Preface: How to use this book

The purpose of the ExamPro Study Guides is to provide practice questions for specific VCE subjects (in this case, Mathematical Methods Units 3&4), and also to provide comprehensive solutions to each of these questions. This book contains only the questions themselves, which have been written to match the style of questions you will get in the official end-of-year VCAA exams (VCAA, the Victorian Curriculum and Assessment Authority, is the organisation responsible for writing and marking final VCE exams, among other things). The solutions manual, freely available for download at www.examproguides.com/solutions, contains marking schemes for each question, as well as model solutions (indicating how each question may be answered in an exam for full marks) and detailed solutions.

The bulk of what the authors have actually written is contained in the detailed solutions section of the solutions manual, where we cover all thought processes used in answering each of the questions in this book. We believe this to be what separates us from the rest of the market, as most of the VCE practice material you will come across will only come with model solutions, and sometimes not even that!

The overall structure of this book can be seen in the table of contents on page 3. We have topic tests in four areas, which are based on the four areas of study listed in the VCAA study design for Methods 3&4. These four areas are "functions and graphs", "algebra", "calculus", and "probability and statistics".

Our topic tests are intended to serve as something like "practice SACs" (SACs are the tests taken at your school that contribute to your final subject marks). Naturally, since different schools will have different SACs, we can't exactly predict the content of your SACs. We have opted for sticking close to the style of the end-of-year exams in these tests, while focusing on one area of study at a time (with some minor exceptions).

As with the areas themselves, our tests in each area will build on knowledge from the previous areas. Occasionally, some of our tests will also require knowledge that is, strictly speaking, from later areas of study. Specifically, some of our tests in the "functions and graphs" area will use some of the knowledge from "algebra". This is because teachers and textbooks will often blend the first two areas of study, as they tend to go hand-in-hand in terms of teaching. Also, some of our "algebra" tests will use some basic calculus knowledge; namely, the use of differentiation in finding stationary points of polynomials, as well as the equations of lines tangent to curves. This amount of calculus knowledge is covered in Units 1&2, and for this reason your SACs have a high chance of including some basic calculus before you formally start learning calculus in Units 3&4.

We mention these discrepancies as a disclaimer to the fact that our tests in each area cover only the content listed in the corresponding area in the VCAA study design. Ultimately, though, all of our tests are within the scope of the study design, and contain content that is very likely to appear in SACs corresponding to the relevant area of study.

As for the actual structure of the tests, we have written each test in the same format as a VCAA exam, but with half the number of marks. The VCAA end-of-year exam formats for Methods are as follows:

- Examination 1: Contains 40 marks' worth of short-answer questions and is to be completed in one hour using only the provided formula sheet (and a device for writing on paper). Contributes 22% to your final mark for Methods.
- Examination 2: Contains 20 multiple-choice questions (1 mark each) and 60 marks' worth of extended-response questions, and is to be completed in two hours using a CAS calculator and a bound reference, as well as the provided formula sheet. Contributes 44% to your final mark for Methods.

As such, our tech-free tests each contain 20 marks' worth of short-answer questions, and are intended to be completed in half an hour using only the formula sheet provided at the end of this book (which is the same as VCAA's formula sheet). Similarly, our tech-active tests contain 10 multiple-choice questions and 30 marks' worth of extended-response questions, and are intended to be completed in one hour using a calculator and a bound reference (and the formula sheet if you wish). More details are given on the front cover of each test.

The second part of this book is easier to describe: it consists of three sets of practice exams. Each set consists of a practice Examination 1 and a practice Examination 2, and these have exactly the same format as the VCAA exams.

Whether you stick to the constraints suggested for each test is up to you, since the book doesn't come with pocket-sized test supervisors. Without adequate practice, though, the VCAA exams can be brutal, especially the second one. Attempting to complete the tests and practice exams in the suggested amount of time, and only with the allowed references, will most likely help you work on your own techniques for answering questions quickly. That said, if you don't get everything done in time, it's vital that you go back and learn as much from your mistakes as you can; any of the top students in each year can attest to this. Apart from explaining how to do every single question in this book, our detailed solutions suggest a lot of time-management techniques, such as techniques for using calculators efficiently, as well as common tricks or formulas that can be used to save time.

A strategy that may be effective is to complete the first of each pair of tests at a leisurely pace, check your answers, read the detailed solutions, and then do the second test of the pair under timed conditions. If you aren't confident in a particular area, then this would be more effective than trying to do the test without knowing any of the content (although it might be better to learn the content from textbooks first, just so you have two whole tests that you can do under timed conditions).

The fact that the final examinations are worth a whopping 66% of your final score makes practice under timed conditions invaluable (not to mention that most of your Methods SACs are likely to also be held under timed conditions, and these contribute the remaining 34%). You should therefore save at least two pairs of our practice examinations for when you are ready to complete them under timed conditions.

If you have time during your exam preparation, you should do any of the questions in our topic tests that you didn't get around to completing during the year, and also read some of the detailed solutions for these tests, as we've scattered a lot of exam advice throughout the book (not just in the solutions to the practice exams).

This second edition of the ExamPro Study Guide for Methods is written for the VCAA study design that was implemented in 2016. If you're reading this within a few years of 2016, you will not have many past VCAA exams from the current study design to work with. In this case, these official past VCAA exams (which are freely available on the VCAA website) will be precious commodities, and should be saved until the last few weeks before your actual exams.

Before attempting relevant past exams, it's worth having a look at the exams from the previous study design (effective in 2006-2015), since a vast majority of the questions will still be relevant. If you're not sure whether something is relevant, you should check with the study design. In fact, it's a good idea to go over the study