} &= 1 \\
-4ke^{-\frac{1}{4}(4\log_e(k)+5-5)} - (4\log_e(k) + 5) - \left(-4ke^{\frac{5}{4}} - 0\right) &= 1 \\
-4k\frac{1}{k} - 4\log_e(k) - 5 + 4ke^{\frac{5}{4}} &= 1 \\
4ke^{\frac{5}{4}} - 4\log_e(k) - 10 &= 0 \\
2ke^{\frac{5}{4}} - 2\log_e(k) - 5 &= 0
\end{aligned}$$

Which we cannot solve by hand so we turn to the cas calculator. So using the solve function, "[Menu] [3] [1]", we can solve straight from the start using the integral function, "[⇑ shift] [+]", making sure we add a restriction for k, using "$k > 0$"



$\therefore k = 0.1286$ or $k = 0.5394$

Hence option **B** is correct.

Question 14 (C)

The general solution to $\cos(x) = b$ is given by

$$x = 2\pi n \pm \cos^{-1}(b)\,,\ n \in \mathbb{Z}$$

So we need to get our expression into that form.

$$\begin{aligned}
2\cos\left(2x - \frac{\pi}{2}\right) - \sqrt{3} &= 0\\
\cos\left(2x - \frac{\pi}{2}\right) &= \frac{\sqrt{3}}{2}\\
2x - \frac{\pi}{2} &= 2\pi n \pm \cos^{-1}\left(\frac{\sqrt{3}}{2}\right),\ n \in \mathbb{Z}
\end{aligned}$$

You could evaluate $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ on the calculator but its best to know your exact values tables.

$$\begin{aligned}
2x - \frac{\pi}{2} &= 2\pi n \pm \frac{\pi}{6},\ n \in \mathbb{Z}\\
2x &= 2\pi n \pm \frac{2\pi}{3},\ n \in \mathbb{Z}\\
x &= \pi n \pm \frac{\pi}{3},\ n \in \mathbb{Z}\\
\therefore x &= \frac{(3n \pm 1)\pi}{3},\ n \in \mathbb{Z}
\end{aligned}$$

So the required answer is **C**.

Question 15 (D)

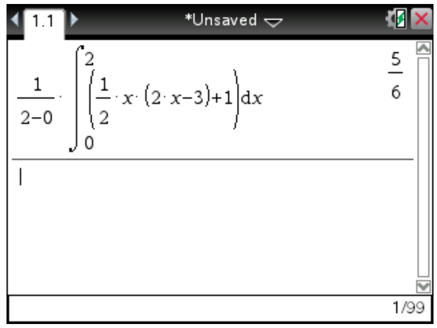
The average value of a function is given by

$$y_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Now since we are finding the average value from $x = 0$ to $x = 2$ our values of a and b are 0 and 2 respectively.

$$\begin{aligned}
y_{avg} &= \frac{1}{2-0}\int_0^2 \left(\frac{1}{2}x(2x-3) + 1\right) dx\\
&= \frac{1}{2}\int_0^2 \left(x^2 - \frac{3}{2}x + 1\right) dx\\
&= \frac{1}{4}\int_0^2 \left(2x^2 - 3x + 2\right) dx\\
&= \frac{1}{4}\left[\frac{2}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_0^2\\
&= \frac{1}{4}\left(\frac{16}{3} - 6 + 4 - (0)\right)\\
&= \frac{1}{4}\left(\frac{16}{3} - \frac{6}{3}\right)\\
&= \frac{5}{6}
\end{aligned}$$

We can also evaluate this on the calculator by using the integral function, that is "[⇑ shift] [+]".



Hence option **D** is correct.

Question 16 (C)

We are asked to find which of the following statements are **false**, so we need to check each option and see if the stated condition holds. We can do this by hand or on the calculator.

Checking option A

$$\begin{aligned} f(x)+f(y) &= 2\log_e(2x)+2\log_e(2y) \\ &= 2\log_e(2x\times 2y) \\ &= 2\log_e(4xy) \\ \therefore f(x)+f(y) &= f(2xy) \end{aligned}$$

Checking option B

$$\begin{aligned} f(x)-f(y) &= 2\log_e(2x)-2\log_e(2y) \\ &= 2\log_e\left(\frac{x}{y}\right) \\ \therefore f(x)-f(y) &= f\left(\frac{x}{2y}\right) \end{aligned}$$

Checking option C

$$\begin{aligned} f(x-y) &= 2\log_e(2(x-y)) \\ &= 2\log_e(2x-2y) \\ f(x+y) &= 2\log_e(2(x+y)) \\ &= 2\log_e(2x+2y) \\ 2\log_e(2x-2y) &\neq 2\log_e(2x+2y) \\ \therefore f(x-y) &\neq f(x+y) \end{aligned}$$

Checking option D

$$\begin{aligned} f(x^y) &= 2\log_e(2x^y) \\ &= 2\log_e(2)+2\log_e(x^y) \\ &= 2\log_e(2)+2y\log_e(x) \\ \therefore f(x^y) &= 2\log_e(2)+2yf\left(\frac{x}{2}\right) \end{aligned}$$

Checking option E

$$\begin{aligned} e^{f(x)} &= e^{2\log_e(2x)} \\ &= e^{\log_e\left((2x)^2\right)} \\ \therefore e^{f(x)} &= 4x^2 \end{aligned}$$

So option **C** is the required answer.

Question 17 (E)

The first note we should make is that the security guard can either "check bags" or "not check bags", that is we have either a success or a failure. We also note that the probability on **any** trial will be 0.8, that is each trial is independent of the other trials. This means we can apply a binomial distribution to the situation. So if we let X be the event that the security guard checks a guests bags then we have

$$\begin{aligned} X &\sim \mathrm{Bi}\left(n, p\right) \\ X &\sim \mathrm{Bi}\left(50, 0.8\right) \end{aligned}$$

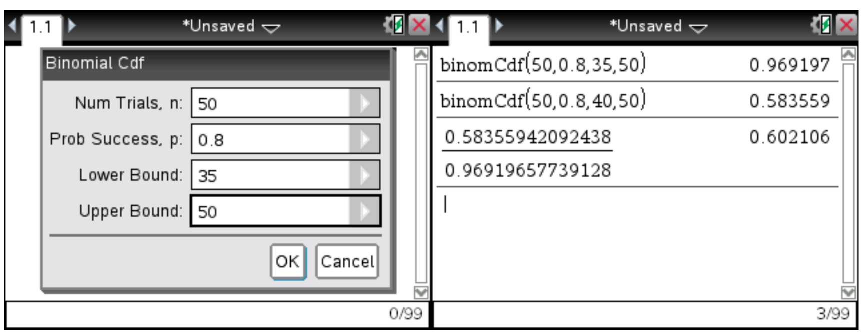
Now we are asked to find *"the probability that the security guard will ask at least* 40 *of them"* to check their bags **given** that *"the security guard checks the bags of at least* 35 *guests*". That is we are looking for $\Pr\left(X \geq 40 | X \geq 35\right)$. So we will need to apply conditional probability to the situation, that is if we had events A and B we would have

$$\Pr\left(A|B\right) = \frac{\Pr\left(A \cap B\right)}{\Pr\left(B\right)}$$

So in our situation we have

$$\begin{aligned} \Pr\left(X \geq 40 | X \geq 35\right) &= \frac{\Pr\left(X \geq 40 \cap X \geq 35\right)}{\Pr\left(X \geq 35\right)} \\ &= \frac{\Pr\left(X \geq 40\right)}{\Pr\left(X \geq 35\right)} \end{aligned}$$

So we need to find the individual probabilities, we can find these using the binomial cumulative distribution function on the calculator, that is "[Menu] [5] [5] [E]".



So then we will have

$$\begin{aligned} \Pr\left(X \geq 40 | X \geq 35\right) &= \frac{0.583559}{0.969197} \\ &= 0.6021 \end{aligned}$$

So the probability that the security guard will check the bags of at least 40 guests, given that he checks the bags of at least 35 guests is 0.6021.

Hence option **E** is correct.

Question 18 (D)

Since we have to deal with long run probability, and we have each event dependent on the previous event, it is best to chose markov chains and transition matrices to deal with this problem. Now we are told to make the event that *"Brenden wins to be a success"*. So lets make the event that Brenden wins the current race to be B and the event that Brenden wins the previous race to be event A. So we will have the transition matrix

$$T = \begin{bmatrix} \Pr(B|A) & \Pr(B|A') \\ \Pr(B'|A) & \Pr(B'|A') \end{bmatrix}$$

Now we are also told that the *"the probabilitiy that that Brenden wins again is x"*. That is

$$\Pr(B|A) = x$$

Next we are told that *"If Thushan beat Brenden in the previous race, then the probability that Thushan wins again is y"*. That is

$$\Pr(B'|A') = y$$

Now if we have the transition matrix below then our steady state probabilities will be as follows

$$\begin{aligned} T &= \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} \\ \Pr(\text{"Success"}) &= \frac{b}{a+b} \\ \Pr(\text{"Failure"}) &= \frac{a}{a+b} \end{aligned}$$

Now we also know that $x + y = 0.8$. So currently we have the transition matrix

$$T = \begin{bmatrix} x & 1-y \\ 1-x & y \end{bmatrix}$$

Now since the long run probability of a success and failure are equal, we have

$$\begin{aligned} \frac{b}{a+b} &= \frac{a}{a+b} \\ b &= a \\ 1-y &= 1-x \\ \therefore x &= y \end{aligned}$$

Now we can substitute this into our previous relationship, that is

$$\begin{aligned} x+y &= 0.8 \\ x+x &= 0.8 \\ 2x &= 0.8 \\ \therefore x &= 0.4 \\ \therefore y &= 0.4 \end{aligned}$$

So now we have our transition matrix

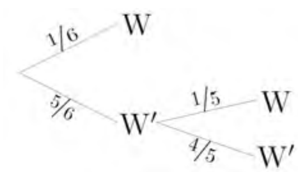
$$T = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}$$

Hence option **D** is correct.

Question 19 (E)

There are two ways to approach this problem, the first looking at what we actually want to find, and the second method which looks at the opposite of what we want to find, then subtracting from 1.

So firstly using the method of looking at what we actually want to find. We want the probability of obtaining no white balls in two draws, so we draw out a tree diagram.



Now the probability of no white will be given by

$$\begin{aligned}\Pr(\text{"No white"}) &= \Pr(W'W') \\ &= \left(\frac{5}{6}\times\frac{4}{5}\right) \\ &= \frac{20}{30} \\ \therefore \Pr(\text{"No white"}) &= \frac{2}{3}\end{aligned}$$

The second method looks for what we don't want, then takes that away from 1, in our case we would be looking for at least 1 white. That is

$$\begin{aligned}\Pr(\text{"No white"}) &= 1-\Pr(\text{"Any white"}) \\ &= 1-(\Pr(W)+\Pr(W'W)) \\ &= 1-\left(\frac{1}{6}+\left(\frac{5}{6}\times\frac{1}{5}\right)\right) \\ &= 1-\left(\frac{5}{30}+\frac{5}{30}\right) \\ &= 1-\frac{10}{30} \\ \therefore \Pr(\text{"No white"}) &= \frac{2}{3}\end{aligned}$$

Hence option **E** is correct.

Question 20 (B)

We are asked to find the *"maximal domain"* of $y=\dfrac{1}{\sqrt{9-x^2}}+\log_e(x+1)$. That is the largest set of values of x for which the curve is defined. There are a few rules to note when finding maximal domains, that is

- We cannot divide by a zero, so that means the **denominator** of a fraction **cannot equal zero**
- Anything under a **square root** must be **equal to or greater than zero**
- Anything inside a **log** has to be **greater than zero**

Looking at our expression, we can break it into two parts, find the maximal domain for each, then the whole curve will be defined for the intersection of the two maximal domains.

So lets first look at $\dfrac{1}{\sqrt{9-x^2}}$, we have a square root as the denominator. So firstly we note that the denominator **cannot** equal zero, that is

$$\begin{aligned}\sqrt{9-x^2} &\neq 0\\ 9-x^2 &\neq 0\\ x^2 &\neq 9\\ x &\neq \pm 3\end{aligned}$$

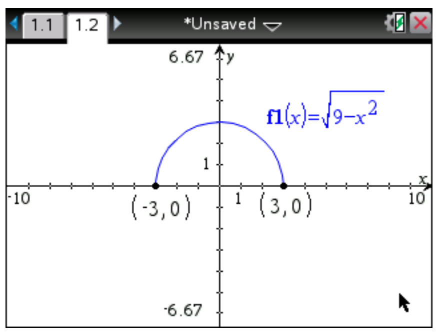
Next we note that what we have inside the square root has to be equal to or greater than zero, that is

$$\begin{aligned}9-x^2 &\geq 0\\ 9 &\geq x^2\\ x^2 &\leq 9\end{aligned}$$

The inequality sign flips when we divide by a negative number. If we had an equals sign this would become

$$x = \pm 3$$

That is we know that $x=-3$ and $x=3$ will be on the edges of our domain, we just don't yet know which parts are the sections we're looking for. Now if we do a quick sketch of the function, we can see when it is greater than zero for $-3 \leq x \leq 3$.



Now this part of the curve will be defined for the intersection of these two restrictions, that is $\mathbb{R}\setminus\{-3,3\}\cap[-3,3]=(-3,3)$.

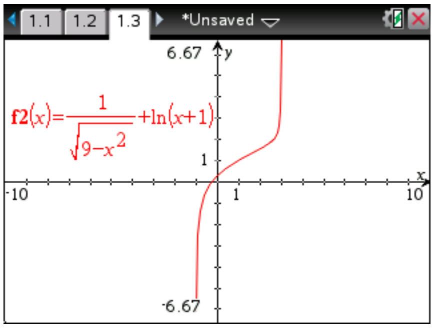
Now we look at the second part of our expression, that is $\log_e(x+1)$. Remembering that whatever is inside the log has to be greater than zero, so we have

$$\begin{aligned}x+1 &> 0\\ x &> -1\end{aligned}$$

So the maximal domain for the entire curve will be given by the intersection of the two sets, $(-3,3)\cap(-1,\infty)$. If you have trouble picturing the situation, draw it out on a number line, remembering that the open (curved) brackets are not included, which are open circles, while the square brackets, that are closed circles are included. So we have

$$(-3,3)\cap(-1,\infty) = (-1,3)$$

So the maximal domain that our curve is defined for is $-1<x<3$. If we graph the original curve, we can see this depicted, you 'could' for a multiple choice question graph the curve, and work out the domain off the graph visually, but for curves with asymptotes, I would advise against it, as it is easy to make mistakes.



So option **B** is the correct answer.

Question 21 (D)

Since we are trying to find out the nature of the stationary points of the graph we will need to do a sign test. We know that since there are two values of x for which $f'(x) = 0$ we will need 5 slots for our table. For the values when $f'(x) \neq 0$ we could pick any values of x between the interval, so we can leave them blank and just note the sign of the gradient.

x		-2		3	
$f'(x)$	$+ve$	0	$+ve$	0	$-ve$
Shape	/	—	/	—	\

Next we need to work out the nature of each stationary point.

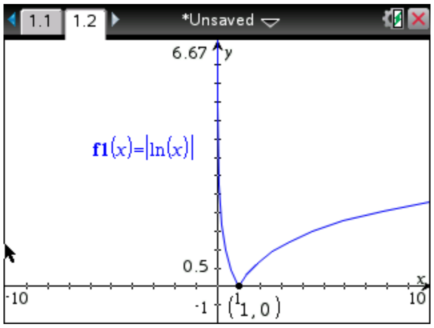
- For maximums we the gradient from left to right, will change from positive, to zero and then to negative.
- For minimums the gradient from left to right, will change from negative. to zero and then to positive.
- For stationary points of inflection the gradient will be the same on both sides of the point at which $f'(x) = 0$

So if we look at the gradient around and at $x = -2$, it goes from positive, to zero and back to positive, so we have a stationary point of inflection at $x = -2$. If we look at the gradient around $x = 3$, the gradient goes from positive, to zero and then to negative, so we have a maximum stationary point at $x = 3$. Looking at the options available we have 'there is a stationary point of inflection at $x = -2$.

Hence option **D** is correct.

Question 22 (C)

Firstly it is best to graph the function on the calculator.



So if we go through the options one by one we can check whether they hold true. Option A states that $f(x)$ is increasing for $x \in \mathbb{R}^+$but as we can see $f(x)$ is decreasing for $0 < x < 1$, thus is incorrect. Option B is incorrect as the function is continuous for $\mathbb{R}^+$, not $\mathbb{R}$. Option C is correct as the derivative will not be defined at the cusp at $x = 1$, as there would be two values for the gradient at this point. So the graph of the derivative would be valid for $\mathbb{R}^+ \setminus \{1\}$. Option D is incorrect since $f(1) = 0$ and option E is incorrect as f does not have any stationary points.

Hence option **C** is correct.

SECTION 2 - Extended Response Questions

Question 1

Part a.

A stationary point of a function will occur when the gradient is zero, that is when our derivative is zero. So firstly we need to find the derivative of f. We can obtain this either using the calculator or by hand, since the question is worth 4 marks, it is probably best to do it by hand, as method marks may be awarded. But we can still check our result on the calculator.

Since we have two groups of brackets involving x's that are being multiplied by each other we are going to need to use the product rule, that is

$$\begin{aligned} \text{If } y &= u \times v \\ \text{then } \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \end{aligned}$$

So we can look at our situation having $u = (3x-a)^2$ and $v = (bx-2)^2$. Now for the derivative of each of these we will need to apply the chain rule, since the coefficient of the x terms is not 1. So remember to multiply by the derivative of the "inside". That is we have

$$\begin{aligned} \frac{d}{dx}\left((3x-a)^2\right) &= 2(3x-a)^1 \times 3 \\ &= 6(3x-a) \\ \frac{d}{dx}\left((bx-2)^2\right) &= 2(bx-2)^1 \times b \\ &= 2b(bx-2) \end{aligned}$$

So we go back to the original expression and apply the product rule.b

$$f'(x) = \frac{1}{16}\left((3x-a)^2 \times 2b(bx-2) + (bx-2)^2 \times 6(3x-a)\right)$$

Now if we look at the expression we have, both terms inside the bracket have $(3x-a)$ and $(bx-2)$ terms. So to simplify this, we will take a factor of both of these out of the brackets. That is

$$\begin{aligned} f'(x) &= \frac{1}{16}(3x-a)(bx-2)\left(2b(3x-a) + 6(bx-2)\right) \\ &= \frac{1}{16}(3x-a)(bx-2)\left(6bx - 2ab + 6bx - 12\right) \\ &= \frac{1}{16}(3x-a)(bx-2)\left(12bx - 2ab - 12\right) \\ &= \frac{1}{8}(3x-a)(bx-2)(6bx - ab - 6) \end{aligned}$$

Now recall that our derivative will be zero at stationary points, that is

$$\begin{aligned} f'(x) &= 0 \\ \frac{1}{8}(3x-a)(bx-2)(6bx-ab-6) &= 0 \end{aligned}$$

Now we apply the null factor law, that is either one, two, or all of the brackets can be zero to get the right hand side to be zero. So we can equate each of the brackets to zero.

$$\begin{array}{lll} 3x - a = 0 \text{ or} & bx - 2 = 0 & \text{or } 6bx - ab - 6 = 0 \\ x = \dfrac{a}{3} \text{ or} & x = \frac{2}{b} & \text{or } x = \dfrac{ab+6}{6b} \end{array}$$

Now we should note that we cannot have $b = 0$ as the denominator of the second two will be 0, so $b \neq 0$. Now we have the x values but we are asked to find the points, so we need the y values as well. So we will need to substitute the points back into the original function. That is

$$\begin{aligned}
f\left(\frac{a}{3}\right) &= \frac{1}{16}\left(3 \times \frac{a}{3} - a\right)^2 \left(b \times \frac{a}{3} - 2\right)^2 - 4 \\
&= -4 \\
f\left(\frac{2}{b}\right) &= \frac{1}{16}\left(3 \times \frac{2}{b} - a\right)^2 \left(b \times \frac{2x}{b} - 2\right)^2 - 4 \\
&= -4 \\
f\left(\frac{ab+6}{6b}\right) &= \frac{1}{16}\left(3 \times \frac{ab+6}{6b} - a\right)^2 \left(b \times \frac{ab+6}{6b} - 2\right)^2 - 4 \\
&= \frac{1}{16}\left(\frac{3ab + 18 - 6ab}{6b}\right)^2 \left(\frac{ab^2 + 6b - 12b}{6b}\right)^2 - 4 \\
&= \frac{1}{16}\left(\frac{6 - ab}{2b}\right)^2 \left(\frac{ab - 6}{6}\right)^2 - 4 \\
&= \frac{1}{2304b^2}(ab - 6)^4 - 4
\end{aligned}$$

So we have stationary points at $\left(\frac{a}{3}, -4\right)$, $\left(\frac{2}{b}, -4\right)$ and $\left(\frac{ab+6}{6b}, \frac{1}{2304b^2}(ab-6)^4 - 4\right)$ for $b \neq 0$.

Part bi.

If we have $b = 2$ then our derivative becomes

$$\begin{aligned}
f'(x) &= \frac{1}{8}(3x - a)(2x - 2)(6 \times 2x - 2 \times a - 6) \\
&= \frac{1}{2}(3x - a)(x - 1)(6x - a - 3) \\
&= 0
\end{aligned}$$

There are two ways to look at this problem, the first should be the path you take since the question is only worth two marks, but I have included the second as another way to look at the problem, but should not be used as it is quite long and not required.

The first method is involves looking at the properties of a quartic. Polynomials of power n can have a maximum of $n - 1$ stationary points, so for quartics we can have a maximum of three stationary points. Now since we have the $x - 1$ term, we already know that there is a stationary point at $x = 1$ and as the power of this bracket is 1, this point will either be a local maximum or a local minimum. So that means we can either have 2 more local maximums/minimums or we can have a stationary point of inflection, but we need to check each case for our exact situation. So if we were to have two more stationary points, then they would be at

$$\begin{aligned}
3x - a = 0 \quad &\text{or} \quad 6x - a - 3 = 0 \\
x = \frac{a}{3} \quad &\text{or} \quad x = \frac{a+3}{6}
\end{aligned}$$

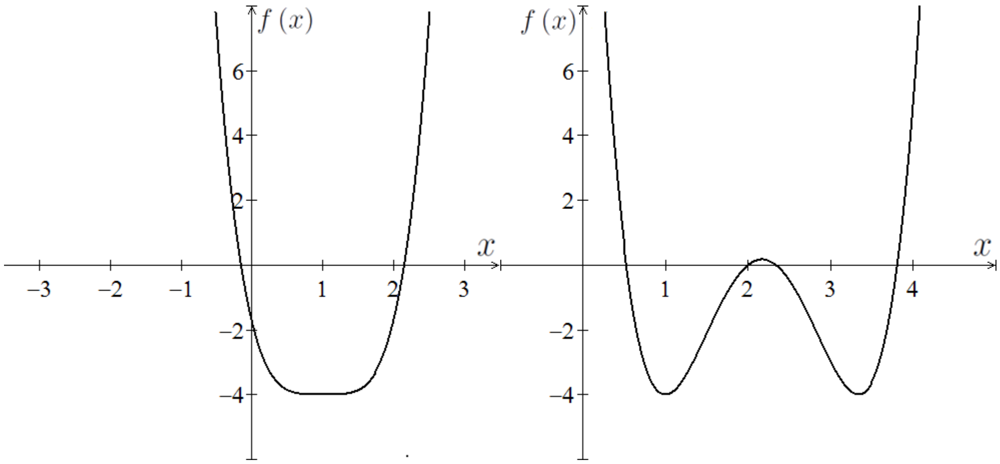
Which are two different points, so that works. But if we want to have only 1 more stationary point, then we need to have these two points being the same, that is we have

$$\begin{aligned}
\frac{a}{3} &= \frac{a+3}{6} \\
2a &= a + 3 \\
\therefore a &= 3 \\
x &= \frac{3}{3} \\
\therefore x &= 1
\end{aligned}$$

The problem that arises here is that we then get the same point as the first turning point, so we really only have one turning point. So **we cannot have two turning points in this situation**, that is we cannot have one local minimum and a stationary point of inflection. So that makes our values of n to be

$$\therefore n \quad = \quad 1,\, 3$$

This corresponds to the two situations below.



The second method is a little tedious, and I would advise against using it, but this is included for further understanding. Now we know that the $x = 1$ term is going to result in a stationary point at $x = 1$, we can combine the other two brackets.

$$f'(x) \quad = \quad \frac{1}{2}(x-1)\left(18x^2 - 9x(a+1) + a^2 + 3a\right)$$

Now we have a quadratic, we can use the discriminate of the quadratic to find out how many solutions to it we can have, and then remembering to add 1 for the other stationary point for $x = 1$. Now if we have $y = ax^2 + bx + c$, our discriminate will be given by

$$\triangle \quad = \quad b^2 - 4ac$$

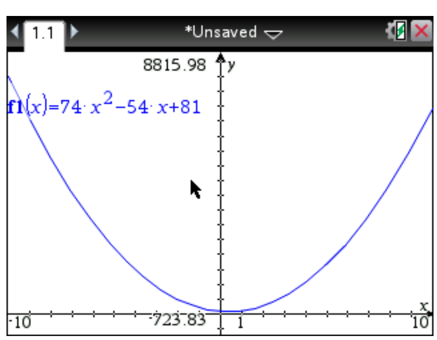
Now if

- $\triangle > 0$ there are two real solutions
- $\triangle = 0$ there is one unique solution
- $\triangle < 0$ there are no real solutions

So in our case we have

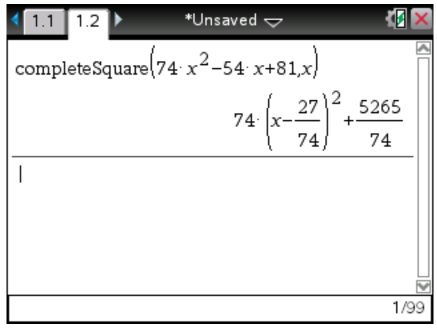
$$\begin{aligned}\triangle &= (-9(a+1))^2 - 4(18)\left(a^2+3a\right)\\ &= 81\left(a^2+2a+1\right) - 7a^2 - 216a\\ &= 74a^2 - 54a + 81\end{aligned}$$

Now from this point we can either graph it or complete the square. When graphing it we will need to change the window size using "[Menu] [4] [A]".



Which we observe that the graph is always greater than 0, so no matter what value of a we use, the discriminate will always be greater than 0.

We could also complete the square using the completeSquare function, "[Menu] [3] [5]".



Now the key result of this is since the leading coefficient is positive, we know that it has a minimum, and since this minimum will be $\dfrac{5265}{74}$, we know that the graph of the discriminate is always greater than 0.

So we have $\triangle > 0$ which means that there are two solutions. Now we add the two solutions to the $x = 1$ case and get three stationary points, but what happens when we obtain $x = 1$ from the second set of solutions? We will then only have one stationary point. So the possible values of n will be 1 and 3.

$$\therefore n = 1,\ 3$$

Part bii.

For f to have three stationary points, then we cannot have the second two solutions being equal, that is

$$\begin{aligned} \frac{a}{3} &\neq \frac{a+3}{6} \\ 2a &\neq a+3 \\ \therefore a &\neq 3 \end{aligned}$$

We also need to check that either of these two points will not be the same as the stationary point at $x = 1$.

$$\begin{aligned} \frac{a}{3} &\neq 1 \\ \therefore a &\neq 3 \\ \frac{a+3}{6} &\neq 1 \\ \therefore a &\neq 3 \end{aligned}$$

We gives us the same result as before. So we will have three stationary points when $a \in \mathbb{R} \backslash \{3\}$.

Part biii.

We now need to find the actual stationary points and their nature. So when $a = 9$ our derivative becomes

$$\begin{aligned} f'(x) &= \frac{1}{2}(3x-9)(x-1)(6x-9-3) \\ &= 9(x-3)(x-1)(x-2) \\ 9(x-3)(x-1)(x-2) &= 0 \end{aligned}$$

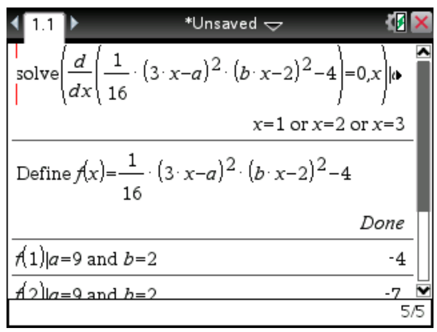
Then using the null factor law

$$\begin{aligned} x-3=0 \quad &\text{or } x-1=0 \text{ or } \quad x-2=0 \\ x=3 \quad &\text{or } x=1 \text{ or } \quad x=2 \end{aligned}$$

Then we need to find the y value for each of these points, which we have in terms of a from a previous question.

$$
\begin{aligned}
\left(\frac{a}{3}, -4\right) &= (3, -4) \\
\left(\frac{2}{b}, -4\right) &= (1, -4) \\
\left(\frac{ab+6}{6b}, \frac{1}{2304b^2}(ab-6)^4 - 4\right) &= \left(\frac{18+6}{6\times 2}, \frac{1}{2304\times 2^2}(18-6)^4 - 4\right) \\
&= \left(2, \frac{9}{4} - 4\right) \\
&= \left(2, -\frac{7}{4}\right)
\end{aligned}
$$

So this means we have stationary points at $(1, -4)$, $\left(2, -\frac{7}{4}\right)$ and $(3, -4)$. We could also have found these solutions using the derivative and solve functions, "[⇑ shift] [−]" and "[Menu] [3] [1]".



So to find the nature of the points we will need to do a sign test. We already know that the gradient is 0 at $x = 1$, 2, 3. So we can fill that in first.

So now we pick any value between each stationary point and the next, and test it in the derivative to find the sign of the gradient in that region. So lets try

$$
\begin{aligned}
f'(0) &= 9(0-3)(0-1)(0-2) \\
&= -ve \\
f'\left(\frac{3}{2}\right) &= 9\left(\frac{3}{2}-3\right)\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right) \\
&= +ve \\
f'\left(\frac{5}{2}\right) &= 9\left(\frac{5}{2}-3\right)\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right) \\
&= -ve \\
f'(4) &= 9(4-3)(4-1)(4-2) \\
&= +ve
\end{aligned}
$$

x		1		2		3	
$f'(x)$	$-ve$	0	$+ve$	0	$-ve$	0	$+ve$
Shape	╲	—	╱	—	╲	—	╱
		Local Minimum		Local Maximum		Local Minimum	

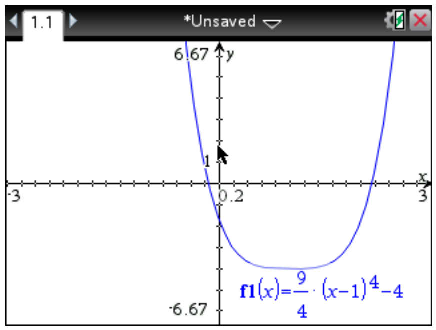
So we have local minimums at $(1, -4)$ and $(3, -4)$ and a local maximum at $\left(2, -\frac{7}{4}\right)$.

Part ci.

For a function to have an inverse function, we need to restrict the original function to be a one-to-one function, that is if we draw a horizontal line anywhere on the axis it will only cut the curve once. So we need to restrict our function. Firstly note that we have new values for a and b. That is

$$\begin{aligned} a = 3,\ b = 2 & \\ f(x) &= \frac{1}{16}(3x-3)^2(2x-2)^2 - 4 \\ &= \frac{9}{4}(x-1)^4 - 4 \end{aligned}$$

So if we look at our graph we have



Now since the upper bound on the domain given is ∞, we will need to find the first turning point from the right hand side. In our situation there is only one turning point, but the domain we want will be to the right of this turning point (including the turning point).

$$\begin{aligned} f'(x) &= 9(x-1)^3 \\ 9(x-1)^3 &= 0 \\ x-1 &= 0 \\ \therefore x &= 1 \\ \therefore c &= 1 \end{aligned}$$

Part cii.

To find the inverse function we need to swap x and y and then solve for y. That is

$$\begin{aligned} x &= \frac{9}{4}(y-1)^4 - 4 \\ \frac{4}{9}(x+4) &= (y-1)^4 \\ y-1 &= \pm\left(\frac{4}{9}\right)^{\frac{1}{4}}(x+4)^{\frac{1}{4}} \end{aligned}$$

We take the positive arm since that is the one that corresponds to the domain restriction we had for the original function.

$$y = \left(\frac{4}{9}\right)^{\frac{1}{4}}(x+4)^{\frac{1}{4}} + 1$$

Now our domain of the inverse will be the range of the original, so we need to find the minimum value. That is

$$f(1) = -4$$

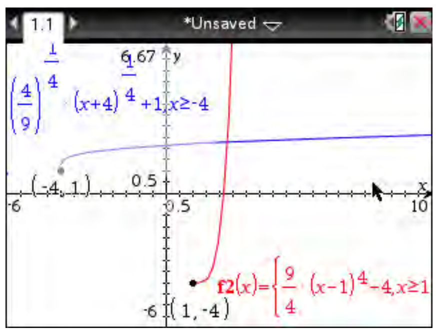
So

$$\begin{aligned} \text{Ran } f &= \left[-4,\infty\right) \\ \text{Dom } f^{-1} &= \text{Ran } f \\ \therefore \text{Dom } f^{-1} &= \left[-4,\infty\right) \end{aligned}$$

$$f^{-1}:\left[-4,\infty\right)\to R,\ f^{-1}\left(x\right) = \left(\frac{4}{9}\right)^{\frac{1}{4}}\left(x+4\right)^{\frac{1}{4}}+1$$

Part ciii.

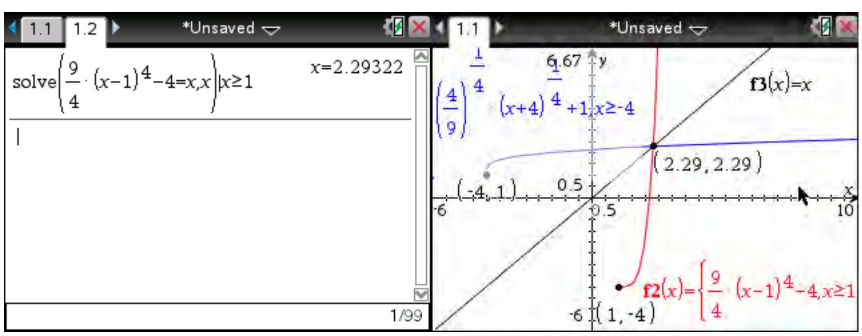
To find the points of intersection of $f\left(x\right)$ and $f^{-1}\left(x\right)$ we solve $f\left(x\right)=f^{-1}\left(x\right)$ for x. For most functions and their inverse the points of intersection will lie on $y=x$ so we could solve $f\left(x\right)=x$ or $f^{-1}\left(x\right)=x$ but there are some exceptions to this case such as $y=-x^3+1$. So make sure you graph the function and its inverse before checking whether you can use $f\left(x\right)=x$. So lets graph our situation.



So we can see that our solutions will lie on $y=x$ in this situation, so we can solve $f\left(x\right)=x$ for x.

$$\begin{aligned} f\left(x\right) &= x \\ \frac{9}{4}\left(x-1\right)^4-4 &= x \end{aligned}$$

Then using the solve function on the calculator, “[Menu] [3] [1]”.

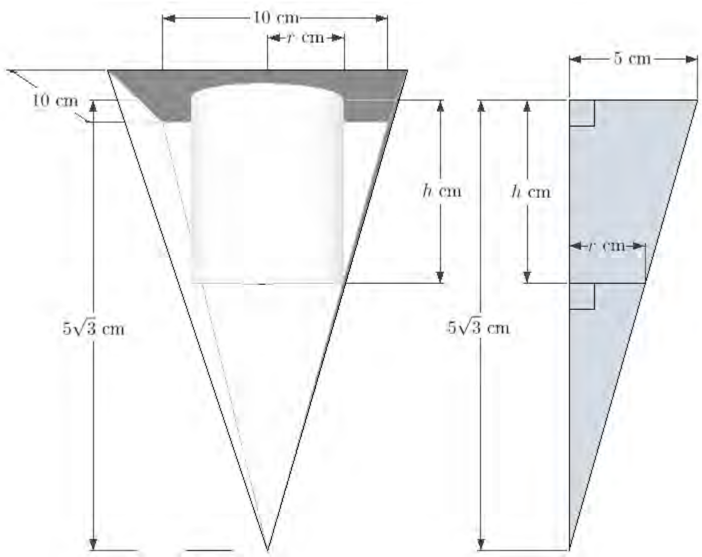


So our point of intersection of $f\left(x\right)$ and $f^{-1}\left(x\right)$ is $(2.29, 2.29)$ correct to 2 decimal places.

Question 2

Part a.

We are asked to find a expression for h in terms of r. So we need to find some kind of relationship between h and r. What we can do is look at the situation from side on, that is we can find two similar triangles in the cone and cylinder. So we draw out the situation below



Now we know that the two triangles are similar since they share a common angle, and they both have the right angle aligned, so the third angle must be the same as well. So this means we can use a ratio of the side lengths of the two triangles to find our relationship between r and h. So we have

$$\begin{aligned}\frac{5}{r} &= \frac{5\sqrt{3}}{5\sqrt{3}-h} \\ 5\sqrt{3}-h &= \sqrt{3}r \\ h &= \sqrt{3}\,(5-r)\end{aligned}$$

Part b.

The volume of a cylinder is given by

$$V = \pi r^2 h$$

We are asked to find the volume in terms of r only, and now have an expression for h in terms of r. We can substitute this in so that V is only in terms of r. We then get

$$V = \sqrt{3}\pi r^2\,(5-r)$$

Now we also need the domain for which this is defined, so we need to look at the "real life situation". That is we know that we can't have negative and/or zero lengths and volumes. From this we obtain

$$\begin{aligned} r &> 0 \\ h &> 0 \end{aligned}$$

We also note that the cylinder has to be inside of the pyramid, that is the radius cannot extend outside of the pyramid. That is

$$\begin{aligned} 2r &< 10 \\ r &< 5 \end{aligned}$$

Thirdly the volume of the cylinder has to be greater than zero, so if we look at our expression for the volume of the cylinder we observed that if the $(5 - r)$ term is less than zero, then the value for the volume obtained would be negative. So we have

$$\begin{aligned} 5 - r &> 0 \\ -r &> 5 \\ r &< 5 \end{aligned}$$

Which is the same restriction we had before. Now the domain that the volume is defined for will be the intersection of these two restrictions, that is

$$r > 0 \cap r < 5$$
$$0 < r < 5$$

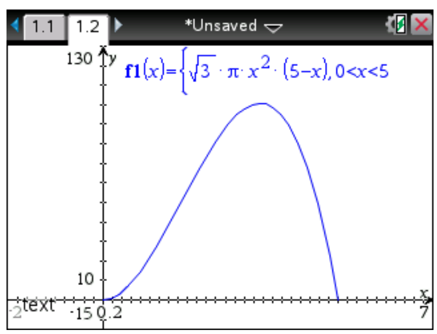
So we now have

$$V : (0, 5) \to R,\ V(r) = \sqrt{3}\pi r^2 (5 - r)$$

Note that since the values of 0 and 5 are not included we used an open, circular bracket as opposed to a closed, square bracket.

Part c.

Firstly, as always it is best to graph the situation to see what we are actually dealing with, so that we don't make any silly mistakes or fall for any tricks. So we graph the volume against the radius (use x when you do this on the calculator), with a domain restriction which we achieve by adding "$|0 < x < 5$" to the end of the function. We then get the curve below



So we observe that our maximum volume will occur at the maximum turning point of the function, that is when the gradient of the function is zero and between $0 < r < 5$. So we need to find the derivative and let it equal zero, then solving for r. We can do this on the calculator using the derivative function, "[⇑ shift] [−]" or we can apply the product rule, as below

$$\begin{aligned} V &= \sqrt{3}\pi r^2 (5 - r) \\ \frac{dV}{dr} &= \sqrt{3}\pi r^2 \times -1 + (5 - r) \times 2\sqrt{3}\pi r \\ &= \sqrt{3}\pi r (2(5 - r) - r) \\ &= \sqrt{3}\pi r (10 - 3r) \\ \frac{dV}{dr} &= 0 \\ \sqrt{3}\pi r (10 - 3r) &= 0 \\ r = 0 \quad &\text{or} \quad 10 - 3r = 0 \\ & \quad\ r = \frac{10}{3} \end{aligned}$$

But $0 < r < 5$, so we can disregard the $r = 0$ solution.

$$\therefore r = \frac{10}{3} \text{ cm}$$

Now we have the value of r for which the maximum occurs, but we need the maximum volume, so we substitute this back into our expression for the volume of the cylinder in terms of r. That is

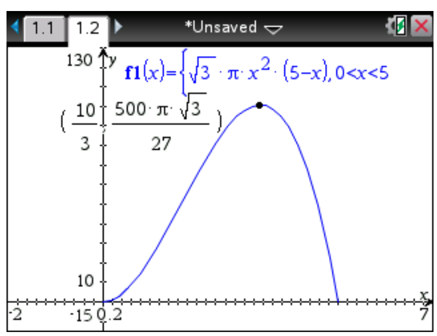
$$\begin{aligned} V_{max} &= V\left(\frac{10}{3}\right) \\ &= \sqrt{3}\pi\left(\frac{10}{3}\right)^2\left(5 - \frac{10}{3}\right) \\ &= \frac{100\sqrt{3}\pi}{9}\left(\frac{15-10}{3}\right) \\ &= \frac{500\sqrt{3}\pi}{27} \text{ cm}^3 \end{aligned}$$

Now since the question did not ask for the answer to a certain amount of decimal places **we leave the answer as an exact value**, otherwise you may lose a mark for not giving it to an exact value when it is specified to in the instructions at the start of section 2.

So the maximum volume of the cylinder is $\dfrac{500\sqrt{3}\pi}{27}$ cm^3 which occurs when $r = \dfrac{10}{3}$ cm.

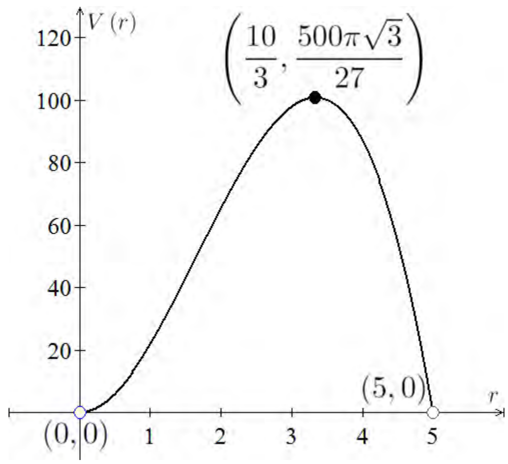
Part d.

We are now asked to graph our function. We need to show the axis intercepts and the maximum volume over an appropriate domain. In this case we have $0 < r < 5$ so **the endpoints are no included** we will need to use **open circles** at our endpoints, while still labeling them. If we had 0 and 5 included we would then use closed circles at the endpoints. To get an idea of the shape of the graph, we plot it on the calculator (if you hadn't already done so from the previous question). We then then get the plot below, this time I've included the maximum value at the turning point.



On a side note it is possible to get exact values when using the trace function on the calculator, as in the graph above. Firstly start "tracing" using "[Menu] [5] [1]". Then click the clickpad to plot any point on the graph. Once you have achieved this click "Esc" to leave "tracing", now you can use the clickpad to hove over the coordinate you have plotted. Click it once until the vertical insert bar comes up, then click again. You can now clear the x value and put in any exact value, and the exact y value will come out.

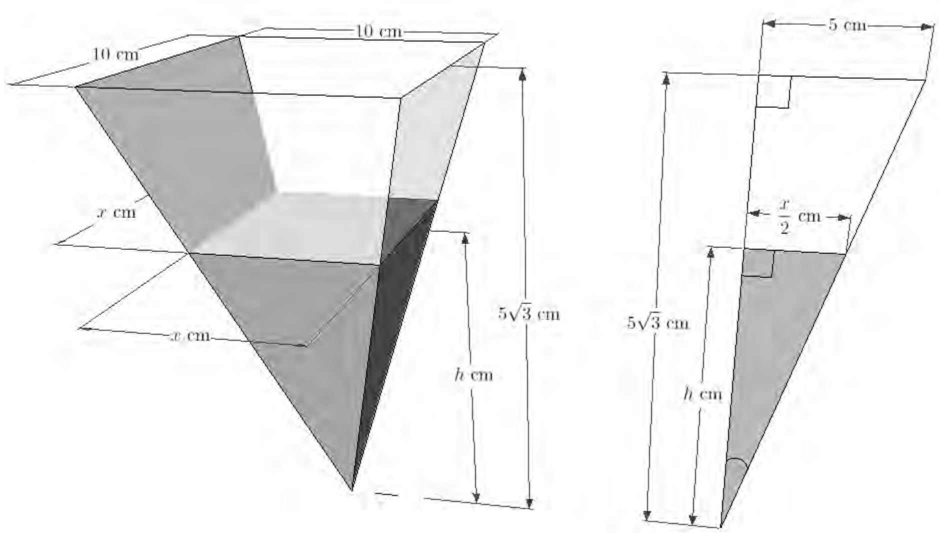
Back to plotting the graph, we will obtain



Note the open circles on the two endpoints and note that we should label the axis r and $V(r)$ **not** x and y as the latter two are not the variables we are dealing with. Also the shape of the graph should show that the graident is zero at $r = 0$.

Part e.

Firstly we note that we have now taken the cylinder out **and the pyramid is the main focus**. So don't make the mistake of using the formula for the volume of a cylinder rather than for the actual pyramid. The pyramid is still inverted, so it will start filling from the tip of the pyramid. So as the water fills up the pyramid, it will take the shape of a smaller pyramid inside the larger one, as in the diagram below.



Now the volume V_p cm^3, of a square based pyramid is given by

$$V_p = \frac{1}{3} \times \text{base area} \times \text{height}$$

But as the water fills up the pyramid, the base area and height both change. But we can find an expression that relates the two, so that we can get the volume in terms of one variable only, just like we did in the previous question except the ratio with the triangles is slightly different. So again we have two similar triangles, and so we can take the ratio of the side lengths.

$$\begin{aligned} \frac{x}{2} \div 5 &= \frac{h}{5\sqrt{3}} \\ x &= \frac{2\sqrt{3}}{3}h \end{aligned}$$

The reason we solve for x instead of h is that since we are looking for $\frac{dh}{dt}$, we will need $\frac{dV_p}{dh}$, so that means we need the volume in terms of h, not x. Then we can substitute this into our formula for the volume.

$$\begin{aligned} V_p &= \frac{1}{3}x^2 h \\ &= \frac{1}{3}\left(\frac{2\sqrt{3}}{3}h\right)^2 h \\ &= \frac{4\times 3}{27}h^3 \\ &= \frac{4}{9}h^3 \end{aligned}$$

Now to find the rate $\frac{dh}{dt}$, we will need to use an application of the chain rule, that is

$$\frac{dh}{dt} = \frac{dh}{dV_p} \times \frac{dV_p}{dt}$$

We already have $\frac{dV_p}{dt}$ so we need to find $\frac{dV_p}{dh}$ by differentiating V_p with respect to h.

$$\begin{aligned} \frac{dV_p}{dh} &= \frac{4\times 3}{9}h^2 \\ &= \frac{4}{3}h^2 \\ \therefore \frac{dh}{dV_p} &= \frac{3}{4h^2},\ h \neq 0 \end{aligned}$$

Now we know that *"Water is now poured into the inverted cone at a rate of* $10\,\text{cm}^3/\text{s}$", that is

$$\frac{dV_p}{dt} = 10\,\text{cm}^3/\text{s}$$

Now when $h = 5\,\text{cm}$

$$\begin{aligned} \frac{dh}{dt} &= \frac{3}{4\times 5^2}\times 10 \\ &= \frac{30}{100} \\ &= \frac{3}{10}\ \text{cm/s} \end{aligned}$$

Now again, since the question did not ask for the answer to a number of decimal places, **we leave it in exact value form**, but $0.3\,\text{cm/s}$ would also be acceptable as this is an exact decimal.

The rate that the height of the water in the pyramid is increasing when $h = 5\,\text{cm}$ is $\frac{3}{10}$ cm/s.

Question 3

Part ai.

Firstly, since this is a probability question, half the battle lies in determining which probability to apply to the situation. The event that we have is "playing an addictive computer game", so we can either have a success or failure to this event, that Rohit can either play the game, or not play the game. Now since whether or not Rohit plays the game on a day **is dependent on the previous day**, we are going to apply markov chains to the situation. So first we need our transition matrix. If we let B be the event that Rohit plays the game on the current day and A be the event that Rohit plays the game on the previous day, then our transition matrix will be

$$T = \begin{bmatrix} \Pr(B|A) & \Pr(B|A') \\ \Pr(B'|A) & \Pr(B'|A') \end{bmatrix}$$

Now we are told that *"The probability Rohit plays an addictive computer game given that the played the game on the previous day is* 0.75", that is we have "B given that A has happened". So this leads to

$$\Pr(B|A) = 0.75$$

We are also told that *"he probability that he does not play on a particular day given that they didn't play on the previous day is* 0.4." That is we have "B' given that A' has happened". That is

$$\Pr(B'|A') = 0.4$$

So our transition matrix becomes

$$T = \begin{bmatrix} 0.75 & \\ & 0.4 \end{bmatrix}$$

Now a required of a transition matrix is that the probabilities in each of the columns must add to 1, this leads to

$$T = \begin{bmatrix} 0.75 & 0.6 \\ 0.25 & 0.4 \end{bmatrix}$$

Now that we have our transition matrix, we need to find our initial state matrix, S_0.

$$S_0 = \begin{bmatrix} \Pr(A) \\ \Pr(A') \end{bmatrix}$$

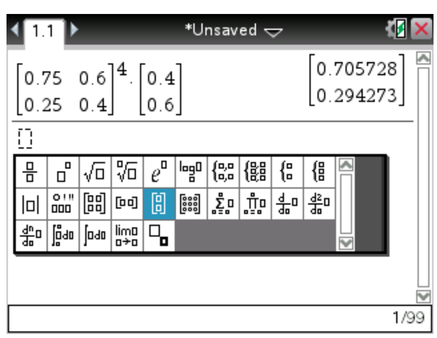
We are told that *"the probability that Rohit doesn't play the game on the first day is* 0.6", that is A'. So our initial state matrix will be

$$S_0 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$$

Now we form out markov chain. Since the first day is our initial state S_0, the second day will be S_1 and the third day will be S_2 and so on. This makes the fifth day S_4. So we have

$$\begin{aligned} S_4 &= T^4 \times S_0 \\ &= \begin{bmatrix} 0.75 & 0.6 \\ 0.25 & 0.4 \end{bmatrix}^4 \times \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \end{aligned}$$

For this we turn to the calculator, we can access the matrix menu by pressing the "$\begin{bmatrix} |\square| & \begin{cases} \square \\ \square \end{cases} \end{bmatrix}$" button on the right hand side of the calculator, then selecting either "$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$" or "$\begin{bmatrix} \square \\ \square \end{bmatrix}$" matrix depending on which one we need.



This gives $S_4 = \begin{bmatrix} 0.7057 \\ 0.2942 \end{bmatrix}$

Now the top row of the matrix corresponds to our success, that is that Rohit plays the game on the fifth day while the bottom row corresponds to our failure, that he does not play the game on the fifth day. So the probability that Rohit plays the game on the fifth day is 0.7057.

Part aii.

If we have the transition matrix $T = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$

Then our steady state probabilities will be

$$\begin{aligned} \Pr(\text{"Success"}) &= \frac{b}{a+b} \\ \Pr(\text{"Failure"}) &= \frac{a}{a+b} \end{aligned}$$

So in our case we have $a = 0.25$ and $b = 0.60$. So our long run probabilities will be

$$\begin{aligned} \Pr(\text{"Success"}) &= \frac{0.60}{0.25+0.60} \\ &= \frac{12}{17} \end{aligned}$$

So the long run probability that Rohit plays the game is $\frac{12}{17}$.

Part bi.

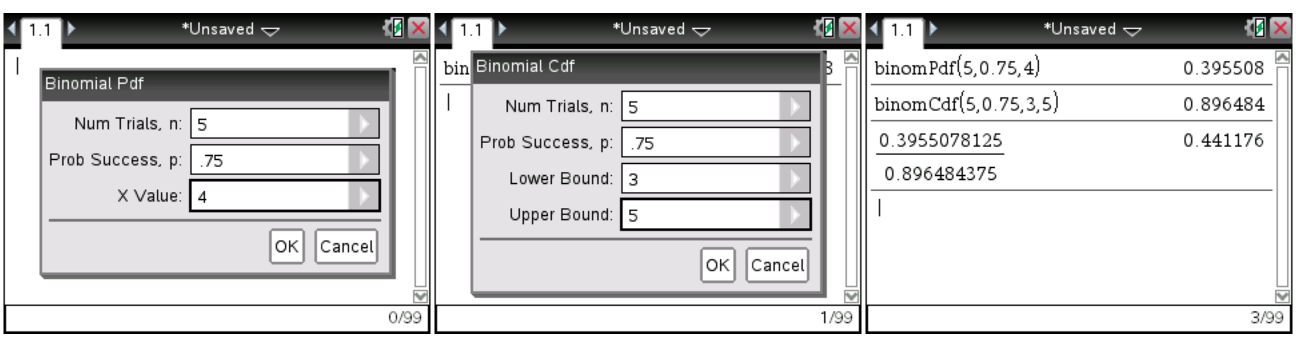
Now we need to note that the type of probability we are going to apply to this situation is different from the part a stems of the question. We now have Steph whose probability of playing on any day **is independent of any other day**. That is the outcomes on the other days have no effect on the outcome on the current day. Now since Steph can either "play the game" or "not play the game", we still have a success and failure situation. So we can apply a binomial distribution to the situation. Lets let X be the number of days on which Steph plays the game. Our probability of success is 0.75 and we are running the situation for 5 days, so that is $p = 0.75$ and $n = 5$. So that is

$$X \sim \text{Bi}(5, 0.75)$$

But we are interested in the probability that she plays for 4 out of the 5 days **given** that she plays for at least 3 days. So we are looking for $\Pr(X = 4 | X \geq 3)$. So we will need to apply conditional probability, that is

$$\begin{aligned} \Pr(X = 4 | X \geq 3) &= \frac{\Pr(X = 4 \cap X \geq 3)}{\Pr(X \geq 3)} \\ &= \frac{\Pr(X = 4)}{\Pr(X \geq 3)} \end{aligned}$$

Now since $\Pr(X = 4)$ is only for one value of X, we can use the binomial probability distribution function, "[Menu] [5] [5] [D]". But for $\Pr(X \geq 3)$ we have more than one value of X, so we will need to apply the binomial cumulative probability distribution function, "[Menu] [5] [5] [E]".



$$\begin{aligned} \Pr(X = 4 | X \geq 3) &= \frac{0.395508}{0.896484} \\ &= 0.4412 \end{aligned}$$

So the probability that Steph plays on 4 days given that she played on at least 3 days is 0.4412.

Part bii.

Now the difference between this part and the previous question is that we are now looking for the value of n, that is the number of trials to make $\Pr(\mathrm{X} \geq 2) > 0.6$. That is we have

$$X \quad \sim \quad \mathrm{Bi}(n, 0.75)$$

Now the problem we run into here is that $X \geq 2$ is unbounded, so we won't be able to solve anything on the calculator. But we can look at the situation in reverse, and we know that $X \geq 0$ as we cannot have negative days. So

$$\Pr(X < 2) \quad = \quad \Pr(X = 0) + \Pr(X = 1)$$

So if we are looking at the reverse situation, we need to change the probability to be for the opposite situation. That is

$$\begin{aligned} \Pr(X \geq 2) &> 0.75 \\ 1 - \Pr(X < 2) &> 0.75 \\ \Pr(X < 2) &< 0.25 \end{aligned}$$

Remember to flip the inequality sign as we are dividing by a negative number. So now we use our result from before

$$\Pr(X = 0) + \Pr(X = 1) \quad < \quad 0.25$$

Now we need to apply the binomial distribution. If we have n trials with the probability of a success being p, and we want $\Pr(X = x)$ then we have

$$\Pr(X = x) \quad = \quad \frac{n!}{x!\,(n-x)!}(p)^x (1-p)^{n-x}$$

So in our case that is

$$\frac{n!}{0!n!}(0.75)^0 (0.25)^n + \frac{n!}{1!\,(n-1)!}(0.75)^1 (0.25)^{n-1} \quad < \quad 0.25$$

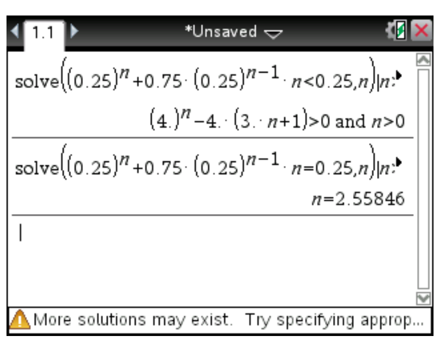
Now if we look at the expressions we have

$$\begin{aligned} \frac{n!}{0!n!} &= 1 \\ \frac{n!}{1!\,(n-1!)} &= \frac{n \times (n-1) \times (n-2) \times (n-3) \times \ldots}{(n-1) \times (n-2) \times (n-3) \times \ldots} \\ &= n \end{aligned}$$

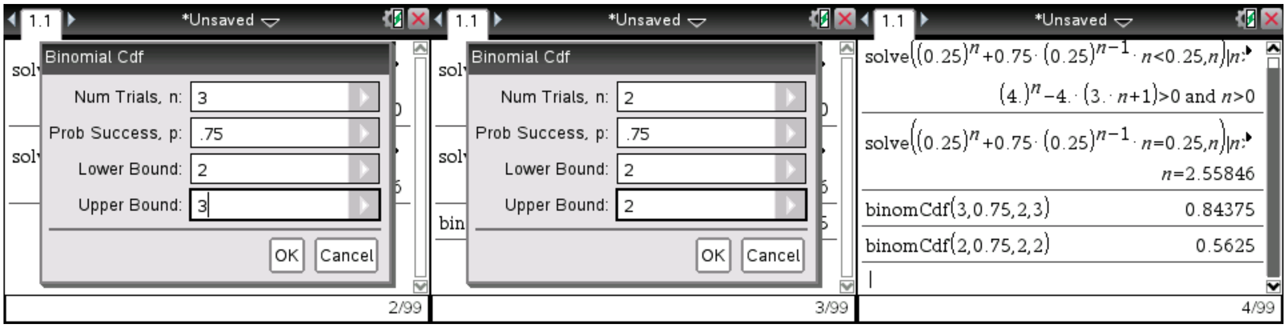
So that means our expression above becomes

$$0.25^n + 0.75 \times 025^{n-1} n \quad < \quad 0.25$$

Now we solve this on the calculator using the solve function, "[Menu] [3] [1]". But we replace the $<$ with an equals sign, otherwise we will not get a solution on the calculator.



Now we need a whole number for our value of n. If we round down then our condition will not be satisfied, but if we round up it will be. We can check this using the binomial cumulative distribution function, "[Menu] [5] [5] [E]".



$$\therefore n = 3$$

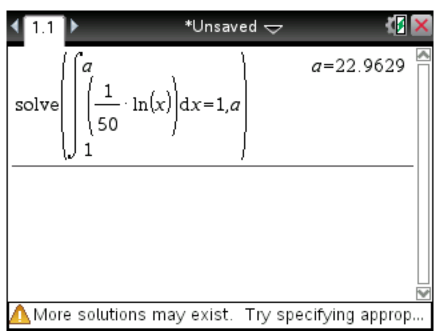
So we will need 3 days for the probability that Steph plays the game for at least 2 days to be greater than 0.75.

Part ci.

There are two key features to a continuous probability distribution are that the curve can never be negative as we cannot have negative probabilities. Also the probabilities have to add to 1, that is the area under the curve has to equal 1. We can use this fact to find the value of a. That is

$$\int_1^a \left(\frac{1}{50}\log_e(x)\right) dx = 1$$

We then use the solve and integral function on the calculator, "[Menu] [3] [1]" and "[⇑ shift] [+]", to find our value of a.



$$\therefore a = 22.9629$$

Part cii.

The mean of a continuous distribution is given by

$$\mu = \int_{-\infty}^{\infty} (xf(x))\,dx$$

While the variance is given by

$$\sigma^2 = \int_{-\infty}^{\infty} \left((x-\mu)^2 f(x)\right) dx$$

But we can expand this out to find an easier form. That is

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} \left(\left(x^2 - 2x\mu + \mu^2\right) f(x)\right) dx \\
&= \int_{-\infty}^{\infty} x^2 f(x) - 2\mu \int_{-\infty}^{\infty} xf(x)\,dx + \mu^2 \int_{-\infty}^{\infty} f(x)\,dx \\
\text{But } \mu &= \int_{-\infty}^{\infty} (xf(x))\,dx \\
\int_{-\infty}^{\infty} f(x)\,dx &= 1 \\
\text{So } \sigma^2 &= \mathrm{E}\left(X^2\right) - 2\mu^2 + \mu^2 \\
\therefore \mathrm{Var}(X) &= \mathrm{E}\left(X^2\right) - \mu^2
\end{aligned}$$

It should be noted that the above working is not required, but is there to let you realise how we come to

$$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2$$

Now we know that the standard deviation of the distribution is the square root of the variance, so

$$\sigma(X) = \sqrt{\mathrm{Var}(X)}$$

So firstly we need to find the mean. That is

$$\begin{aligned}
\mu &= \int_{1}^{22.9629} \left(x \times \frac{1}{50}\log_e(x)\right) dx \\
&= 13.8933
\end{aligned}$$

Then to find the standard deviation, we also need $\mathrm{E}\left(X^2\right)$, that is

$$\begin{aligned}
\mathrm{E}\left(X^2\right) &= \int_{-\infty}^{\infty} \left(x^2 f(x)\right) dx \\
&= \int_{1}^{22.9629} \left(x^2 \times \frac{1}{50}\log_e(x)\right) dx \\
&= 226.0640
\end{aligned}$$

So our variance is then given by

$$\begin{aligned}
\sigma^2 &= 226.0640 - 13.8933^2 \\
&= 33.0425
\end{aligned}$$

Thus our standard deviation is

$$\begin{aligned}
\sigma &= \sqrt{33.0.425} \\
&= 5.7483
\end{aligned}$$

So the mean and standard deviation of the distribution are 13.89 and 5.75 respectively.

Question 4

Part a.

The first step we take here is to find $f(x)$ so that we can find the curve of the battle line. That is we need to integrate $f'(x)$, but when we do this we find a family of antiderivatives, that is we need to add a constant, C.

$$\begin{aligned}
\int f'(x)\,dx &= \int \left(\frac{1}{30}x\left(4x^2 - 3(c+10)x + 20c\right)\right)dx \\
&= \frac{1}{30}\int \left(4x^3 - 3(c+10)x^2 + 20cx\right)dx \\
&= \frac{1}{30}\left(x^4 - (c+10)x^3 + 10cx^2\right) + C
\end{aligned}$$

Now we know that the curve passes through $(0,0)$, so we can use this to find our value of C.

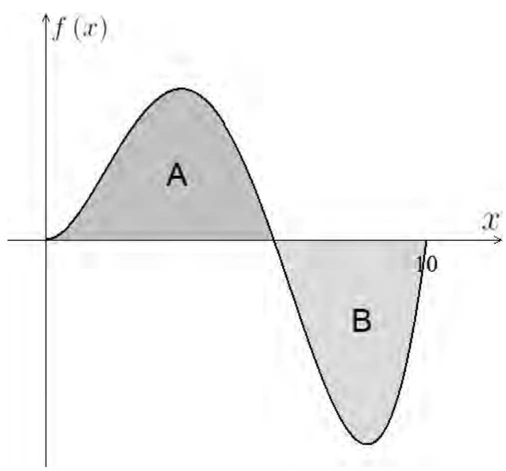
$$\begin{aligned}
f(0) &= 0 \\
0 &= C \\
\therefore C &= 0 \\
\therefore f(x) &= \frac{1}{30}\left(x^4 - (c+10)x^3 + 10cx^2\right)
\end{aligned}$$

Part b.

Now we need to equate the area of the two areas, but for this we need the terminals for our integrals. To do that we need to find the first intercept. So factorising $f(x)$ and equating to zero gives

$$\begin{aligned}
\frac{1}{30}\left(x^4 - (c+10)x^3 + 10cx^2\right) &= 0 \\
x^2\left(x^2 - (c+10)x + 10x\right) &= 0 \\
x^2(x-10)(x-c) &= 0 \\
x = 0 \quad \text{or} \quad x - 10 &= 0 \text{ or } x - c = 0 \\
x = 0 \quad \text{or} \quad x &= 10 \quad \text{or} \quad x = c
\end{aligned}$$

So if we plot the curve we can work out which intercepts we are needing to look at.



So we can tell that the middle intercept here will be c. So the terminals for the integral for region will be from $x = 0$ to $x = c$. Now since $f(x) < 0$ for $c < x < 10$, if we were to integrate in this region then we would get a negative area. So we need to put a minus in from of the integral for region B. So our terminals for region B will be from $x = c$ and $x = 10$. So we have

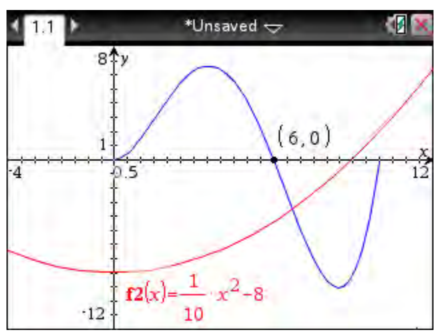
$$\begin{aligned}
\frac{1}{30}\int_0^c \left(x^4 - (c+10)x^3 + 10cx^2\right) dx &= -\frac{1}{30}\int_c^{10} \left(x^4 - (c+10)x^3 + 10cx^2\right) dx \\
\left[\frac{1}{5}x^5 - \left(\frac{c+10}{4}\right)x^4 + \frac{10}{3}cx^3\right]_0^c &= -\left[\frac{1}{5}x^5 - \left(\frac{c+10}{4}\right)x^4 + \frac{10}{3}cx^3\right]_c^{10} \\
\frac{1}{5}c^5 - \frac{1}{4}(c+10)c^4 + \frac{10}{3}c^4 - 0 &= -\left(\frac{1}{5}(10)^5 - \frac{10^4}{4}(c+10) + \frac{10^4}{3}c - \left(\frac{1}{5}c^5 - \frac{1}{4}(c+10)c^4 + \frac{10}{3}c^4\right)\right) \\
20000 - 2500(c+10) + \frac{1000}{3}c &= 0 \\
\frac{2500}{3}c &= 5000 \\
\therefore c &= 6
\end{aligned}$$

We can also solve this on the calculator using the define function and the solve function, "[Menu] [1] [1]" and "[Menu] [3] [1]".

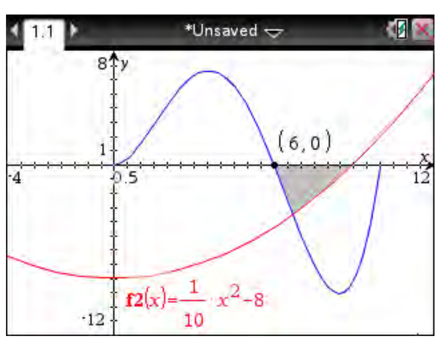
Part c.

Now firstly, again we need to picture our situation again, but now that we know our value of c we can plot $f(x)$ on the calculator as well as $g(x)$.

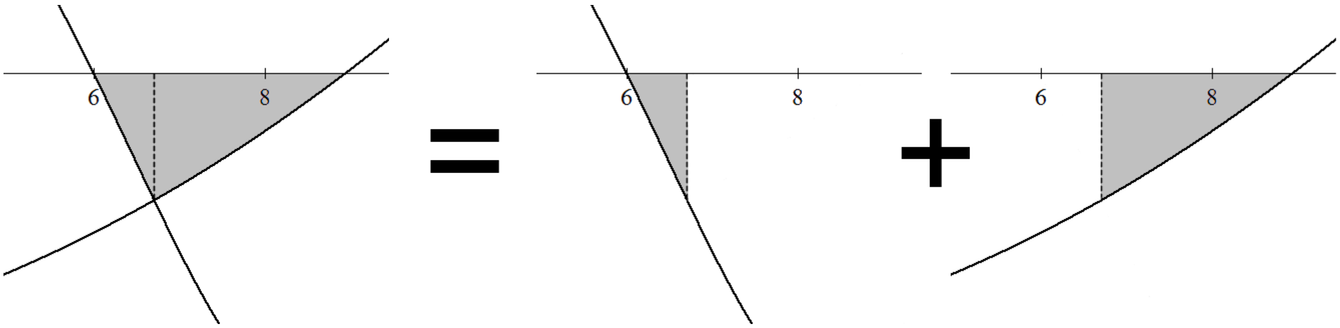
$$\begin{aligned}
f(x) &= \frac{1}{30}x^2(x-10)(x-6) \text{ for } 0 \le x \le 10 \\
g(x) &= \frac{1}{10}x^2 - 8
\end{aligned}$$



Now the area of the region we are interested in is the region that is bounded by the x-axis, the curve $f(x)$ and the curve $g(x)$. That is the shaded region below



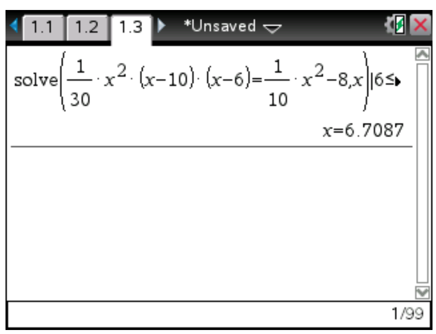
Now to find this region, we will need to split our integral up into two sections. The first will integrate the function that is closest to the axis from the x intercept at $(6,0)$ up to the intersection of the two curves, while the second is from this intersection up until the x intercept of $g(x)$. The reason we integrate the curve that is closest to the x axis is depicted in the diagram below. Also as the regions are below the x-axis we will need a minus in front of the integrals to that we get a positive area.



So now we need to find our terminals, we already have the first one, $x = 6$. Next we need to find the intersection of the two curves, that is

$$\begin{aligned}
\frac{1}{30}x^2(x-10)(x-6) &= \frac{1}{10}x^2 - 8 \\
x^4 - 16x^3 + 60x^2 &= 3x^2 - 240 \\
x^4 - 16x^3 + 57x^2 + 240 &= 0 \\
\therefore x &= 6.7087
\end{aligned}$$

We can solve this on the calculator by using the solve function, "[Menu] [3] [1]" and adding a domain restriction using "$|6 \leq x \leq 10$".



We also note that the final terminal will be the positive intercept of the parabola. That is

$$\begin{aligned}
\frac{1}{10}x^2 - 8 &= 0 \\
x &= \pm\sqrt{80}
\end{aligned}$$

But in this case we are looking at the positive intercept

$$\therefore x = 4\sqrt{5}$$

So for now we have

$$\begin{aligned}
\text{Area} &= -\int_6^{6.7087} \frac{1}{30}\left(x^4 - 16x^3 + 60x^2\right) dx - \int_{6.7087}^{4\sqrt{5}} \left(\frac{1}{10}x^2 - 8\right) dx \\
&= 1.2349 + 4.0977 \\
&= 5.3326
\end{aligned}$$

Again, we calculate this using the integral function on the calculator, that is "[⇑ shift] [+]"

Now that is the area of the enemies territory that they can fire in, but we are asked for the **percentage** of the enemies territory. So we need the total area of the enemies territory. That is

$$\begin{aligned} \mathrm{A}_2 &= -\int_6^{10} \frac{1}{30}\left(x^4 - 16x^3 + 60x^2\right) dx \\ &= 23.0400 \end{aligned}$$

So the percentage is

$$\begin{aligned} \%\ \text{Area} &= \frac{5.3326}{23.0400} \times 100 \\ &= 23.15\% \end{aligned}$$

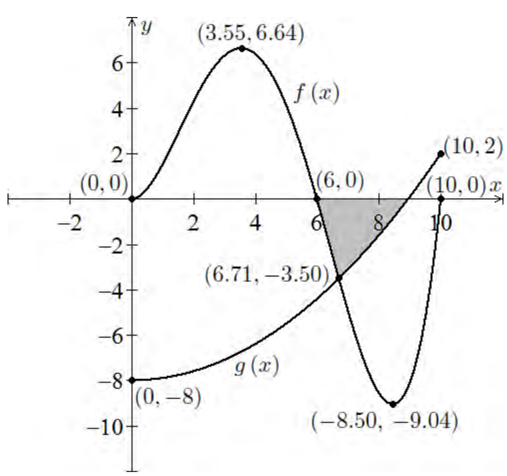
That is they can fire into 23.15% of the enemies territory.

Part d.

We have most of the information we need to graph the two functions, except we need to find the stationary points of f. We are given the derivative right before part a, and since the gradient of a curve at a stationary point is 0, we can equate this to zero and solve for x.

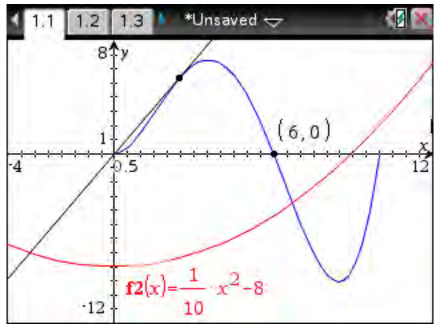
$$\begin{aligned} f'(x) &= \frac{1}{30}x\left(4x^2 - 48x + 120\right) \\ &= \frac{2}{15}x\left(x^2 - 12x + 30\right) \\ \frac{2}{15}x\left(x^2 - 12x + 30\right) &= 0 \\ x\left(x^2 - 12x + 6^2 - 6^2 + 30\right) &= 0 \\ x\Big((x-6)^2 - 6\Big) &= 0 \\ x\left(x - 6 - \sqrt{6}\right)\left(x - 6 + \sqrt{6}\right) &= 0 \\ x &= 0,\ 6 \pm \sqrt{6} \\ &= 0,\ 3.55,\ 8.50 \\ f(3.55) &= 6.64 \\ f(8.50) &= -9.04 \\ g(0) &= -8 \\ g(10) &= 2 \end{aligned}$$

When we plot the two curves, we need to draw the curve in a way that shows that the gradient is zero for both curves at $x = 0$.



Part e.

Since our linear line starts at the origin, we can represent it as $y = mx$ where m is the gradient of the line. Now since m is our gradient, we can equate it to the derivative of f, and then we are looking for the point of intersection between this line and the curve for f.



$$\begin{aligned} xf'(x) &= f(x) \\ \frac{2}{15}x\left(x^2-12x+30\right) &= \frac{1}{30}x^2(x-10)(x-6) \\ \frac{1}{15}\left(2x^3-24x^2+60x\right) &= \frac{1}{30}\left(x^4-16x^3+60x^2\right) \\ \therefore x &= 0,\ 2.4274,\ 8.2393 \end{aligned}$$

Now since we know that the peasants started in the region for $0 \leq x \leq 6$, we know that they must re-enter their own region, so we can disregard the first and the last solutions, leaving only $x = 2.4274$. Now this is not the gradient at that point, we still need to find that, so we have

$$\begin{aligned} f(x) &= \frac{1}{30}(2.4274)^2(4.4274-10)(4.4274-6) \\ &= 5.3136 \\ & (2.43, 5.31) \end{aligned}$$

Now to find the distance that they run, we make use of the distance formula between two points (x_1, y_1) and (x_2, y_2).

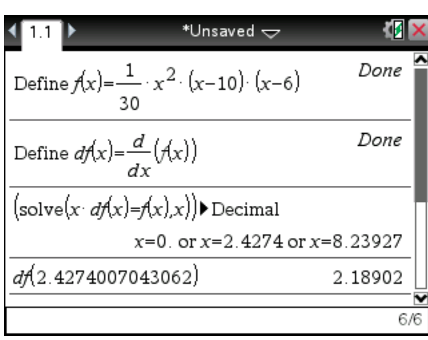
$$|AB| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

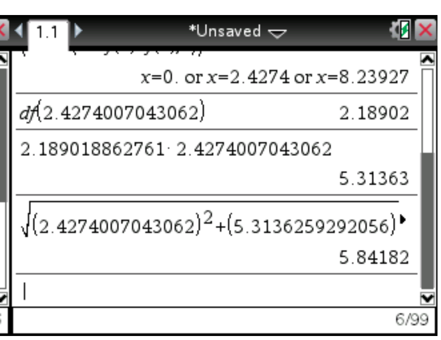
But since one of our points is at the origin, this becomes

$$\begin{aligned} |OB| &= \sqrt{x^2+y^2} \\ \text{distance} &= \sqrt{2.43^2+5.31^2} \\ &= 5.84 \text{ km} \end{aligned}$$

So the peasants travel a distance of 5.84 km to the point $(2.43, 5.31)$

This could all be achieved on the calculator, making use of defining a function "[Menu] [1] [1]", the derivative function "[⇑ shift] [−]", the solving function "[Menu] [3] [1]" and the copy function "[ctrl] + [c]", as below.





SET 2 EXAM 1

DETAILED SOLUTIONS

Question 1

Part a.

This is a standard opening question on an Exam 1, they usually start off with a differentiation question. If we look at what we're being asked to differentiate, we can see that it is a **composite function**, because we have a function, $x\log_e(x)$, inside a cos(...). Out of the three differentiation rules - the **chain rule, product rule** and **quotient rule**, the chain rule is the one which is used for composite functions. We need to recall the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

What this rule tells us is that the derivative of a composite function is simply the 'derivative of the outside function' times the 'derivative of the inside function'. Here, the outside function is a cos(...) function, so we should let whatever is inside it be u.

$$\text{Let}\, u = x\log_e(x)$$

Now we have our two functions, so we can start off with finding the derivative of the outside function:

$$\frac{d}{du}(\cos(u)) = -\sin(u) = -\sin\left(x\log_e(x)\right)$$

Then we can find the derivative of the inside function, but as we're about to see, that is going to be a bit of a challenging task. What we want is:

$$\frac{d}{dx}\left(x\log_e(x)\right)$$

However, this is not a simple function, it is actually a **product** - the product of x and $\log_e(x)$ so we are going to have to use the **product rule** in order to solve it. Recall the product rule, remembering that the u here is different to the u used before in the chain rule:

$$\frac{d}{dx}(u \times v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

The simple way to remember this is the derivative of one function times the other, plus the derivative of the other times the initial one. Thus, using this rule, we can solve the question:

$$\frac{d}{dx}\left(x\log_e(x)\right) = x \times \frac{1}{x} + 1 \times \log_e(x)$$

Remember that the derivative of the natural logarithm function can be found on the supplied formula sheet, but of course, it is best to memorise the basic derivatives and integrals for efficiency. From here, we can go back and substitute it all in, multiplying the 'derivative of the outside function' by what we just found (i.e. the 'derivative of the inside function').

$$\frac{d}{dx}\left(\cos\left(x\log_e(x)\right)\right) = -\sin\left(x\log_e(x)\right)\left(\log_e(x) + 1\right)$$

With these questions, experience and speed will help us solve them more quickly. Once we become proficient, it is no longer necessary to even think about 'chain rule' or 'product rule', we will instinctively know what to do. Looking at this question, it would be best to recognise the composite function, find the derivative of the outside function, recognise the inside function requires the product rule, so use it and then multiply the two.

Part b.

We get another differential calculus question here. This time we are being asked to find the normal and we are given the x-coordinate only. So first off, we should know that sooner or later, we will need to find the corresponding y-coordinate so we should start off with finding that. Personally I find it nice to start off with the easier tasks and then progress to the harder ones as it minimises our margin for error.

So we will need to substitute $x = \pi$ into our equation in order to get our y-value:

$$y = 2\sin^3(x) + 4x + 11$$

$$x = \pi,\ y = 4\pi + 11$$

Now we know that the coordinate required is $(\pi,\ 4\pi + 11)$ so we can continue on and find the derivative. We should know that for tangent and normal questions, the gradient is required. Thus, we will need to find the gradient, which is given by the **derivative**. This is a function with many terms added together, so we will need to go through and differentiate each term.

The first term is the trickiest because it requires the chain rule, which we looked at in the previous question. We know that the chain rule is required because we can see a composite function. What $\sin^3(...)$ means is that it is the $\sin(...)$ to the power of three. Thus, the 'outside function' here is the cubed and the 'inside function' here is the sin. So, use our chain rule:

$$\frac{d}{dx}(2\sin^3(x)) = 6\sin^2(x) \times \cos(x)$$

Now we can just differentiate the other terms and add in order to get our derivative function. Remember that the derivative of a constant is zero. We should also know how to differentiate polynomials quickly.

$$\frac{dy}{dx} = 6\sin^2(x)\cos(x) + 4$$

Substitute in the relevant x-value in order to get the gradient at the point. Note that this is the gradient of the curve, which is the gradient of the **tangent** and not the gradient of the **normal** which is what we're after:

$$x = \pi,\ \frac{dy}{dx} = 4$$

We need to memorise our exact values for the trigonometric ratios for Exam 1 because they are not provided. What we can do now is use the fact that perpendicular gradients multiply to give -1 to find the gradient of the normal. Recall that the tangent and the normal at a point are perpendicular:

$$m_N \times \frac{dy}{dx} = -1$$

$$\therefore m_N = -\frac{1}{4}$$

From here, there are two methods to solve this question. We should be familiar with both the methods to find the equation of a straight line. We can substitute a point and the gradient into the formula and solve for c:

$$y = mx + c$$

Alternatively, it is probably slightly faster to use the point-gradient formula:

$$y - y_1 = m(x - x_1)$$

If we use that method, we will substitute in our point, $(\pi,\ 4\pi + 11)$, and the gradient of the normal we just found. So substituting in and simplifying:

$$y - 4\pi - 11 = -\frac{1}{4}(x - \pi)$$

$$y = -\frac{1}{4}x + \frac{17\pi}{4} + 11$$

Question 2

This question requires us to find the original function, given the derivative. We should be familiar with the concept of antidifferentiation being the opposite process of differentiation. What this means is that in order to find the original function, we will need to integrate the derivative function. So let's start with that:

$$\begin{aligned} g'(x) &= \cos(2x) - 2\sin(x) \\ \therefore\ g(x) &= \int (\cos(2x) - 2\sin(x))dx \\ &= \frac{1}{2}\sin(2x) + 2\cos(x) + c \end{aligned}$$

It is extremely important to remember to include the constant of integration every time we are performing an integration, unless it is a definite integral. Even though the constant of integration may end up working out to be zero, never assume that would be the case. So now that we have the original function, we will need to use the point we have been given in order to solve for c:

$$g(0) = 1$$

$$\therefore 1 = 2 + c$$

$$\therefore c = -1$$

Now we have our final answer of:

$$g(x) = \frac{1}{2}\sin(2x) + 2\cos(x) - 1$$

Let's take a moment, however, to have a look at the implications of the constant of integration. What the constant of integration tells us is whether any vertical translations have occured. Have a think about a common curve such as $y = x^2$. Now, imagine shifting it up and down vertically, all shifted graphs will be a different function, however at any particular x-value, they will have the same gradient. Thus, they have the same derivative, despite being different functions. That's why we need the extra piece of information to work out our constant and figure out the exact function.

Question 3

Part a.

Inverse function questions are very common and generally are not difficult, however, we need to be extremely particular about the notation. This function here is a logarithmic function. We should recall that logarithmic functions and exponential functions are inverse pairs, so we should make a mental note to expect an exponential function after we have found the inverse. In order to find the inverse, it is always easier to let the function be y:

$$\text{Let } y = f(x) = 2\log_e(5x) - 2$$

Next, we will need to swap x and y in order to find the inverse. Think about what is happenning, the inverse is essentially the graph reflected in the line $y = x$, meaning that what was once the x-value at a point is now the y-value and what was once the y-value is now the x-value. So, we can swap the x and y to get:

$$x = 2\log_e(5y) - 2$$

From here, we will need to solve for y:

$$\begin{aligned} x &= 2\log_e(5y) - 2 \\ \implies \frac{x+2}{2} &= \log_e(5y) \\ \implies e^{\frac{x+2}{2}} &= 5y \\ \implies y &= \frac{1}{5}e^{\frac{x+2}{2}} \end{aligned}$$

However, the question was in terms of $f(x)$ and we were asked to find $f^{-1}(x)$, so it is best to revert back to that notation for the answer:

$$f^{-1}(x) = \frac{1}{5}e^{\frac{x+2}{2}}$$

Part b.

We should know that the domain of an exponential function is $\mathbb{R}$, the set of all real numbers. We can find this by imagining an exponential function. We can see that it extends indefinitely in both the left and the right direction. We should know from our methods knowledge that the domain of f^{-1} is the same as the range of f and the range of f^{-1} is the same as the domain of f.

It is also possible to consider that we can have e to the power of anything, negative numbers, zero, positive numbers, large numbers, small numbers... etc. They will all give an answer, never will we get an undefined answer. The domain of f^{-1} is $x \in \mathbb{R}$.

Part c.

First of all, there are two ways we can approach this question, we can do it using first principles or we can do it using recognition. If we decide to do it using first principles, we will have to change x and y in the transformed function into x' and y'. We should get:

$$y' = \frac{1}{5}e^{\frac{x'+2}{2}}$$

We can then rearrange it to get:

$$5y' = e^{\frac{x'+2}{2}}$$

Thus, if we compare it to the function $y = e^x$, we can see that we will have:

$$\begin{aligned} 5y' &= y \\ \frac{x'+2}{2} &= x \end{aligned}$$

Rearranging, we will get:

$$\begin{aligned} y &= \frac{y'}{5} \\ x &= 2x' - 2 \end{aligned}$$

From the first equation, we can see a dilation by a factor of $\frac{1}{5}$ away from the x-axis. For the second equation, we can see that it is a dilation and a translation. Dilation by a factor of 2 away from the y-axis and translation of 2 units horizontally to the left.

We can also do this using recognition, seeing that the $\frac{1}{5}$ in front means that it has been dilated by a factor of $\frac{1}{5}$ away from the x-axis. We will now need to tackle the x-values, we can see there's a dilation by a factor of 2 away from the y-axis, recall that a dilation by a factor of n means x is multiplied by $\frac{1}{n}$. What we see now is the translation of 2 units left. Remember that a positive means left and negative means right.

- Dilation by a factor of $\frac{1}{5}$ away from the x-axis.
- Dilation by a factor of 2 away from the y-axis.
- Translation of 2 units horizontally to the left.

Question 4

A general solution is one that is not restricted over a domain. A trigonometric function is periodic in that it will extend indefinitely towards both the left and right side, repeating itself over and over again. What this means is that it will have an **infinite** number of x-axis intercepts, provided that it has at least one. Of course, we can have certain functions which will have no x-axis intercepts - e.g. $f(x) = \sin(x) + 3$.

For this question here, we are asked to find the general solution of a tricky-looking trigonometric function. My most important piece of advice regarding equations which involve both sin and cos is that they either must all be converted to sin or all be converted to cos, or possibly a tan (by division).

Here, we can see that we have a $\sin^2(x)$ and a $\cos(x)$. This should signal to us that the Pythagorean identity is required in order to convert that $\sin^2(x)$ term into a $\cos^2(x)$ term. After this, it should be apparent that the trigonometric equation will present itself as a quadratic:

$$\begin{aligned} 2\sin^2(x) - 1 &= \cos(x) \\ \implies 2(1-\cos^2(x)) - 1 &= \cos(x) \\ \implies 1 - 2\cos^2(x) &= \cos(x) \\ \implies 2\cos^2(x) + \cos(x) - 1 &= 0 \end{aligned}$$

Since we can now see that it is a quadratic, what we want to do is leave the trigonometric bits for now and focus on solving the quadratic. So this is a quadratic in terms of $\cos(x)$. We can begin by trying to replace it by another variable so we isolate the quadratic on its own. After that, it becomes easy to solve the quadratic.

$$\text{Let } A = \cos(x)$$

$$\begin{aligned} 2A^2 + A - 1 &= 0 \\ \implies (2A-1)(A+1) &= 0 \\ \implies A &= \frac{1}{2} \text{ or } -1 \end{aligned}$$

Here, what this tells us is that $\cos(x)$ can either equal $\frac{1}{2}$ or -1. So we have two easier trigonometric equation to solve. First of all, let's solve the first case - we need to remember that the general solution for a cos function will be:

$$\begin{aligned} \cos(x) &= a \\ x &= 2k\pi \pm \cos^{-1}(a) \end{aligned}$$

Remember that of course, k is an integer, so that must be stated as $k \in \mathbb{Z}$. We can now solve the first case:

$$\cos(x) = \frac{1}{2} \implies x = \pm\frac{\pi}{3} + 2k\pi \;\; k \in \mathbb{Z}$$

For the second case, we don't need to apply the same concept because we will know that $\cos(x)$ will equal -1 for all odd multiples of π - i.e. π, 3π, 5π..., thus, we can represent this by saying:

$$\cos(x) = -1 \Rightarrow x = (2k+1)\pi \;\; k \in \mathbb{Z}$$

So our answer is:

$$\therefore x = 2k\pi \pm \frac{\pi}{3} \text{ or } (2k+1)\pi \;\; k \in \mathbb{Z}$$

Question 5

Part a.

This question requires us to know that the area under a probability density function is 1. This arises because we know, from earlier years, that if we add up the probabilities of all possible events, we will end up with 1. We should also know that the area under a PDF between two intervals gives the probability that the random variable will fall between those two intervals. Looking at the hybrid function, we can see that what we are interested in is the $a\sin(x)$ from 0 to π part, because the PDF is zero elsewhere, meaning that it will not contribute to the area (i.e. the random variable cannot fall in any of those areas). Thus, we will need to evaluate the area under $a\sin(x)$ from 0 to π and equate that to 1 to get our answer.

We should recall that the anti-derivative of the $\sin(x)$ function is $-\cos(x)$. This can be found on the formula sheet if required. There are two ways which we can deal with the negative sign. We can either put the negative sign in once we have found the anti-derivative, or we can simply swap the terminals, which is the same as negating. Personally, I prefer to swap the terminals rather than to put the negative sign in because it's less work, not having to deal with the negative at the end. From here, we can just evaluate by substituting in our terminals, then letting that equal 1 and solve for a.

$$\begin{aligned}
\text{Area} &= \int_0^\pi (a\sin(x))dx \\
&= \big[a\cos(x)\big]_\pi^0 \\
&= a(\cos(0) - \cos(\pi)) \\
&= a(1+1) \\
\therefore 2a &= 1 \\
a &= \frac{1}{2}
\end{aligned}$$

Part b.

This question continues on from the last part, again, recall that the probability that the random variable X falls below $\frac{\pi}{3}$ is given by the area under the PDF from $-\infty$ to $\frac{\pi}{3}$ however, like we discussed in the previous part, there is no need to actually evaluate anything except the sin function because the probability is zero elsewhere, so we only really need to consider the inverval from 0 to $\frac{\pi}{3}$. We already have the anti-derivative from the last part, so all we need to do is substitute in the new bounds and work through to get the answer. Remember that we already know our value of a which was found in the last question.

$$\begin{aligned}
\Pr\left(X < \frac{\pi}{3}\right) &= \int_0^{\frac{\pi}{3}} \left(\frac{1}{2}\sin(x)\right)dx \\
&= \frac{1}{2}\left(\cos(0) - \cos\left(\frac{\pi}{3}\right)\right) \\
&= \frac{1}{4}
\end{aligned}$$

Question 6

Part a.

If we look at what we're being asked to differentiate, we can see that it is a **composite function**, because we have a function, $\log_e(x)$ inside a $\sin(...)$. Out of the three differentiation rules - the **chain rule, product rule** and **quotient rule**, the chain rule is the one which is used for composite functions. We need to recall the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

What this rule tells us is that the derivative of a composite function is simply the 'derivative of the outside function' times the 'derivative of the inside function'. Here, the outside function is a $\sin(...)$ function, so we should let whatever is inside it be u.

$$\text{Let } u = \log_e(x)$$

Now we have our two functions, so we can start off with finding the derivative of the outside function:

$$\frac{d}{du}(\sin(u)) = \cos(u) = \cos\left(\log_e(x)\right)$$

Then we can find the derivative of the inside function. What we want is:

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$$

That derivative is on the formula sheet if required. Now we just need to multiply the two in order to get our final answer:

$$\frac{dy}{dx} = \frac{1}{x}\cos\left(\log_e(x)\right)$$

Part b.

This is an integration by recognition. The giveaway here is the **hence**, meaning that the result from the previous part must be used in order to complete this question. However, the problem here is that we have a sin instead of a cos. However, there is conveniently an identity which we can use in order to solve this, it's called the **complementary** identity:

$$\sin\left(\frac{\pi}{2} - A\right) = \cos(A)$$

This identity comes from the considering the graphs of the two functions. We can see that if we take a $\sin(x)$ function, reflect it in the y-axis and then translate it by $\frac{\pi}{2}$ units to the left, we will actually get a $\cos(x)$ function! So now that we know this, what we can do is use this identity in order to change the sin function into a cos function. It should immediately be apparent that it is now asking us to antidifferentiate what was found in the previous part. So we should recall here that antidifferentiation and differentiation are almost like reverse processes - just like addition and subtraction. In order to antidifferentiate, what we are doing is sort of like **undoing** the differentiation. So we will end up getting the same answer as the original function. However, we must remember to add the constant of integration.

$$\begin{aligned}
&\int\left(\frac{1}{x}\sin\left(\frac{\pi}{2} - \log_e(x)\right)\right)dx \\
=\ &\int\left(\frac{1}{x}\sin\left(\frac{\pi}{2} - \log_e(x)\right)\right)dx \\
=\ &\sin\left(\log_e(x)\right) + c
\end{aligned}$$

Let's take a moment, however, to have a look at the implications of the constant of integration. What the constant of integration tells us is whether any vertical translations have occured. Have a think about a common curve such as $y = x^2$. Now, imagine shifting it up and down vertically, all shifted graphs will be a different function, however at any particular x-value, they will have the same gradient. Thus, they have the same derivative, despite being different functions. That's why we need the extra piece of information to work out our constant and figure out the exact function.

Part c.

For this question, we need to know that the average value of $f(x)$ between a and b is given by:

$$\text{Avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

It is important to remember that the function must be continuous for this formula to work. What the average value tells us is how "high" a rectangle with a "width" from a to b would have to be for it to have the same area as $f(x)$ from a to b. This is an important concept to understand, because if we don't remember the formula, we can derive it using first principles. So if we know that the area of a rectangle is given by:

$$\text{Area} = \text{width} \times \text{height}$$

Now, we know that the area must equal the area under $f(x)$ from a to b and that the width would be $b - a$, we can say that:

$$\text{height} = \frac{\text{Area}}{\text{width}} = \frac{1}{b-a} \times \int_a^b f(x)dx$$

This is the formula for the average value. But now, we can substitute everything in, remember that we already found the indefinite integral in the last question, so really, this question only requires some substitution:

$$\begin{aligned}\text{Avg} &= \frac{1}{\pi-1}\int_1^{\pi}\left(\frac{1}{x}\sin\left(\frac{\pi}{2}-\log_e(x)\right)\right)dx\\ &= \frac{1}{\pi-1}\left[\sin\left(\log_e(x)\right)\right]_1^{\pi}\\ &= \frac{\sin\left(\log_e(\pi)\right)}{\pi-1}\end{aligned}$$

Question 7

This is a pretty tough question which requires both good algebraic skills as well as good foresight and problem solving skills. The first port of call for many students would be to try and do something with the equation $x^2 = 4ay$, the second might be to sit there and wonder how they are going to solve this problem when they only have one bit of information and they are trying to find two variables. This is the type of question where the more time you take sitting and looking at it, the less chance you will have of completing it correctly. The best way to approach this question would be to put pen to paper and do some working, even if it's not in the right direction initially.

So if we ignore the equation $x^2 = 4ay$ for the moment, let's look at what we've been given. We have two points (in terms of a, p and q) and we have the fact that the line joining the two points passes through the point $(0, a)$. This is actually a very important and interesting point. In an exam situation, when we see the words "straight line" and "passes through the point" we should already be thinking about things such as "gradient", "equation of a straight line", so that seems like a pretty natural place to start. To try and figure out the equation of this line, we have three points we can work with, so we can find the gradient using the first two points that we're given:

$$m = \frac{aq^2 - ap^2}{2aq - 2ap} = \frac{a(q+p)(q-p)}{2a(q-p)} = \frac{q+p}{2}$$

And now again, we can substitute in another point in order to find the equation of the line.

$$y - y_1 = m(x - x_1)$$

$$y - ap^2 = \frac{(q+p)(x-2ap)}{2}$$

From here, all we have to do is substitute the point $(0, a)$ into the equation and manipulate it algebraically in order to get our answer:

$$\begin{aligned}a - ap^2 &= \frac{(q+p)(-2ap)}{2}\\ a(1-p^2) &= -apq - ap^2\\ 1 - p^2 &= -pq - p^2\\ pq &= -1\end{aligned}$$

This question definitely is the hardest question on this exam, requiring the most foresight out of all of the questions. If you didn't quite reach the answer in time, the key points to take away from this question is to think of "gradient" and "equation of a straight line" when seeing words such as "straight line" and "passes through the point" because more often than not, they're the keys to solving the question.

Question 8

This is a normal distribution question. They rarely come up in the first exam because they often require the use of a CAS in order to evaluate the probabilities and intervals using the normCDF and invNorm functions. This question initially looks very much like a CAS question and it is very easily solved using a CAS. However, we don't have access to technology in this exam, so we need to utilise a work-around in order to get to our answer.

With the normal distrbution, we can compare values using a process called standardisation. What this aims to do is convert normally distributed random variables to ones which have a mean of 0 and standard deviation of 1. This will allow us to sort of "level the playing field" and get a better look at the two bakeries in this question. In order to normalise a random variable, we need to know the formula:

$$Z = \frac{X - \mu}{\sigma}$$

Consider the Hettige Bakery and apply this formula. First of all, X is the critical point we are after - i.e. 750 g. The mean, we are told, is 760 g and the standard deviation is 7 g. So if we substitute in these values, we can find what is called our z-score:

$$Z = \frac{X - \mu}{\sigma} = \frac{750 - 760}{7} = -\frac{10}{7}$$

However, what does this actually mean? In fact, what it means is:

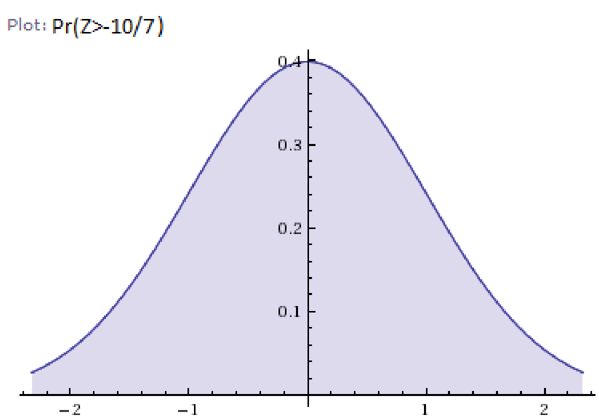
$$\therefore \Pr(X > 750) = \Pr(Z > -\frac{10}{7})$$

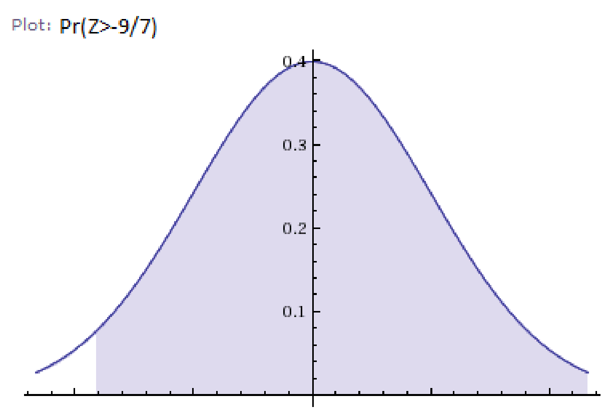
Since the z-value is standardised, we can compare it to other z-scores in order to know where abouts it lies on the standard normal distribution curve. Consider the Levy Bakery:

$$Z = \frac{X - \mu}{\sigma} = \frac{750 - 768}{14} = -\frac{9}{7}$$

$$\therefore \Pr(X > 750) = \Pr\left(Z > -\frac{9}{7}\right)$$

Now, we have two standardised normal values, $-\frac{10}{7}$ and $-\frac{9}{7}$. What we need to do now is figure out which interval is larger as this would mean that the probability (i.e. the area under the curve) will be greater. The way we can do this is to picture a standard normal distribution curve and have a think about where the values $-\frac{10}{7}$ and $-\frac{9}{7}$ will actually lie. The following two diagrams will give a better view of these intervals.





We will see that the interval from $-\frac{10}{7}$ to infinity is greater than the interval from $-\frac{9}{7}$ to infinity, thus, we can conclude that:

$$\Pr\left(Z > -\frac{10}{7}\right) > \Pr\left(Z > -\frac{9}{7}\right)$$

This means that there is a greater chance of getting a loaf of greater than 750 g from the Hettige Bakery.

Question 9

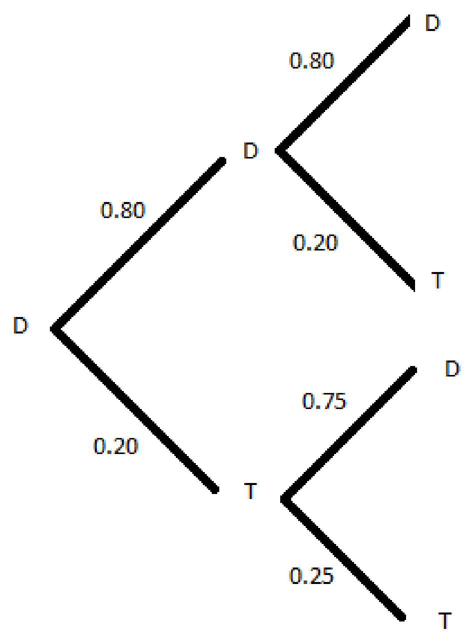
This is a Markov chain question. There are generally two ways to do this. We can either use a tree diagram, or we can choose to just use our standard probability techniques in order to solve it. First of all, the most important step with probability questions is to interpret the question and make sure that we know exactly what it is asking.

Probability is sometimes known for being confusing, however, the mathematics behind it isn't actually as confusing as the words which are presented to us. So the question asks us the probability that Trevor will drive to university exactly once on the next two days, given that he has driven to university on Monday.

So, what exactly does "drive to university exactly once on the next two days" actually mean? Let's simplify down what the question is asking here, there are two possible cases that can actually happen:

- Trevor drives on Monday, Trevor drives on Tuesday, Trevor catches the train on Wednesday
- Trevor drives on Monday, Trevor catches the train on Tuesday, Trevor drives on Wednesday

No other combinations are possible without violating the conditions which are presented in the question. So now let's consider our two cases individually. Let's, first of all, draw a tree diagram to represent our situation:



From the tree diagram, we can see that the pathways we are interested in is DTD and DDT, giving us a probability of 0.15 and 0.16 respectively. All we have to do now is add the probabilities to get 0.31 which is the correct answer.

Otherwise, we can do it using our old probability rules. Let's consider case DTD first. So Trevor starts off driving on Monday, this is a certainty as Trevor has already driven, so we will put that down as 1. Now on the next day, the question says that the probability Trevor will drive given that he drove the last day is 0.80, meaning the probability he catches the train is 0.20. Thus, our next step would be to multiply 1 by 0.20. Now for the last step, we are after the transition from train to drive, which is 0.75, so multiply 1 by 0.20 by 0.75:

$$\Pr(DTD) = 1 \times 0.20 \times 0.75 = 0.15$$

Similarly, we can apply a similar process for DDT and get a probability as well. All we need to do from here is add it all up and get our final answer of 0.31:

$$\Pr(DDT) = 1 \times 0.80 \times 0.20 = 0.16$$

$$\text{Total Probability} = 0.31$$

Question 10

General Comments - Question 10 is a related rates question. What we generally find with related rates questions in VCAA exams is that they tend to be multi-part, so that even if the final answer is not obtained, intermediate marks for the earlier parts will be given. Even in questions which are not divided up into parts, it is important to be clear with your working out so that you obtain as many partial marks as you can.

Part a.

We need to know the equation for the total surface area for a cylinder for this part. We need to remember that a cylinder will have a top base and a bottom base, each with an area of πr^2 and it will have a rectangle wrapping around it - height of h and width of $2\pi r$, the circumfrence of the circular base. So if we sum up these parts, we get:

$$S = 2\pi r^2 + 2\pi rh$$

We are told that the height is 80 mm, thus, what we can do is substitute that into our equation in order to get:

$$S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 160\pi r$$

Part b.

This part is the related rates part and requires us to use the chain rule in order to get our answer. Let us look at what we have, we are told in the question that:

$$\frac{dr}{dt} = 0.5 \text{ mm/hr}$$

From the above part, we can work out the rate of change of the surface area with respect to the radius:

$$\frac{dS}{dr} = 4\pi r + 160\pi$$

We are asked to find the rate of change of the surface area with respect to **time**, so here, recall the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Now we can change it to look more like the question here and what we're after, so change the variables for surface area and radius:

$$\frac{dS}{dt} = \frac{dS}{d\square} \times \frac{d\square}{dt}$$

So what we need to do here is find that intermediate variable, we can see from the information we have that the intermediate variable is r, the radius, so if we substitute that in and the information we have, we can find the formula for the rate of change of S with respect to t.

$$\frac{dS}{dt} = 0.5(4\pi r + 160\pi) = 2\pi r + 80\pi$$

From here, it's a simple substitution in order to get the answer, we are asked what the rate of change is when $r = 10$mm, so that is what we will substitute in.

$$r = 10,\ \frac{dS}{dt} = 2\pi \times 10 + 80\pi = 100\pi \text{ mm}^2/\text{hr}$$

It is absolutely important that units are included here in the answer, because other answers, such as π cm^2/hr, are also acceptable because it is equivalent, so make sure the units are included and are correct.

SET 2 EXAM 2

DETAILED SOLUTIONS

SECTION 1 - Multiple-Choice Questions

Question 1 (D)

Questions involving transformations seem to constantly be a pitfall for most students. The trick to most transformation questions is to break them down into basic components that are easy to deal with. In this case, we can simply go step-by-step with each transformation, writing down the function after each one in the correct order.

Firstly, a translation of 1 units in the negative direction of the y-axis

$$\cos(3x) \to \cos(3x) - 1$$

Secondly, a reflection in the x-axis

$$\begin{aligned} \cos(3x) - 1 \quad &\to \quad -(\cos(3(x) - 1) \\ &= - \quad \cos(3x) + 1 \end{aligned}$$

Thirdly, a translation of $\frac{\pi}{2}$ units in the positive direction of the x-axis

$$-\cos(3x) + 1 \to -\cos\left(3\left(x - \frac{\pi}{2}\right)\right) + 1$$

Hopefully doing one transformation at a time as such is not too difficult - if it is, you must go back to the textbooks and go over transformations.

The problem now is that none of the answers match this one correctly. Only two of them involve a cosine function - **B** is incorrect because of the incorrectly reflected translation in the y-axis. **C** is incorrect because the reflection has been applied in the y-axis instead of x as the question specifies.

So how does the cosine function change its form into a sine function? The big trigger that all students should look out for is seeing $\pm\frac{k\pi}{2}$ within the angle of a trig function. It is useful because of the complimentary identity $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$. This is an important rule because not only does it help you change forms to one you might desire in a question, it can also greatly simplify your expressions and make it easier for calculations. In this question, we can simplify the transformed function from $-\cos\left(3\left(x - \frac{\pi}{2}\right)\right) + 1 = -\cos\left(3x - \frac{\pi}{2} - \pi\right) + 1$ to $\sin(3x) + 1$. Hence, the answer is **D**.

Question 2 (E)

The phrase "decreasing" is one that often confuses students. A set that is STRICTLY decreasing is usually defined as, $[a, b]$ such that $f(a) > f(b)$ and $b > a$. However, the question only asks for when it is "decreasing" and hence the definition then includes the equals sign i.e. $f(a) \geq f(b)$.

This question looks like a trick because at no point is the gradient less than 0. But remember, the definition of decreasing makes no mention of gradients or derivatives.

The next question to ask yourself is whether the lack of continuity in the function raises a problem. Again, refer to the definition - look at one of the discontinous points $(3, 3)$. $(3, 2)$ doesn't exist (since it has an open circle) but at the next x-value (infitissimally small next to $x = 3$, the value of the function is less than 3. So yes, it does fit the definition. Hence the function is decreasing as defined in the diagram across [0,4). Hence, the answer is **E**.

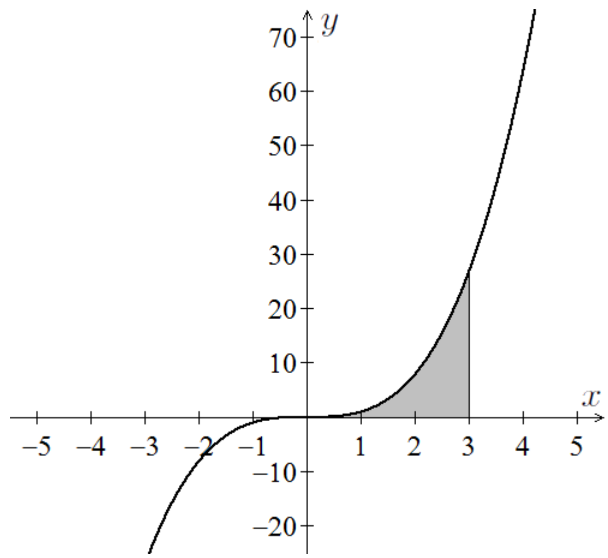
Question 3 (C)

$$\int_{-2}^{-5} -2f(-(x+2))dx \quad = \quad -2\int_{-2}^{-5} f(-(x+2))dx$$

From this point on, (most) students will have to think analytically about this question. It is perhaps easiest to think of this in terms of transformation and use visual analysis to determine what is happening.

Consider the following curve, $f(x) = x^3$.
The transformations that have occured to get $f(-(x+2))$ are a reflection in the y-axis followed by a translation of 2 units in the negative direction of the y-axis.

We can do quick sketches, marking in the area under the curve from 0 to 3 to examine this.



The question is - what area does this new curve take? If only translations have occurred we can trace the interval of 0 to 3 through the transformations and find that the interval that takes up the same area is $[-5,-2]$. Fortunately, this is the interval given in the intergral in the question! Now we know that the area is 5 according to the question despite the transformation. Hence, $\int_{-5}^{-2} f(-(x+2))dx = 5$.

However, beware! Look at the terminals of the definite integral in the question. Are the same? No, but fortunately we know that $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$. Hence, $\int_{-2}^{-5} f(-(x+2))dx = -5$

Therefore,

$$\begin{aligned} -2\int_{-2}^{-5} f(-(x+2))dx &= -2(-5) \\ &= 10 \end{aligned}$$

Therefore, the answer is **C**.

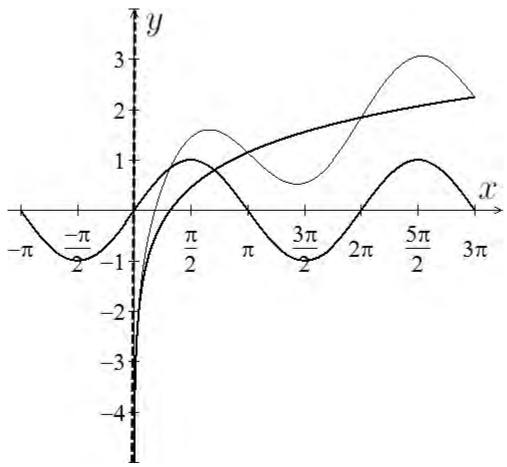
Note: You may be curious as to why "(most) students" is written above - for those of you also studying Specialist Mathematics, you will be familiar with substitution in integrals which can be used to answer this question relatively simply.

Question 4 (D)

A bit of theory regarding addition of ordinates is covered in Q16 of this multiple choice section. It is recommended that students read this first. What you will learn from that question is how a function has its domain defined by the domains of the two functions being added. This question clearly indicates that there is an asymptote along $x = 0$, hence clearly $x = 0$ must not be defined in one of the functions being added - more specifically, the behaviour of the graph suggests it must be an aysymptote as well. Knowing this only removes option **C** which lacks a function with an asymptote at $x = 0$.

However, there is more information to be drawn from this asymptote. Another conclusion to be made is that the function with the asymptote must also arise from $-\infty$ at $x = 0$ since any function (not approaching infinity as well) will have no effect on $f(x)$. For clarity, refer to the diagram below to understand this. From this we can also disregard option **A**.

The next hint to the question is in the shape of $f(x)$. There clearly seems to be some form of sinusoidal wave occuring. Determining which one is perhaps a little troubling but with any doubt, you would be wise to graph the functions and see which one suits the given $f(x)$ graph best. In fact, you can even set the window on the calculator to match the axes given in the question.



You will find that $\log_e(x)+\sin(x)$ produces the best fit for $f(x)$.

Hence, the correct answer is **D**.

Question 5 (D)

This question is a classic example of one that tests a student has an assured knowledge of content, specifically the first principles of finding a derivative. The multiple choice answers look to make sure that students are reading carefully and that they are sure of what the rule is:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

From this point, inputting the information given should be simple (with some care).

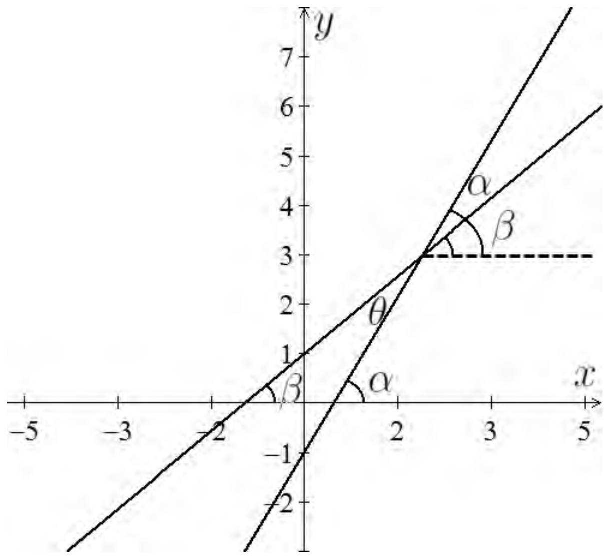
Hence the answer is $\lim_{h\to 0} \dfrac{\log_e(k(a+h)^2) - \log_e(ka^2)}{h}$, **D**.

For this question, the big trap is in the x^2 and ensuring that the k is not squared and that $(a+h)$ as a whole is squared and expanded correctly.

Question 6 (B)

In the Mathematical Methods course, there is no requirement in the study design for students to find angles between intersecting curves - however, students need to be able to find angles between intersecting lines. while the question does ask for the intersection of curves, it also demonstrates quite simply how to calculate them and this question does provide the two lines required. Simply find the tangents at that point and proceed as normal to find the angle between those lines.

Consider the following diagram



Try to understand this diagram well - it's important that you can recognise why the angles around the intersection are as they are. The end result you should get is that $\theta = \alpha - \beta$ So in order to find the angle between the two lines, we must first find the angle that each line makes with the x-axis. A bit of geometry tell us that $\tan(\theta) =$ gradient - this can easily be shown if you think of gradient as "$\frac{\text{rise}}{\text{run}}$" and you construct a right angled triangle with the height as "rise" and the base as "run".

Therefore in this question,

$$\begin{aligned} \tan(\alpha) &= 2 \\ &\text{and} \\ \tan(\beta) &= 1 \end{aligned}$$

Since $\theta = \alpha - \beta$ we simply need to calculate

$$\begin{aligned} \theta &= \tan^{-1}(2) - \tan^{-1}(1) \\ &= 18.43^\circ \end{aligned}$$

But wait! while this is an option available, you must read the question carefully and realise that it asks for the "largest angle" formed by the intersection. Two angles are formed by the intersection - the other is the supplementary angle, $180^\circ - 18.43^\circ = 161.57^\circ$ which is the larger angle of the two.

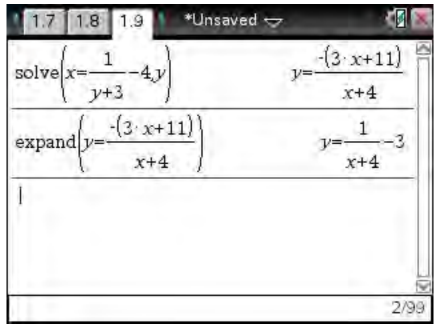
Therefore, the answer is **B**.

Also note that $\tan(\alpha - \beta) = \dfrac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$. Thus, in this case $\tan(\alpha - \beta) = \dfrac{2-1}{1+2(1)} = \dfrac{1}{3}$. $\tan^{-1}\left(\dfrac{1}{3}\right) = 18.43^\circ$ as above.

Question 7 (D)

Students may choose to tackle this question in one of two ways, find the inverse of the function given and decide which option is false, or with enough understanding of what inverses are, determine the answer from the original function.

Finding the inverse is simple enough on the calculator. Simply input the function but with y and x swapped and solve for y - the exact same thing you would do by hand if required.



From a basic understanding of hyperbolas, it is clear that this inverse has a vertical asymptote of $x = -4$ and thus the correct answer is **D**.

However, it is probably quicker to consider the second way of answering this question. A simple understanding of inverses can go a long way. For the most part, all you have to realise is that inverses are formed by a reflection in the $y = x$ line, thus why when we find the inverse, we swap the y and x. If we are doing this swap, we can expect all the properties involving x and y to swap. For example, the original function has a vertical asymptote of $x = -3$. Hence, we know that $y = -3$ is an asymptote in the inverse - we simply substituted the x with a y.

Similarly, the horizontal aysymptote was $y = -4$ and thus the vertical asymptote will be $x = -3$ in the inverse.

Again, the correct answer is **D**.

Question 8 (E)

Finding the average value of a function is a standard multiple choice question in VCAA exams - it should become a free mark. The average value of a function, $f(x)$, for the interval $[a, b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\frac{1}{10-0}\int_0^{10} f(t)\,dt$$

Hence, the answer is **E**.

Refer to Section 2: Q1 b) ii) for a further explanation on average values and the underlying theory.

Note: Most questions imply that time is given in terms of second, but ensure you read carefully as in this case time is given in terms of minutes.

Question 9 (D)

This question tests a student's knowledge of the basic definition of terms in probability, specifically, those involved in discrete random variables. You will find that each of the options given represent a very simple calculation and one that you should be able to write off quickly.

Examine the options given.

A requires the student to be able to find the mean or "expected value". The expected value is given as the sum of the product of each x-value and the probability of X being that value.

Put it simply,

$$E(X) = \sum x \Pr(X = x)$$

This is result that commonly confuses students, and it's probably due to a misguided understanding of how "averages" are actually calculated. Consider a six-sided die - students will recognise that the probability of landing on a single side is $\frac{1}{6}$. However if they are then interested in finding the average, they will likely add all the values from 1 to 6 then divide by 6 as one does when finding the average of numbers. i.e.

$$\text{Mean} = \frac{1+2+3+4+5+6}{6}$$

What many students fail to realise is that the actual calculation you're doing is

$$\begin{aligned} \text{Mean} &= \frac{1}{6}\times 1 + \frac{1}{6}\times 2 + \frac{1}{6}\times 3 + \frac{1}{6}\times 4 + \frac{1}{6}\times 5 + \frac{1}{6}\times 6 \\ &= \sum x \Pr(X = x) \end{aligned}$$

Once this is understood, you can calculate expected values for distributions that have different probabilities for different values.

In this question,

$$\begin{aligned} \mu &= a \times -1 + b \times 1 \\ &= -a + b \end{aligned}$$

Therefore, **A** is an incorrect answer

Next, let us consider option **B**

A rule of probability that is often neglected by students is that the sum of probabilities must equal to 1 i.e. $\sum \Pr(x) = 1$. Ironically, this is a very simple rule to be forgotten, and one that is often very useful in questions - at a basic level, it is a free equation given to you for all probability questions should you need it.

In this question,

$$\begin{aligned} \sum \Pr(x) &= 1 \\ a + b &= 1 \end{aligned}$$

Once again, this option is incorrect and will probably be selected by those who are confusing their rules since the expected value of this distribution is $-a + b$.

Options **C** and **E** essentially test the same thing with variance. Variance (and standard deviation) measures the spread of probabilities around a distribution's mean.

It is generally defined as,

$$\text{Var}(X) = E\left[(x-\mu)^2\right]$$

From this definition, you can see that variance is essentially the expected values of distances away from the mean. i.e. How far away a value is from the mean $x - \mu$. Students often question why it is squared - for the purposes of this course, it is enough to understand that variance is an arbitrary construction of spread and because we think about "distance" away from the mean, that value must alway be positive, thus why the distance itself is squared.

There is a very handy alternative to this rule which is more pratical in most cases (however, it's not very useful in helping to explain what variance is),

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

This can be shown very easily and if you're interested, you should refer to your textbooks for how this alternative is derived from the definition.

In any case, this question can serve as an example of how to go about finding $E(X^2)$ and hence variance. Take note of how the X is squared.

$$\begin{aligned}
\text{Var}(X) &= E(X^2) - [E(X)]^2 \\
&= \left[a \times (-1)^2 + b \times (1)^2\right] - (-a+b)^2 \\
&= a + b - (-a+b)^2 \\
&= a + b - a^2 + 2ab - a^2
\end{aligned}$$

The third line shows that **E** is incorrect and the fourth line shows that **C** is incorrect.

Knowing the above, we can safely conclude that **D** is the correct answer but let us look into it to be sure.

The definition of standard deviation is given as follows,

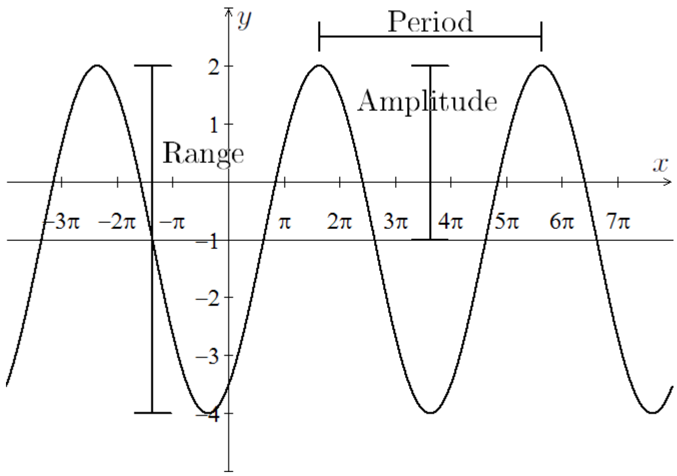
$$\begin{aligned}
\sigma &= \sqrt{\text{Var}(x)} \\
\text{Therefore, } \sigma &= \sqrt{a + b - (-a+b)^2} \\
&= \sqrt{a + b - (b-a)^2}
\end{aligned}$$

Therefore, **D** is the correct answer

Question 10 (D)

This is a typical VCAA question that simply relies on a student understanding what the various definitions of terms relating to trignometric functions are.

Below is a diagram that depicts each of the terms in the question.



Finding each of these are relatively simple and will be required for many questions of greater difficulty.

Consider the function, $f(x) = a\sin(b(x-c)) + d$

The amplitude is very simply a. The amplitude is NOT the range - it is half this distance for a sine function. Physicists beware! In this course, "amplitude" does not mean "peak to peak amplitude". It is simply "peak" amplitude as shown in the diagram above.

The period can be worked out just by remembering what the period of the basic function is (in this case $y = \sin(x)$ which has a period of 2π) and then apply the appropriate dilation along the x-axis. For the general function above, there is a dilation of factor $\frac{1}{b}$ along the x-axis, hence the period is $\frac{2\pi}{b}$. Note: The translation along the x-axis does not affect the period in any way.

Range has the same meaning as it does in any function and can be found easily by noting that the average value of the function is d and the amplitude of the wave goes to the negative and positive of this average. i.e. The range is $[d-a, d+a]$.

Hence for this question, the period is 4π, the amplitude is 3 and the range is $[-4, 2]$. Therefore the correct answer is **D**.

Question 11 (C)

For a matrix equation $\mathbf{AX} = \mathbf{B}$, we are looking for no solution for $\mathbf{X}$. If we arrange (with correct matrix arithmetic) we get $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. Obviously, if $\mathbf{B}$ is given to us, there must be some problem with finding the inverse, $\mathbf{A}^{-1}$. The question is when does the inverse not exist?

Let us first ask another question - it is beneficial for students to consider how a matrix equation is formed. For the simple matrix equation $\mathbf{AX} = \mathbf{B}$, the simple answer can be found by conducting the multiplication of $\mathbf{AX}$. In this question, we get the following result

$$\begin{aligned} \begin{bmatrix} a & 2 \\ 3 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} ax+2y \\ 3x+by \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

Hence, two equations are formed in order to find the solutions for two variables, x and y.

$$\begin{aligned} ax + 2y &= 1 \\ &\text{and} \\ 3x + by &= 2 \end{aligned}$$

Students should be familiar with this form and should be capable of solving these simultaneous equations. In this form, what causes there to be no solution? It is when the lines are parallel and unique.
Let us first rearrange the equations to make it easier to see

$$y = \frac{1-ax}{2} \qquad (1)$$
$$y = \frac{2-3x}{b} \qquad (2)$$

The equations will be unique and parallel when $\frac{1}{2} \neq \frac{2}{b}$ and $-\frac{a}{2} = -\frac{3}{b}$
The second of these produces the result $ab - 6 = 0$.

Let us go back to $\mathbf{A^{-1}}$ and ask again when this inverse does not exist. For $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{A^{-1}} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
Consider what would make this inverse non-existent. Simple understandings of a function like $f(x) = \frac{1}{x}$ will tell you that the function does not exist when $x = 0$. Clearly the inverse does not exist when $ad - bc = 0$. As it turns out $ad - bc$ is the determinant of a matrix. Hence, the inverse does not exist when $det(A) = 0$.

Going back to the question,

$$\begin{aligned} \det\left(\begin{bmatrix} a & 2 \\ 3 & b \end{bmatrix}\right) &= 0 \\ ab - 6 &= 0 \end{aligned}$$

Familiar? Yes it's the same result as the one produced by finding when the equations are parallel and unique.

Hence, the correct answer is **C**.

Students who rely on very accute memory should be extra careful when it comes to reading a question. while students are probably used to seeing $\det = ad - bc$ when the matrix is in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the matrix in this question has "b" in the place of "d" in the rule they have learnt. It's a common trick used by VCAA to trip up students and one that should keep you cautious.

Question 12 (C)
This question is probably not as difficult as it may seem - all the skills required to answer this question are embedded in the ability to draw all the functions within the Methods Course - what you have to ask yourself is what are the key structures to the function? As you would know, if it's a sine function, it has a sinusoidal wave with a particular period and amplitude. If it's a truncus, it has two asymptotes and approaches infinity as $x \rightarrow^{\pm} 0$. These properties may appear second nature to you now, but it's a critical understanding of them that needs to be brought out in order to answer this question.

Firstly, a note about composite functions and their existence - there are often trick questions about whether a composite function exists, but the only thing you have to remember is the following:

For $f(g(x))$ to exist,

$$\text{ran}(g) \subseteq \text{dom}(f)$$

In other words, the range of $g(x)$ must fit within the domain of $f(x)$. Why? Because $f(x)$ only has a certain set of x-values for which it exists - if you input a value outside this set, $f(x)$ is undefined. For a composite function, instead of a "set of x-values", the x is substituted for $g(x)$. Therefore, the $g(x)$ values (its range) must be within the possible maximum domain of $f(x)$ in order for $f(g(x))$ to exist.

With this in mind, **B** and **E** can be disregarded because the domain of the first function is restricted to $x > 0$ but the range of the second functions clearly include values equal to 0 and less.

Let us look at another significant property of $f(g(x))$ - the approach to infinity when $x \to^{+} 0$ (meaning as x approaches 0 from the positive end of the x-axis). For the options not disregarded, $f(x)$ appears to be a function that at least behaves like e^x. It's clear that in order for this function to approach infinity at $x = 0$ then $g(x)$ must be a very large number around that point as well - this is only true for **A** and **C**. However, **A** appears to be a truncus for $g(x)$ - due to its symmetry, this means that for $f(g(x))$, it should also also be positive around $x = 0$ and approach infinity from the negative end of the x-axis but it does not.

It becomes apparent that $f(g(x))$ is a function like $e^{\frac{1}{x}}$ since it approaches 1 when x approaches both infinity and negative infinity. i.e. $e^{\frac{1}{\text{a very large number}}} = e^{\text{a very small number}} = 1$.

Hence, the answer is **C**

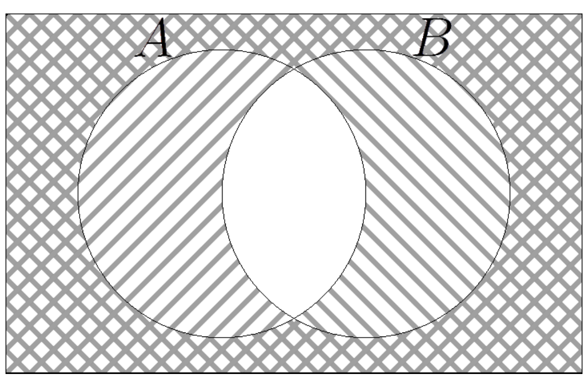
Question 13 (C)

This question should be tackled as systematically as possible - if you are careful and methodical in your approach, these questions become very simple. Venn diagrams are extremely useful for questions like this - they are quick to produce, and as always, the visual aid is significantly helpful.

However, before we can understand how to approach this question, we have to understand the terms - students should get to know these well because they come up very often. In this question there are only two things of main concern: 1) The dash above the A and B and the whole bracket, e.g. "A'", and 2) the "$\cap$". The dash indicates that we are required to focus on the probability of everything that is not a particular event. So the probability that of not being A and not being B. The "$\cap$" indicates an "intersection" - Event A might arise by drawing a certain coloured ball, but that colour might also exist in Event B. That probability is shared and is hence involved in the intersection of the two events. Now that the terms are out of the way, we can look at the question.

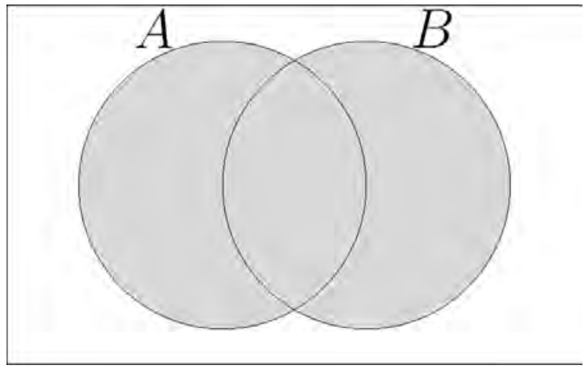
Let us begin with $(A' \cap B')$, the intersection of not-A and not-B. Firstly shade everything that is not-A with some pattern. Then shade everything is not-B with another pattern. Then you merely have to shade the areas of the Venn Diagram to have the intersection.

You should end up with a diagram similar to the following:



Everything that is "not-A" is given by diagonal lines going down from left to right. Everything that is "not-B" is given by diagonal lines going down from right to left. The intersection is shown by the checkered areas.

But wait! Carefully read the question to realise there is another dash and we in fact want $(A' \cap B')'$. This simply means we want to shade everything that is not checkered in our previous diagram. The result is below.



Hence, the answer is C.

Question 14 (B)

This question tests a niche piece of information in the Mathematical Methods course.
Essentially, for a left-endpoint approximation is calculated by the following sum. Note: summations are not tested in the Mathematical Methods course (it is just used here for teaching purposes).

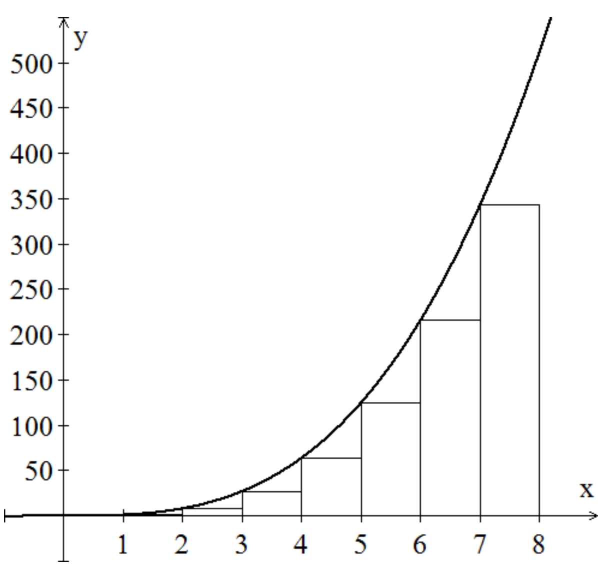
$$\sum_{x=0}^{n-1} f(x)\delta x, \text{ where } \delta x \text{is the width of the rectangles}$$

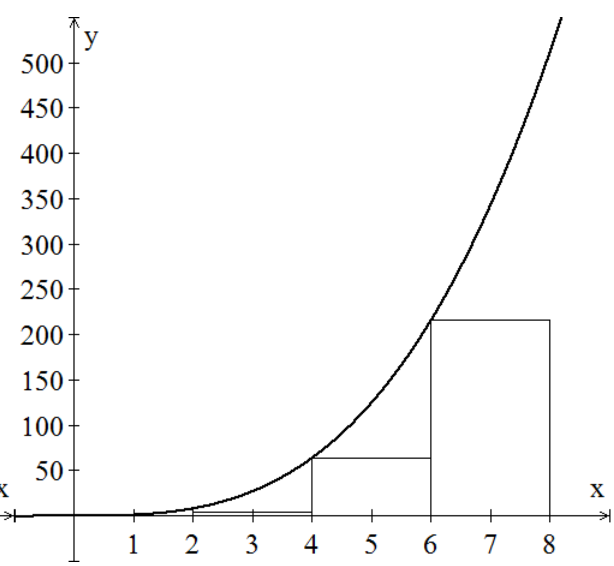
Or to put is simply, it is the sum of rectangle areas which is given as width $\times$ height and height is $f(x)$.

There is an important word in the question - "best". A few of the answers will provide a correct approximation, however, the best approximation implies that it is the most accurate approximation area, i.e. the area closest to the exact area under the curve (but still a left-point approximation). To see how the best approximation is acquired requires an understanding of the definition of areas under a curve by intergration.

The interval of areas is given by the question and hence unchanging. Therefore, the only way to make the width of each rectangle as small as possible is to increase the number of rectangles. If we increase the number of rectangles, we must decrease the width so that they fit within the interval.

Consider the two diagrams below.





We can see from this that the first approximation, with more rectangles is a much better approximation and covers more of the area underneath the curve.

Out of the options, the sum with supposedly the largest number of rectangles is either **B**, **D**, or **E** with 8 rectangles. With 8 rectangles, the width must be 1 unit which means we can disregard **E** which seems to suggest that the width is 2. Moreover, to be a left-endpoint it must start from $x = 0$ and lead to the x value of 7 so that the height of the last rectangle is found by $f(7)$, the left hand corner of the last rectangle (refer to the first approximation diagram). But option **D** actually demonstrates the right-endpoint approximation since its rectangle areas start with a height of 1^3 and end with a height of 8^3 - while actually a better approximation than the left-endpoint approximation, it is not what we are looking for.

Therefore the sum that provides the best estimate is $1 + 8 + 27 + 64 + 125 + 216 + 343 + 512$. Hence, the correct answer is **B**.

Question 15 (E)

This question looks at the behaviour of graphs according to their gradients. The best way to look at stationary points and such is with the use of a sign graph as shown below.

x		-1		2	
$f'(x)$	$-$ve	0	$+$ve	unknown	$-$ve
Shape	╲	—	╱		╲

The “unknown” poses a question as to what COULD happen, as opposed to what “must” happen as the question asks for. This is a subtle ommission in the question - most of the time you will be told that $f'(2) = 0$ but we don’t know that in this question.

So what could the behaviour around $x = 2$ be? The most obvious possibility is a maximum. But this isn’t the only possibility - it could also be a cusp in which case the gradient at that point is undefined. Therefore we cannot state that either of these options “must” describe the behaviour at $x = 2$.

Hence, since we can’t conclude what occurs at $x = 2$ we can only state there is a local minimum at $x = 1$. Therefore, the correct answer is **E**.

Question 16 (C)

The domain for an addition of ordinates function can be found by working out the domains of the functions being added together. To put it simply, the function can only be defined for the values of x that are defined for the individual functions - both functions must be defined at that certain x value in order to be a defined addition (otherwise it’s like $y+$ undefined and what is that? Undefined).

Domain of $\sqrt{x-3} + 1$ is

$$\begin{aligned} x - 3 &\geq 0 \\ x &\geq 3 \end{aligned}$$

Domain of $-\log_e(-x + 4)$ is

$$\begin{aligned} -x + 4 &> 0 \\ x &< 4 \end{aligned}$$

It is important that students can work out these restricted maximal domains quickly. Anything under a square root sign must be greater than or equal to 0 and anything within a logarithmic function must be greater than 0.

The maximal domain of the whole function will simply be the values of x for which both functions exist - i.e. the intersection of their domains.

Therefore, the domain is $3 \leq x < 4$ or [3,4).

Hence, the correct answer is **C**

Question 17 (D)

This question is a relatively simple linear approximation relying on the linear approximation rule,

$$f(x+h) \backsimeq f(x) + hf'(x)$$

In most of these questions, you simply have to define $f(x)$, h and x. The question tell us that $x = a$, $h = h$ and that $f(x) = \sqrt[3]{(x-3)^2}$.

We only have to find what the question is asking for - in this case the "approximate change" which is found by calculating $hf'(x)$. Why is this the approximate change? It's possible to rearrange the linear approximation rule to,

$$hf'(x) \backsimeq f(x+h) - f(x)$$

However! $f(x+h) - f(x)$ is the exact difference, but $hf'(x)$ is the approximation unless $h \to 0$ as in the first principles of derivatives rule. $hf'(x)$ is used in finding the approximation. Students selecting **B** will be incorrect, not because the change is incorrect, but because the question specifically asks for the "approximate change".

Back on track, we simply have to calculate $hf'(x)$.

$$hf'(x) = \frac{2h}{3(a-3)^{\frac{1}{3}}}$$

Hence, the correct answer is **D**

Question 18 (D)

The mean of a continuous random variable is given by,

$$\mu = \int_{-\infty}^{\infty} xf(x)\,dx$$

The concern in this question is how to deal with the two functions and the two sets of domains. Students should become familiar with the sum of independent variables - the rule is $E(X+Y) = E(X) + E(Y)$.

Hence if you find the sum of the average of both functions in the hybrid function over the given domains, you find the average of the whole distribution. Therefore,

$$\begin{aligned} \mu &= \int_{-20}^{10} xke^x\,dx + \int_{10}^{50} x(-kx+10)\,dx \\ &= 156905k + 12000 \end{aligned}$$

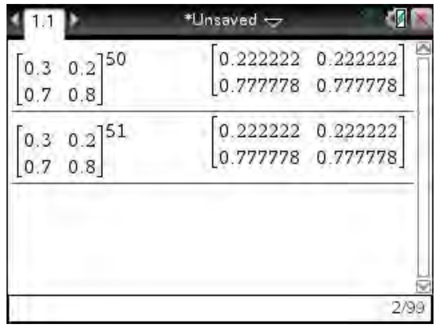
Hence the correct answer is **D**

Question 19 (B)

With most questions involving Markov Chains, the first thing a student should look to do is define a transition matrix and an initial matrix. For questions involving steady-state matrices, it should be understood that the initial matrix is irrelevent in terms of working out the proportion of people/things in the long term. This question obviously involves proportions since the question asks for a "percentage of students".

$$T = \begin{bmatrix} 0.3 & 0.2 \\ 0.7 & 0.8 \end{bmatrix}$$

The steady state matrix is generally defined as $S_{n\to\infty}$ which relies on $T^{n\to\infty}$. The CAS does not recognise T^∞ but the idea of the "steady" state might give an idea as to how to find the steady-state matrix. The point is that the proportion of people preferring science or biomedicine does not change from one point in time to another. Generally, finding T^{50} (or to the power of something over 50) will be sufficient. HOWEVER, be aware that taking T to the power of a number over 99 will generally result in your calculator reaching its limit and going into an endless loop trying to calculate the matrix. Basically, you can be assured that you have the steady-state matrix when $T^n = T^{n+1}$.



It is clear that this produces the steady-state matrix. From this transition matrix we need to select the correct percentage - The trick is to consider the transition matrix and the value you desire is read from left to right. You should label your matrices as such

$$T = \begin{array}{cc} & \begin{array}{cc} S & B \end{array} \\ \begin{array}{c} S \\ B \end{array} & \begin{bmatrix} 0.3 & 0.2 \\ 0.7 & 0.2 \end{bmatrix} \end{array}$$, where S represents the preference of science and B represents the preference of biomedicine.

Always read across for your values - i.e. if you want a portion for science, you read the top row of the matrix (30% continuing to prefer science and 20% preferring science who used to prefer biomedicine.

Hence the percentage of students in the long run that prefer biomedicine over science is 78%. The answer is **B**

Note: There is an alternative solution that students should be aware of when considering long-term transition matrices. It is generally considered the method to be used in the absence of a calculator - but it is actually a much simpler calculation (inputted into the calculator) than taking the transition matrix to a certain power.

Consider the transition matrix, $T = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$, steady state proportion for whatever the first row represents is $\dfrac{b}{a+b}$ and the steady state proportion for the second row is $\dfrac{a}{a+b}$.

In this question, we are interested in the second row that looks at the biomedicine preferences i.e. we want to find $\dfrac{0.7}{0.7+0.2} = 0.78 = 78\%$. The same result as what we found using a calculator.

Question 20 (B)

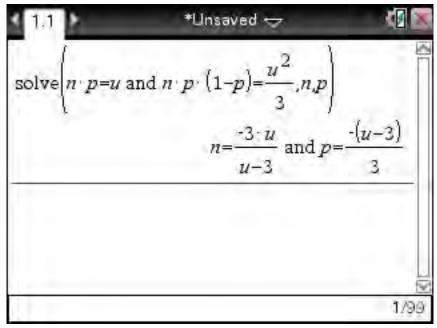
When looking at the binomial distribution and having to define it in some way (whether it be finding mean/variance or the probability and number of trials), you should only have to refer to two rules

$$\begin{aligned} np &= \mu \\ &\text{and} \\ np(1-p) &= \sigma^2 \end{aligned}$$

So the two equations we know in this case are

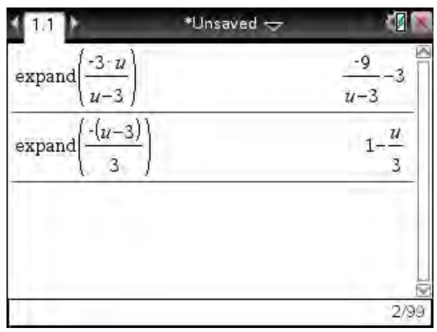
$$\begin{aligned} np &= \mu \\ &\text{and} \\ np(1-p) &= \frac{\mu^2}{3} \end{aligned}$$

These equations can be entered into the CAS as such



The answers the CAS provides look very similar to a couple in the options given. However, you must look very carefully to ensure the answers are in fact the same as what you got on the calculator. This question requires you to consider alternate forms to the one given in the calculator - this is not a free mark to be solved by the CAS.

The CAS is capable of changing the form of expressions, particularly factorising, expanding and changing fraction forms. None of the answers seem like they will be produced by a factorisation - so it is likely an expansion or different fraction will be the answer. If we use the expansion function, we get the following result



Hence **B** is the correct answer.

Question 21 (B)

Most students will tackle this question by simply inputting each of the functions given in the options to see if it follows the rule given and this is a perfectly viable method and probably the most straightforward in this instance.

It isn't easy for most students to intuitively know which answers seem unreasonable. $f(x) = x^2$ seems highly unlikely with a quick thought. Otherwise, the rest seem to require a bit of fiddling around with simplification.

Why can't you just use a calculator? The problem with the CAS is that it can easily determine when numbers are equal, but it struggles to determine when functions are equal for any value of x. It is even more complicated because the logarithmic function and trigonometric functions can be simplified in intricate ways that the calculator cannot readily recognise. (However, there are trig expansions and collect functions that are very useful at times).

If we were to input $f(x) = \log_e(x^2)$ into the rule,

$$\begin{aligned}
\text{RHS} &= \log_e(x^2) + \log_e(y^2) \\
\text{LHS} &= \log_e[(xy)^2] \\
&= \log_e(x^2y^2) \\
&= \log_e(x^2) + \log_e(y^2) \text{ as required}
\end{aligned}$$

Therefore, $f(xy) = (f(x) + f(y))$ when $f(x) = \log_e(x^2)$. The above solution required a knowledge of "log laws", in particular $\log(a) + \log(b) = \log(ab)$. This should have been the most obvious answer from those available - the rule required is an exact copy of the log rule for $f(x) = \log_e(x)$. Students only had to be careful not to immediately select **A**. Hence, the correct answer is **B**.

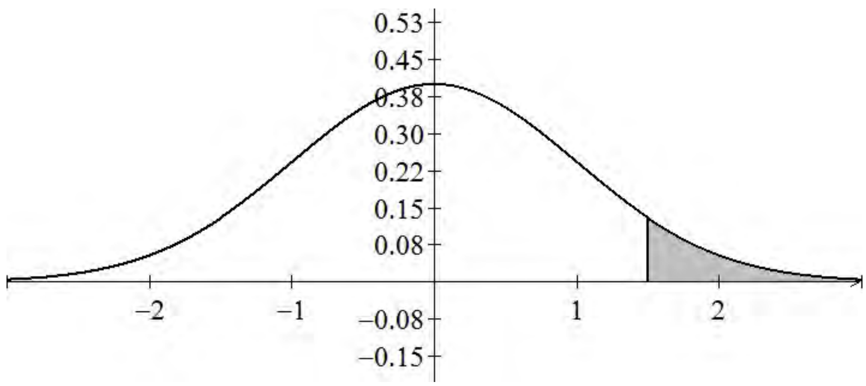
Question 22 (B)

This question involves a relatively typical standardisation followed by an understanding of the normal curve geometry. If we look to the options available in the answers, the "Z" should be the immediate indication that standardisation to a z-score is required.

$$\begin{aligned} X &> \mu + \frac{3\sigma}{2} \\ Z &> \frac{(\mu + \frac{3\sigma}{2}) - \mu}{\sigma} \\ &= \frac{3}{2} \\ &= 1.5 \end{aligned}$$

Hence, we want an answer equivalent to $Z > 1.5$, however, this is not an option to select. We can immediately disregard **A** and **B** since $Z > 2.5$ (or any variation) will not yield the required probability.

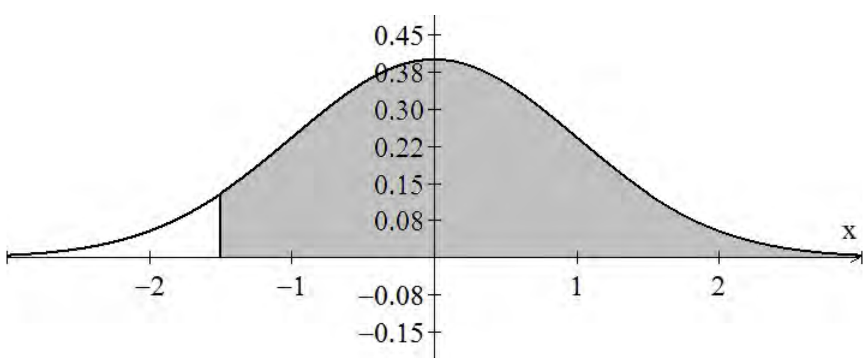
Let us consider this result in the context of the normal curve.



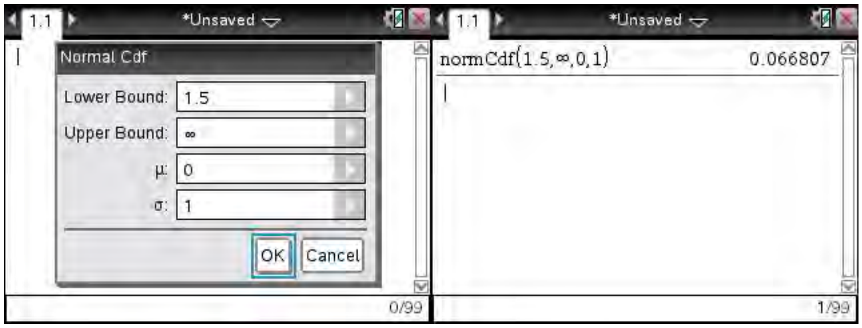
Most questions involving the normal distribution can be easily handled with a quick sketch of a normal curve, marking the mean and standard deviation and shading what you desire. Students should readily make use of this method because it is extremely quick and having a sketch to visualise what you need makes questions significantly easier.

We can note that **D** cannot be correct since it is the area not shaded in the diagram above. In fact, $\Pr(Z < 1.5) = 1 - \Pr(Z > 1.5)$. AND, $\Pr(Z < 1.5) = \Pr(Z > -1.5)$ so **C** also incorrect.

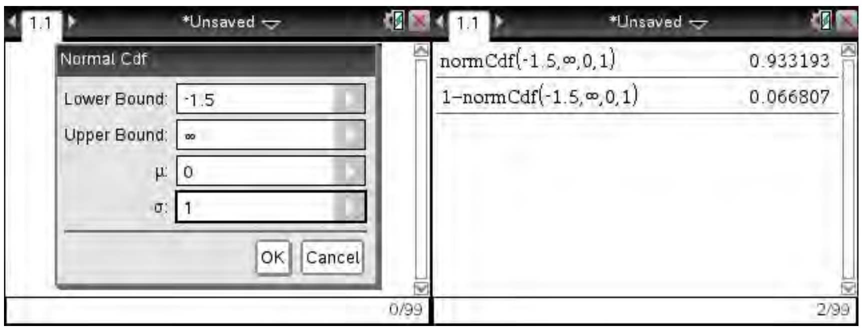
Hence, the correction answer should be **D** (and it is). This can easily be checked by shading in the area of $\Pr(Z > -1.5)$ and realising that everything not shaded is equal to the area required, i.e. $Z > 1.5$.



You can also make use of the normal CDF function on the CAS - [Menu] [5] [5] [2] and finding the probability for $Z > 1.5$, then checking the multiple choice options to find the same result.



Testing option **D** (which we now know is correct) to see if it gets the same result as what we desire.

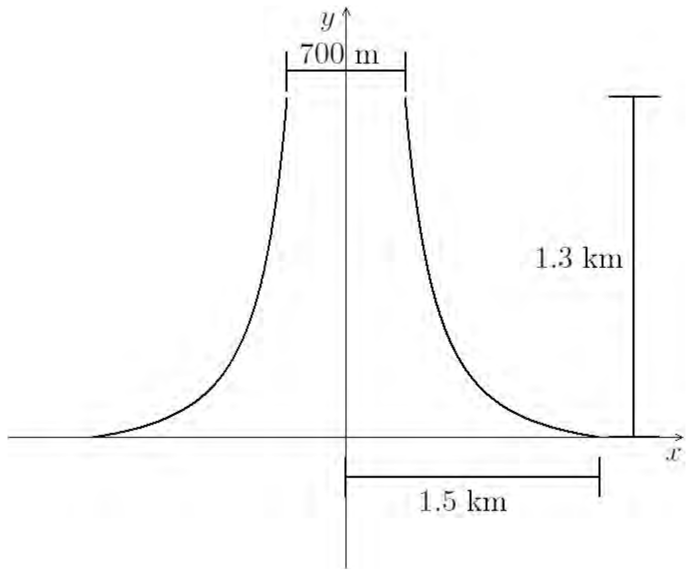


SECTION 2 - Extended-Response Questions

Question 1

a.

This first part of the question is a standard modelling question - i.e. defining a function that describes the situation from the given information. The difficulty with this question only lies in the student's ability to recognise the shape the volcano takes - we have been given the the form of $y = \dfrac{a}{x^2} + c$ so it's worth students at least drawing a quick truncus to demonstrate what it looks like and if they can picture it correctly, it should look like this:



This illustrates that there are various known points we should be aware of (though the information in the question has sufficient information to provide the points). Going back to the model we've been given, $y = \dfrac{a}{x^2} + c$, there are two unknowns so we're going to need two points on the volcano in order to use simultaneous equations and solve for a and c.

From the last piece of information in the stem, we know that the base ($y = 0$) of the volcano ends 1.5 km from its centre ($x = 0$). In other words, this end is either 1.5 km to the right or left of the centre).

Hence, when $x = 1.5$, $y = 0$. We can substitute this point into the given model in order to gain one equation.

$$0 = \frac{a}{1.5^2} + c \ (1)$$

The other piece of information given is in regards to the diameter of the crater. If the diameter is 700 m, then its radius from the centre of ($y = 0$) will be 0.350 km either side of the middle of the crater. The other thing known about this crater is that it is at a height of 1.3 km.

Hence, when $x = 0.350$, $y = 1.3$. Again we will substitute this value into the given model.

$$1.3 = \frac{a}{0.350^2} + c \ (2)$$

Now that we have achieved two equations, we can solve for the two unknowns a and c. We are looking to get rid of one of the unknowns through simultaneous equations - the easiest way seems to be to subtract the two equations and hence remove the c then proceed to solve for a.

$(2) - (1)$

$$\begin{aligned} 1.3 - 0 &= \frac{a}{0.350^2} + c - \left(\frac{a}{1.5^2} + c\right) \\ 1.3 &= 7.719a \\ a &= 0.168 \end{aligned}$$

Now if we substitute this value of a into equation 1, we get the following result

$$\begin{aligned} 0 &= \frac{0.168}{1.5^2} + c \\ c &= -0.075 \end{aligned}$$

Now we can substitute these values into the general model given to achieve our final function, $y = \dfrac{0.168}{x^2} - 0.075$. But be careful! Read the question carefully and you will note that it specifically asks for the appropriate domain to accompany the function. Have a look at the diagram above - the domain will obviously include the width of the base $[-1.5, 1.5]$ but the shape of the volcano also shows that the function is not continuous between the ridges of the crater i.e. there is a hole in the volcano that means the volcano isn't defined there. That crater has a width defined by $[-0.350, 0.350]$ if you include the actual edge of the crater. But when we exclude values, we don't want to exclude the edge of the crater because that is still apart of the volcano, so we don't include the values on the end.

Hence the function required is $y = \dfrac{0.168}{x^2} - 0.075$ for $x \in [-1.5, 1.5] \setminus (-0.350, 0.350)$.

b. i.

This is a typical related rates question - related rates is often deemed a difficult portion of calculus application, however there are some very simple steps students can follow in order to solve these questions.

Firstly, in most questions, a student is given one of the rates required within the information given. In this case we are told that the radius of the lava decreases at 1 m/s as it rises. Before we go into how to write this as a rate, careful thought when trying to interpret the question is required here - students may struggle with the idea of "the rate at which the radius of the lava decreases". The volcano has a radius which is described by x-values so if you're looking at a rate that x changes, you essentially want the change in x over a certain time, $\dfrac{dx}{dt}$, where the x is in km and t is the time in seconds.

Hence, the "radius" of the lava is the distance "x" from the y-axis. Hence, $\dfrac{dx}{dt} = -1$ m/s$= -0.001$ km/s. Take note of the negative! x is decreasing according to the information so it must be a negative rate.

The next part to answering these questions involves knowing what you're looking for. We what the rate of change in depth - i.e. a change in y over a certain time. $\dfrac{dy}{dt}$. These "rates" are always given as a change in something over a change in time.

We now need to have a method of finding $\dfrac{dy}{dt}$. The best way to do this is to set up an equation that uses the rate given. Use: $\dfrac{dy}{dt} = A \times \dfrac{dx}{dt}$. Now think about what "$A$" must be? If you solve it, you'll see that $A = \dfrac{dy}{dx}$. Therefore, $\dfrac{dy}{dt} = \dfrac{dx}{dt} \times \dfrac{dy}{dx}$. Do we have a function that relates x and y? Indeed, it is model of the volcano we found in **part a.**

$$\frac{dy}{dx} = -\frac{0.337}{x^3}$$

After this point, we just substitute this into our equation to find $\dfrac{dy}{dx}$ and we will have our required rate.

$$\begin{aligned} \text{Therefore, } \frac{dy}{dt} &= -0.001 \times -\frac{0.337}{x^3} \\ &= \frac{0.337 \times 10^{-3}}{x^3} \text{ km/s} \end{aligned}$$

ii.

This is a trick question that tests whether the student is reading the question correctly - the "average rate of change" in a function is found by finding the gradient of the line formed by two points on a function. The question seemingly guides you towards this idea by telling you to find the average between the floor and the ledge. If you start thinking along these lines you'll hit a roadblock - the function for the floor and ledge is given by the volcano model function we found in **part a.** - NOT the rate of change in lava as the question specifies. We don't even have a function for "lava depth" as it stands to find the average rate of change. What we do have is its rate of change. In actual fact the question asks for the average value of this rate of change - you MUST read the question carefully. The formula given for the average value of a function, $f(x)$ between the interval $[a, b]$ is

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Some work needs to be done before we can apply this rule to our given question. As said above, $f(x)$ is the function that describes the rate of change in lava depth, i.e. $f(x) = \frac{dy}{dt}$. Since $\frac{dy}{dt}$ is given in terms of x, it is necessary to find the x-values of when the lava is at the floor and when it is at the ledge to find out interval.

When $y = 0$, $x = \pm 1.5$ as given by the base of the volcano. The ledge is 5 m below the top of the volcano i.e. $1.3 - 0.005 = 1.295\,\text{km}$. By substituting this value of y into the model, we can find the value of the corresponding x. When $y = 1.295$,

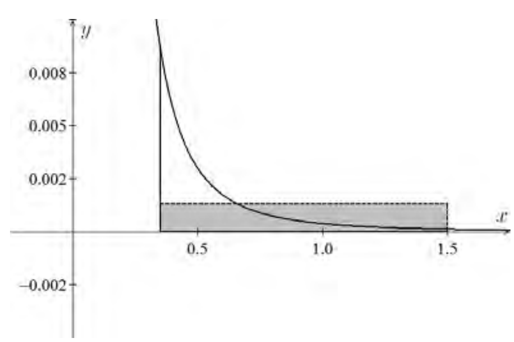
$$\begin{aligned} 1.295 &= \frac{0.168}{x^2} - 0.075 \\ x &= \pm 0.351 \end{aligned}$$

Hence for the average value of the function which describes the rate of change in lava depth,

$$\begin{aligned} \text{Average value of the rate of change in lava depth} &= \frac{1}{1.5 - 0.351}\int_{0.351}^{1.5} \frac{0.337 \times 10^{-3}}{x^3}\,dx \\ &= 0.00113\,\text{km/s} \end{aligned}$$

Students will notice that the values of y used in the question each had two corresponding values of x - we would get the same average value even if we used them as expected since the depth would rise at the same rate between the two halves of the truncus.

Students may ponder where the rule for average values come from. The best way to work this out is to look at the rule and work backwards. The rule consists of the area underneath the curve, divided by the width of the interval. If we were to represent this area as a rectangle along the curve, it becomes clear that the average value is in fact the height, or the y-value of that area. So this height is a constant value, that when multiplied by the interval width results in the same area as under the curve - the average value of the function. Below shows the average value for the rate of change in lava depth (the function $f(x) = \frac{dx}{dt}$) between the x-values of 0.350 and 1.5. It is an illustration to demonstrate the lesson on average values. Note that the shaded area is equal to the area under the curve - the y-value of the rectangle is the average value of the function between that interval.



iii.

This question can easily be worked out by simply finding the time it takes for Mr. Williams to escape and the time it takes for the lava to rise to the top of the crater. If Mr. Williams is quicker to reach the top, he will have survived the rising lava.

At 5 cm/s (= 0.05 m/s), Mr. Williams must travel 5 m, hence the time it takes him to get out is

$$\begin{aligned} t &= \frac{d}{s} \\ &= \frac{5}{0.05} \\ &= 100 \text{ seconds} \end{aligned}$$

Every student should be aware of such a simple rule as $d = t \times s$ where d stands for distance, t stands for time, and s stands for speed. It should be an intuitive rule considering that we can think about speed as being the distance covered over a certain time, i.e. $s = \frac{d}{t}$.

Now the more tricky part to this question is finding how long it takes for the lava to rise to the top of the volcano. You would expect to use a function that connects depth and time such that we can substitute the value of depth as 1.3 km (the top of the crater) and find the corresponding time that it takes to reach that height. This is not an easy thing to do with the information given because there isn't any direct relationship between y and t that has been found yet. However! What if we had a function that related x and t? We have a relationship that connects x and y so we can find the corresponding value of x to our desired depth of 1.3 km so having a relationship between x and t is very close to having a relationship between y and t. But do we have this function? We do, but it's somewhat hidden within the information "the rate at which the radius of the lava decreases is 1 m/s" i.e. $\frac{dx}{dt} = -0.001$.

If we think of x as a function of time, and found its derivative with respect to t we would have the result $\frac{d}{dt}(x)$. We know $\frac{dx}{dt}$, so if we found its anti-derivative, we would have the function of x in terms of t, time.

$$\begin{aligned} x &= \int -0.001 \, dt \\ &= -0.001t + c \end{aligned}$$

Ensure that you do not forget the addition of the constant when you are evaluating a indefinite integral. But now we need one point of x and t that will help us to find the one unknown c. We do know that the lava begins to rise from the base of the volcano. Hence at $t = 0$, $y = 0$. But remember, we are dealing with x not y so we need to find the corresponding x value. But we have two corresponding x-values, ± 1.5.

It is actually easier to use $x = -1.5$ because as t increases, y increases only as x increases towards 0. It becomes messy if we start thinking of t and y increasing as x decreases which is the case on the right-hand side of the truncus.

Therefore we can substitute the point $x = -1.5$ and $t = 0$ into our x-t relationship found above and solve for c.

$$\begin{aligned} -1.5 &= 0 + c \\ c &= 1.5 \\ \text{Therefore, } x &= -0.001t + 1.5 \end{aligned}$$

The lava reaches the top once $y = 1.3$ which has a corresponding x-value of $x = -0.350$, i.e. the radius of the crater. Ensure you use the x-value for the right-hand side of the truncus because that is what we substituted into the function to find the c-value.

The time once the reaches the top can thus be found by

$$\begin{aligned} -0.350 &= -0.001t + 1.5 \\ t &= 1850 \text{ seconds} \end{aligned}$$

This time is significantly longer than the time it takes for Mr. Williams to get out of the volcano. Hence, the lava takes longer to reach the top of the crater and Mr. Williams survives.

Question 2

a.

There is no need to overcomplicate this question - simply write out the function as it is given. There is no need to contextualise this within transformations or the like, though if you believe it will help, then do whatever is best for you. Students typically get confused by substituting something such as $(2x-1)$ into a function when the function was already given in terms of just x. There's no reason why you can't substitute $(2x-1)$ into the function in the same manner in which you would substitute a value for x in $f(x)$. See below to realise that it is just the same

$$\begin{aligned} g(x) &= -f(2x-1)+3 \\ &= -[((2x-1)-1)(2x-1)^2-k)]+3 \\ &= -8x^3+16x^2+(2k-10)x-2k+2 \end{aligned}$$

Ensure that your answer is given in the form as specified in the question and these are some easy marks to pick up.

b. i.

When finding the tangent to a function, you merely have to find the gradient of the function at the given point of tangent then substitute that point into the general form of a line with the calculated gradient.

In order to find the gradient of the function at a certain point, we must find the derivative of the function.

$$f'(x) = 3x^2-2x-k$$

With the derivative found, we only have to substitute the value of x in question. We wish to find the tangent for when $x=1$.

$$f'(1)=1-k$$

Lastly, we need a point to substitute in our equation of a straight line - we know one point which is the one in question at $x=1$. However, we need to find the y-value at this value of x.

We just have to substitute $x=1$ into $f(x)$.

$$f(1)=0$$

Hence the point known is $(1,0)$.

Now we can use the very useful equation of a straight line $y-y_1=m(x-x_1)$ where (x_1,y_1) is the known point and m is the gradient of the line. Using this rule is generally quicker than finding the value of c in the form $y=mx+c$.

Therefore the tangent at $x=1$ is

$$\begin{aligned} y-0 &= (1-k)(x-1) \\ y &= (1-k)(x-1) \\ &= (1-k)x+k-1 \end{aligned}$$

As such, we have found the tangent at $x=1$ for $f(x)$

ii.

Finding the normal to a curve at a point on a function utilises the same method as finding the tangent. However, the gradient will be found in a different way. The normal is defined as being perpendicular to the tangent and follows the rule that $m_1 m_2 = -1$, where m_1 is the gradient of the tangent at a point and m_2 is the gradient of the normal at that same point. This can be rearranged to conclude that the gradient of the normal is $-\dfrac{1}{m}$ where m is the gradient of the function at that point (i.e. the gradient of the tangent).

So if our gradient of the normal is dependent on the gradient of the tangent, we should find this tangent gradient first. It will be done in the same method used in **part b.i.** in this question.

$$\begin{aligned} g'(x) &= -24x^2 + 32x + 2bk - 10 \\ g'(3) &= 2k - 130 \end{aligned}$$

Hence the gradient of the normal is $-\dfrac{1}{m} = -\dfrac{1}{2k-130}$. Now we will go through the same steps of finding the point at which this normal is being found and substituting it into the general rule of the straight line.

When $x = 3$, $g(3) = 4k - 100$. Therefore, our known point is $(3, 4k - 100)$. Therefore, the normal at $x = 3$ is

$$\begin{aligned} y - (4k - 100) &= -\frac{1}{2k-130}(x-3) \\ y &= -\frac{1}{2k-130}(x-3) + 4k - 100 \end{aligned}$$

c.

It is worth reading the Detailed Solutions for **Question 2 b.** in Algebra Tech-Free Test 1 for an in-depth look at the nature of intersections and how to distinguish them. For the purposes of this question, we are only interested in when there is no intersection for the two lines found in **part b.** of this question. There is no intersection when the gradients of the two lines are the same but the y-intercepts are different. These are the only two things we will have to concern ourselves with for the meantime.

Firstly, we'll look at the gradients. The gradient of the tangent found was "$1 - k$" and the gradient of the normal found was "$-\dfrac{1}{2k-130}$". Therefore, the gradients are equal when

$$\begin{aligned} (1-k) &= -\frac{1}{2k-130} \\ k &= \frac{66 \pm \sqrt{4098}}{2} \end{aligned}$$

These values of k may refer to two types of solutions - no solutions or infinite solutions. The difference between these two types intersections is their y-intercept. The point of no solutions is that the lines are parallel but different so that they never cross at any point. When there are infinite solutions, it's because the lines are not just parallel but also the same and hence have the same y-intercept. We want to ensure that for the k-values found above, that neither produce two lines which are the same - we want the y-intercepts to be different.

The y-intercept of the tangent is $y = k - 1$ and the y-intercept of the normal is $y = \dfrac{3}{2k-130} + 4k - 100$. Hence, the y-intercepts will only be equal when,

$$\begin{aligned} k - 1 &= \frac{3}{2k-130} + 4k - 100 \\ k &= \frac{98 \pm \sqrt{1022}}{2} \end{aligned}$$

These k-values are not equal to either of the k-values when the gradients are the same. Hence, the original k-values found do not produce infinite solutions.

Hence, the k-values that produce no intersection are $k = \dfrac{66 + \sqrt{4098}}{2}$ and $k = \dfrac{66 - \sqrt{4098}}{2}$.

Question 3

Probability questions in Mathematical Methods Exams are generally considered relatively easy compared to the other areas of the course tests - this is largely in part due to the lack of scope in the types of questions that can be asked, and, especially in an Exam 2, the ability of the calculator allows students to take these questions in their stride. It is worth becoming very familiar with the CAS to speed up your answering of these questions (though it shouldn't be at the cost of understanding the concepts). For the most part, this is a typical question regarding Markov Chain application (until **part c.**). Refer to **Question 1 b.** in the Detailed Solutions of Probability Tech-Free Test 2 for a lengthier description of how to tackle such questions and the underlying theory.

a. i.

We wish to find the probabilities of watches and wallets sold in the month of April. The question begins in January, and hence we will refer to January as S_1. And since April is the fourth month in the year (yes, students should know this), it will be assigned S_4. If we are to find the state of April, we will have to use the expression $S_4 = T^3 S_1$.

We shall let the first row of our transition matrix represent watch sales and the second row will be wallet sales.

Our transition matrix is as follows: $T = \begin{array}{c c} & \begin{array}{cc} \text{Watches}_0 & \text{Wallets}_0 \end{array} \\ \begin{array}{c} \text{Watches}_1 \\ \text{Wallets}_1 \end{array} & \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \end{array}$

This is constructed from the information that tells us 90% of people who buy watches will buy watches the following month - hence, from Watches_0 to Watches_1, which corresponds to the top left corner in the table (in matrix form). The same is applied for the information regarding wallet buyers. We then just have to ensure that the columns add to 1 because from the people who initially bought watches, they will definitely buy either a watch or wallet, so there's no probability outside of this. The same goes for people who initially bought wallets.

As usual, we have to take care when defining our initial state matrix because the information given is for January, the FIRST month of the year. If we were given information about the month prior to the start of the year, we would use S_0 but in this case we will use $S_1 = \begin{bmatrix} 150 \\ 60 \end{bmatrix}$. Again, the top row corresponds to the watches and the second row corresponds to the wallets - it is crucial that these match with the transition matrix you defined above.

Now that we have both the transition matrix and the initial state matrix, we just have to find S_4 from those results. You have a calcualtor, so do not waste time trying to multiply the matrices out by hand.

$$\begin{aligned} S_4 &= T^3 S_1 \\ &= \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}^3 \begin{bmatrix} 150 \\ 60 \end{bmatrix} \\ &= \begin{bmatrix} 0.781 & 0.438 \\ 0.219 & 0.562 \end{bmatrix} \begin{bmatrix} 150 \\ 60 \end{bmatrix} \\ &= \begin{bmatrix} 143.43 \\ 66.57 \end{bmatrix} \end{aligned}$$

At first, this result seems a bit strange - and it should because you can't have 0.43 of a person buy a watch and 0.57 of a person buy a wallet! Students should recognise that the number of people buying should be whole numbers, so the numbers should be integers. Usually, when our answers are not whole, we always round down because in having say 29.8 people, we actually only have 29 people, not 30 complete people despite 29.8 being closer to 30. BUT, when it comes to proportions as commonly seen in these probabilities, particularly Markov Chain, questions, we have to retain the number of people, 210 in this question. Hence we would round the numbers appropriately to the closest integer. $143.43 \to 143$ and $66.57 \to 67$

Hence, Clare will need to stock 143 watches and 67 wallets in April.

ii.

This question is as simple as it looks - too often students look to complicate the situation because they expect a question to be more difficult than it really is. We know the price of the items and we know how many of them are bought - if we sell 5 apples for \$1 each, we know that we just made \$5. In other words, we just have to multiply the number of wallets sold by its price and add it to the multiple of watches sold and their price.

$$\begin{aligned}\text{Sales} &= 143 \times \$80 + 67 \times \$40 \\ &= \$14120\end{aligned}$$

Further students will have seen similar questions that utilise a matrix to answer this question. Yes, that is possible, but it is literally the same arithmetic involved in finding $\begin{bmatrix} 80 & 40 \end{bmatrix} \begin{bmatrix} 143 \\ 66 \end{bmatrix}$ as that shown above. Don't get bogged down in course content just because you know it - more often than not, questions can be thought of in a more intuitive way than what the course depends on (and this is how mathematics should be).

b.

The application of Markov Chains is very limited so students should become quite accustomed to the type of questions involved with them - trigger words will help you a long way. This question involves the use of steady states, and that should be obvious by the phrase "in the long run". Anything that seems to refer to the vague idea of the distant future without specifying an exact time should be a trigger for students to launch straight into steady state matrices.

The steady state matrix is simply given as T^t as $t \to \infty$. This makes sense because we are looking for a time in the future at which the transition matrix no longer changes for two consecutive values of t, i.e. $T^t = T^{t+1} = T^{t+2}$... So if we look to the transition matrix as t is approaching infinity, that will be sufficient to find the steady state matrix. However, as mentioned previously in MCQ19 of this exam, do not input the expression $T^{t \to \infty}$ into your calculator because it will cause an infinite loop (as you would expect) and then you will have to interrupt the calculation or restart your CAS. Usually, placing the matrix to the power of something greater than 50 will be more than sufficient (in this question, it reaches its steady state by about the 15th month) BUT when writing the solution, allude to the idea of a time infinitely away.

The other thing to take note of is that the transition matrix is given as proportions, not the actual number of sales for watches and wallets. Hence, it will still need to be multiplied by the initial state matrix. However, in saying that, it is an important concept that students recognise that it doesn't matter what the initial state matrix is with regards to the steady state matrix - it is steady regardless of what matrix you multiply into it. As long as the matrix S you choose still adds up to 210, the numbers will be the same. Feel free to test this in your calcualtor.

Hence in the long run, Clare will sell wallets and watches according to

$$T^{t \to \infty} S_1 = \begin{bmatrix} 140 \\ 70 \end{bmatrix}$$

There is an alternative method for finding steady state matrices which is done by hand (or by the calculator very quickly) - for the sake of checking your answer, you might as well use both methods because they are both extremely quick.

Consider the transition matrix, $T = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$. The steady state proportion for whatever the first row represents is $\dfrac{b}{a+b}$ and the steady state proportion for the second row is $\dfrac{a}{a+b}$.

As such, the proportion of watch buyers in the long term is

$$\frac{0.2}{0.2+0.1} = \frac{2}{3}$$

Hence, the number of watch buyers in the town in the long run (out of 210) is $\dfrac{2}{3}(210) = 140$.

Therefore, the number of wallet buyers in the long run is $210 - 140 = 70$.

For those interested in showing the above result, you wish to use the fact that

$$\begin{bmatrix}\Pr(X_{n+1} = A)\\ \Pr(X_{n+1} = B)\end{bmatrix} = \begin{bmatrix}\Pr(X_n = A)\\ \Pr(X_n = B)\end{bmatrix}, \text{ for two events A and B}$$

This is done by knowing that $\begin{bmatrix}\Pr(X_{n+1} = A)\\ \Pr(X_{n+1} = B)\end{bmatrix} = T \times \begin{bmatrix}\Pr(X_n = A)\\ \Pr(X_n = B)\end{bmatrix}$ and substituting that into the above equation. Then solve for either $\Pr(X_n = A)$ or $\Pr(X_n = B)$. It is not necessary to know how to do this, but it's always nice to see where certain rules come from.

In any case, in the long term, Clare will sell 140 watches and 70 wallets every month. Now we can just use the same method as seen in **part a.ii.** to find the sales made for each month in the long run. Therefore, per month she will make

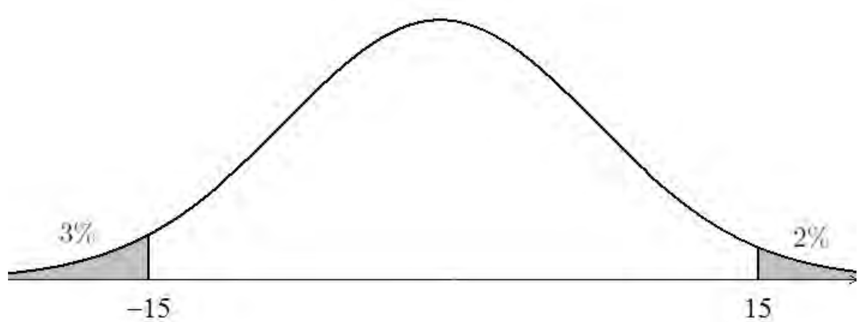
$$\begin{aligned}\text{Sales} &= 140 \times \$80 + 70 \times \$40\\ &= \$14120\end{aligned}$$

c.

This question is made difficult by the information given - it might be obvious that a normal distribution is used, but how do we construct it?

For now, let us state that $X \sim \mathrm{N}(\mu, \sigma^2)$, just a general form of the normal distribution. From here we can try to piece everything together. When dealing with normal distribution questions, it never hurts to draw a quick bell curve and see what information we can put on it. Error is given as either "too fast" or "too slow". That means that the value of error for when the time is perfect is 0, i.e. it is 0 seconds fast (or slow). 2% of watches become too fast over a month - so the watches that have an error greater than 15 seconds make up 2% of all the watches OR $\Pr(X > 15) = 0.02$. We can mark this on our normal curve. Watches that are "too slow" will be behind in seconds, so they will have negative values of error. Then similarly with the watches too slow, we know that 3% are 15 seconds too slow or less OR $\Pr(X < -15) = 0.03$.

The distribution can be quickly sketched as such:



The question asks for the "average error" - the word "average" should be enough of a hint that we wish to find the mean of the error in watches i.e. the mean of the normal distribution above. It is almost always harder to work backwards from probabilities and find the mean and standard deviation of a distribution. There is no easy to way to simply find these values on the calculator - some thought will be required.

Like a lot of mathematics beyond the VCE course, sometimes you have to think outside the box - write down what you know, explore and see if something comes out nicely. It is not always the case that VCE prepares you with a method for any type of question that can be written (though once you have seen something similar, it should be easier in later cases).

But it's not just random exploration - we want an expression that involves μ since this is ultimately what we want to find.

So far, all that we know is $\Pr(X > 15) = 0.02$ and $\Pr(X < -15) = 0.03$. If you look up chapter summaries of textbooks, you'll actually find there isn't a lot we can do with the Normal Distribution by hand to get equations and expressions. In fact, going by the Essentials textbook, there is only one expression in the summary (besides the function for the normal distribution which cannot be tested) - it is the standardisation expression $z = \frac{x-\mu}{\sigma}$. Standardisation is required in almost every normal distribution question that is done by hand.

The z-score for $X = -15$ is given by $\dfrac{-15-\mu}{\sigma}$ and the same can be done for $X = 15$.

But now that we know that $z = \dfrac{-15-\mu}{\sigma}$ what do we do? We don't know what the corresponding z-value is yet to have an actual equation.

So now we need to find the z-value that gives the same probability of 0.03 on the left tail because this will match with $X = -15$. We will have to use a calculator and the inverse normal function to do so.

$$\begin{aligned} \text{For } \Pr(Z &< c_1) = 0.03 \\ & \quad c_1 = -1.88 \\ \text{Therefore, } \frac{-15-\mu}{\sigma} &= -1.88 \ (1) \end{aligned}$$

This equation involves two unknowns so that means that we will need another similar equation that will allow us to find both unknowns. Fortunately, we have been given another probability on the normal curve to find a z-score for and get another equation by the same method.

So now we will do the same for $X = 15$, where $z = \dfrac{15-\mu}{\sigma}$.

$$\begin{aligned} \text{For } \Pr(Z &> c_2) = 0.02 \\ & \quad c_2 = 2.05 \\ \text{Therefore, } \frac{15-\mu}{\sigma} &= 2.05 \ (2) \end{aligned}$$

Now that we have two equations, we can use simultaneous equations to find the value of μ. Dividing equations is not commonly done by students with most preferring to just add or subtract, however in this case it is useful because if we divide the equations, we immediately remove σ and leave us with just μ to find - it shouldn't matter, though, as long as your method is valid.

$$\begin{aligned} \frac{(2)}{(1)} \quad \frac{2.05}{-1.88} &= \frac{\frac{15-\mu}{\sigma}}{\frac{-15-\mu}{\sigma}} \\ -1.09 &= \frac{\mu-15}{\mu+15} \\ \mu &= -0.66 \end{aligned}$$

Question 4

a. i.

Remembering distance formulas should be a given by the stage students are doing practice exams. Even if you haven't memorised them, you can always rely on the geometry of the Pythagorean theorem after sketching the points roughtly on a quick set of axes. You only have to find the hypotenuse of a triangle according to two points, (x_1, y_1) and (x_2, y_2), which is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Therefore the distance to be found is,

$$\begin{aligned} AC &= \sqrt{(5-1)^2 + (2-3)^2} \\ &= \sqrt{17} \text{ units} \end{aligned}$$

ii.

Again, this part of the question is fairly simple (in an attempt to set you up for the next part) - it only varies from the regular "find the normal" question in that we haven't got a function to find a derivative and gradient. But in essence, we do have a function because AC forms a line for which we can set up an equation. For now, we only need to concern ourselves with the gradient of AC such that we can find the normal to it.

The gradient, m, of the line between two points, (x_1, y_1) and (x_2, y_2), is given by $m = \dfrac{\text{rise}}{\text{run}} = \dfrac{y_2 - y_1}{x_2 - x_1}$ where as convention $x_2 > x_1$. The gradient of the normal is given as $-\dfrac{1}{m}$ where m is the gradient of the line.

$$\begin{aligned} \text{Gradient of } AC &= \frac{2-3}{5-1} \\ &= -\frac{1}{4} \\ \text{Gradient of normal to } AC &= -\frac{1}{-\frac{1}{4}} \\ &= 4 \end{aligned}$$

From here we only have to ensure that the normal goes through point B, so we will substitute point B into general form of a straight line with gradient 4.

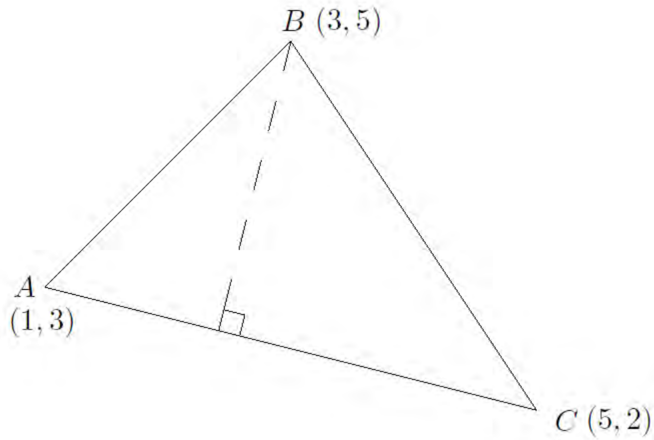
$$\begin{aligned} y - 5 &= 4(x - 3) \\ y &= 4x - 7 \end{aligned}$$

This is the equation of our normal.

iii.

This is clearly a geometry question, and, with almost all geometry questions, we should draw a diagarm. There's a lot of information given in this question so we just have to break it down as best we can - the visual aid will help.

From the information we have been given, and from what we have found in previous parts, we can construct the triangle in question. Take note of the line that forms the normal to AC and goes through point B - it forms a right angle to AC (as per the definition of the normal).



Does this diagram look familiar? It probably brings back memories of primary school mathematics. Finding the area of a triangle seems too simple for VCE mathematics that some students might even forget how to do so! But just remember: Area of triangle $= \dfrac{\text{base } \times \text{height}}{2}$. The height of a right angled triangle is usually pretty obvious - it forms a right angle with the base. The same applies to all traingles, the height is given by the line that forms a right angle with the base and reaches the top point - in this question, the normal to AC that goes through B. We only have two components to find in this case, the base and the height. Let us break down the question by looking at each individually - always look to break up a question wherever you can because you don't want to be involved in a juggling act with information, numbers and equations everywhere. After this, then we can find the area.

Part a.i. already gave us the length of the base, so all that is left is the distance of the height. We know that distance can be found by knowing two points. One of the points on the normal is point B, but what about a second point? It will be the intersection of the normal with the line AC. To find the intersection of two lines, we first need to find the equation of the straight line AC (and we found the equation of the normal in **part a.ii.**).

As found in **part a.ii.** the gradient of AC is $-\dfrac{1}{4}$. Hence we now only have to substitute either point A or point C into the general equation of straight to find the line AC.

Let us substitute point A into the equation.

$$\begin{aligned} y - 3 &= -\frac{1}{4}(x - 1) \\ y &= -\frac{1}{4}x + \frac{13}{4} \end{aligned}$$

Now we can find the point of intersection between the normal and the line AC by equating their values of y.

$$\begin{aligned} 4x - 7 &= -\frac{1}{4}x + \frac{13}{4} \\ x &= \frac{41}{17} \end{aligned}$$

Now we can substitute this x-value either into the normal or the line AC to find the y-value of this point.

$$\begin{aligned} \text{and } y &= 4\left(\frac{41}{17}\right) - 7 \\ &= \frac{45}{17} \end{aligned}$$

Hence, the point of intersection is $\left(\frac{41}{17}, \frac{45}{17}\right)$. Let us call this point D. Therefore the height of the triangle is given by the line DB.

We can now find the height of the triangle by once again using the distance formula.

$$\begin{aligned} \text{Height of triangle} &= DB \\ &= \sqrt{\left(3 - \frac{41}{17}\right)^2 + \left(5 - \frac{45}{17}\right)^2} \\ &= \frac{10\sqrt{17}}{17} \text{ units} \end{aligned}$$

Now that we all the components that are required to find the area of a triangle, we can do the appropriate calculations. As such,

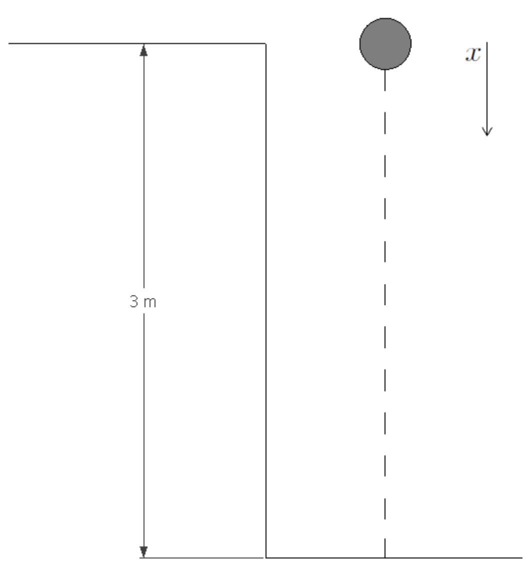
$$\begin{aligned} \text{Area of } \Delta ABC &= \frac{\text{height} \times \text{base}}{2} \\ &= \frac{\frac{10\sqrt{17}}{17} \times \sqrt{17}}{2} \\ &= 5 \text{ units} \end{aligned}$$

b.

Students should realise upon reading the question that this whole question can be done without having answered **part a.** This emphasises an important lesson in reading the questions during the reading period of an exam. Don't underestimate and waste the reading time - it can save you valuable time and in this, save you from ignoring 8 marks - 10% of the whole exam (and not a particularly difficult 8 marks to achieve either). VCAA does not make a great amount of marks inaccessible to students just because they cannot do one part (especially not 10% of an exam). Look carefully to see if you can still answer questions and scrounge marks - it is particularly obvious when there are "show that" questions because even if you can't show the result, you can use the result in a later part because it has been given to you in the question.

None of the parts of this question (i.e. **4b.**) make any reference to the triangle above, its points or its area.

Again, I highly recommend the use of a diagram to help visualise everything given in the question information. There isn't much to show so the diagram can be quite simple. An example of what you might draw is given below:



i.

This question only tests whether a student is capable of realising that within application questions, there are subtle points which are not necessarily given to you but are provided by the nature of the situation. In this case, we need a point for (t, v) so that only a remains within the function and we can find its value. That point is given by the information "He now beings to drop marbles from the very top of the hole". If we are considering this point in time, it is surely at $t = 0$ because this is when he drops it. The harder question is what the velocity is? The hint is in the word "drop" because just as the marble is dropped, it must start at $v = 0$ and the pick up speed from there. In the instances of an object being thrown, the question will usually indicate the speed at which it is thrown and that will be the initial speed at $t = 0$ (though if it is somehow "thrown" without moving the arm, it must have started at $v = 0$ too). In most questions, going to the point of when $t = 0$ is usually a good start to helping you find a point.

Hence we know that when $t = 0$, $v = 0$. We can substitute this into our given velocity rule,

$$\begin{aligned} 0 &= -2e^0 + b \\ b &= 2 \end{aligned}$$

ii.

This question involves relatively basic kinematics and subtle use of a differential equation (which is not necessarily part of this course, but a similar use of differential equations has occurred in a recent exam) which is really only seen within kinematics because the relationships between displacement, velocity and acceleration is quite accessible to students.

You may recall from Question 1 of this exam that distance = speed $\times$ time and thus speed $= \dfrac{\text{distance}}{\text{time}}$. In other words, speed is the change in distance OVER a certain time interval which can be represented as $\dfrac{dx}{dt}$.

Therefore, $v(t) = \dfrac{dx}{dt}$. But we also know that our derivative of the displacement function $x(t)$ is given as $\dfrac{dx}{dt}$ so if we need to find $x(t)$ we can find the anti-derivative of $\dfrac{dx}{dt}$.

Putting two and two together, we should now realise that the anti-derivative of the velocity function is the displacement function.

The above can be applied to acceleration too. In fact, $a(t) = \dfrac{d}{dt}(v(t)) = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right)$. Just realise that the derivative of $x(t)$ is $v(t)$ and its derivative is $a(t)$ - you can go backwards in this order by finding antiderivatives.

Hence, we can find our function $x(t)$ as such:

$$\begin{aligned} x(t) &= \int v(t)\, dt \\ &= \int -2e^t + 2\, dt \\ &= -2e^t + 2t + c \end{aligned}$$

However, we're in a similar position to where we were at the start of **part b.i.** because we have to find the constant c. But now we need a point for t and x. Again, think about when $t = 0$ - at what position is the ball? It is at the top of the hole.You can best define the top of the hole as either $x = 0$ or $x = 3$ (though technically you can make it anything since displacement is relative).

Let us define the top of the hole as $x = 0$.

Therefore we know that when $t = 0$, $x = 0$ and we can substitute this into the function for $x(t)$ and find c.

$$\begin{aligned} 0 &= -2 + 0 + c \\ c &= 2 \end{aligned}$$

Thus, writing out the whole function,

$$x(t) = -2e^t + 2t + 2$$

iii.

We were given the velocity function in **part b.i.** but we can't find the velocity without a specific time. So we must find when the marble hits the bottom of the hole.

The marble hits the bottom of the hole at $x = -3$ (since in the previous part we defined the top of the hole as $x = 0$ and the ball travels downwards).

We can substitute this value of x into our displacement function and hence find the corresponding time at which the marble hits the bottom of the hole.

$$\begin{aligned} -3 &= -2e^t + 2t + 2 \\ t &= 1.35 \text{ seconds} \end{aligned}$$

Now we just have to find the velocity at $t = 1.35$ by substituting it into the velocity function.

$$\begin{aligned} v(1.35) &= -2e^{1.35} + 2 \\ &= -5.69 \ \text{ m/s} \end{aligned}$$

Note: Velocity is a vector quantity, which, for the purposes of this course, means that it will have a sign depending on the direction it is travelling. It is travelling in the negative direction of x (from 0 to -3) and so it must be travelling with a negative velocity to be going in a downwards direction.

SET 3 EXAM 1

DETAILED SOLUTIONS

a. This is the standard derivative question that appears as the first question on exam 1 every year (since 2007, anyway). It always involves use of either the chain rule, the product rule, or the quotient rule, and is really nothing more than a test on the application of these relatively simple rules. These are two marks that shouldn't really be lost, as these rules are a fundamental part of the calculus area of study. Of course, even if you know how to use the rules, there's always room for arithmetic error, and as we don't have a calculator at our disposal we'll have to find other ways to minimise these errors. Those well-accustomed to using the differentiation rules will be able to complete the problem in two or three lines, but there are ways to go about the problem in a more careful way if you aren't confident with mental arithmetic, or are just under the pressure of the exam.

Before differentiating, you need to know which rule you're actually going to use. You'll get a function in one of three forms: $f(g(x))$, $f(x)g(x)$, or $\dfrac{f(x)}{g(x)}$. Once you recognise which of these forms the given expression is in, you can use the appropriate rule. In the case of this question, we have a fraction, which might suggest use of the quotient rule, but the trick here is recognising that $\dfrac{1}{\sqrt{1-x^2}}$ can be re-written as $(1-x^2)^{-\frac{1}{2}}$ (since the square root function is just $x^{\frac{1}{2}}$, and the reciprocal of it is just a negative power), so we can see it is of the form $f(g(x))$, which calls for use of the chain rule. We COULD use the quotient rule, but then we'd get to the point where we need to differentiate the expression $\sqrt{1-x^2}$, which is barely easier to differentiate than the original expression. Now, on to the actual problem:

$$\begin{aligned}
\text{Let} \quad u &= 1-x^2 \\
\text{and} \quad y &= \frac{1}{\sqrt{1-x^2}} \\
&= \frac{1}{\sqrt{u}} \\
&= u^{-\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
\text{Then} \quad \frac{du}{dx} &= -2x \\
\text{and} \quad \frac{dy}{du} &= -\frac{1}{2} \times u^{-\frac{1}{2}-1} \\
&= -\frac{1}{2}u^{-\frac{3}{2}}
\end{aligned}$$

Before we go further: if you use this drawn-out method, pay close attention to the '$\dfrac{dy}{du}$'s and whatnot. You have three variables now, so it can get a little confusing, but just remember: we want to end up with $\dfrac{dy}{dx}$, and in the intermediate steps we want $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$, as multiplying them together will give $\dfrac{dy}{dx}$ (the 'du's will 'cancel' upon multiplication).

Also, remember what an expression like $\dfrac{dy}{du}$ actually means - it refers to the derivative of a function of u, with respect to u. Writing something like $y = u^{-\frac{1}{2}} \implies \dfrac{dy}{dx} = -\frac{1}{2}u^{-\frac{3}{2}}$ is incorrect, because there's no 'x' involved at all. In short, be careful not to confuse variables with each other - before you write down 'x', make sure you actually mean to write 'x' - we use it so often that we can sometimes write it in place of another variable like t or u.

Anyway, we have

$$\begin{aligned}
\frac{du}{dx} &= -2x \\
\text{and} \quad \frac{dy}{du} &= -\frac{1}{2}u^{-\frac{3}{2}} \\
\therefore \quad \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
&= \left(-\frac{1}{2}u^{-\frac{3}{2}}\right) \times (-2x) \\
&= x \times u^{-\frac{3}{2}} \\
&= x(1-x^2)^{-\frac{3}{2}} \\
&= \frac{x}{(1-x^2)^{\frac{3}{2}}}
\end{aligned}$$

Either of those last two forms are acceptable answers.

Remember to put u back into the expression. Whenever you use the chain rule, the 'inner' function (in this case, $1 - x^2$) should usually still be in the final answer.

As mentioned in the model solutions, whenever you introduce new variables, you should indicate this with "Let $y = ...$" or similar. This particular question only has x given to you, so if you use y and u you need to define them.

This type of question should be practiced to the point where it can be done in a few steps. In order to do this I usually have the relation $\dfrac{d}{dx}\left(f(g(x))\right) = g'(x)f'(g(x))$ in my head, and try to keep track of everything at once:

$$\begin{aligned}
\frac{d}{dx}\left((1-x^2)^{-\frac{1}{2}}\right) &= (-2x) \times -\frac{1}{2}(1-x^2)^{-\frac{3}{2}} \\
&= \frac{x}{(1-x^2)^{\frac{3}{2}}}
\end{aligned}$$

Two things to notice here. One, notice how the form of the first line is more or less identical to $\dfrac{d}{dx}\left(f(g(x))\right) = g'(x)f'(g(x))$, except I just replaced everything with the particular functions we're dealing with. Also, the use of '$\dfrac{d}{dx}(...)$' avoids writing the extra line of "Let $y = ...$" or "Let $f(x) = ...$".

In the end, you want to get this question out of the way in less than a minute, since other questions will certainly require more attention. If you find this type of question takes too long, then just keep practicing differentiation rules, perhaps make up your own functions and check your answers with a calculator. The aim is to correctly apply the chain, quotient or product rule in just a few lines, but to also be able to do it a bit more carefully if you aren't sure of your answer. Check the first question on each past examination 1 from 2007 onwards to get an idea of the type of functions you will be asked to differentiate.

b. My discussion about the use of differentiation rules here would be mostly the same as in **part a.**, so have a read of the last section for more detail on these.

This question is more or less the same concept as the first, except for this one they usually ask you to evaluate the derivative at a particular value of x. Here we clearly have a product of two functions, so we should immediately think to use the product rule. A bit of a challenge here might be remembering derivatives of the sine and cosine functions (in particular the negative signs), so have your formula sheet at the ready if you forget. You might like to set out your solution like this:

$$\begin{aligned}
\text{Let } u &= \sin(2x) \\
\text{and } v &= \cos(2x) \\
\\
\text{Then } u' &= 2\cos(2x) \\
\text{and } v' &= -2\sin(2x) \\
\\
\therefore\ f'(x) &= (u')v + u(v') \\
&= (2\cos(2x)) \times \cos(2x) + \sin(2x) \times (-2\sin(2x)) \\
&= 2\left(\cos(2x)\cos(2x) - \sin(2x)\sin(2x)\right) \\
&= 2\left(\cos^2(2x) - \sin^2(2x)\right) \\
\\
\Longrightarrow\ f'\left(\frac{\pi}{8}\right) &= 2\left(\cos^2\left(2 \times \frac{\pi}{8}\right) - \sin^2\left(2 \times \frac{\pi}{8}\right)\right) \\
&= 2\left(\cos^2\left(\frac{\pi}{4}\right) - \sin^2\left(\frac{\pi}{4}\right)\right) \\
&= 2\left(\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2\right) \\
&= 0
\end{aligned}$$

Remember that $\cos^2(2x)$ means the whole function squared, that is, $\cos^2(2x) = (\cos(2x))^2$, so $\cos^2\left(\frac{\pi}{4}\right) = \left(\cos\left(\frac{\pi}{4}\right)\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$.

As with **part a.**, you should aim be able to differentiate the given function in a few lines, plus a few more lines for evaluation of the derivative.

I won't go into the use of the double formula for this question beyond what is written in the model solutions, for two reasons: one, I don't really recommend using it, since using the product rule is what VCAA would expect you to do and it isn't much more difficult; and two, it's a straightforward substitution that doesn't require much explanation, and those who even think to use it in the first place will know how. You should use the obvious method, and if you have time, and if you happen to remember it, use a double angle formula as a check.

Question 2

a. This is your standard anti-derivative question, and the function you need to antidifferentiate is usually of the form $f(ax+b)$, where f is your standard log, circular, exponential or power function. The means to antidifferentiate these expressions are usually on the formula sheet, so they shouldn't be much trouble. The only challenge with this question would be having the correct sign at the end of it. The general anti-derivative of $\sin(ax)$ is given on the formula sheet as $-\frac{1}{a}\cos(ax)+c$, and this can be used for a simple horizontal shift, i.e. the general anti-derivative of $\sin(ax+b)$ is $-\frac{1}{a}\cos(ax+b)+c$ (think about why - if you shift a function horzontally, it moves the gradient along with it). Here, we have $a=-2$ and $b=4$, so

$$\begin{aligned}\int \sin(4-2x)dx &= -\frac{1}{(-2)}\cos(4-2x)\\ &= \frac{1}{2}\cos(4-2x)\end{aligned}$$

The "$+c$" isn't necessary if they ask for **an** anti-derivative, since as the word 'an' suggests they don't need the general anti-derivative, they just want one, as long as the derivative of your answer is the original expression they gave you. For example, you could pick $c=-941$ and write your answer as $\frac{1}{2}\cos(4-2x)-941$, since its derivative is $\sin(4-2x)$, but obviously $c=0$ is simplest. You can leave the "$+c$" in your answer; it's not wrong to include it, but it isn't necessary.

As with both parts of **Question 1**, this is not a mark you should lose, and should not take more than about thirty seconds.

b. This question is one of the slightly unusual questions that they sometimes have near the beginning of the exam, and it's usually the point where students start to get differentiated based on their skill. They're not exceedingly difficult, but they can throw a few people. Here we have an instruction to evaluate the expression, even though we have an arbitrary constant a present, so this can be a little confusing, but in doubt you should just proceed to perform the integration:

$$\begin{aligned}\int_0^a \frac{1}{x+a}\,dx &= \left[\log_e\left(|x+a|\right)\right]_0^a\\ &= \log_e\left(|a+a|\right)-\log_e\left(|a|\right)\\ &= \log_e\left(|2a|\right)-\log_e\left(|a|\right)\end{aligned}$$

Here we have the modulus functions still present, which can be a little awkward, but we are given that $a > 0$ so they can be removed:

$$\begin{aligned} \log_e(|2a|) - \log_e(|a|) &= \log_e(2a) - \log_e(a) \quad (\text{as } a > 0) \\ &= \log_e\left(\frac{2a}{a}\right) \\ &= \log_e(2) \end{aligned}$$

So despite a variable in the integrand and one of the terminals, the integral is in fact a constant value. It is actually also equal to $\log_e(2)$ for $a < 0$, but the question would be made more complex if we had to consider both cases, so this was left out.

A general reminder when dealing with antiderivatives of $\frac{1}{x}$: firstly, don't forget the modulus function; and secondly, don't remove it without justification (see above; I wrote "as $a > 0$" at the end of the line where I removed the modulus function). If you had forgotten the modulus in this problem you would have fortuitously gotten the right answer regardless, but it wouldn't have been a correct method. I can't say whether VCAA would mark you down for it - it depends on the particular examiner to some extent - but you should be mindful of it. I can say that an answer containing a would not be marked correct, since the instruction 'evaluate' means to find a numerical value, and having a in your answer does not exactly specify one. Part of this question was a test on manipulating logarithms.

Question 3

a. The concept behind this question is fairly simple - finding the maximal domain of a given expression, i.e. what the allowable values of x are - but actually finding the values can be tricky if you aren't familiar with logarithms. They usually have a question testing this kind of familiarity in some form or another, so you should aim to be ready for it.

Here we have a logarithm within a logarithm, $f(x) = \log_e(\log_e(x))$. Clearly x needs to be positive, since $\log_e(x)$ only operates on positive values in $\mathbb{R}$. But we are also taking the logarithm of $\log_e(x)$, so this needs to be positive itself. Some may see straight away that it is when $x > 1$, but the inequality $\log_e(x) > 0$ can actually be solved using the exponential function (the inverse).

We should be familiar with the fact that e^x is increasing for increasing x, but perhaps its use in solving inequalities is unfamiliar. The fact that e^x is increasing means that if $a > b$, then $e^a > e^b$ (draw a quick diagram if you aren't sure), and so:

$$\begin{aligned} \log_e(x) &> 0 \\ \implies e^{\log_e(x)} &> e^0 \quad (\text{as } e^x \text{ is increasing}) \\ \implies x &> 1 \end{aligned}$$

This method of solving inequalities may seem unfamiliar, but we use the notion of increasing and decreasing functions to solve inequalities all the time without realising it. For example, if we multiply both sides of an inequality by a positive constant a, then we are applying the function $f(x) = ax$ to both sides, and this is an increasing function, so the direction of the inequality sign is preserved. If we multiply both sides of an inequality by a negative constant b, then we are applying a decreasing function $g(x) = bx$ to both sides, and so the direction of the inequality sign is reversed.

This is why we cannot, for example, apply the x^2 function to both sides of an inequality, because it is not consistently increasing or decreasing for increasing x. We know $-2 < 1$, but squaring both sides gives $4 < 1$, which is false. But squaring both sides of the inequality $-1 < 2$ gives a true statement. The only way to safely square both sides of an inequality is if we know that both sides are non-positive, or both sides are non-negative. This is because x^2 is decreasing for $(-\infty, 0]$ and increasing for $[0, \infty)$, so if we know $0 \le x < y$ then we can say that $x^2 < y^2$, and if we know $x < y \le 0$, then $x^2 > y^2$.

At this point I should clarify: I included 0 in the above intervals, even though x^2 doesn't have a positive or negative gradient there, because I mean strictly increasing/decreasing, as VCAA likes to call it - it is different from when the gradient is positive/negative, and in fact doesn't require a function to be differentiable everywhere. The definition of a function $f(x)$ being strictly increasing on an interval I is the following: if $a, b \in I$ and $a > b$, then $f(a) > f(b)$, with the definition of strictly decreasing being similar: $a, b \in I$ and $a > b$ implies $f(a) < f(b)$. This is true for points of zero gradient - even though the gradient of x^2 is zero at $x = 0$, we can still safely say that the square of 0 is less than the square of any positive or negative number, so therefore x^2 is strictly decreasing for $(-\infty, 0]$ and strictly increasing for $[0, \infty)$. A major use for the notion of strictly increasing and decreasing functions is how we used them just now - for solving inequalities. A few more examples: $\frac{1}{x}$ is strictly decreasing on $(-\infty, 0)$ and on $(0, \infty)$; the modulus function is strictly decreasing for $(-\infty, 0]$ and strictly increasing for $[0, \infty)$.

Although there are similarities between a function being strictly increasing/decreasing and having a positive/negative gradient, they are ultimately different concepts and shouldn't be confused. VCAA will not examine students on the distinction between these definitions, but it's good to know them (well, I should say "has not examined" - you never know).

Anyway, I should go on with the actual question. Another way to solve the inequality $\log_e(x) > 0$ is a simple graphical approach. The graph of $y = \log_e(x)$ has an x-intercept at $(1, 0)$, is above the x-axis for $x > 1$, and below the x-axis for $0 < x < 1$. So $\log_e(x)$ is positive for $x > 1$. As far as solving this inequality in an examination goes, you will be expected to simply know that $\log_e(x)$, and logarithms of other bases greater than 1, are positive for $x > 1$.

We aren't done with answering the question, however. The question asks us to find D, and although we have found $x > 1$, we must write $D = (1, \infty)$ as our final answer. Be careful with the inclusive and exclusive brackets - you cannot include 1, because we would get $\log_e(\log_e(1)) = \log_e(0)$, which is undefined.

b. This is a fairly standard "ran$(g) \subseteq$ dom(f)" type question that arises from composition of functions, which should be familiar to you.

The idea behind this is that if a function f only operates on certain values, and we compose f and g to get $f(g(x))$, then the values of $g(x)$ must be within this set of values that f will accept. And also, the values of x must be values that g will accept, because we are taking $g(x)$. It's a bit like a chain - x goes through g, which goes through f. Stated more formally, this translates to: If $x \in \text{dom}(g)$, and $\text{ran}(g) \subseteq \text{dom}(f)$, then $f(g(x))$ is defined for all of these values of x.

The tricky part in dealing with this statement is converting it to something we can use. $\text{ran}(g) \subseteq \text{dom}(f)$ can be rewritten as $g(x) \in \text{dom}(f)$; it means essentially the same thing. We want the g to be in the domain of f - that is, we want the values of $g(x)$ to be values that f will accept.

We have $g(x) = \frac{1}{4}x^2$, and $\text{dom}(f) = (1, \infty)$. Hence, $g(x) \in \text{dom}(f)$ is equivalent to $\frac{1}{4}x^2 \in (1, \infty)$, or just $\frac{1}{4}x^2 > 1$. So we now have an inequality we can solve.

$$\begin{aligned} g(x) &> 1 \\ \implies \frac{1}{4}x^2 &> 1 \\ \implies x^2 &> 4 \\ \implies x > 2 &\quad \text{or} \quad x < -2 \\ \implies x &\in (-\infty, -2) \cup (2, \infty) \end{aligned}$$

Now, normally this would be a correct set of values of x for which the composition is defined, but we are given the domain of g in the form (b, ∞), and the set $(-\infty, -2) \cup (2, \infty)$ can't be expressed in this form, because there is a 'gap' in the interval. So we must have the domain of g as being $(2, \infty)$, and thus $b = 2$.

b could be any number larger than 2, however. For example, $f(g(x))$ would be defined if the domain of g was $(4, \infty)$, since $g(x)$ would still be greater than 1 over this domain, but the question asks for the smallest value of b, so we have $b = 2$ as our answer.

Question 4

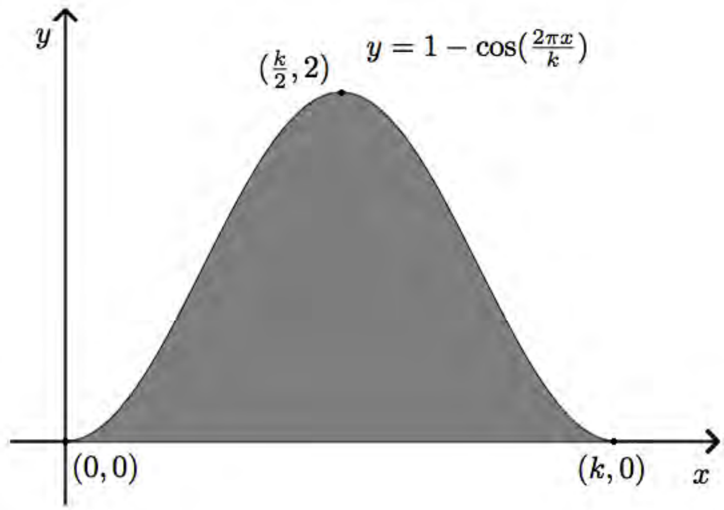
This is one of the simpler questions on this paper. It's essentially a test on knowledge of the fact that the area under the graph of a probability density function is equal to 1. Using this fact, all you need to do set up an appropriate integral and equate it to 1.

$$\begin{aligned}
\int_0^k \left(1-\cos\left(\frac{2\pi x}{k}\right)\right) dx &= 1 \\
\implies \left[x-\frac{1}{\left(\frac{2\pi}{k}\right)}\sin\left(\frac{2\pi x}{k}\right)\right]_0^k &= 1 \\
\implies \left[x-\frac{k}{2\pi}\sin\left(\frac{2\pi x}{k}\right)\right]_0^k &= 1 \\
\implies k-\frac{k}{2\pi}\sin\left(\frac{2\pi k}{k}\right)-0+\frac{k}{2\pi}\sin(0) &= 1 \\
\implies k-\frac{k}{2\pi}\sin(2\pi) &= 1 \\
\implies k-0 &= 1 \\
\implies k &= 1
\end{aligned}$$

This question of course involves antidifferentiating the circular function correctly with respect to x, which can be a little tricky with the two pronumerals involved. It could be somewhat easy to mistake k for x while writing (I did a few times while writing the question...) so that's another reason to do this question carefully. Fortunately the solution here is as simple as they come, so once you've found it you can evaluate $\displaystyle\int_0^1 (1-\cos(2\pi x))\,dx$ and check that it is equal to 1. Questions like these tend to lend themselves to relatively simple checks, so you you should always use them.

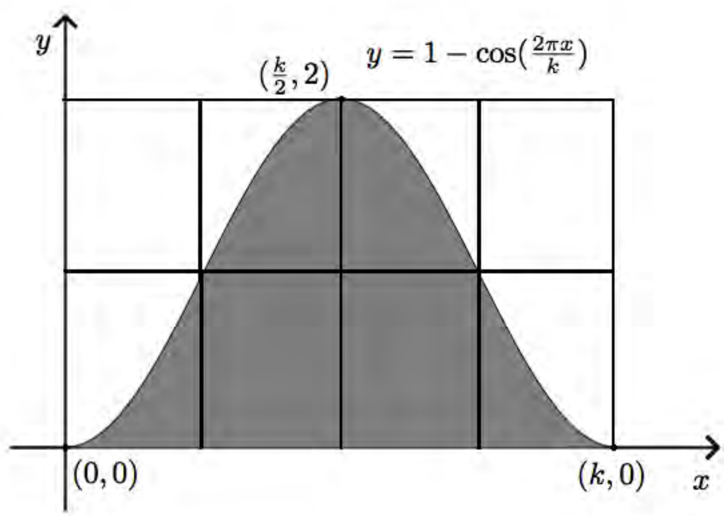
Now, for the graphical approach to this question. This method can be a little difficult to articulate in an exam, so I would recommend the standard integration method, but I will go over this method anyway, since if you pull it off it can save a bit of time - it is rather quick.

You shouldn't have too much trouble determining the period of the cosine function, as well as the other transformations applied to it. You should be familiar with the general formula for the period of the sine and cosine functions: the periods of $\sin(nx)$ and $\cos(nx)$ are both $\dfrac{2\pi}{|n|}$ (and the period of $\tan(nx)$ is $\dfrac{\pi}{|n|}$) so for this function the period is $\dfrac{2\pi}{\left(\frac{2\pi}{k}\right)}=k$. This part of the rule applies for $x\in[0,k]$ so we are in the fairly simple situation where the non-zero part of the graph covers exactly one period. The vertical transformations of the cosine function should be fairly obvious: a reflection in the x-axis, followed by a shift of 1 unit up. Or, if you like, a shift of 1 unit down followed by a reflection in the x-axis. Either way, this, along with the period, should be enough information to quickly sketch $p(x)$:



Your knowledge of the cosine function and transformations should be enough to tell you where the graph touches the x-axis, and the location of the turning point, but even so you can easily evaluate the function at $x = 0$, $\frac{k}{2}$, k to determine the y-values. At this point I should say that if this kind of approach feels like it will take too long, you should stick to the standard integration method.

Now, all you need to do is draw a rectangle and split up the area so you can determine the symmetry (and remember, the reason we can use symmetry is because it's a sinusoidal curve - most other functions do not have symmetry this exploitable):



Take a look at the above diagram for a moment. There are a few ways to see that the shaded region takes up exactly half the area of the large rectangle. The two shaded regions on the top half of the large rectangle are the same size as the two white regions on the bottom half, so if you swapped their colours you'd have the top half of the rectangle unshaded and the bottom half shaded. Or if you just look at the right or left side of the large rectangle separately, you can see how the curve cuts the 2×2 blocks in half on both sides.

In any case, this wouldn't require much justification on the exam - if you spot the symmetry and draw a quick sketch, all you have to do is let them know you've spotted it and then proceed to write an equation for k. So all you'd really need to write, if you answered the question this way, is something close to this (along with the above sketch of the graph):

The area under the graph is exactly half the area of a $2 \times k$ rectangle. Therefore,

$$\begin{aligned}\frac{1}{2} \times 2 \times k &= 1 \\ \implies k &= 1\end{aligned}$$

This is quite a quick solution, and you will get the full 3 marks for an approach like this, as long as you communicate it properly, because it's efficient and shows some creativity. The hardest part about this method is figuring out what the graph looks like, so if you can work that out quickly then this is the way to go. But, again, if this sort of thing does come up, and this kind of method feels too time-consuming, go for the integration.

Question 5

(Reminder: the notation $\sin^n(x)$ means $(\sin(x))^n$)

At least two of the last few tech-free exams from VCAA have included a question like this; that is, one where you need to divide both sides by $\cos(x)$ (or $\cos^2(x)$ in this case) to get $\tan(x)$ (although this particular one can be solved by other means which I will also discuss). Any equation of the form $a\sin^n(x) = b\cos^n(x)$, where $a \neq 0$, can be put in the form $\tan^n(x) = \frac{b}{a}$, and this question is a particularly simple case.

Before proceeding, another reminder: be careful when dividing both sides of an equation by a function. You need to be sure that it cannot equal zero. In this case, we know that $\cos^2(x)$ and $\sin^2(x)$ cannot be equal to zero simultaneously (their sum is equal to 1), so no solutions can arise from the case where $\cos^2(x) = 0$. Hence, we can divide by $\cos^2(x)$:

$$\begin{aligned}\sin^2(x) &= \cos^2(x) \\ \implies \frac{\sin^2(x)}{\cos^2(x)} &= 1 \\ \implies \left(\frac{\sin(x)}{\cos(x)}\right)^2 &= 1 \\ \implies \tan^2(x) &= 1 \\ \implies \tan(x) &= \pm 1\end{aligned}$$

Or alternatively you could use a factorisation method:

$$\begin{aligned}\tan^2(x) &= 1\\ \implies \tan^2(x)-1 &= 0\\ \implies (\tan(x)-1)(\tan(x)+1) &= 0\\ \implies \tan(x) &= \pm 1\end{aligned}$$

It's important not to forget the negative solution, and that was the main point of this question.

At this point we have the equations $\tan(x)=1$ and $\tan(x)=-1$, which, for clarity, we will solve separately, starting with $\tan(x)=1$. We should be familiar with the fact that $\tan\left(\frac{\pi}{4}\right)=1$, so we have one solution to the equation, $x=\frac{\pi}{4}$. How many more should there be? Well, our interval is $[0,2\pi]$, and the period of $\tan(x)$ is π, so it covers two periods. There should therefore be two solutions, due to $\tan(x)$ covering each of its values only once in each period (as opposed to $\sin(x)$ and $\cos(x)$, which can double up on values). The other solution should occur after exactly one period, so adding π to our first solution gives $\frac{\pi}{4}+\pi=\frac{5\pi}{4}$. Quickly check that this is in $[0,2\pi]$, and go on to the next equation. (Disclaimer: although two periods should mean two solutions for $\tan(x)=c$, in the case where we had the equation $\tan(x)=0$, we would have three solutions over the interval $[0,2\pi]$; think about why - the 2π at the end of the interval can be considered the start of a third period).

Using more or less the same approach to solve $\tan(x)=-1$, we can use $\tan\left(\frac{3\pi}{4}\right)=-1$ to get our first solution of $\frac{3\pi}{4}$, and adding π to this gives $\frac{7\pi}{4}$. Again, there should only be two solutions, and in total we get four, $x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$. If you're on the ball you can probably churn out the four solutions just from looking at $\tan(x)=\pm 1$.

Now, for the second method of answering this question, which uses the identity $\sin^2(\theta)+\cos^2(\theta)=1$. This isn't on the formula sheet, so I recommend memorising it. Rearranging the identity gives $\cos^2(\theta)=1-\sin^2(\theta)$, and so:

$$\begin{aligned}\sin^2(x) &= \cos^2(x)\\ \implies \sin^2(x) &= 1-\sin^2(x)\\ \implies 2\sin^2(x) &= 1\\ \implies \sin(x) &= \frac{1}{2}\\ \implies \sin(x) &= \pm\frac{1}{\sqrt{2}}\\ \implies x &= \frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}\end{aligned}$$

The solutions $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ come from the equation $\sin(x)=\frac{1}{\sqrt{2}}$, and the solutions $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ come from $\sin(x)=-\frac{1}{\sqrt{2}}$.

Alternatively, we can rearrange $\sin^2(\theta) + \cos^2(\theta) = 1$ to give $\sin^2(\theta) = 1 - \cos^2(\theta)$, and we get:

$$\begin{aligned} \sin^2(x) &= \cos^2(x) \\ \implies 1 - \cos^2(x) &= \cos^2(x) \\ \implies 1 &= 2\cos^2(x) \\ \implies \cos^2(x) &= \frac{1}{2} \\ \implies \cos(x) &= \pm\frac{1}{\sqrt{2}} \\ \implies x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

This time, the solutions $\frac{\pi}{4}$ and $\frac{7\pi}{4}$ come from the equation $\cos(x) = \frac{1}{\sqrt{2}}$, and the solutions $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$ come from $\cos(x) = -\frac{1}{\sqrt{2}}$.

Now, in the model solutions I described the method using the double angle formula, $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$, which reduces the problem to just the one simple equation, $\cos(2x) = 0$. Again, I won't go over the use of the double angle formula, since it isn't required and is a straightforward substitution anyway, but I will use the resulting equation, $\cos(2x) = 0$, to demonstrate a point.

You may have been taught to list an excessive amount of solutions to an equation of the form $\cos(ax + b) = 0$, solve for x, and get rid of solutions that aren't in the domain, but this is inefficient, and there are better ways to go about it.

If you do get an equation like $\cos(2x) = 0$, for $x \in [0, 2\pi]$, then you should manipulate the inequality $0 \leq x \leq 2\pi$ like so:

$$\begin{aligned} & 0 \leq x \leq 2\pi \\ \iff \quad & 0 \leq 2x \leq 4\pi \end{aligned}$$

So if you let $\theta = 2x$, then $0 \leq \theta \leq 4\pi$, and you can solve $\cos(\theta) = 0$ for $\theta \in [0, 4\pi]$:

$$\begin{aligned} \cos(\theta) &= 0 \\ \implies \theta &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \quad \text{(all solutions in } [0, 4\pi]) \\ \implies 2x &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \\ \implies x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

By doing this, you don't list any uncessesary solutions, and you don't miss any, for that matter. Needless to say, practicing this method by making up a few examples, and checking your answers on your calculator (after you've given them a good go of course), will go a long way.

Question 6

This is a common worded probability question that is designed to be different from the stock standard probability questions, so as to test students on whether they are actually thinking about the problem, or just regurgitating formulas. Some might expect a binomial distribution, or a conditional probability question, but this is neither. The probability of James hitting a bull's-eye does not depend on whether he hit the previous throw, so it is not conditional, and there isn't a set number of throws, so it isn't binomial, even though it is a success/fail situation. What kind of distribution is it? This is somewhat detached from the question, but I will address it anyway, for those interested. But first, the actual question.

Let X be the number of consecutive bull's-eyes that James throws. The event of James winning more than \$20 corresponds to $X > 2$. But since X could potentially be any number greater than 2, it would be impossible to sum all of the probabilities (at least within the scope of this course). So we will use the fact that $\Pr(X > 2) = 1 - \Pr(X \leq 2)$, and find $\Pr(X \leq 2)$, which is just $\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)$.

$X = 0$ corresponds to the event that he misses on the first throw, and this event has a probability of $1 - 0.1 = 0.9$, i.e. $\Pr(X = 0) = 0.9$.

In the event that he hits exactly once, it will be a hit followed by a miss, and so, $\Pr(X = 1) = 0.1 \times 0.9 = 0.09$.

In the event that he hits exactly twice, it will be two hits followed by a miss (remember, they must all be consecutive, so there isn't the option of, say, a hit followed by a miss followed by another hit), so we have $\Pr(X = 2) = 0.1 \times 0.1 \times 0.9 = 0.009$.

Thusly, $\Pr(X \leq 2) = 0.9 + 0.09 + 0.009 = 0.999$, and hence $\Pr(X > 2) = 1 - 0.999 = 0.001$, or $\dfrac{1}{1000}$. This method of complements should be familiar, as well as the differences in how to use them in discrete versus continuous distributions.

A good way to tell if you're on the right track is to determine whether all of the probabilities will sum to 1. Here, we can perhaps see the general pattern: $\Pr(X = n) = 0.9 \times (0.1)^n$. If you sum all of these together, you will get $0.99999... = 1$.

You can also prove this using the converging geometric series formula. This isn't part of the course, but I will demonstrate it for those who know about it, or would like to know about it. The general formula for converging infinite geometric series (i.e. infinite geometric series that approach a certain value) is $1 + r + r^2 + r^3 + \cdots = \dfrac{1}{1 - r}$, wherever $|r| < 1$. The sum of our probabilities is $0.9\left(1 + 0.1 + (0.1)^2 + (0.1)^3 + \cdots\right)$, so in this case we have $r = 0.1$. Using the formula, the sum of our probabilities is $0.9\left(1 + 0.1 + (0.1)^2 + (0.1)^3 + \cdots\right) = 0.9\left(\dfrac{1}{1 - 0.1}\right) = 0.9\left(\dfrac{1}{0.9}\right) = 1$.

Question 7

a. I always prefer to do this sort of question algebraically, because it's more fool-proof than a descriptive approach. But as you might be able to tell, since we are dealing with points in this question, and not curves, a descriptive approach (e.g. "a dilation by a factor of...") will be practically impossible to pull off. I would encourage algebraic methods anyway, since they don't take very long if you've practiced them enough.

The starting point of this question is the matrices. In particular, you need to know how to multiply them out. The matrix product $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ gives $\begin{bmatrix} ax + 0y \\ 0x + by \end{bmatrix}$, which is just $\begin{bmatrix} ax \\ by \end{bmatrix}$, and adding $\begin{bmatrix} c \\ d \end{bmatrix}$ to this gives $\begin{bmatrix} ax + c \\ by + d \end{bmatrix}$. Each of these two elements are the transformed coordinates, so if we let x' represent the transformed x-coordinate, and y' represent the transformed y-coordinate, we get $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax + c \\ by + d \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$, and therefore

$$\begin{aligned} x' &= ax + c \\ \text{and} \quad y' &= by + d \end{aligned}$$

We are finding the four values, a, b, c, and d, so we need at least four equations. We have four points, so this is likely where we will have to get them from.

Now, we know the point $(1, 1)$ is mapped to the point $(-2, 2)$, so we can use the two equations we found above to generate two more equations. The trick here is to consider each ordinate separately. In other words, from these two points, we can see that $x' = -2$ when $x = 1$, and that $y' = 2$ when $y = 1$. So, using our first two equations, we get two more, which we will label, since we will refer back to them.

$$\begin{aligned} x' &= ax + c \\ \implies -2 &= a + c \quad (1) \qquad (\text{using } x = 1 \text{ and } x' = -2) \\ \\ \text{and} \quad y' &= by + d \\ \implies 2 &= b + d \quad (2) \qquad (\text{using } y = 1 \text{ and } y' = 2) \end{aligned}$$

Using exactly the same approach with the points $(2, 2)$ and $(3, 1)$, we get

$$\begin{aligned} x' &= ax + c \\ \implies 3 &= 2a + c \quad (3) \qquad (\text{using } x = 2 \text{ and } x' = 3) \\ \\ \text{and} \quad y' &= by + d \\ \implies 1 &= 2b + d \quad (4) \qquad (\text{using } y = 3 \text{ and } y' = 1) \end{aligned}$$

So, we now have four fairly simple equations in the four unkowns. But notice that only the equations resulting from the x and x' values contain a and c, and only the equations resulting from the y and y' values contain b and d. It makes sense that these variables would be independent of each other, because whether you dilate, reflect or translate an x-coordinate, it will not affect the value of the y-coordinate, and vice versa.

Solving this set of four equations is fairly simple once you realise that it is really two sets of two equations, in two unknowns each. We have

$$\begin{aligned} a+c &= -2 \quad (1) \\ \text{and} \quad 2a+c &= 3 \quad (3) \\ \\ \text{Eq(3)} - \text{Eq(1) gives} \qquad (2a+c)-(a+c) &= 3-(-2) \\ \Longrightarrow a &= 5 \\ \\ \therefore 5+c &= -2 \quad (\text{from (1)}) \\ \Longrightarrow c &= -7 \end{aligned}$$

And using the other two equations:

$$\begin{aligned} b+d &= 2 \quad (2) \\ \text{and} \quad 2b+d &= 1 \quad (4) \\ \\ \text{Eq(4)} - \text{Eq(2) gives} \qquad (2b+d)-(b+d) &= 1-2 \\ \Longrightarrow b &= -1 \\ \\ \therefore -1+d &= 2 \quad (\text{from (2)}) \\ \Longrightarrow d &= 3 \end{aligned}$$

So we have our four values, $a = 5$, $b = -1$, $c = -7$ and $d = 3$.

b. This question is the more standard type of transformation question, where you simply apply a known transformation to an equation. If you use the previous part of the question, there are two ways to tackle this part, one being the systematic, fool-proof approach, which I will go through now, and the other being simply reading numbers off the matrix and writing down an equation, which I will also explain how to do.

The systematic approach is as follows. From **part a.**, we know that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -7 \\ 3 \end{bmatrix}$, so we get the two equations:

$$\begin{aligned} x' &= 5x - 7 \\ \text{and } y' &= -y + 3 \end{aligned}$$

Our given equation is $y = x$, but we need the equation of the image of the line, so we need to get rid of x and y to get an equation with x' and y' only. We do this by transposing the two equations above to make x and y the subjects, and it is usually a simple transposition:

$$\begin{aligned} x' &= 5x - 7 \\ \iff x &= \frac{x' + 7}{5} \\ \\ \text{and } y' &= -y + 3 \\ \iff y &= -(y' - 3) \end{aligned}$$

Once we have these, wherever we see x and y in our old equation, we replace them with whatever they are in terms of x' and y'. So, since our equation is $y = x$, we get

$$\begin{aligned} y &= x \\ \implies -(y' - 3) &= \frac{x' + 7}{5} \\ \iff y' &= -\frac{x'}{5} - \frac{7}{5} + 3 \\ \iff y' &= -\frac{x'}{5} + \frac{8}{5} \end{aligned}$$

This systematic approach to applying transformations usually ends up being the most reliable, but with enough experience it's possible to write down a transformed equation just from looking at the matrices in the equation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -7 \\ 3 \end{bmatrix}$. You can see that $y' = -y + 3$, and you can mentally transpose $x' = 5x - 7$ to get $x = \dfrac{x' + 7}{5}$. Using these, and your original equation $y = x$, you can do the following:

$$\begin{aligned} y' &= -y + 3 \\ \implies y' &= -x + 3 \qquad \text{(from the original equation, } y = x\text{)} \\ \implies y' &= -\left(\frac{x' + 7}{5}\right) + 3 \\ \implies y' &= -\frac{x'}{5} + \frac{8}{5} \end{aligned}$$

The general method to quickly apply a transformation of the form $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$ to an equation $y = f(x)$ is to remember that the application of the transformation will give $y' = bf\left(\dfrac{x'-c}{a}\right) + d$ (try proving this on your own as an exercise, it will help you to remember it). However, in a tech-free exam, if you aren't completely sure of how to use this method, or aren't sure if you remembered it correctly, you should use the more careful method, with all the steps, as we did before. I recommend practicing applying transformations of the form $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$ to curves, first by using the drawn-out method, and until you can do them in a few steps, or at least until you can use the systematic method in a small amount of time. Also, since you may get a transformation that deviates slightly from this general form, for example $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right)$, you should become familiar with simple matrix multiplication and addition, so you can get it in the form you're used to (here, you would just have to distribute over the brackets, but any transformation should be able to be put in the form $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$). This kind of familiarity and confidence will save time on both exams, not just the tech-free one.

This particular question has another way of being solved, and this method is a result of the fact that we are given a line, rather than anything else, and also the fact that this line passes through the un-transformed points we are given, $(1, 1)$ and $(2, 2)$.

If a transformation like this is applied to a line, it will still be a line afterwards. And any point on the original line will be mapped to a point that is on the image of the line. Therefore, the two transformed points, $(-2, 2)$ and $(3, 1)$, will lie on the transformed line, and as we are familiar with, if you know two points on a line, you can find its equation.

The gradient of the line through $(-2, 2)$ and $(3, 1)$ is $\dfrac{\Delta y'}{\Delta x'} = \dfrac{1-2}{3-(-2)} = -\dfrac{1}{5}$, so the equation of the image of the line is $y' = -\dfrac{x'}{5} + c$. To find c, use either of the points it passes through:

$$\begin{aligned} y' &= -\frac{x'}{5} + c \\ \therefore 2 &= -\frac{(-2)}{5} + c \quad \text{(using the point } (-2, 2)) \\ \implies c &= 2 - \frac{2}{5} \\ &= \frac{8}{5} \end{aligned}$$

Hence the equation of the transformed line is $y' = -\frac{x'}{5} + \frac{8}{5}$. You could also use the fact that, if the point (x', y') lies on the transformed line, then the gradient of the line through (x', y') and $(-2, 2)$ (or $(3, 1)$, if you like) is equal to $-\frac{1}{5}$, so therefore

$$\begin{aligned} \frac{y'-2}{x'-(-2)} &= -\frac{1}{5} \\ \implies y'-2 &= -\frac{1}{5}(x'+2) \\ \implies y' &= -\frac{x'}{5} - \frac{2}{5} + 2 \\ \implies y' &= -\frac{x'}{5} + \frac{8}{5} \end{aligned}$$

VCAA isn't too strict on whether you have primed or unprimed variables (that is, x' versus x), but I like to leave the primes on the variables if I'm referring to transformed ordinates, it avoids confusion for the most part.

You can check that your equation is correct by substitution the values for the points $(-2, 2)$ and $(3, 1)$ into the equation you get. If they both satisfy the equation then your line is the correct one.

Note that you cannot use this part of the question to answer **part a.** (though you shouldn't have to answer previous parts of a question with later parts anyway), as there are actually an infinite amount of transformations that could take the line $y = x$ to the line $y' = -\frac{x'}{5} + \frac{8}{5}$, so you could not get the values of a, b, c, and d from the transformed equation alone. Most curves will be ambiguous when it comes to transformations, anyway, so trying to find the transformation applied to a curve from its original and transformed equation usually isn't possible without some other piece of information (such as being given one of the four values).

Question 8

a. Both parts of this question are simple and common types of probability questions that really only test basic knowledge of probability and set theory. Given $\Pr(A \cap B)$ and $\Pr(A \cap B')$, you should be able to calculate $\Pr(A)$ by just adding these two together. The fact that the sum of these gives $\Pr(A)$ can be determined from a Venn diagram or Karnaugh map, but it also makes sense intuitively. If something is in a set A, then it is either in A and in B, or it is in A but not in B.

Carrying out this addition, we get

$$\begin{aligned} \Pr(A) &= \Pr(A \cap B) + \Pr(A \cap B') \\ &= \frac{1}{4} + \frac{5}{8} \\ &= \frac{2}{8} + \frac{5}{8} \\ &= \frac{7}{8} \end{aligned}$$

Now, we are given that A and B are independent so, $\Pr(A) \times \Pr(B) = \Pr(A \cap B)$. We know $\Pr(A)$ and $\Pr(A \cap B)$, so we can calculate $\Pr(B)$:

$$\begin{aligned}
\Pr(B) &= \frac{\Pr(A \cap B)}{\Pr(A)} \\
&= \frac{\left(\frac{1}{4}\right)}{\left(\frac{7}{8}\right)} \\
&= \frac{1}{4} \times \frac{8}{7} \\
&= \frac{2}{7}
\end{aligned}$$

Generally this type of question can be solved easily with a Venn diagram or Karnaugh map. And, again, these types of questions only test basic knowledge, so they are marks you shouldn't really lose if you take care with the arithmetic and your diagrams.

b. We should be familiar with the fact that two events are independent if and only if the product of their probabilities is equal to the probability of their intersection. Thus, to show that A and B' are independent, we must first find $\Pr(B')$, and then show that $\Pr(A) \times \Pr(B') = \Pr(A \cap B')$. The way we can do this is as follows:

$$\begin{aligned}
\Pr(B') &= 1 - \Pr(B) \\
&= \frac{7}{7} - \frac{2}{7} \\
&= \frac{5}{7} \\
\therefore\ \Pr(A) \times \Pr(B') &= \frac{7}{8} \times \frac{5}{7} \\
&= \frac{5}{8} \\
&= \Pr(A \cap B')
\end{aligned}$$

Therefore, A and B' are independent, as required.

This is a straightforward question that doesn't require much explanation, but there are a few things I should discuss. If you weren't getting the result that $\Pr(A) \times \Pr(B') = \Pr(A \cap B')$ this would have been an indication that you had incorrecly found one or more of your values of $\Pr(A)$, $\Pr(B)$, and $\Pr(B')$. Also, in a 'show that' question such as this, you usually need to have some form of $RHS = LHS$ set-up. That is, having "$\Pr(A) \times \Pr(B') = ...$" eventually lead to "$... = \Pr(A \cap B')$", or the other way around. Take this for example:

$$\begin{aligned}
\Pr(A) \times \Pr(B') &= \frac{7}{8} \times \frac{5}{7} \\
&= \frac{5}{8}
\end{aligned}$$

If you had only done this in your proof, without any conclusion linking this to $\Pr(A \cap B')$, it is unlikely you would be awarded full marks. You should be as thorough as possible in 'show that' questions, since the fact that they give you a result and ask you to show it is true means that the only thing they are giving you marks for is your understanding of the derivation of the result. You want them to know that you understand what is going on.

It is actually possible to show that the independence of A and B necessarily implies A and B' are independent, because it is generally true. Provided that $\Pr(A) \times \Pr(B) = \Pr(A \cap B)$, we have

$$\begin{aligned}
\Pr(A) \times \Pr(B') &= \Pr(A) \times (1 - \Pr(B)) \\
&= \Pr(A) - \Pr(A) \times \Pr(B) \\
&= \Pr(A) - \Pr(A \cap B) \qquad (A \text{ and } B \text{ are independent}) \\
&= \Pr(A \cap B')
\end{aligned}$$

Therefore, A and B' are independent, as required. This kind of approach would be acceptable, particularly if you had made a mistake elsewhere and were unable to prove the specific case, but again, you need to be clear with your working.

Question 9

a. It should be noted from the start that you cannot approach this question by assuming that $a = 1$ and $p = 3$ and substituting these values in. You cannot assume the given result of a 'show that' question in order to prove the result. Here, the instruction 'show that' means you need to go about finding the values of a and p as if they were not given to you. The only reason the values are given to you is so the second part of the question can still be done if you cannot complete the first part.

Now, there are three ways to go about this question, but two of them are similar in most ways. The first method (which I would recommend using) is to differentiate $f(x)$ and use the fact that the derivative of $f(x)$ at $x = p$ is equal to the gradient of $g(x)$, and therefore the gradient of the line segment OP.

O is the point $(0, 0)$ and P is the point $(p, f(p))$, so the gradient of OP is $\dfrac{f(p) - 0}{p - 0} = \dfrac{f(p)}{p}$. This is equal to the derivative of $f(x)$ at $x = p$, and the derivative of $f(x)$ is $3x^2 - 12x + 8$, so

$$\begin{aligned}
\frac{f(p)}{p} &= f'(p) \\
\implies \frac{p^3 - 6p^2 + 8p}{p} &= 3p^2 - 12p + 8 \\
\implies p^2 - 6p + 8 &= 3p^2 - 12p + 8 \\
\implies 0 &= 3p^2 - p^2 - 12p + 6p + 8 - 8 \\
\implies 2p^2 - 6p &= 0 \\
\implies p(p - 3) &= 0 \\
\implies p &= 3 \quad (\text{as } p > 0)
\end{aligned}$$

Since the equation gives two potential values of p, you should give a reason for picking one over the other, especially in a 'show that' question. We were given that p is a positive number, so $p = 0$ is not a solution.

Now, there are two ways to find a. Since $-a$ is the gradient of $g(x)$, it is equal to the gradient of OP, as well as $f'(p)$, so you can use either of these:

$$\begin{aligned} -a &= f'(3) \\ \implies a &= -(3 \times 3^2 - 12 \times 3 + 8) \\ &= -27 + 36 - 8 \\ &= 1 \end{aligned}$$

Or:

$$\begin{aligned} -a &= \frac{f(3)}{3} \\ \implies a &= -\left(\frac{3^3 - 6 \times 3^2 + 8 \times 3}{3}\right) \\ &= -9 + 6 \times 3 - 8 \\ &= -9 + 18 - 8 \\ &= 1 \end{aligned}$$

Once you have found both values, you should end your solution with something to the effect of "therefore $a = 1$ and $p = 3$, as required."

Now, the second method you can use is one where you find the equation of the tangent at P, in terms of p. This way you can let the y-intercept of the tangent equal 0 and solve for p.

The gradient of the tangent at P is $f'(p)$, so the equation of the tangent is $y = f'(p) \cdot x + c$. To find c in terms of p, we will use the point $(p, f(p))$ (you can use $(0, 0)$, but you would just get $c = 0$ and get nowhere). We have

$$\begin{aligned} y &= f'(p) \cdot x + c \\ \therefore\ f(p) &= f'(p) \cdot p + c \qquad \text{(using } (p, f(p))\text{)} \\ \implies c &= f(p) - f'(p) \cdot p \end{aligned}$$

So the equation of the tangent is $y = f'(p) \cdot x + f(p) - f'(p) \cdot p$. But since we know the y-intercept is 0, we get

$$\begin{aligned} f(p) - f'(p) \cdot p &= 0 \\ \implies f(p) &= f'(p) \cdot p \\ \implies \frac{f(p)}{p} &= f'(p) \end{aligned}$$

This is exactly what we had before, so the rest of the solution is the same. Though, I should say that being able to find the generalised tangent to a graph (at, say, $x = m$) is a useful skill for either exam.

The third method is to use the fact that $g(x)$ intersects $f(x)$ exactly twice. If a was slightly bigger or smaller, there would be either 1 or 3 points of intersection, so we can use the fact that there are 2 points of intersection to our advantage. We do this by equating $f(x)$ and $g(x)$ and trying to figure out how there can be two solutions to the equation. We have

$$\begin{aligned} f(x) &= g(x) \\ \iff x^3 - 6x^2 + 8x &= -ax \\ \iff x^3 - 6x^2 + 8x + ax &= 0 \\ \iff x(x^2 - 6x + 8 + a) &= 0 \end{aligned}$$

Now, this is a cubic equation, so there can be anywhere from 1 to 3 solutions. Our knowledge of cubics tells us that there are two solutions whenever we have a linear factor that occurs twice, plus one other distinct factor. In this case, it is possible for x to be a repeated factor, and that is when $8 + a = 0$, since we would get the equation $x(x^2 - 6x) = x^2(x - 6) = 0$. But this cannot be the case, for two reasons. One, the equation $8 + a = 0$ gives $a = -8$, and we know that a is positive. Two, even if a could be negative, the solutions to the equation $x^2(x - 6) = 0$ would be $x = 0$ and $x = 6$, but we can see from the graph and the factorised rule for f that $2 < p < 4$, so there must be a solution in this range, which neither 0 nor 6 are.

It might seem intuitive or obvious that x is not the repeated linear factor, or you may not have noticed the possibility at all, so if you didn't consider this case it most likely wouldn't be penalised, but this is why I would recommend the first method, since it usually makes all possible solutions obvious. Generally, though, you should always look for all possible solutions to a problem like this, since part of these questions is showing that the values given are the only possible values. You should have your goal of arriving at the given values in mind, but you should look out for other values, and reasons why they can't also be solutions.

Moving on, the only other way there can be a repeated linear factor of $x(x^2 - 6x + 8 + a)$ is if the quadratic factor $x^2 - 6x + 8 + a$ is a perfect square. The are a few ways to determine the value of a for which it is a perfect square. You could simply recognise that it would have to be the perfect square $x^2 - 6x + 9$ due to first two terms, and therefore that $8 + a = 9$, so $a = 1$. Or you could consider the discriminant of the quadratic:

$$\begin{aligned} \Delta &= (-6)^2 - 4(8 + a) \\ &= 36 - 32 - 4a \\ &= 4 - 4a \end{aligned}$$

This must equal 0 for the quadratic to be a perfect square. Solving $4 - 4a = 0$ for a gives $a = 1$.

Either way, we get the equation $x(x - 3)^2 = 0$, which gives us the x-coordinates of the points of intersection of the graphs of f and g. These x-coordinates are $x = 0$ and $x = 3$. We know f and g intersect at O and P, and since $x = 0$ is obviously the x-coordinate of O, $x = 3$ must be the x-coordinate of P, and so $p = 3$. Thus, we have $a = 1$ and $p = 3$.

b. This is a straightforward integration question that could be done even if **part a.** was not completed. This question requires use of the general formula for the area between two curves f and g that intersect at $x = a$ and $x = b$, and for which $f(x) > g(x)$ for $a < x < b$. The area enclosed by the curves for this interval is given by $\int_a^b (f(x) - g(x))\,dx$. In this case, the two curves intersect at $x = 0$ and $x = 3$, with $f(x) > g(x)$ for $0 < x < 3$, and so the area is given by:

$$\begin{aligned}
\int_0^3 (f(x) - g(x))\,dx &= \int_0^3 \left(x^3 - 6x^2 + 8x - (-x)\right) dx \\
&= \int_0^3 \left(x^3 - 6x^2 + 9x\right) dx \\
&= \left[\frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2\right]_0^3 \\
&= \frac{3^4}{4} - 2 \times 3^3 + \frac{9 \times 3^2}{2} \\
&= \frac{3^4}{4} - 2 \times 3^3 + \frac{3^4}{2} \\
&= 3^3\left(\frac{3}{4} - 2 + \frac{3}{2}\right) \\
&= 27\left(\frac{3}{4} - \frac{8}{4} + \frac{6}{4}\right) \\
&= 27\left(\frac{1}{4}\right) \\
&= \frac{27}{4}
\end{aligned}$$

Hence, the area of the shaded region is $\frac{27}{4}$ units^2. Although you would not be penalised for omitting the "units^2" or "square units", it's better to include it.

The arithmetic required for this question is about as hard as it gets for a tech-free exam, so if you manage this, you should be able to deal with everything on the arithmetic side of things. A general tip I have is to try and take common factors out, since it usually avoids adding or subtracting overly large numbers. It probably goes without saying that you should take a lot of care, and also that you should check your answer is reasonable. In this case, a negative answer would have been a sign that you made an error somewhere.

Question 10

a. VCAA will usually have their final question on their tech-free exam as something a little out of the ordinary, so as to separate the top students. It will usually require, at some point, some kind of method that is different to the ones you would be used to using during the year, and requires you to think on the spot, so it is really a test of your ability to adapt your methods and knowledge to new situations. However, you may already be familiar with the methods required anyway, and this is also deserving of marks.

The first thing you need to be able to do here is correctly interpret the piece of information that each edge of the pyramid is equal to x metres in length. Namely, the fact that this refers to the four slanted edges, AB, AC, AD, and AE, as well as the four edges of the square base. It does not include AM, firstly because it isn't an edge - it lies within one of the triangular faces - and secondly because it would be impossible for it to be the same length as AC, AD and CD, the three edges that surround it. On that note, these three edges are all equal in length, and so ACD is an equilateral triangle, meaning each of the three angles within the triangle are equal to $\frac{\pi}{3}$ radians or 60 degrees. Hence, we may use simple trigonometric ratios to find AM. You could use any of the sine, cosine or tangent ratios to do this, but here we will use sine, since it the simplest one to use here:

$$\begin{aligned} \frac{AM}{AC} &= \sin\left(\frac{\pi}{3}\right) \\ \implies \frac{AM}{x} &= \frac{\sqrt{3}}{2} \\ \implies AM &= \frac{\sqrt{3}}{2}x \end{aligned}$$

At this point, you need to recognise that AO is perpendicular to the base of the pyramid, meaning AOM is a right-angled triangle. So, using our favourite theorem, we get:

$$\begin{aligned} AO &= \sqrt{AM^2 - OM^2} \\ &= \sqrt{\left(\frac{\sqrt{3}}{2}x\right)^2 - \left(\frac{1}{2}x\right)^2} \\ &= \sqrt{\frac{3}{4}x^2 - \frac{1}{4}x^2} \\ &= \sqrt{\frac{2}{4}x^2} \\ &= \sqrt{\frac{1}{2}x^2} \\ &= \frac{1}{\sqrt{2}}x \end{aligned}$$

So we have $AM = \frac{\sqrt{3}}{2}x$ and $AO = \frac{1}{\sqrt{2}}x$.

An alternative way to approach this is to find AO first, and then find AM. If you remember your standard $1 : 1 : \sqrt{2}$ right-angled triangle, you can use the fact that the hypotenuse of the triangle BCD (or one of the other triangles within the square base) is $\sqrt{2}$ times either of the other edges:

$$\begin{aligned} BD &= \sqrt{2}x \\ \implies OD &= \frac{\sqrt{2}}{2}x \\ &= \frac{1}{\sqrt{2}}x \end{aligned}$$

$$\begin{aligned}
\therefore AO &= \sqrt{AD^2 - OD^2} \\
&= \sqrt{x^2 - \left(\frac{1}{\sqrt{2}}x\right)^2} \\
&= \sqrt{\frac{2}{2}x^2 - \frac{1}{2}x^2} \\
&= \sqrt{\frac{1}{2}x^2} \\
&= \frac{1}{\sqrt{2}}x
\end{aligned}$$

$$\begin{aligned}
\therefore AM &= \sqrt{AO^2 + OM^2} \\
&= \sqrt{\left(\frac{1}{\sqrt{2}}x\right)^2 + \left(\frac{1}{2}x\right)^2} \\
&= \sqrt{\frac{1}{2}x^2 + \frac{1}{4}x^2} \\
&= \sqrt{\frac{2}{4}x^2 + \frac{1}{4}x^2} \\
&= \sqrt{\frac{3}{4}x^2} \\
&= \frac{\sqrt{3}}{2}x
\end{aligned}$$

b. This is a straightforward question that just requires use of the pyramid volume formula (given on the formula sheet) and is a mark that would be awarded conesquentially if the answer in **part a.** for AO in terms of x is incorrect. The area of the base is x^2, and the height of the pyramid is $AO = \frac{1}{\sqrt{2}}x$, and so:

$$\begin{aligned}
V &= \frac{1}{3} \times x^2 \times \left(\frac{1}{\sqrt{2}}x\right) \\
&= \frac{1}{3\sqrt{2}}x^3 \\
&= \frac{\sqrt{2}}{6}x^3
\end{aligned}$$

c. Before I explain this question, I'd like to discuss why I included it in the first place, and there are a few reasons.

This question can be done easily by appropriately using the concept of similar triangles, which is something that has appeared on a question in a tech-free exam before, despite not explicitly being in the study design. This is because it is knowledge that students are expected to have retained from earlier years of study of mathematics. This question was intended to emphasise that the concept is examinable.

This question also uses the fact that OY being the shortest distance from O to AM coincides with OY being perpendicular to AM. This idea was also able to be used on another past VCAA question. Those who had already done past the past VCAA exams prior to this one would have encountered the aforementioned questions, and therefore hopefully would have been able to apply the two concepts here, and this would have been a nice 3-mark reward for doing the past exams, which is definitely encouraged.

I also wanted to bring up the idea that not every minimisation or maximisation problem needs to use calculus, particularly when there are other much more efficient means available. Understanding calculus not only requires you to understand when to use it, but also when not to use it. This question COULD be solved using calculus, but the geometric method was much more efficient, time-wise. For completeness I will explain the calculus method, but first I will go over the geometric method.

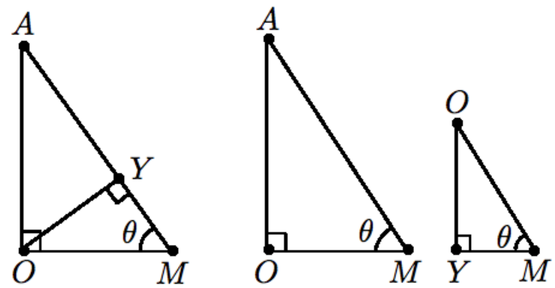
Firstly, the claim that OY being shortest distance from O to AM coincides with OY being perpendicular to AM may require some proof on my part if it is not clear to you. Assume that OY is perpendicular to AM, and suppose there is another point, P, on the line segment AM that is distinct from Y. We want to prove that OY is shorter than OP.

Since we are assuming OY is perpendicular to AM, OY is also perpendicular to YP, since YP is part of the line AM. Thus, OYP is a right-angled triangle, with a right angle at Y.

Since $YP^2 > 0$, we may then say that $YP^2+OY^2 > OY^2$, and since $YP^2+OY^2 = OP^2$, it follows that $OP^2 > OY^2$. Since these lengths are positive, $OP > OY$, and therefore Y is closer to O than any other point on AM, if OY is perpendicular to AM.

Note that there has to actually be a perpendicular line segment for this result to be true. For example, imagine three collinear points, A, B, and C, with B in the middle. The shortest distance from C to the line segment AB is the length of BC, and this line segment is obviously not perpendicular to AB. But, we should be able to tell from the diagram we were given, or from drawing our own, that it is possble for OY to be perpendicular to AM.

Now that we've established that OY is perpendicular to AM, we should draw some triangles, so we can find a way to relate the length OY to the others:



Here, I redrew the triangle contained within the pyramid, and then drew the two triangles AOM and OYM next to each other, so that they have the same orientation in terms of the angles they share. By drawing them this way it is easy to see which edges correspond to which. All three angles in one triangle need to be the same as they are in the other, and since both triangles contain the angle labelled θ, as well as a right angle, the third angle must also be the same for both triangles. Hence, the two triangles AOM and OYM are simiilar.

Now, we need three other lengths to find the length of OY. The edge OY corresponds to AO, so we will have to use AO. We don't know the length of YM, so we will have to use OM (equal to $\frac{1}{2}x$, since it is half the width of the square base of the pyramid), which corresponds to AM, so we have our three lengths. Using common ratios:

$$\begin{aligned}
\frac{OY}{OM} &= \frac{AO}{AM} \\
\implies OY &= \frac{OM \times AO}{AM} \\
\implies OY &= \frac{\left(\frac{1}{2}x\right) \times \left(\frac{1}{\sqrt{2}}x\right)}{\left(\frac{\sqrt{3}}{2}x\right)} \\
&= \frac{1}{2} \times \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{2}}x \\
&= \frac{1}{\sqrt{6}}x
\end{aligned}$$

So, using our given value $x = \frac{\sqrt{30}}{2}$ m, we have $OY = \frac{1}{\sqrt{6}} \times \frac{\sqrt{30}}{2}$ m $= \frac{\sqrt{5}}{2}$ m. We could have substituted this value in before, but it would have been somewhat more tedious. Also, note that you wouldn't have to specify units (m) here, but as I've said before it's better to include them.

Now, for the method using calculus. For this, we can use the x-y plane, with the origin being O, A being a point on the y-axis, and M being a point on the x-axis (it is a different x here, just to clarify). The length of OA is $\frac{1}{\sqrt{2}} \times \frac{\sqrt{30}}{2} = \frac{\sqrt{15}}{2}$, so A is the point $\left(0, \frac{\sqrt{15}}{2}\right)$, and the length of OM is $\frac{1}{2} \times \frac{\sqrt{30}}{2} = \frac{\sqrt{30}}{4}$, so M is the point $\left(\frac{\sqrt{30}}{4}, 0\right)$. If we find the line through these two points, then we can find the point on this line that is closest to the origin using the distance formula.

The gradient of the line through A and M is $\dfrac{\left(0-\dfrac{\sqrt{15}}{2}\right)}{\left(\dfrac{\sqrt{30}}{4}-0\right)}=-\dfrac{\sqrt{15}}{2}\times\dfrac{4}{\sqrt{30}}=-\dfrac{2}{\sqrt{2}}=-\sqrt{2}$. The y-intercept is $\left(0,\dfrac{\sqrt{15}}{2}\right)$, so the equation of the line is $y=\dfrac{\sqrt{15}}{2}-\sqrt{2}x$, and thus every point in the line is given by $\left(x,\dfrac{\sqrt{15}}{2}-\sqrt{2}x\right)$. The distance l of this point from the origin is then:

$$\begin{aligned}
l &= \sqrt{x^2+y^2} \\
&= \sqrt{x^2+\left(\frac{\sqrt{15}}{2}-\sqrt{2}x\right)^2} \\
&= \sqrt{x^2+\frac{15}{4}-2\cdot\frac{\sqrt{15}}{2}\cdot\sqrt{2}x+2x^2} \\
&= \sqrt{3x^2-\sqrt{30}x+\frac{15}{4}}
\end{aligned}$$

At this point, it is best to consider the square of the distance, rather than the distance itself. If you find the smallest value of the square of the distance, you can just take the square root of this to find the shortest distance.

The square of the distance is given by $l^2=3x^2-\sqrt{30}x+\dfrac{15}{4}$, and so to find the minimum value of l^2:

$$\begin{aligned}
\frac{d(l^2)}{dx} &= 0 \\
\implies 6x-\sqrt{30} &= 0 \\
\implies x &= \frac{\sqrt{30}}{6}
\end{aligned}$$

This value could have also been obtained from the general formula for the x-value of the stationary point of a quadratic, $x=-\dfrac{b}{2a}$, but I felt the need to use calculus at some point here. Anyway, our minimum value of l^2 is then

$$\begin{aligned}
l^2 &= 3\left(\frac{\sqrt{30}}{6}\right)^2-\frac{\sqrt{30}\times\sqrt{30}}{6}+\frac{15}{4} \\
&= \frac{3\times 30}{36}-\frac{30}{6}+\frac{15}{4} \\
&= \frac{30}{12}-\frac{60}{12}+\frac{45}{12} \\
&= \frac{15}{12} \\
&= \frac{5}{4}
\end{aligned}$$

Hence, the minimum value of l is $\sqrt{\dfrac{5}{4}}=\dfrac{\sqrt{5}}{2}$, and thus the length of OY is $\dfrac{\sqrt{5}}{2}$ m. Once again, I would avoid using this approach if there is a simpler method available, since it can often be tedious to do it this way, particularly in a tech-free exam. But if there's no other option, you should know how to use this method if you need to.

SET 3 EXAM 2

DETAILED SOLUTIONS

SECTION 1 - Multiple-Choice Questions

Question 1 (D)

In order to find the maximal domain of the product of two functions, we must first find the maximal domains of the individual functions.

$f(x)$ is defined so long as $9 - x^2$ is not negative, i.e. $9 - x^2 \geq 0 \iff x^2 \leq 9 \iff -3 \leq x \leq 3$, so the maximal domain of $f(x)$ is $[-3, 3]$.

Similarly, $g(x)$ is defined provided that $3 - x \geq 0 \iff x \leq 3$, and also that $\sqrt{3-x} \neq 0 \iff x \neq 3$, as we cannot have a zero denominator. Combining $x \leq 3$ and $x \neq 3$ gives $x < 3$, so the maximal domain of $g(x)$ is $(-\infty, 3)$.

When you take the product of two functions, the maximal domain of the product is the intersection of the individual domains of the two functions. This is because x has to be in the domains of both functions in order for the product of the functions to be defined, and this is what the intersection is - the set of values common to both domains (this also applies to the sum of two functions - the domain of the sum is the intersection of the individual domains).

The maximal domain of $f(x)g(x)$ is therefore $[-3, 3] \cap (-\infty, 3) = [-3, 3)$, which is option **D**.

Question 2 (C)

The general anti-derivative of $-2\sin(2x)$ is $\cos(2x) + c$, and so $f(x) = \cos(2x) + c$ for some value of c. Using the fact that $f(0) = 1$, we get $\cos(0) + c = 1 \implies 1 + c = 1 \implies c = 0$, and so $f(x) = \cos(2x)$, which is option **C**.

In a multiple-choice situation such as this, you could simply differentiate each of the five given possibilities. In this particular question, option **C** was the only option for which the derivative was equal to the given derivative, $f'(x) = -2\sin(2x)$. It should be easy to verify that this option satisfies $f(0) = 1$, so this method would have quickly given you the correct answer.

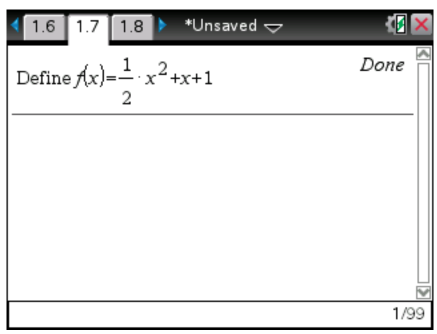
Question 3 (E)

This question can be done using by-hand methods, but since it's a multiple-choice question I will only describe the method using technology, since it is significantly faster, and of course working doesn't need to be shown.

Firstly, if the transformations are not all in the form $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix}$, then you should manipulate them so they are. The transformation in option **A** becomes $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}-1\\\frac{1}{2}\end{bmatrix}$, and the transformation in option **D** becomes $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}$. This is just a simple distribution over brackets.

The five transfromations are now all in the form $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix}$. This form of transformation always maps a curve $y=f(x)$ to the curve $y=bf\left(\frac{x-c}{a}\right)+d$ (you should have this relationship in your reference, or simply have it memorised if reasonable).

If you define $f(x)=\frac{1}{2}x^2+x+1$ using CAS, you can use this to find $bf\left(\frac{x-c}{a}\right)+d$, by reading each of the four values off the matrices in the transformations given. Remember, we are looking for the transformation that will give $bf\left(\frac{x-c}{a}\right)+d=x^2$. To define $f(x)$, use [Menu] [1] [1]:



Now, to apply each transformation. The (expanded) transformation in option **A** has $a=1$, $b=\frac{1}{2}$, $c=-1$, and $d=\frac{1}{2}$, so substituting these into $bf\left(\frac{x-c}{a}\right)+d$ gives:

$$\frac{1}{2}f(x+1)+\frac{1}{2} \quad = \quad \frac{1}{4}x^2+x+\frac{7}{4}$$

Option **B** gives:

$$f\left(\frac{x+1}{2}\right)+\frac{1}{2} \quad = \quad \frac{1}{8}x^2+\frac{3}{4}x+\frac{17}{8}$$

Option **C**:

$$f\left(\frac{x+1}{4}\right)+\frac{1}{2} \quad = \quad \frac{1}{32}x^2+\frac{5}{16}x+\frac{57}{32}$$

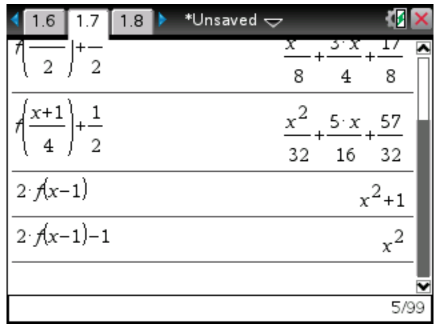
Option **D** (expanded form from above):

$$2f(x-1) \quad = \quad x^2+1$$

And finally, option **E**, which we can now presume is the correct one:

$$2f(x-1) - 1 \quad = \quad x^2$$

Hence option **E** is the correct answer. The procedure for finding these transformed functions on CAS is simple if you've defined $f(x)$:



Note that the transformation given in option **A** is actually the transformation that takes the curve $y = x^2$ to the curve $y = \dfrac{1}{2}x^2 + x + 1$, which was included deliberately. You should be mindful of which way the transformation is going.

Question 4 (B)
This question is fairly simple but it may trip up those who don't fully understand what is happening.

The fact that $a - x$ is raised to the power of four means the result will be positive, except when $a - x = 0$, i.e. when $x = a$. Since we may take the logarithm of any positive number, but not zero, the only value of x for which $f(x)$ is not defined is $x = a$. Hence, the maximal domain of f is $\mathbb{R} \setminus \{a\}$, which is option **B**.

As a side note, you may recall the rule $\log(m^n) = n\log(m)$, but this only applies when m is positive. If it isn't, and n is even, then m can also be negative, since raising m to an even power will give a positive number, and thus the logarithm of m^n is defined for negative values of m. Hence, so long as $m \neq 0$, and n is even, we have $\log(m^n) = n\log(|m|)$. The reason I bring this up is due to potential confusion that may arise from this particular question, namely that it would appear that $\log_e((a-x)^4) = 4\log_e(a-x)$, which would suggest that x must be less than a, but this is in fact incorrect - the correct rule is $\log_e((a-x)^4) = 4\log_e(|a-x|)$, since we have an even power.

Question 5 (C)

The concept behind this question is simple, but it will take a lot of care to find the right answer, so it's best to write a few things down instead of doing the whole thing in your head.

Each of the points given to us enables us to find an equation in the five variables. From the point $(-3, 61)$ we get the equation:

$$\begin{aligned} p(-3) &= 61 \\ \implies a(-3)^4 + b(-3)^3 + c(-3)^2 + d(-3) + e &= 61 \\ \implies 81a - 27b + 9c - 3d + e &= 61 \quad (1) \end{aligned}$$

Using the other four points we get four more equations:

$$\begin{aligned} 16a - 8b + 4c - 2d + e &= 11 \quad (2) \qquad (\text{using } (-2, 11)) \\ 16a + 8b + 4c + 2d + e &= 31 \quad (3) \qquad (\text{using } (2, 31)) \\ a - b + c - d + e &= 1 \quad (4) \qquad (\text{using } (-1, 1)) \\ a + b + c + d + e &= 5 \quad (5) \qquad (\text{using } (1, 5)) \end{aligned}$$

You may be able to do these mentally, but it's a good idea to write them down and label them. This way you can write the same labels next to the rows in each option whenever a row matches one of your equations, and you can eliminate an option when you find a row that doesn't match any of your equations. The option which matches all five of yours is the correct one, so once you've found it you can move on.

Carefully examining option **C** will show it is the correct equation:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & -27 & 9 & -3 & 1 \\ 16 & -8 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 31 \\ 61 \\ 11 \end{bmatrix} \begin{array}{l} (5) \\ (4) \\ (3) \\ (1) \\ (2) \end{array}$$

Note that matrix equations in more than three variables have not appeared on any VCAA exams under the current study design (as of writing this), but the study design still allows for matrix equations in up to five variables to be examinable.

Question 6 (D)

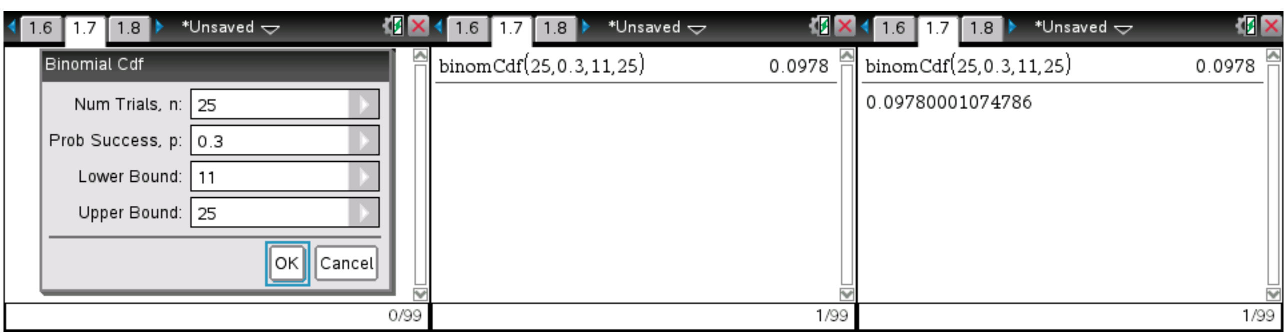
Since both functions are positive, we can safely using logarithm rules to expand out the composite function before differentiating:

$$
\begin{aligned}
\frac{d}{dx}\left(\log_e\left(\frac{2f(x)}{g(x)}\right)\right) &= \frac{d}{dx}\left(\log_e(2) + \log_e(f(x)) - \log_e(g(x))\right) \\
&= \frac{d}{dx}\left(\log_e(2)\right) + \frac{d}{dx}\left(\log_e(f(x))\right) - \frac{d}{dx}\left(\log_e(g(x))\right) \\
&= 0 + \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)} \\
&= \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}
\end{aligned}
$$

Which is option **D**. Generally, if you are used to manipulating logarithms, a question like this shouldn't pose much of a problem. There isn't much to say on this, but some suggestions: if you can see a way to simplify an expression before differentiating, you always should - if we had not expanded the expression here we sould have had to use the chain rule and quotient rule which would have been time-consuming. Also, be careful when trying to prove the result - you might think one of the incorrect options is correct and have that in mind while you're trying to prove it, and then mistakenly arrive at that result.

Question 7 (C)

This is a standard CAS-based binomial question. Here we are given the probability of obtaining heads as 0.7, but it is simpliest to use the probability of obtaining tails, which is 0.3. If we let X be the number of tails observed, we have $X \sim \text{Bi}(25, 0.3)$. The probability required is $\Pr(11 \leq X \leq 25)$ (more than 10, so it doesn't include 10), which can be evaluated to four decimal places as 0.0978, option **C**. There isn't much more to say on this question, but as a general reminder, take care with upper or lower bounds in these types of questions, and keep track of whether you're considering heads or tails, or whatever the success/fail situation might be. The binomial cumulative density function can be accessed by [Menu] [5] [5] [E]. If the given result doesn't have enough decimal places, you can copy the value into a new calculation, which should show more decimal places.



An alternative way to approach this question would have been to have $X \sim \text{Bi}(25, 0.7)$, and evaluate $\Pr(0 \leq X \leq 14)$, since 'more than 10 tails' coincides with "less than 15 heads".

Question 8 (B)

The term "stationary points" should indicate that we require consideration of the derivative, which is $p'(x) = 3ax^2 + 2bx + c$. A function having no stationary points means the derivative is never equal to 0, so we require that there are no solutions to the equation $3ax^2 + 2bx + c = 0$.

Our knowledge of quadratic equations tells us that there are no solutions when the discriminant is negative, i.e. when

$$\begin{aligned} (2b)^2 - 4(3a)(c) &< 0 \\ \iff 4b^2 &< 12ac \\ \iff b^2 &< 3ac \end{aligned}$$

Which is option **B**.

Note that option **A** cannot be correct, because the question states that $p(x)$ is a cubic function. If a and b were both equal to 0, we would have a linear function. Although it is true that it would not have any stationary points in this case, it would not be a cubic function, which would contradict the given information.

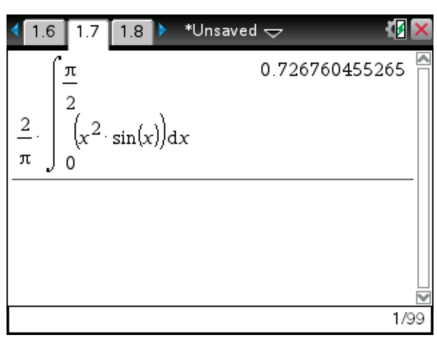
Question 9 (D)

This is one of the few types of multiple-choice questions that has appeared on every single tech-active exam under the current study design (as of writing this), and that is finding the average value of a function over an interval, given the function and the interval.

The average value of a function $g(x)$ over an interval $[a, b]$ is given by the net area under the graph of g, which is $\int_a^b g(x)dx$, divided by the size of the interval, which is $b - a$. That is, the average value of $g(x)$ over $[a, b]$ is $\frac{1}{b-a}\int_a^b g(x)dx$. Here, the interval is $\left[0, \frac{\pi}{2}\right]$, and hence the average value of $g(x)$ over this interval is:

$$\begin{aligned} \frac{1}{\left(\frac{\pi}{2} - 0\right)} \int_0^{\frac{\pi}{2}} x^2 \sin(x)dx &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x^2 \sin(x)dx \\ &= 0.726760... \end{aligned}$$

Which is closest to option **D**. It probably goes without saying that this should be done on CAS, since it's faster, and the means to perform the integration by hand are outside this course anyway. The integral can be accessed by [⇑ shift] [+], and [ctrl] [enter] will give a decimal approximation:



This concept should not be confused with the average rate of change of a function, but the two concepts are related.

Consider some function $F(x)$, and let $F'(x) = f(x)$. The average **rate of change** of $F(x)$ over an interval $[a, b]$ is given by $\frac{F(b) - F(a)}{b - a}$ (a formula you should also be familiar with, as well as the reasoning behind it). But since $f(x)$ **is equal to** the rate of change of $F(x)$, then we can say the average **value** of $f(x)$ is $\frac{F(b) - F(a)}{b - a}$. And furthermore, since $F(b) - F(a)$ is in fact equal to $\int_a^b f(x)dx$ (yet another formula you should be familiar with), it follows that the average value of $f(x)$ is $\frac{1}{b - a}\int_a^b f(x)dx$, which is where that formula comes from.

Question 10 (D)

This is almost a standard z-score type of question, except for one small alteration - instead of the standard deviation, you are given its square, the variance. Hence, the standard deviation of Y is 2.

Converting a value of a normally distributed variable to its equivalent z-value should be a familiar process, simply requiring the formula $z = \frac{y - \mu}{\sigma}$. This should be memorised, but more importantly understood for what it is (which generally helps memorisation anyway). The quantity $y - \mu$ is how far y is **above** the mean. This is why z-values below the mean are negative. Dividing this quantity by the standard deviation σ tells you how many standard deviations above the mean a given value of y is. And since Z is normally distributed with a mean of 0 and standard deviation of 1, knowing how many standard deviations above the mean a given value of y is will tell you its corresponding value of z. For example, if $z = 1$, it is one standard deviation above the mean, and if $\frac{y - \mu}{\sigma} = 1$, then y will also be one standard deviation above the mean.

Our given value of y to consider is 160, and this corresponds to a z-value of $\frac{160 - 164}{2} = -2$. Hence, $\Pr(Y > 160) = \Pr(Z > -2)$. But this option isn't given, so we need to use the symmetry of the z-distribution. It should be clear from a quick diagram (which is the recommended approach to this type of question, after you've found the appropriate z-value) that $\Pr(Z > -2) = \Pr(Z < 2)$, so the correct answer is **D**. You could also figure this out from the fact that $1 - \Pr(Z > -2)$ is clearly not equal to $\Pr(Z > -2)$, eliminating option **E**, and since none of the other options have anything to do with 2, those can be eliminated too, leaving just option **D**. But drawing a diagram or two is a more fool-proof approach.

Question 11 (C)

This question is best approached by testing all of the options, since they shouldn't take too long.

The first option is $p = 0.25$. We can find the value of p easily. The sum of the five probabilities must equal 1, so

$$\begin{aligned} p + 0.15 + 0.15 + 0.45 + 0.05 &= 1 \\ \implies p + 0.8 &= 1 \\ \implies p &= 0.2 \end{aligned}$$

So option **A** is incorrect.

The second option is the expected value, which is given by sum of each of the values of x multiplied by their individual probabilities, i.e.

$$\begin{aligned} E(X) &= (0 \times 0.2) + (1 \times 0.15) + (2 \times 0.15) + (3 \times 0.45) + (4 \times 0.05) \\ &= 2 \end{aligned}$$

Which is not equal to $p + 2$ since $p \neq 0$. Hence option **B** is incorrect.

The median value of X is the value of x for which all of the values above this have a 0.5 probability of occurring, and all of the values below this value have a 0.5 probability of occurring. Here we can see that $\Pr(X \geq 3) = 0.45 + 0.05 = 0.5$ and $\Pr(X \leq 2) = 0.2 + 0.15 + 0.15 = 0.5$. In this situation, the median is taken as halfway between 2 and 3, so the median here is 2.5. Hence option **C** is correct.

For completeness I will consider the other two options, starting with **E**, since the variance is calculated more easily before the standard deviation. Using the formula $\text{Var}(X) = E(X^2) - (E(X))^2$, we get

$$\begin{aligned} E(X^2) &= (0^2 \times 0.2) + (1^2 \times 0.15) + (2^2 \times 0.15) + (3^2 \times 0.45) + (4^2 \times 0.05) \\ &= 5.6 \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{Var}(X) &= 5.6 - 2^2 \qquad (\text{as } E(X) = 2) \\ &= 5.6 - 4 \\ &= 1.6 \end{aligned}$$

So the value in option **D** is in fact the variance, not the standard deviation, while the value in option **E** is $E(X^2)$ rather than the variance, so both options **D** and **E** are incorrect.

Question 12 (E)

This question is intended to be somewhat of a trick question, since it is asking about relations, and not functions. A function will only have an inverse if it is a one-to-one function, but there is no such requirement for a relation to have an inverse. In simple terms, a relation between two variables, as the name would suggest, is simply something relating the two. It could be an inequality (e.g. $y \geq x$), or an equation (e.g. $x^2 + y^2 = 1$), or even just a set of points (e.g. $\{(1, 2), (2, 4), (56, 3)\}$). The inverse of a relation between x and y is obtained by simply swapping x and y, so in the case of $x^2 + y^2 = 1$, the inverse relation is exactly the same. You can swap x and y in any relation between the two variables, so every relation has an inverse. Hence, all of the graphs shown are specified by a relation with an inverse, which means option **E** is correct.

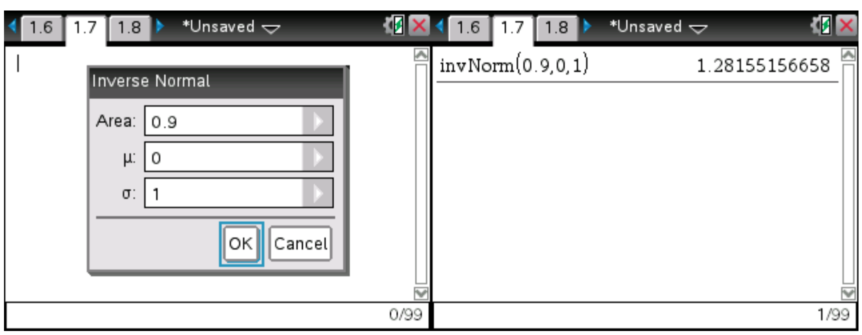
Question 13 (C)

Most of the concept here is similar to **Question 10**, so read the solution above for more detail.

As we are given a probability, we need to use the inverse normal function on CAS. However, since this function only finds values of a such that $\Pr(X < a) = p$ for a given mean and standard deviation, we need to first convert the equation $\Pr(X > 2.5) = 0.1$ to $\Pr(X < 2.5) = 0.9$, which is easily done by using complements, or just a simple diagram.

There is one more factor, however. We don't know the standard deviation of X, so we cannot directly use the distribution of X in the inverse normal function. We need to find the corresponding value of z. This is equal to $\dfrac{2.5 - 0}{\sigma} = \dfrac{2.5}{\sigma}$. If we let $a = \dfrac{2.5}{\sigma}$, then we can use the inverse normal function to find a such that $\Pr(Z < a) = 0.9$.

So, using the inverse normal function, with area 0.9, mean 0, and standard deviation 1, we get $a \approx 1.28155156658...$ (some calculators won't allow you to copy this value, so in this case it's best to write it down to as many decimal places as the calculator gives you if you need to do further calculations with it, which we do here). The inverse normal function can be accessed by [Menu] [5] [5] [3]:

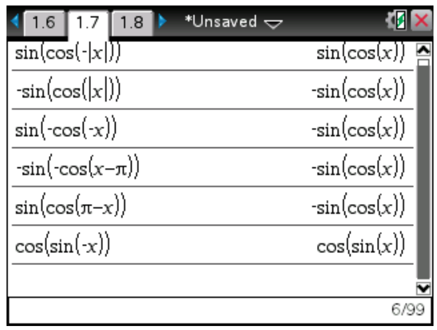


So, since $a = \dfrac{2.5}{\sigma}$, we have $\sigma = \dfrac{2.5}{a} \approx \dfrac{2.5}{1.28155156658} \approx 1.95076$. This gives us the approximate standard deviation of X, but we require the variance. The variance is the square of the standard deviation, and so, the variance is approximately equal to $(1.95076)^2 \approx 3.81$, which is closest to option **C**.

Question 14 (B)

This question can be done in two ways; one being using your calculator, the other by using odd/even function properties.

$\sin(x)$ is an odd function, which means it satisfies the property that $f(-x) = -f(x)$, whereas $\cos(x)$ is an even function, so it satisfies the property that $g(-x) = g(x)$. Using these facts, and having a quick look at the options, it should be fairly easy to see that $f(-g(-x)) = f(-g(x)) = -f(g(x))$, so option **B** is correct. If you type $\sin(-\cos(-x))$ into your calculator, it should give $-\sin(\cos(x))$, and the other options can be checked this way as well.



On a related note, if $g(x)$ is even, then $g(|x|) = g(x)$. This is because the property $g(-x) = g(x)$ implies that, regardless of the sign of x, the function will evaluate to the same quantity, and so it doesn't matter if the sign of x is neglected. Hence, $g(-|x|) = g(|x|) = g(x) = g(-x)$, where $g(x)$ is even. The modulus function itself is also even.

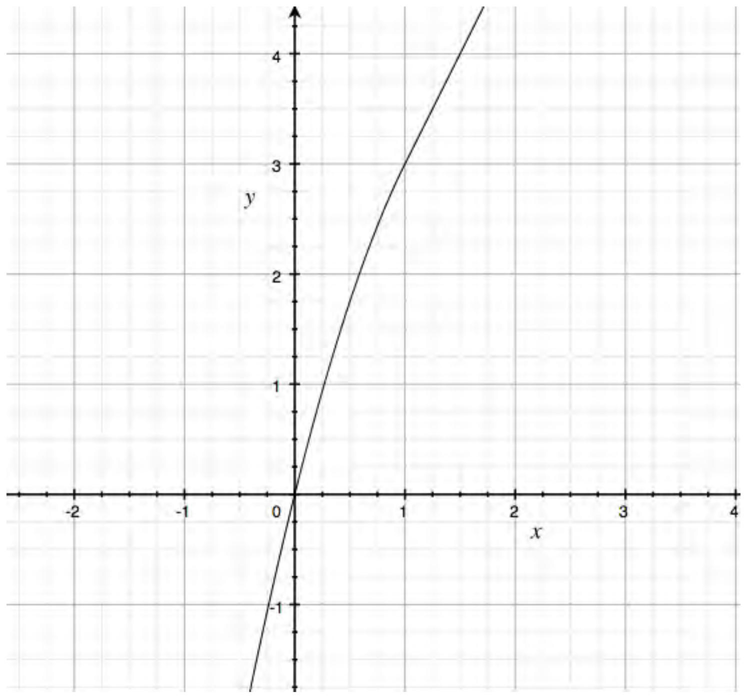
The reason for the terms "odd function" and "even function" is mostly outside of the course, but it can be related to polynomial functions easily. Consider the polynomial $p(x) = x^5 + x^3 + x$. This function satisfies $p(-x) = (-x)^5 + (-x)^3 + (-x) = -x^5 - x^3 - x = -p(x)$, so it is an odd function. This corresponds to all of the powers of x being odd (5, 3, and 1). Similarly, a function like $q(x) = x^4 + x^2 + 1$ is an even function, and all of the powers of x are even (4, 2, and 0 for the constant term, which is $1 \times x^0$). So if a polynomial has only even powers of x, it is an even function, and it is odd if all of the powers of x are odd. It should be noted that there are many functions that are neither odd nor even, but if you happen to notice a function is odd or even, their symmetry properties can often be exploited.

Question 15 (D)

$f(1)$ is clearly defined, since 1 is in the domain of the function, so we will need to draw the graph of f to test the other options. The graph of f is quicker to sketch by hand than on CAS, since the individual parts of the function are quite simple, so it is best to use this approach.

For the part of the rule that applies for $x < 1$, we have a parabola. It should be easy to find and verify that it passes through the origin, and that this part of the rule ends at the point $(1, 3)$, which is before the stationary point would occur at $(2, 4)$, so f does not have a stationary point for $x < 1$.

The other part of the graph begins at the point $(1, 3)$, and is a straight line with gradient 2. The important thing to note here is the function is continuous - the second part begins where the first ends. The graph of $y = f(x)$ will look like this:



From this alone, we can eliminate three more options. Option **B** is clearly a false statement, since the range of f is $\mathbb{R}$.

Option **C** is also a false statement, because $f(x) < 0$ for $x < 0$, and consequently $|f(x)| > 0$ for $x < 0$. Thus, the quantity $f(x) - |f(x)|$, which is necessarily a negative number minus a positive number, cannot be equal to 0.

Option **E** is almost true, but the case where $x = 0$ makes the statement false. At $x = 0$, the modulus graph would have a cusp, and therefore it would not be differentiable at that point, while f would be.

This leaves option **D**, and it is in fact true. Clearly f is differentiable for $x < 1$ and for $x > 1$; the question is whether it is differentiable at $x = 1$. If we differentiate the rule for the function for $x < 1$, we get the expression $0 - 2(x - 2) = 4 - 2x$. As x approaches 1 from the left hand side, $4 - 2x$ approaches 2, and this is equal to the gradient of the linear part of the function. Hence, the derivative is in fact defined at $x = 1$, since the derivatives of both rules of the function approach the same value there, **and** f is continuous there. Hence option **D** is a true statement.

Although the written use of limits is not required for this study, much less a multiple choice question, they can provide a useful and more sound proof that f is differentiable at $x = 1$.

Recall that a limit is said to exist if both the left hand side limit and the right hand side limit exist, and are equal to each other. In mathematical notation:

$$\begin{array}{rrcl} \text{If} & \lim_{x\to c^+} f(x) & = & L \\ \text{and} & \lim_{x\to c^-} f(x) & = & L \\ \text{then} & \lim_{x\to c} f(x) & = & L \end{array}$$

The notation $x \to c^-$ means that x is approaching the value c from the left hand side (the negative side), while $x \to c^+$ means x is approaching c from the right hand side (the positive side). So, if a function $f(x)$ is approaching the same value as x approaches c from either side, then $\lim_{x\to c} f(x)$ exists.

The requirement for a function to be differentiable at a point $x = c$ is that the limit $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists. Here we want to verify that the function f is differentiable at $x = 1$, so we must determine if the limit $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$ exists. We know $f(1) = 3$, so this becomes $\lim_{h\to 0} \frac{f(1+h)-3}{h}$.

We must consider both the left and right limits, so we will start with the right hand limit. For $h \to 0^+$, the quantity $1 + h$ is greater than 1, so the rule $f(x) = 2x + 1$ will apply for this value.

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(1+h)-3}{h} &= \lim_{h\to 0^+} \frac{2(1+h)+1-3}{h} \\ &= \lim_{h\to 0^+} \frac{2+2h-2}{h} \\ &= \lim_{h\to 0^+} \frac{2h}{h} \\ &= \lim_{h\to 0^+} 2 \\ &= 2 \end{aligned}$$

Now, for $h \to 0^-$, the quantity $1 + h$ is less than 1. This means the rule $f(x) = 4 - (x-2)^2$ applies, and so:

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(1+h)-3}{h} &= \lim_{h\to 0^-} \frac{4-((1+h)-2)^2-3}{h} \\ &= \lim_{h\to 0^-} \frac{1-(h-1)^2}{h} \\ &= \lim_{h\to 0^-} \frac{1-(h^2-2h+1)}{h} \\ &= \lim_{h\to 0^-} \frac{h^2+2h}{h} \\ &= \lim_{h\to 0^-} h+2 \\ &= 0+2 \\ &= 2 \\ &= \lim_{h\to 0^+} \frac{f(1+h)-3}{h} \end{aligned}$$

Hence, $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$ exists, and so f is differentiable at $x = 1$.

To reiterate, this method of using limits to prove differentiability is not required, but an intuitive understanding of it will serve you well in approaching these sorts of differentiability questions. The graphical interpretation of a hybrid function being differentiable at a “joining point” is that the two different parts of the graph have the same slope at the point where they join, so there is a specific tangent at that point. This is in contrast to a point such as a cusp, which is “sharp”, and so it is possible to draw an infinite number of tangents to a graph at that point.

Question 16 (A)

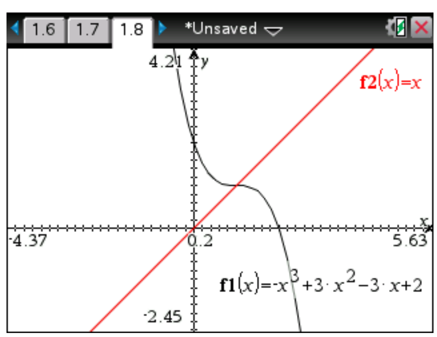
The first thing you should do here is consider option **E**, since it is the natural starting point for determining where a function intersects its inverse.

In the event that your calculator does not invert the function given, you shouldn’t assume it does not have an inverse; your calculator simple may not have the facilities to do so in this case. For a cubic to have an inverse function, it needs to have less than two stationary points, which would make it one-to-one. Here we have $p'(x) = -3x^2 + 6x - 3 = -3(x^2 - 2x + 1) = -3(x-1)^2$, so there is exactly one stationary point, at $x = 1$. Hence, p is invertible.

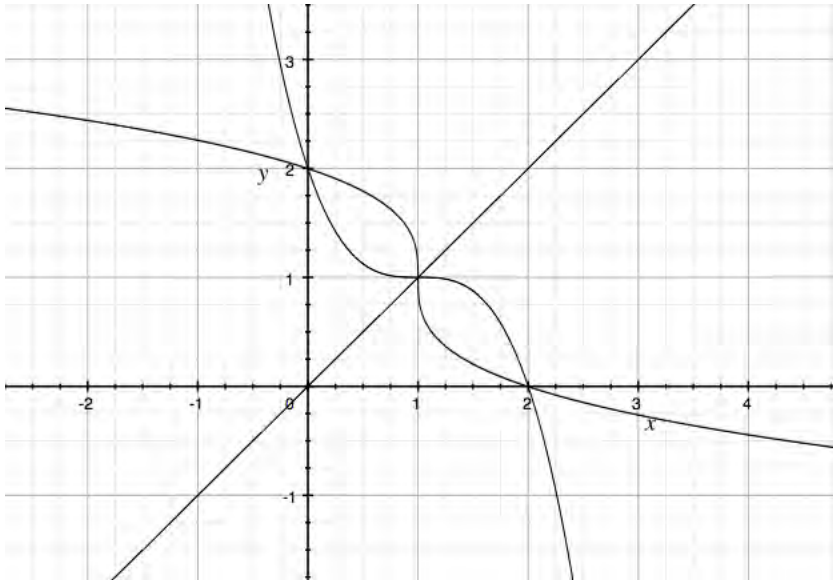
If you’re observant you will realise that the derivative is a perfect square, so the function itself must be some sort of perfect cube. Indeed, we can see that

$$\begin{aligned} p(x) &= -x^3 + 3x^2 - 3x + 2 \\ &= -x^3 + 3x^2 - 3x + 1 + 1 \\ &= -(x^3 - 3x^2 + 3x - 1) + 1 \\ &= 1 - (x-1)^3 \end{aligned}$$

Which shouldn’t be a problem to recognise if you are familiar with the cubic expansion formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. But even if you don’t recognise this (and you don’t have to in order to find the correct answer), you still have the facilities to graph $y = p(x)$, and along with the line $y = x$, it will look like this:



Copying this graph into your paper, you should be able to sketch the inverse function:



So we can see that there are three points of intersection.

The interesting thing to note here is that there are intersections that do not lie on the line $y = x$. You may have been taught to solve the equation $p(x) = x$ to find the x-coordinates of the points of intersection of a function with its inverse, but this does not always work. If you had solved $p(x) = x$ there would have been one solution, which would have led you to believe that option **C** was correct, but as we can see from the graph, this is not the case. The approach of sketching the graph would have led you to the correct answer, but I will go on with the algebraic method.

We know that $p(x) = 1 - (x-1)^3$, and its range and domain are both $\mathbb{R}$, which means for all $x \in \mathbb{R}$,

$$\begin{aligned} p(p^{-1}(x)) &= x \\ \iff 1 - (p^{-1}(x) - 1)^3 &= x \\ \iff (p^{-1}(x) - 1)^3 &= 1 - x \\ \iff p^{-1}(x) - 1 &= (1-x)^{\frac{1}{3}} \\ \iff p^{-1}(x) &= 1 + (1-x)^{\frac{1}{3}} \end{aligned}$$

And so, solving the equation $1 - (x-1)^3 = 1 + (1-x)^{\frac{1}{3}}$ on CAS, we will get three solutions, $x = 0, 1, 2$. Hence option **A** is correct.

To summarise, in order to answer this question, you would have needed to first determine the invertibility of the function, which in the case of a cubic can be done by determining if there are less than two stationary points. If you could not invert the function by algebraic or other means, then sketching the graph on CAS, copying it down, and sketching the inverse by reflecting the graph in the line $y = x$ would have led you to the correct answer.

Question 17 (A)

Simply by looking at the available answers, we should be able to see that we will need to use the conditional probability relation, $\Pr(B \mid A) = \dfrac{\Pr(B \cap A)}{\Pr(A)}$, in order to arrive at one of the answers. So,

$$\begin{aligned} \Pr(B \mid A) &= \frac{1}{2}\Pr(B' \mid A) \\ \implies \frac{\Pr(B \cap A)}{\Pr(A)} &= \frac{\Pr(B' \cap A)}{2\Pr(A)} \\ \implies 2\Pr(B \cap A) &= \Pr(B' \cap A) \end{aligned}$$

At this point, we will need to use the fact that $\Pr(B \cap A) + \Pr(B' \cap A) = \Pr(A)$, and this fact can be seen by drawing a Venn diagram, or remembering that the structure of a Karnaugh map has these two quantities adding to give $\Pr(A)$. It can also be understood intuitively - if something is in A, then it is either in A and B, or it is in A but not in B.

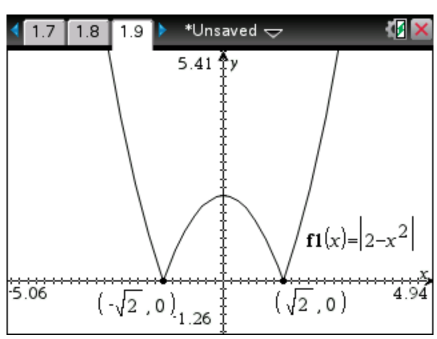
We will need to substitute either $\Pr(B' \cap A) = \Pr(A) - \Pr(B \cap A)$ or $\Pr(B \cap A) = \Pr(A) - \Pr(B' \cap A)$, so we will try the former and see if it works:

$$\begin{aligned} 2\Pr(B \cap A) &= \Pr(B' \cap A) \\ \implies 2\Pr(B \cap A) &= \Pr(A) - \Pr(B \cap A) \\ \implies 3\Pr(B \cap A) &= \Pr(A) \\ \implies \Pr(B \cap A) &= \frac{1}{3}\Pr(A) \\ \implies \Pr(A \cap B) &= \frac{1}{3}\Pr(A) \end{aligned}$$

And this is option **A**. Beyond this, there isn't much to say, other than the obvious advice to take care with your working, and make sure you use the correct relations.

Question 18 (E)

Analysing the graph of $|-f(x)|$ is the simplest way to determine where the gradient is positive, and since it is a fairly simple function, its graph can be sketched by hand reasonably quickly. Observing that $|-f(x)| = |f(x)|$ can simplify this procedure, but the graph of $y = |2 - x^2|$ isn't much more difficult to sketch anyway. Sketching the graph on CAS may be a more efficient option, however. In either case, the graph will have cusps at $(\pm\sqrt{2}, 0)$ and will look like this:



It is fairly easy to see that the gradient is positive for $x \in (-\sqrt{2}, 0) \cup (\sqrt{2}, \infty)$, which is option **E**. The only thing to be wary of is including $\pm\sqrt{2}$ in this set of values; remembering that the derivative is not defined at cusps, you should not have a problem determining that **C** is incorrect.

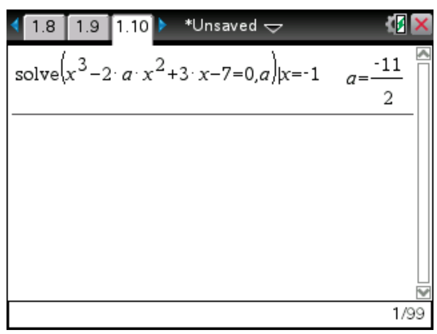
Question 19 (B)

This is a simple factor theorem question. The factor theorem is this: If $x - a$ is a factor of a polynomial $p(x)$, then $p(a) = 0$. This should be somewhat intuitive; you would be familiar with the factorised forms of a polynomial, and how the x-intercepts can be easily seen from this form. For example, if we had $q(x) = (x+1)(x-2)(x-5)$, the x-intercepts of $y = q(x)$ would be at $x = -1, 2, 5$, and they are the values of x for which $q(x) = 0$.

Here we have $x + 1$ as a factor of $p(x) = x^3 - 2ax + 3x - 7$, so by the factor theorem:

$$\begin{aligned} p(-1) &= 0 \\ \implies (-1)^3 - 2a(-1)^2 + 3(-1) - 7 &= 0 \\ \implies -1 - 3 - 7 - 2a &= 0 \\ \implies 2a &= -11 \\ \implies a &= -\frac{11}{2} \end{aligned}$$

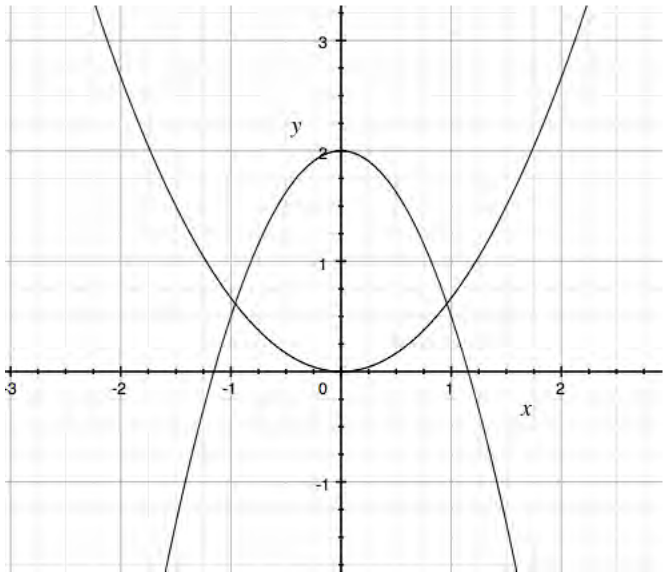
Which is option **B**. This can be solved on CAS just as easily:



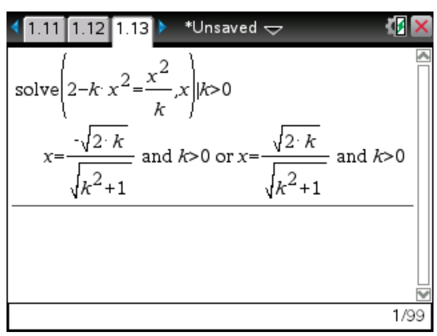
Question 20 (D)

This is certainly one of the more challenging questions in this section, and the main challenge is in the efficiency of using your calculator to solve the problem. But before you do this, you need to make some graphical observations about what is happening. A graph was not provided for you because part of the question was being able to quickly see what the two graphs will look like.

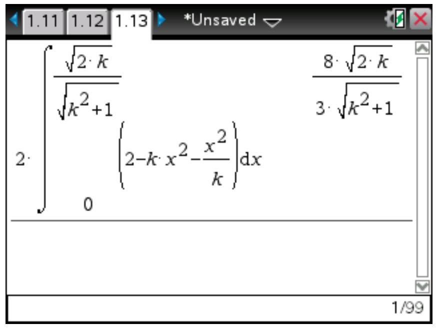
The fact that k is positive will allow you to determine the nature of both of the graphs. The graph of $y = 2 - kx^2$ is a negative parabola that has its turning point at (0,2). The graph of $y = \dfrac{x^2}{k}$ is a positive parabola that has its turning point at the origin. Drawing a quick sketch of the general situation, we get:



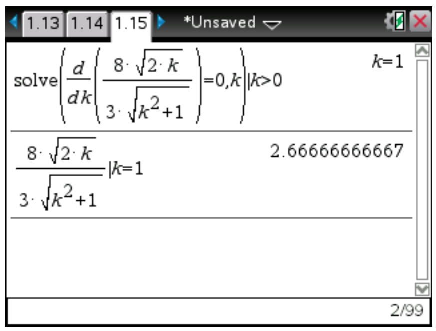
We know the general formula for the area between two curves that intersect at $x = a$ and $x = b$, with $a < b$, is $A = \int_a^b (f(x) - g(x))dx$, provided $f(x) > g(x)$ over this interval. But the first thing to do here is find the x-coordinates of the points of intersection of the two curves in terms of k. You can solve $2 - kx^2 = \frac{x^2}{k}$ on CAS, but I suggest also adding the constraint that $k > 0$, since your calculator would give a messy set of solutions otherwise. Be sure to put a multiplication sign between k and x^2, since CAS will usually consider kx to be a single variable if not separated. Solving the equation $2 - kx^2 = \frac{x^2}{k}$ for x will give $x = \pm\frac{\sqrt{2k}}{\sqrt{k^2+1}}$ as shown:



Now we can find the area in terms of k by using $A = \int_{-\frac{\sqrt{2k}}{\sqrt{k^2+1}}}^{\frac{\sqrt{2k}}{\sqrt{k^2+1}}} \left((2 - kx^2) - \left(\frac{x^2}{k}\right)\right) dx$, but by observing that the area is symmetrical about the y-axis, you can simply take the area to the right of the y-axis and double it, i.e. $A = 2\int_0^{\frac{\sqrt{2k}}{\sqrt{k^2+1}}} \left((2 - kx^2) - \left(\frac{x^2}{k}\right)\right) dx$. This should save typing out $\frac{\sqrt{2k}}{\sqrt{k^2+1}}$ twice. Evaluating this integral in terms of k using CAS will give $A = \frac{8\sqrt{2k}}{3\sqrt{k^2+1}}$.



To find the maximum value of A, you can solve $\dfrac{dA}{dk} = 0$ for $k > 0$ ([⇑ shift] [−] will allow you to access the derivative feature). This will give $k = 1$, and so the maximum area is $A = \dfrac{8\sqrt{2}}{3\sqrt{1+1}} = \dfrac{8\sqrt{2}}{3\sqrt{2}} = \dfrac{8}{3}$ which is 2.67, correct to two decimal places, so the correct answer is option **D**.



This question can be completed in a minimum of five steps:

- Sketching the two graphs, or at least working out what they look like
- Solving $2 - kx^2 = \dfrac{x^2}{k}$ for $k > 0$ on CAS to get $x = \pm\dfrac{\sqrt{2k}}{\sqrt{k^2+1}}$
- Evaluating $A = 2\displaystyle\int_0^{\frac{\sqrt{2k}}{\sqrt{k^2+1}}} \left((2 - kx^2) - \left(\frac{x^2}{k}\right)\right) dx$ on CAS to give $A = \dfrac{8\sqrt{2k}}{3\sqrt{k^2+1}}$
- Solving $\dfrac{d}{dk}\left(\dfrac{8\sqrt{2k}}{3\sqrt{k^2+1}}\right) = 0$ for $k > 0$ on CAS to give $k = 1$
- Evaluating A for $k = 1$ to give $A = \dfrac{8}{3} \approx 2.67$

Notice that this can be done in a minimum of only three calculations on CAS, if you're able to evaluate A for $k = 1$ by hand quickly, which is reasonable. While it may have seemed like a long question, being efficient in your computations can save a lot of time, which was again the challenge of this question.

Question 21 (E)

Letting $f(x) = \dfrac{1}{\sqrt{x}}$, the first thing to do here is to find $f'(x)$, which is equal to $\dfrac{-1}{2x^{\frac{3}{2}}}$. Now, since we are using the tangent at $x = 1$, our value of h must be -0.3 in order to approximate $\dfrac{1}{\sqrt{0.7}}$. Hence,

$$\begin{aligned} f(1-0.3) &\approx f(1) - 0.3f'(1) \\ &= 1 - 0.3 \times (-0.5) \\ &= 1.15 \end{aligned}$$

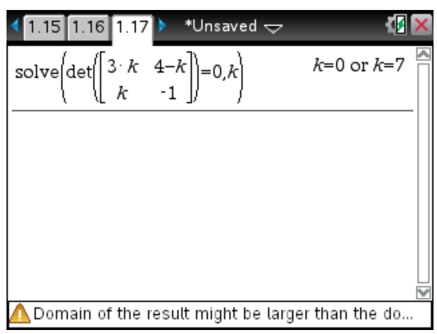
The value of $\dfrac{1}{\sqrt{0.7}}$ is, more accurately, 1.1952286 (to 7 decimal places), so the approximation of 1.15 is around 0.0452286 **less** than the exact value. To get the percentage of the exact value, simply take $\dfrac{0.0452286}{1.1952286} \times 100 \approx 3.78$, so the approximation is 4% less than the exact value, to the nearest percent, and hence option **E** is correct.

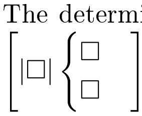
Question 22 (B)

This is a very common multiple-choice question that has appeared on almost every exam under the current study design. If you had not been given a set of answers, the most efficient way to go about this question would be by writing down the equivalent matrix equation:

$$\begin{bmatrix} 3k & 4-k \\ k & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ k-3 \end{bmatrix}$$

There will be a unique solution as long as the determinant of the square matrix is non-zero. The determinant of the square matrix is $(3k)(-1) - (4-k)(k) = -3k - 4k + k^2 = k(k-7)$, which is zero when $k = 0$ or $k = 7$, so there will be a unique solution when $k \in \mathbb{R} \setminus \{0, 7\}$. These values can be found on CAS:



The determinant function can be accessed through [Menu] [7] [3], and a 2×2 matrix can be created by pressing .

The question is when there will be an infinite amount of solutions, and this will be when the two lines are exactly the same line.

Considering the two cases separately, the case where $k = 0$ gives the two equations $4y = 12$ and $-y = -3$. These are both the same line, $y = 3$, so there are infinitely many solutions for this case (x can be any real number).

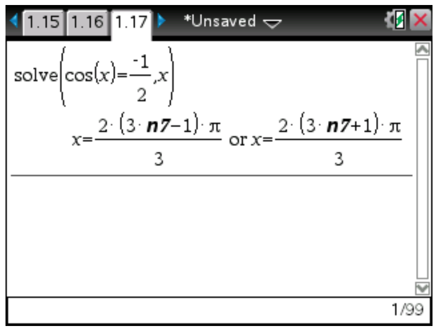
For $k = 7$, we get the two lines $21x - 3y = 12$ and $7x - y = 4$, and, again, these are both the same line, so there are also infinitely many solutions for $k = 7$, and hence option **B** is correct.

A shortcut method is to simply look at the options given to you and work from there. You will see that the answers will all contain either 0 or 7, and you can easily verify that the two lines will be the same for both of these values. You know that these are the only two values, because the determinant of the coefficient matrix is a quadratic in k. This means there cannot be more than two values of k for which it is zero, and so there must be a unique solution otherwise.

SECTION 2 - Extended-Response Questions

Question 1

a.i. There are quite a few ways to go about this question. The most efficient method is probably to do it by hand, since it results in a more compact general solution, though if you are familiar with your calculator's output then it is reasonable to use technology to give a general solution. If you use CAS, you will need to "translate" the output to standard mathematical notation - for example, the solution given by CAS may involve a parameter such as "@n1" or similar, which you should just write down as n or k, and specify what values it can take (which would be integer values). Finding the general solution on CAS gives:



Which should be written down as:

$$\begin{aligned} x &= \frac{2(3n+1)\pi}{3} \\ \text{or} \quad x &= \frac{2(3n-1)\pi}{3}, \ n \in \mathbb{Z} \end{aligned}$$

Or more simply as:

$$x = \frac{2(3n \pm 1)\pi}{3}, \ n \in \mathbb{Z}$$

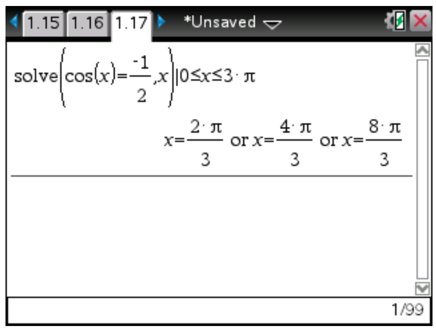
If you do this question by hand (which shouldn't take any longer than typing the equation out on your calculator), you'll first need two non-equivalent angles for which $\cos(x) = -\frac{1}{2}$. By "non-equivalent angles," I mean angles that aren't an integer multiple of 2π apart - you wouldn't be able to generate every solution from, for example, $\frac{2\pi}{3}$ and $\frac{8\pi}{3}$ as your base angles, since you'd miss other solutions such as $\frac{4\pi}{3}$. However, there are a few exceptions to this: in the case where $\cos(x) = 0$ or $\cos(x) = \pm 1$, you wouldn't need two angles, since these particular equations have more simple solutions. It's a good idea to learn these three solutions, and understand why they are less complicated, either by visualising the graph of $y = \cos(x)$ or visualising the unit circle. On that note, visualising graphs or the unit circle is usually a good idea for most questions involving circular functions.

Familiarity with exact values will tell you that $\frac{2\pi}{3}$ is a solution to $\cos(x) = -\frac{1}{2}$, and you can verify quickly on your calculator that $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$. We need another angle though, and the way I like to find the second, when dealing with cosine, is to use the fact that $\cos(x) = \cos(-x)$, which is extremely useful in solving equations of the form $\cos(x) = c$. The property that $\cos(x) = \cos(-x)$ tells us that if $\frac{2\pi}{3}$ is a solution, then the negative of it, $-\frac{2\pi}{3}$, is also a solution. This is useful on two levels, since we can abbreviate this as $\pm\frac{2\pi}{3}$.

We now have two non-equivalent solutions, and so the rest can be generated by adding or subtracting 2π to these. Hence, the general solution is $x = 2n\pi \pm \frac{2\pi}{3}$, where $n \in \mathbb{Z}$. Again, it's important to state that n is an integer. Also, notice that this solution is the same as the solution obtained from CAS above, except the CAS solution is over one denominator.

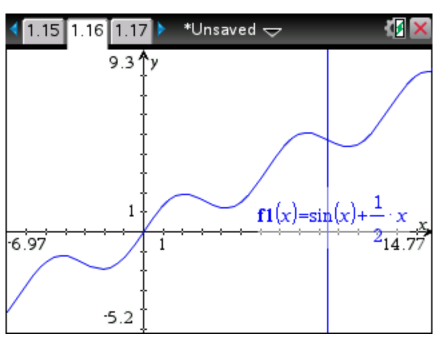
a.ii. Here, the first thing you should do is find $f'(x)$, which shouldn't take very long - it can be done by hand quite quickly, but even so, you still have your calculator.

We have $f'(x) = \cos(x) + \frac{1}{2}$, and this is equal to 0 when $\cos(x) = -\frac{1}{2}$. This is the same equation as in the first part of the question, so you are able to use this if it will help, although the three solutions can be found straight away by solving $\cos(x) = -\frac{1}{2}$ over the domain $[0, 3\pi]$ on CAS. This will give $x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$, as shown:



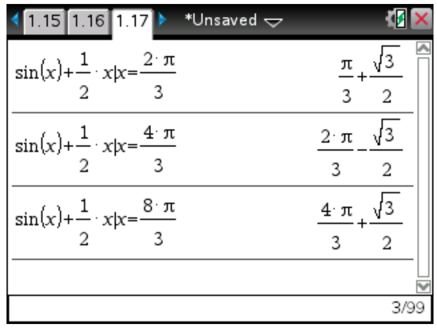
If you use the general solution, $x = 2n\pi \pm \frac{2\pi}{3}$, you would simply plug in reasonable values of n - here, any value of n besides 0 or 1 would not give a solution, since, for example, if $n = 2$ you will get the solutions $4\pi \pm \frac{2\pi}{3}$, which are both greater than 3π, and for $n = -1$ you get $-2\pi \pm \frac{2\pi}{3}$, which are both less than 0.

For $n = 1$, you get the two solutions $x = 2\pi \pm \frac{2\pi}{3}$, or more simply, $x = \frac{4\pi}{3}, \frac{8\pi}{3}$. These are both within the required domain. For $n = 0$, you get $x = \pm\frac{2\pi}{3}$, but only $\frac{2\pi}{3}$ is within the required domain, and so in total we have three solutions, $x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$. Once you obtain these solutions, it's certainly a good idea to graph f on CAS, since you will be able to see that you have the correct number of solutions, as well as the fact that you will have to sketch the graph in **part b.** anyway. The graph of $y = \sin(x) + \frac{x}{2}$ is shown below, along with the line $x = 3\pi$:



We can see that there are three stationary points for $0 \leq x \leq 3\pi$, so we have the correct number of solutions.

a.iii. Using the previous part of the question, this should be very easy to answer - you have found the x-coordinates of the stationary points (i.e. the values of x for which $f'(x) = 0$) and so to find the y-values, you must simply evaluate $f(x)$ for each of these values:

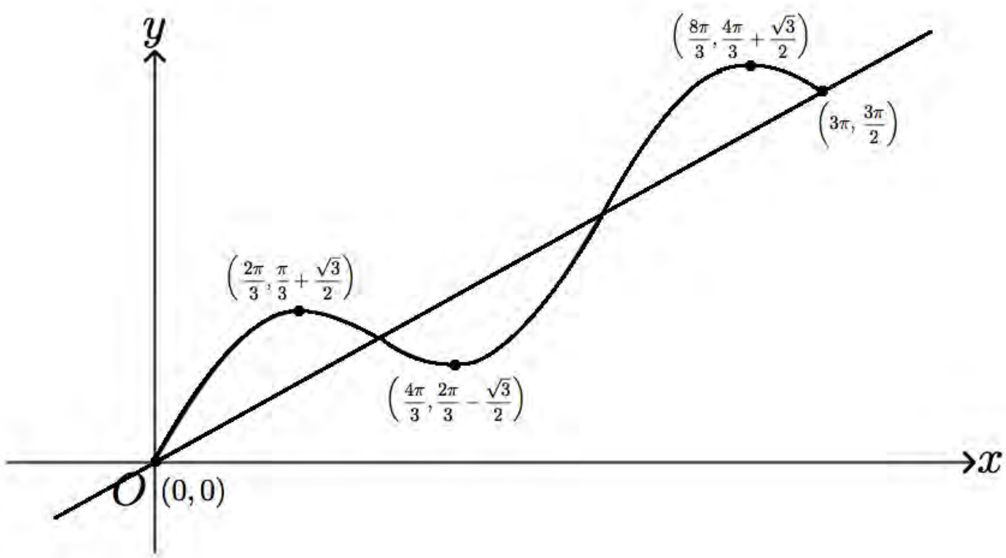


And hence the coordinates are $\left(\frac{2\pi}{3}, \frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$, $\left(\frac{4\pi}{3}, \frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$, and $\left(\frac{8\pi}{3}, \frac{4\pi}{3} + \frac{\sqrt{3}}{2}\right)$.

A common error in questions asking for coordinates is to only write down the x- or y-values - this is not answering question, and so you will not get full marks unless you write down the coordinates when the question requires them. And, as always, since decimal approximations are not asked for, coordinates must be given as exact values.

b. Before sketching anything, you should ask yourself why they would ask you to sketch the line $y = \frac{x}{2}$ along with f. This line is part of the rule for f, so it must have some significance. The graph of f actually oscillates about this line - $\sin(x)$ ranges between -1 and 1, and so the graph of f will stick close to the line $y = \frac{x}{2}$. In particular, f will intersect this line at every integer multiple of π (since $\sin(x) = 0$ for these values of x), and so both of the endpoints of f are on this line, which you would need to show in your sketch.

You should sketch both f and the line $y = \frac{x}{2}$ on CAS before drawing them on paper, so you know how they will look together. Also, it's a good idea to draw the line first, and then draw the curve around it, since it would be easier than the other way around. In any case, your graph should look like this:



The curve given by f must be restricted to its domain, but since the function $y = \dfrac{x}{2}$ does not have a specified domain, it does not need to be restricted, since it is implied that its domain is $\mathbb{R}$. As per the instructions, both endpoints and all three stationary points must be labelled. You may also label the other two points of intersection, but they are not explicitly asked for and so do not need to be labelled. An important part of sketching questions is making sure you've labelled everything that they've asked you to label, since full marks can't be given if you haven't, even if your curves are the correct shape.

For the sketching itself, this should be done methodically - a good idea is to draw the curve lightly at first, then draw over it properly once the shape is on the page, and then clean it up with an eraser if need be. You don't want to waste time rubbing the whole curve out if the shape is wrong. Since these types of questions aren't worth a great deal of marks, you should try and get them done fairly quickly. On a minutes-per-mark basis, you should aim to spend no more than 4.5 minutes on a 3-mark sketching question such as this, though if you've practiced sketching enough, 4.5 minutes should be more than enough time to get it done.

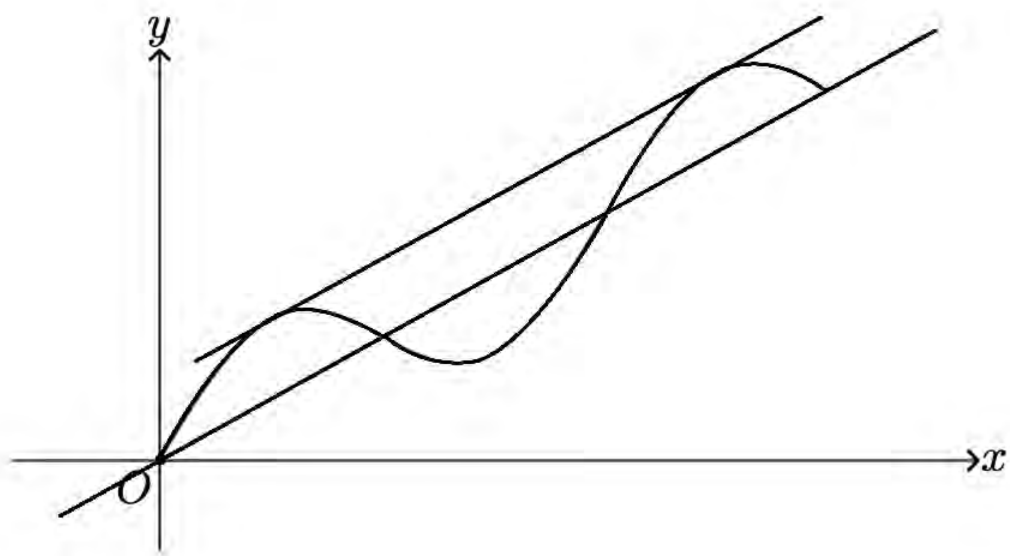
c.i. The harder part of this question is starting your proof off - the algebra itself isn't overly difficult. We are given quite a bit of information, but we need to decide exactly which piece(s) of information we need to use. We will most likely not need to use the ranges of values that a and b can take, since this does not appear to affect the expression for m. Also, we will not need to use the derivative of f just yet - we can see that the expression for m involves sines rather than cosines, and sines are present in the rule for f, rather than its derivative. Therefore, we should use the points A and B to begin the proof, since the coordinates of these feature $\sin(a)$ and $\sin(b)$, which are present in the final expression for m.

The gradient of the line through A and B is easily calculated using $\dfrac{\Delta y}{\Delta x}$, and indeed we obtain an expression that already resembles the final one:

$$\begin{aligned}
m &= \frac{f(b) - f(a)}{b - a} \\
&= \frac{\sin(b) + \frac{b}{2} - \sin(a) - \frac{a}{2}}{b - a} \\
&= \frac{\sin(b) - \sin(a)}{b - a} + \frac{\frac{b}{2} - \frac{a}{2}}{b - a} \\
&= \frac{\sin(b) - \sin(a)}{b - a} + \frac{1}{2}\left(\frac{b - a}{b - a}\right) \\
&= \frac{\sin(b) - \sin(a)}{b - a} + \frac{1}{2}, \quad \text{as required}
\end{aligned}$$

As a general reminder, you want to have some sort of "LHS = ... = RHS" set-up in "show that" questions, although in this case there shouldn't be too much difficulty linking m to the right hand side expression with a few "=" signs.

c.ii. At this point, the coordinates of A and B alone will not be able to give us a relationship between a and b, and so we will have to use some of the other information. A good starting point is to envision what such a tangent might look like with a quick sketch (which you should draw elsewhere - drawing it on the graph at the top of the page wouldn't be the best idea unless you rub it out afterwards):



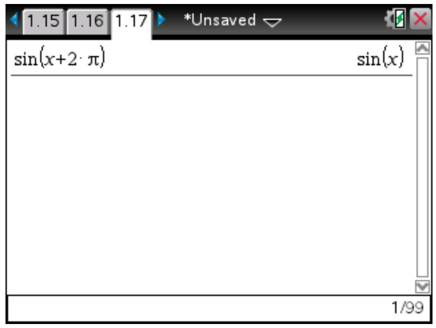
From this, you can get the sense that we will have to use the periodicity of circular functions to get a relationship between the two values. $f'(x)$ is periodic in x, so its values will repeat every 2π units in the x-direction. From the graph, it should be clear that the section of the curve where $x \in [0, \pi]$ is identical to the section where $x \in [2\pi, 3\pi]$, other than a difference in vertical height. From this point of view, we can see that the two values, a and b, are 2π units apart, since they correspond to the same point in the "cycle" of the derivative. Thus, $b = a + 2\pi$, since b must be greater than a. Being a 1-mark question, you would only need to write this down to get the mark.

c.iii. As the instruction "hence, or otherwise" might suggest, there is more than one way to go about this question, but I'll start with the "hence" method.

Given that we now have an expression for m in terms of a and b, and b in terms of a, we can easily find m in terms of just one of the values a and b. We have the rather convenient $b = a + 2\pi$, which is easily substituted in the expression for m:

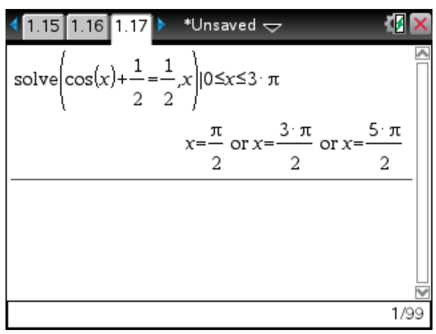
$$\begin{aligned} m &= \frac{\sin(b) - \sin(a)}{b - a} + \frac{1}{2} \\ &= \frac{\sin(a + 2\pi) - \sin(a)}{a + 2\pi - a} + \frac{1}{2} \\ &= \frac{\sin(a) - \sin(a)}{2\pi} + \frac{1}{2} \\ &= 0 + \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

The only real trick here was to recognise that $\sin(a+2\pi) = \sin(a)$. The rest of the problem becomes straightforward after that. Most of the more basic symmetry properties of sine and cosine can be verified on CAS:



An alternative approach to this question is to simply observe that the tangent at A and B must also be tangent to f every 2π units, due to the periodic nature of the function and its derivative. Indeed, from our sketch of the tangent, we can see that the tangent is parallel to the line $y = \frac{x}{2}$, and so must have the same gradient of $\frac{1}{2}$. This method may be somewhat harder to articulate than the "hence" method, which is the recommended method, but the graphical method is still available if you weren't able to complete the previous part.

d. Here, we need to use the fact that the derivative of f at $x = a$ and $x = b$ is equal to $\frac{1}{2}$. The best approach here is to solve $f'(x) = \frac{1}{2}$ and decide which solution is a and which is b.



Now, we have three solutions, but only two desired values. This is where the given range of values for a and b come into play. The solution $\frac{3\pi}{2}$ is not an element of either $[0, \pi]$ or $[2\pi, 3\pi]$, so this value cannot be a nor b. However, $\frac{\pi}{2} \in [0, \pi]$, and $\frac{5\pi}{2} \in [2\pi, 3\pi]$, so it follows that $a = \frac{\pi}{2}$ and $b = \frac{5\pi}{2}$.

Note that, at this point, you can easily verify you answer to **part c.ii.** - you can simply subtract a from b to get 2π. In fact, you can use this part to answer **part c.ii.** if you weren't able to before, particularly since it was a 1-mark question that didn't require working. In general, though, it shouldn't be necessary to answer previous parts of a question with parts that follow.

Now, given our values of a and b, we can use one of these to find the equation of the tangent. $f\left(\frac{\pi}{2}\right) = 1 + \frac{\pi}{4}$, and so the tangent passes through $\left(\frac{\pi}{2}, 1 + \frac{\pi}{4}\right)$. Using this point, and the value of m, we get

$$\begin{aligned} y &= \frac{1}{2}\left(x - \frac{\pi}{2}\right) + 1 + \frac{\pi}{4} \\ \iff y &= \frac{1}{2}x - \frac{\pi}{4} + 1 + \frac{\pi}{4} \\ \iff y &= \frac{1}{2}x + 1 \end{aligned}$$

For those unfamiliar with this method, we are taking the line $y = \frac{1}{2}x$ and translating it $\frac{\pi}{2}$ units to the right and $1 + \frac{\pi}{4}$ units vertically upwards, which gives us the required line (since the translation doesn't affect the gradient, but it now passes through a point we know it should pass through).

After you find this tangent, it is highly recommended that you draw it on CAS, along with f, to verify visually that it is actually tangent to f at these two points, especially in a particularly simple situation such as this. Most likely you would still have the graph of f saved, so it wouldn't take too long. As well as this, you can verify your answers to **part c.** and **part d.** - that is, the values of a, b, and m, as well as your answer for b in terms of a.

Question 2

a. There is a slight trick to this question, and that is recognising that it is asking for a general proof - that is, you don't need to actually find $D(x)$ or $D'(x)$ in order to show the required result. The main indicator of this is that you find them in later parts of the question - that should suggest you don't need to find them in this part.

Before dealing with this specific question, the general form needs to be explained. We have a "Show that Y when X" type of question. What this requires you to do is to substitute whatever X is, and show that Y is true. In this particular case, we need to substitute in $D'(x) = 0$ into $\frac{d}{dx}\left(\sqrt{D(x)}\right)$ and show that it is equal to 0. This is different from equating $\frac{d}{dx}\left(\sqrt{D(x)}\right)$ to 0 and showing that $D'(x) = 0$ (this would be the reverse of what you're supposed to do), though it is a subtle difference in terms of how the question is phrased.

A good example to illustrate this difference is this:

Show that $(x-3)(x-4)(x-7) = 0$ when $x = 4$.

We can see that this is equal to 0 when $x = 3, 4, 7$, but we are asked to show it for a particular case. If we solved $(x-3)(x-4)(x-7) = 0$ to give three solutions this wouldn't exactly be answering the question, even though $x = 4$ is a solution. You'd have to substitute in $x = 4$:

$$\begin{aligned}(4-3)(4-4)(4-7) &= 1 \times 0 \times (-3) \\ &= 0\end{aligned}$$

Hence, $(x-3)(x-4)(x-7) = 0$ when $x = 4$.

So, for the question itself, you should proceed as follows:

$$\begin{aligned}\frac{d}{dx}\left(\sqrt{D(x)}\right) &= \frac{d}{dx}\left((D(x))^{\frac{1}{2}}\right) \\ &= \frac{1}{2} \times (D(x))^{-\frac{1}{2}} \times D'(x) \\ &= \frac{D'(x)}{2\sqrt{D(x)}}\end{aligned}$$

Therefore, when $D'(x) = 0$,

$$\begin{aligned}\frac{D'(x)}{2\sqrt{D(x)}} &= \frac{0}{2\sqrt{D(x)}} \\ &= 0\end{aligned}$$

Hence, $\frac{d}{dx}\left(\sqrt{D(x)}\right) = 0$ when $D'(x) = 0$, as required.

This result is generally true for any differentiable function $D(x)$, except for the case when $D(x)$ and $D'(x)$ are both 0 simultaneously, since you would have a zero denominator. But in the context of this question, $D(x)$ is clearly never 0, since the point $(5, 3)$ does not lie on the parabola, so the result that $\frac{d}{dx}\left(\sqrt{D(x)}\right) = 0$ when $D'(x) = 0$ is definitely true. You wouldn't need to mention this in your solution, since it would be implied, but zero denominators are always something you should be mindful of.

Another thing to note here - you should always think about how early parts of an extended-response question can help with later parts. Here, the "message" of this question is that you don't need to minimise the distance if it's easier to minimise the **square** of the distance. Finding $\frac{d}{dx}\left(\sqrt{D(x)}\right)$ would have been more complicated if we had actually found $D(x)$ first, but if you realise you only need to find $D'(x)$, you can avoid finding $\frac{d}{dx}\left(\sqrt{D(x)}\right)$ explicitly.

As with all "show that" questions, you are being marked on whether you understand how to arrive at the given result. Although one mark here would be for correctly differentiating $\sqrt{D(x)}$, the other mark requires clear communication. Your response should be logical, and writing a conclusion is recommended, mainly as a reminder to yourself of what the question was, but it's good form in any case. Generally, if your mathematical communication skills are evident, then your examiners may be somewhat lenient if you make slight mistakes elsewhere. You should aim to convince them that you know what you're doing, not just in "show that" questions, but everywhere you can.

b.i. This requires application of the distance formula, which is a simple concept, but in this case care should be taken, since we're going to end up with a lot of terms.

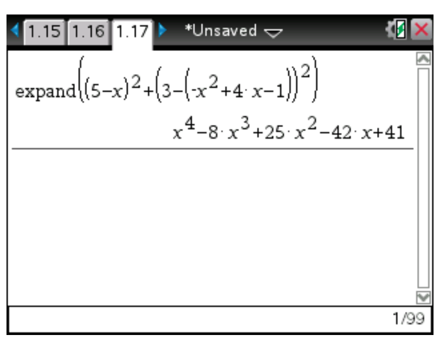
You need to interpret from the information given that $D(x)$ is the square of the distance between the points (x, y) and $(5, 3)$. This means that

$$D(x) \quad = \quad (\Delta x)^2 + (\Delta y)^2$$

Where Δx is the difference in the x-coordinates, and Δy is the difference in the y-coordinates. So,

$$\begin{aligned} D(x) &= (5-x)^2 + (3-y)^2 \\ &= (5-x)^2 + (3-(-x^2+4x-1))^2 \end{aligned}$$

Remember that y is the y-coordinate of any point on the parabola, and so you need to substitute in $y = -x^2+4x-1$. At this point, you should expand the expression using CAS, since it eliminates most chance of error. You can generally expand using [Menu] [3] [3], but sometimes it will expand the expression automatically after you type it out.



Hence, $D(x) = x^4 - 8x^3 + 25x^2 - 42x + 41$. You'd only need to write these three lines out to get the marks. Once you've set up an expression for $D(x)$ in terms of x based on the distance formula, then going straight to the expanded form is the best way to proceed. Expanding by hand isn't recommended, since there's more chance for error, and it would be unnecessarily time-consuming.

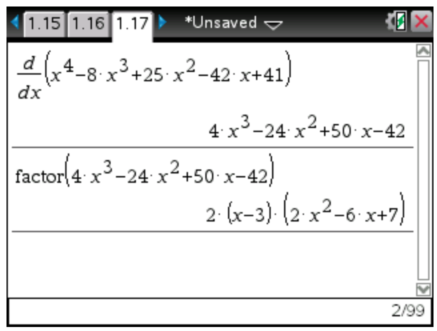
b.ii. This part clearly follows from the first and can be done by hand or on CAS fairly easily. As with all 1-mark differentiation questions, you'd only have to write the one line to get the mark.

$$D'(x) \quad = \quad 4x^3 - 24x^2 + 50x - 42$$

If you like, you can write it down in factorised form (only if you factorise it using technology - doing it by hand is obviously too time-consuming for a 1-mark question), since the factorised form would save some time in answering the next part of the question:

$$D'(x) \quad = \quad 2(x-3)(2x^2 - 6x + 7)$$

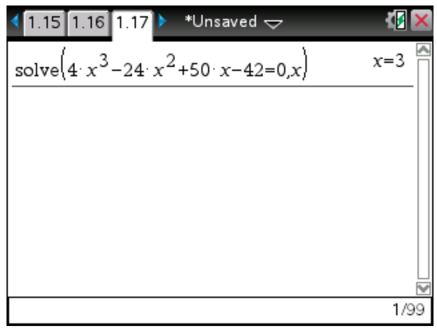
This is, of course, optional, and you wouldn't be expected to write one form over the other. You can factor the polynomial using [Menu] [3] [2], but it will not always completely factorise the polynomial, particularly when the roots are irrational; if it doesn't factorise nicely in a question like this (worth 1 mark) then you shouldn't bother with it.



c. Here we need to use the derivative we just found, and find when it is equal to zero. Using the factorised form:

$$\begin{aligned} D'(x) &= 0 \\ \implies 2(x-3)(2x^2-6x+7) &= 0 \\ \implies x &= 3 \end{aligned}$$

It should be fairly easy to verify that $2x^2 - 6x + 7$ is never equal to zero, and so there is one solution. Even if you don't have the factorised from, though, you can easily solve $4x^3 - 24x^2 + 50x - 42 = 0$ on CAS to give $x = 3$. You wouldn't have to show much working for this part - it would be sufficient to write down anything that indicates that solving $D'(x) = 0$ for x gives $x = 3$.



For the next part, you'd have to show all steps, but there aren't really that many - it is simple arithmetic. In order to proceed, though, you need to remember that P lies on the parabola, and that we have just found the x-coordinate of P, which is 3. By substituting $x = 3$ into the equation for the parabola, we get:

$$\begin{aligned} y &= -(3)^2 + 4(3) - 1 \\ &= -9 + 12 - 1 \\ &= 2 \end{aligned}$$

Hence the y-coordinate of P is 2, as required.

Note that we didn't need to actually find the value of $\sqrt{D(3)}$ or even $D(3)$; we only needed to use $D(x)$ to find the coordinates of P. Although it might make sense that they would ask you to find $\sqrt{D(3)}$, you should be mindful of whether they've actually asked you to find something. You shouldn't waste time finding it unless they ask you to, or if it will help in answering the question, which in this case it wouldn't.

Being given the y-coordinate of P, this part of the question is able to serve as a check for the previous two parts, and it also allows you to answer the next parts if you got stuck somewhere - letting $-x^2 + 4x - 1 = 2$ and finding the more positive solution would give you the x-coordinate of P if you had been unable to find it by other means, and so you can proceed to answer the next parts of the question.

d.i. Finding the equation of a line given two points should be a familiar process, though it is a good idea to write down the coordinates of P first. We implicitly found in the previous part of the question that P is the point $(3, 2)$, and so the gradient of the line is given by

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{3-2}{5-3} \\ &= \frac{1}{2}\end{aligned}$$

And, using the point $(3, 2)$, we get the equation:

$$\begin{aligned} y &= \frac{1}{2}(x-3)+2 \\ \iff y &= \frac{1}{2}x - \frac{3}{2} + 2 \\ \iff y &= \frac{1}{2}x + \frac{1}{2}\end{aligned}$$

Given that we know two of the points it passes through, these can serve as a definite check as to whether we have the correct line. If your line passes through the points $(3, 2)$ and $(5, 3)$, then it is the correct line, since there is only one line that passes through these points. It's always a good idea to check your answer there's a reasonably quick process, like there is in this case.

d.ii. To show that the line is normal to the parabola, we need to consider the derivative of the parabola at the point P, since this will tell us whether they intersect at right angles. For the parabola, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(-x^2+4x-1) \\ &= -2x+4\end{aligned}$$

And at the point P, where $x = 3$, we have

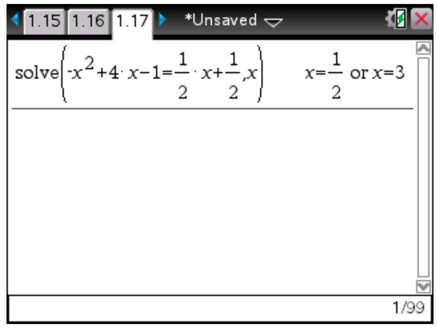
$$\begin{aligned}\frac{dy}{dx} &= -6+4 \\ &= -2\end{aligned}$$

So the tangent at P would have a gradient of -2. In general, if the product of the gradients of two non-vertical lines is -1, then the two lines are perpendicular. Thus, we must show that the product of the derivative at P and the gradient of the line through P and $(5, 3)$ is equal to -1. The gradient of the line is $\frac{1}{2}$ and the derivative is -2, and so we have

$$\frac{1}{2} \times (-2) = -1$$

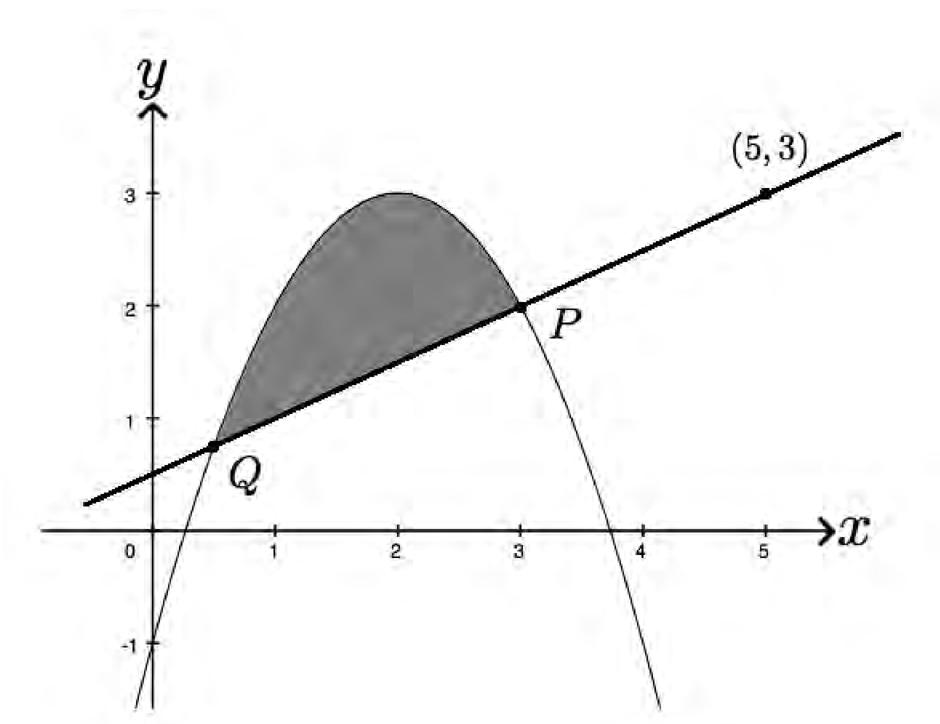
And therefore the line is normal to the parabola at P. Two of the marks here are allocated to finding the derivative of the parabola at P. In order to get the third mark, you'd only have to show that the product of the gradients is -1 and follow this up with a conclusion stating the line is normal to the parabola at P. You wouldn't need to explain that $m_1 m_2 = -1$ means the lines are perpendicular, since it's assumed knowledge - you'd only have to show the result and state a conclusion.

d.iii. To find the x-coordinate of Q, you'd simply need find where the line and the parabola intersect:



So we have $x = \frac{1}{2}$ or $x = 3$ as the x-coordinates of the points of intersection between the line and the parabola. Since 3 is the x-coordinate of P, the x-coordinate of Q must be $\frac{1}{2}$.

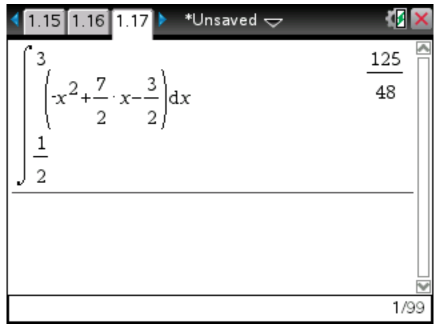
Now, we are asked for the area of a region, so it's a good idea to visualise this region before proceeding:



The general formula for the area between two curves that intersect at $x = a$ and $x = b$ is $A = \int_a^b (f(x) - g(x))dx$, where $f(x) > g(x)$ for $a < x < b$. This is a formula you should be familiar with. So, given that the curves intersect at $x = \frac{1}{2}$ and at $x = 3$, we have

$$
\begin{aligned}
A &= \int_{\frac{1}{2}}^{3} \left((-x^2 + 4x - 1) - \left(\frac{1}{2}x + \frac{1}{2}\right)\right) dx \\
&= \int_{\frac{1}{2}}^{3} \left(-x^2 + \frac{7}{2}x - \frac{3}{2}\right) dx
\end{aligned}
$$

The rest should be done on CAS:

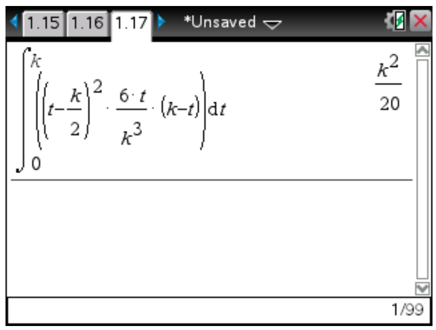


So $A = \dfrac{125}{48}$ square units. An exact answer is required here, as a decimal approximation is not asked for. To get full marks for this question, you don't need to write a great deal - for calculating the area, you'd simply have to write down a correct integral and what it evaluates to in order to get the marks.

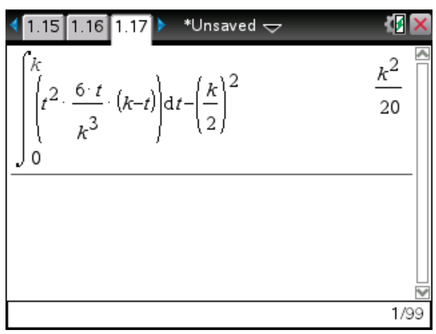
Question 3

a. Here, you can use the formula $E(T) = \int_{-\infty}^{\infty} t \cdot p(t)dt$, but a more simple approach is to use the fact that the graph of $p(t)$ is a parabola for $0 \le t \le k$. Since the intercepts are at $t = 0$ and $t = k$, we know the stationary point occurs at $t = \frac{k}{2}$, due to the symmetry of the parabola. This is also where the mean value occurs, and hence, $E(T) = \frac{k}{2}$. Evaluating $\int_0^k t \cdot \frac{6t}{k^3}(k-t)dt$ would give the same answer, but in any case you'd only have to write down the answer to get the mark.

b.i. As per usual, in order to find the standard deviation, you'll need to find the variance first. Using the formula $\text{var}(T) = \int_{-\infty}^{\infty} (t-\mu)^2 p(t)dt$, we get



Alternatively, you could use the equivalent formula, $\text{var}(T) = E\left(T^2\right) - \mu^2$, which gives



Now, the standard deviation is the square root of this, which is $\frac{k}{\sqrt{20}} = \frac{\sqrt{5}k}{10}$. If you took the square root of the variance on CAS, you'd get $\frac{\sqrt{5}|k|}{10}$, but since we know that k is positive (implied by $0 \le t \le k$), we don't need to write the modulus signs.

b.ii. Given that we have an expression for the standard deviation, finding when it is equal to 2 is fairly simple:

$$\begin{aligned} \frac{\sqrt{5}k}{10} &= 2 \\ \implies k &= \frac{20}{\sqrt{5}} \\ &= 4\sqrt{5} \end{aligned}$$

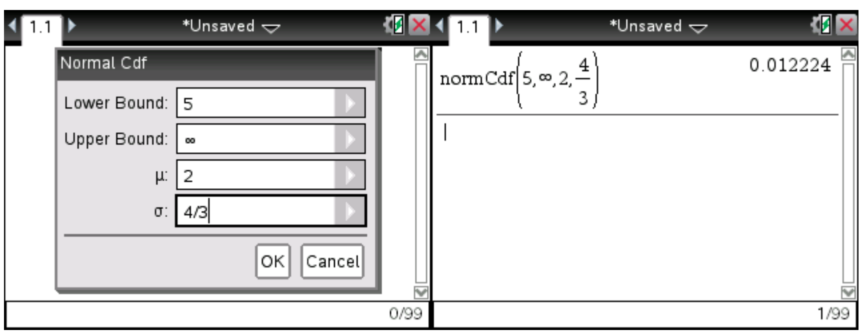
b.iii. The smallest value of $E(T)$ corresponds to the value of k we have just found. This is because the standard deviation, k, and $E(T)$ all increase and decrease simultaneously if the value of k increases or decreases, which can be easily observed from the relations $\sigma = \dfrac{\sqrt{5}k}{10}$ and $E(T) = \dfrac{k}{2}$. Since the smallest value of the standard deviation is given as 2, the smallest value of k is $4\sqrt{5}$, and therefore the smallest value of $E(T)$, using the formula found in **part a.**, is given by

$$\begin{aligned} E(T) &= \frac{4\sqrt{5}}{2} \\ &= 2\sqrt{5} \end{aligned}$$

To one decimal place, this is equal to 4.5 minutes.

c. Firstly, we need to convert the given standard deviation to minutes, so that the units are consistent in our calculations. In minutes, the standard deviation is $\dfrac{80}{60} = \dfrac{4}{3}$. Hence the time after 8:00 am that Thomas arrives at the station is normally distributed with a mean of 2 minutes and a standard deviation of $\dfrac{4}{3}$ minutes. If we let X represent this time, then we have $X \sim \mathrm{N}\left(2, \dfrac{16}{9}\right)$ (recall that the general written notation is $X \sim \mathrm{N}(\mu, \sigma^2)$, where μ is the mean and σ^2 is the variance of the given normal distribution, even though CAS input usually requires σ instead of σ^2).

We know the distribution of X, so now we need to figure out what probability we require. Thomas will miss the train if he arrives after 8:05 am, or rather, if he arrives more than 5 minutes after 8:00 am. Therefore, we require $\Pr(X > 5)$:



Hence, the probability that Thomas will miss the train is 0.0122, correct to four decimal places.

d.i. For both parts of this question, you will need to correctly interpret the phrase "he will be sure to catch the train the next day." This means the probability that Thomas catches the train on any day during the week, given that he missed the train on the previous day, is equal to 1. The word "otherwise" refers to the alternative - the probability that he will catch the train, given that he caught the train on the previous day, is equal to 0.95. This indicates we are in a situation with transitional probabilities, and most likely that we will need to use a transition matrix at some point, but for this part we don't need to, since we know the exact sequence of events.

If he misses the train on day 1, then he will catch it on day 2 - this is given. Catching the train on day 3, then, has a probability of 0.95, since he will have caught the train on the previous day, and similarly, him catching the train on day 4 and day 5 will have probabilities of 0.95, if this trend continues. So, if he misses the train on day 1, then the probability of him catching the train on the next four days - that is, days 2, 3, 4, and 5 - is equal to $1 \times 0.95 \times 0.95 \times 0.95 = (0.95)^3$, which is equal to 0.8574, correct to four decimal places.

d.ii. For this part, we only consider day 1 and day 5, since for the days in between, he can either catch or miss the train, and we need to account for all of the combinations. Setting up a transition matrix allows us to do this.

Let C_n be the probability that he catches the train on day n, and C'_n be the probability that he misses the train on day n. The general form of our 2×2 transition matrix is:

$$T = \begin{bmatrix} \Pr(C_k \mid C_{k-1}) & \Pr(C_k \mid C'_{k-1}) \\ \Pr(C'_k \mid C_{k-1}) & \Pr(C'_k \mid C'_{k-1}) \end{bmatrix}$$

We know that $\Pr(C_k \mid C_{k-1}) = 0.95$, and $\Pr(C_k \mid C'_{k-1}) = 1$, and therefore that $\Pr(C'_k \mid C_{k-1}) = 0.05$ and $\Pr(C'_k \mid C'_{k-1}) = 0$, since the elements of each column add to 1. So our transition matrix is:

$$T = \begin{bmatrix} 0.95 & 1 \\ 0.05 & 0 \end{bmatrix}$$

Now, since he missed the train on day 1, our initial state matrix is:

$$S_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

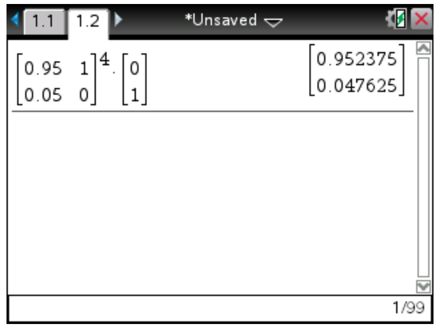
And in general,

$$S_n = \begin{bmatrix} \Pr(C_n) \\ \Pr(C'_n) \end{bmatrix}$$

To calculate the probability that he will miss the train on day 5, which is $\Pr(C'_5)$, we will use the relationship $S_n = T^n \times S_0$, which is what is written on the formula sheet, but since we are numbering our initial state as 1 rather than 0, we can change this to $S_n = T^{n-1} \times S_1$ to suit our purposes. Using this, we have:

$$\begin{aligned} \begin{bmatrix} \Pr(C_5) \\ \Pr(C'_5) \end{bmatrix} &= \begin{bmatrix} 0.95 & 1 \\ 0.05 & 0 \end{bmatrix}^{5-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.95 & 1 \\ 0.05 & 0 \end{bmatrix}^{4} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

Evaluating this on CAS gives:

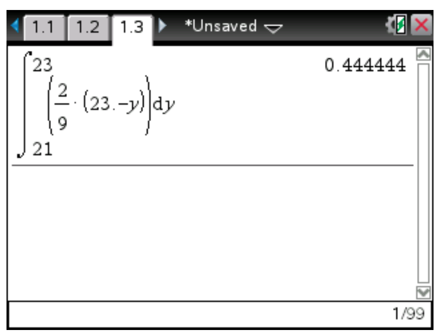


So, correct to four decimal places, $\Pr(C_5') = 0.0476$.

e. Although the wording of this question is somewhat lengthy and complicated, what it's actually asking is fairly simple, which is common to a lot of probability questions. This question is mostly just a matter of figuring out what is being asked.

We want to know the probability he will have to run to work, and he will have to run to work if he has less than 3 minutes to make it there by 8:30 am. So, if he arrives at the station later than 8:27 am, he will have to run to work, since if he arrives any time after that, there will be less than 3 minutes before 8:30 am. Therefore, we want the probability that he will arrive after 8:27 am.

The train departs at 8:06 am, and we have to assume that it is exactly at the start of 8:06 am, or there wouldn't be a definite answer to the question. If the train arrives at 8:27, this will correspond to $Y = 21$, since 8:27 am is 21 minutes after 8:06 am. Hence, we want to find $\Pr(Y > 21)$, which can be evaluated on CAS:



So, the probability that he will have to run to work is 0.44, correct to two decimal places.

Question 4

a.i.

$$f'(x) = 2\cos(2x) + 2\cos(x)$$

This is a simple 1-mark differentiation question that doesn't require much explanation. A question like this wouldn't require any working to be shown, so it's fine to just write down the derivative and move on.

a.ii. Being a "show that" question, this question requires clear communication, since the only thing they are marking you on is whether or not you understand how to arrive at the result they have given you. There are a few ways to approach this type of question, and while the "LHS = ... = RHS" set-up is common, it's possible for you to "meet halfway" if it's more convenient. We can start by substituting in the equivalent expression for $\cos(2x)$ that was given to us:

$$\begin{aligned} \text{LHS} &= 2\cos(2x) + 2\cos(x) \\ &= 2(2\cos^2(x) - 1) + 2\cos(x) \\ &= 4\cos^2(x) + 2\cos(x) - 2 \end{aligned}$$

Now, at this point, you should be able to guess that this expression will factorise to resemble the factorised form of $p(x)$. But rather than attempting to factorise this quadratic function of $\cos(x)$, you can simply expand $p(x)$:

$$\begin{aligned} p(x) &= 2(2x-1)(x+1) \\ &= 2(2x^2 + x - 1) \\ &= 4x^2 + 2x - 2 \end{aligned}$$

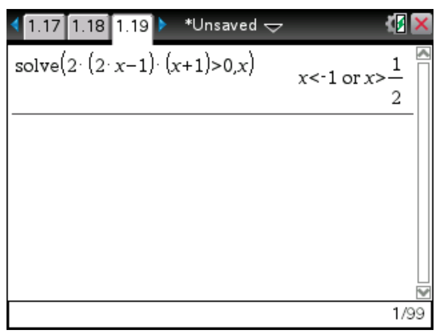
$$\begin{aligned} \therefore \text{RHS} &= p(g(x)) \\ &= 4(\cos(x))^2 + 2(\cos(x)) - 2 \\ &= 4\cos^2(x) + 2\cos(x) - 2 \\ &= \text{LHS} \end{aligned}$$

Therefore $f'(x) = p(g(x))$, as required.

The method of manipulating both sides of the equation, instead of just one, is useful when both sides have some sort of neatness to them. Here, the expression for $p(x)$ was factorised. Factorising a quadratic is often more time-consuming than expanding one, so rather than attempting to factorise the quadratic in $\cos(x)$, it's perfectly fine, and more convenient, to expand $p(x)$ first, since it's logically equivalent. It should be emphasised, though, that you need to clearly indicate that you have understood how both sides of the equation are equal. Above, I simplified the left hand side expression first, but ultimately I ended up with "RHS = ... = LHS", which is how you should aim to write out your proofs in questions like these. It doesn't hurt to write a conclusion (e.g. "Therefore $f'(x) = p(g(x))$, as required", as above) to further show the examiners that you have understood the question.

b.i. Being a 1-mark question, a highly recommended approach is to graph $y = \cos(x)$ over the given interval (either on CAS or by hand) and determine the range visually. The endpoints of the graph (which are not included) are $\left(-\frac{\pi}{3}, \frac{1}{2}\right)$ and $\left(\frac{\pi}{3}, \frac{1}{2}\right)$, which should be easily determined if you know your exact values, but can be verified with technology. You should be able to determine from a quick sketch, or on CAS, at least, that the graph of g is a $\cap$-shaped graph with its maximum at $(0, 1)$, so the endpoints (again, not inclusive) are where the minimum values of g occur, and hence the range of g is $\left(\frac{1}{2}, 1\right]$. This interval is all you need to write to get the mark, so you should use any means to find the range that will give you the answer in the shortest amount of time. Generally, recalling what the graphs of $y = \cos(x)$ and $y = \sin(x)$ look like will be beneficial in questions like these, in terms of how much time you'll save. The fact that the graph of $y = \cos(x)$ is symmetrical about the y-axis may also help.

b.ii. As with the previous part of the question, you will only need to state a set of values to get the mark, so it doesn't particularly matter how you go about it, as long as you write down the correct answer. Your CAS should be able to solve this inequality, but you can do it quickly by hand, or in your head, if you're on the ball. The x-intercepts of the graph of $y = p(x)$ occur when $2x - 1 = 0$ or $x + 1 = 0$, i.e. when $x = -1, \frac{1}{2}$. Since $p(x)$ is a quadratic whose x^2 term has a positive coefficient, $p(x)$ will be less than or equal to zero for $-1 \leq x \leq \frac{1}{2}$, and positive elsewhere. Hence, $p(x) > 0$ when $x < -1$ or $x > \frac{1}{2}$. Or, in interval notation, when $x \in (\infty, -1) \cup \left(\frac{1}{2}, \infty\right)$. This can be verified on CAS:



When it comes to quadratic inequalities, you should picture the graph of the parabola in your head to make sure your answer makes sense. For a parabola such as this one that has a minimum turning point and two intercepts, the values of x for which it is less than or equal to zero should be a single interval, since there is only one section of the graph that is below the x-axis. It will be positive elsewhere, so you will have a union of intervals for these values of x, rather than a single interval.

b.iii. Here, you will need to use the fact that $g(x) > \frac{1}{2}$. It's not particularly important that $g(x) \leq 1$. The idea is that $g(x)$ is in the set of domain values for which $p(x) > 0$, which means $p(g(x)) > 0$. To make this easier to understand, it may be useful to let $u = g(x)$. Then we have $u > \frac{1}{2}$, because $g(x) > \frac{1}{2}$.

If $p(x) > 0$ when $x > \frac{1}{2}$ (and also when $x < -1$, but this won't apply here), we can also say that $p(u) > 0$ when $u > \frac{1}{2}$, since it shouldn't matter what we call the variable; our choice of x in the first place is arbitrary. What we really mean by saying "$p(x) > 0$ when $x > \frac{1}{2}$" is that for any number greater than $\frac{1}{2}$, the function p outputs a number greater than 0. So whether we call this number x or u, the statement is still true.

So, if $p(u) > 0$ when $u > \frac{1}{2}$, and $u = g(x)$, then $p(g(x)) > 0$ when $g(x) > \frac{1}{2}$, which is true for all values in the domain of g, which is the given interval $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$. Hence, $p(g(x)) > 0$ for all $x \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Now, there is one last step. We need to connect this to $f'(x) > 0$. This is the easier part of the question, since we know $f'(x) = p(g(x))$, we can simply say that $p(g(x)) > 0$ means $f'(x) > 0$.

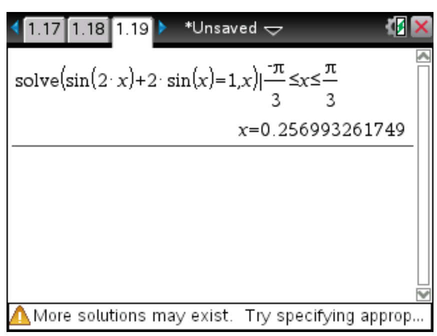
Although this was a lengthy explanation, you wouldn't need to write a great deal for this question. I would write something like this:

$$\begin{aligned}
\text{If} \quad x &\in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) = \text{dom}(g), \quad \text{then} \\
g(x) &> \frac{1}{2} \\
\implies p(g(x)) &> 0 \qquad (\text{as } p(x) > 0 \text{ when } x > \frac{1}{2}) \\
\implies f'(x) &> 0 \qquad (\text{as } f'(x) = p(g(x)))
\end{aligned}$$

Which is fairly concise but still shows understanding of the problem. You should mention somewhere that $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ is the domain of g, which is what makes the result valid.

As a reminder, the word "hence" indicates that you need to use the previous results of **part b.i.** and **part b.ii.**, and while the word "hence" does not refer to **part a.ii.** in this sense, you should still use this result, as long as you also use the previous two results. Note that, regardless of whether or not you were able to prove the result given in **part a.ii.**, you are still expected to use it in later parts of the question if you need to.

c.i. This is mostly a straightforward CAS question, though you will need to restrict the values of x to within the domain of f in order to only get one solution. Otherwise you will get a general solution, which isn't of use here.



So the required solution is $x = 0.257$, correct to three decimal places. It should be noted that in a simple question such as this, which does not require an exact value and is worth 1 mark, no attempt should be made to solve the equation by hand. As with most of the 1-mark questions here, you only need to write down your answer of $x = 0.257$ to get the mark.

c.ii. There a few ways to go about answering this, since there are a few ways to understand why there is only one solution. Firstly, it should be noted that f is not differentiable at $x = -\dfrac{\pi}{3}$ or $x = \dfrac{\pi}{3}$, since these are the endpoints of the graph, and it impossible to find a unique tangent line at such points. This is why we have considered $f'(x)$ over $\left(-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$ rather than $\left[-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right]$, since f is only differentiable over $\left(-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$.

The fact that $f'(x) > 0$ means there are no local minima or maxima on the graph of f. This is because either of these types of stationary points would require f to have a negative gradient at some point. If the gradient of f is never negative, we can never have a horizontal line intersect the graph of $y = f(x)$ more than once, and this is what the question is asking - why the graph of $y = f(x)$ intersects the graph of $y = c$ at exactly one point, so long as c is not outside of the range of f. It's because f is one-to-one.

In other words, if $f'(x) > 0$, and f is continuous (which it is - while it is not differentiable at its endpoints, it is still continuous at those points), it follows that f is a one-to-one function, and therefore any horizontal line would intersect the graph of $y = f(x)$ at most once.

Another way to look at this is that the fact that $f'(x) > 0$ and f is continuous implies that f is a strictly increasing function. The definition of a function f being strictly increasing over a domain D is this:

$$\begin{array}{rcl} \text{If} \quad a, b & \in & D \\ \text{and} \quad a & < & b \\ \text{then} \quad f(a) & < & f(b) \end{array}$$

Or, in words, as x increases, $f(x)$ also increases. The definition of strictly decreasing is similar, except you would have $a < b \implies f(a) > f(b)$, rather than $f(a) < f(b)$. Note that it doesn't matter if f has zero gradient, or is not differentiable at some point. For example, the cubic function $y = x^3$ has zero gradient at the origin, but the cube of a negative number is less than the cube of 0, and the cube of a positive number is greater than the cube of 0, so the cubic function $y = x^3$ is strictly increasing over its entire domain of $\mathbb{R}$, since any two real values will satisfy $a < b \implies a^3 < b^3$.

In particular, we have two points where f is not differentiable, but we know that since f is continuous, and $f'(x) > 0$ for $\left(-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$, that $f\left(\dfrac{\pi}{3}\right) > f(x)$ for any other domain value, and that $f\left(-\dfrac{\pi}{3}\right) < f(x)$ for any other domain value. Therefore, f is a strictly increasing function.

In terms of the problem at hand, what we may say from the fact that f is strictly increasing is this:

$$\begin{array}{rrcl} & a < b & \implies & f(a) < f(b) \\ \therefore & a \neq b & \implies & f(a) \neq f(b) \end{array}$$

A function satisfying $a \neq b \implies f(a) \neq f(b)$ is actually the definition of a one-to-one function. This should be understood intuitively - we know a one-to-one function mustn't have more than one x-value for a given y-value. The function $y = x^2$ is not one-to-one because there are, for example, two x-values for which $y = 1$. So we cannot say that $a \neq b$ means $a^2 \neq b^2$, because there can very well be two distinct real numbers with the same square. If the domain of the square function is restricted to, for example, $(-\infty, 0]$, then it will be a one-to-one function, as you may be familiar with.

However, to use the example of $y = x^3$ again, each y-value is unique for this function. There are no two distinct real numbers that have the same cube. So, if $a \neq b$, we can say that $a^3 \neq b^3$, and so the function is one-to-one.

It should be easy to see from this that if a function is strictly increasing or strictly decreasing, then it is must also be one-to-one. Therefore, by simply observing that f is strictly increasing, you can conclude that f is one-to-one.

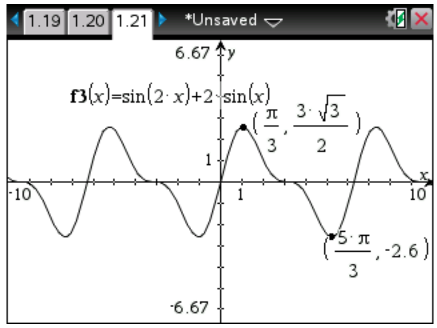
There is one more link to make here, and that is using this information to explain why there will only be one solution to $f(x) = c$ for $c \in \text{ran}(f)$. If f is one-to-one, then f has an inverse (as a side-note, the converse is also true - if a function is invertible, then it is one-to-one). Using this fact, it is easy to prove that there is one solution to $f(x) = c$ for $c \in \text{ran}(f)$. The solution is simply $x = f^{-1}(c)$, because f has an inverse function. And since $\text{ran}(f) = \text{dom}(f^{-1})$, c is always in the domain of f^{-1}, and so the solution $x = f^{-1}(c)$ always exists and is unique.

Again, this is a lengthy explanation, but once again the extent to which you have to answer this question is much less than what I've written here. You would simply have to write something akin to the following:

f is continuous, and $f'(x) > 0$, which implies that f is strictly increasing, and is therefore one-to-one. Hence, f has an inverse function, and the solution to the equation $f(x) = c$ is $x = f^{-1}(c)$, which is unique and defined for all values of $c \in \text{ran}(f) = \text{dom}(f^{-1})$.

You may not have to even write this much, but it would be best to include all relevant information. The must-haves here would be using the fact that $f'(x) > 0$, and concluding from this that there can only be one intersection with the horizontal line $y = c$, preferably by considering the properties of one-to-one functions. This is the bare minimum, but again, you should use as much relevant information as possible. Mentioning the continuity of f can probably be omitted since it's somewhat obvious.

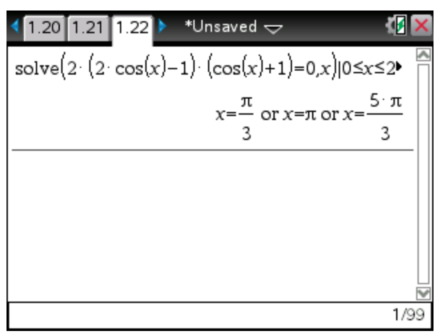
d. This question uses the results, or at least the concepts, of all of the previous parts of the question, in some form or another. Firstly, we will have to use the notion that the function h will be invertible if and only if it is a one-to-one function, and we will be able to see when this is the case most easily if we graph it using CAS.



We are interested in the stationary points that occur for $x \geq \frac{\pi}{3}$, since these will indicate when h is one-to-one, and therefore invertible. There is a stationary point of inflection at $x = \pi$, and a local minimum at $x = \frac{5\pi}{3}$. There is also a stationary point at $x = \frac{\pi}{3}$, but this isn't particularly useful information - we only want to know where the function stops increasing, or decreasing, as the case may be. The x-coordinates can be found by hand, by using the factorised expression for $f'(x)$ (since $h'(x)$ has the same rule):

$$\begin{aligned} 2(2\cos(x) - 1)(\cos(x) + 1) &= 0 \\ \implies \cos(x) &= \frac{1}{2} \\ \text{or} \quad \cos(x) &= -1 \\ \implies x &= \frac{\pi}{3}, \pi, \frac{5\pi}{3}, \cdots \end{aligned}$$

Or on CAS, over a reasonably small domain:



We can determine from these values, and the graph, that h will be one-to-one so long as $d \leq \frac{5\pi}{3}$. This is because h is strictly decreasing over the interval $\left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$ (remember that the zero gradient at $x = \pi$ does not mean h is not strictly decreasing - we only require that $a < b$ implies $h(a) > h(b)$ for all $a, b \in \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$, and we can see that this is the case). However, if we allow d to be greater than $\frac{5\pi}{3}$, h will not be one-to-one, since, as we can see from the graph, it would be possible for a horizontal line to intersect the graph at more than one point. Hence, h will not have an inverse for $d > \frac{5\pi}{3}$.

In order to get full marks for this, you should list both of the x-coordinates of the stationary points at $x = \pi, \dfrac{5\pi}{3}$, and you would need to express that h is one-to-one provided that $d \leq \dfrac{5\pi}{3}$, and is not invertible otherwise. You wouldn't have to explicitly state that there is a point of inflection at $x = \pi$, but it would be better to include it. You would not have to justify the nature of any of the stationary points. Doing a quick sketch of the graph would most likely assist you in explaining your answer, though you should be careful when drawing the stationary point at $x = \pi$ so it is clear that h is strictly decreasing for $x \in \left[\dfrac{\pi}{3}, \dfrac{5\pi}{3}\right]$. You need to write $d > \dfrac{5\pi}{3}$ or $d \in \left(\dfrac{5\pi}{3}, \infty\right)$ as your final answer, and you need to use exact values.

Author biographies

Trevor Batty graduated dux of Copperfield College in 2011 and achieving a study score of 45 raw in Mathematical Methods. He is currently studying a double Bachelor of Aerospace Engineering and Science at Monash University, maintaining an outstanding high distinction average.

Will Hoang graduated from Melbourne Grammar School in 2011, achieving an ATAR of 99.90. His score included a perfect 50 in Mathematical Methods and he received a Premier's Award in the subject for placing among the top 5 students in the state. Will is now studying a Bachelor of Biomedicine (Chancellor's Scholars Program) at the University of Melbourne.

Tim Koussas achieved an impressive trifecta of study scores of 50, 48 and 47 in Mathematical Methods, Further Mathematics and Specialist Mathematics when he graduated from Parade College in 2011. He also received a Premier's Award in Mathematical Methods for placing among the top 5 students in the state. He is now studying a Bachelor of Science at La Trobe University and as an undergraduate has already completed research projects for senior academics at the University.

William Swedosh scored a perfect 50 in Mathematical Methods in his graduating year of 2007 at McKinnon Secondary College. He has since completed a double Bachelor of Engineering and Science with an average of 94% across his Mathematics units.

Nathaniel Lizak graduated dux of Bialik College in 2011 with an ATAR of 99.90 which included a perfect 50 in Specialist Mathematics and a Premier's Award for placing amongst the top 5 students in the state. In that same year, Nathaniel also achieved the impressive feat of scoring the top mark in the University of Melbourne Extension Program for Mathematics (which allows year 12 students to undertake university studies while still in high school). Nathaniel is currently studying for an MBBS at Monash University.

Thushan Hettige is the author of the highly popular ExamPro Chemistry Units 3&4 Study Guide. He graduated dux of Scotch College in 2011 with an ATAR of 99.95, which included 5 perfect 50s (one of which included Mathematical Methods) and Premier's Awards in Chemistry, Biology, English Language and Mathematical Methods. This means that Thushan was in the top 5 students in the state for four of the subjects he studied. Thushan also won the silver medal at the International Chemistry Olympiad in Turkey. His achievements earned him the Australian Students' Prize for 2011 which is awarded to the top 500 students from across the country. Thushan is currently studying for an MBBS at Monash University.

Daniel Levy is a founder of ExamPro and graduated from Bialik College in 2007 with an ATAR of 99.45. Daniel is currently completing a double degree in Aerospace Engineering and Science (Applied Mathematics) part-time at Monash University while in a full-time role with ExamPro. Daniel was previously employed by Monash University as a subject tutor for units taught by the Faculty of Mathematics.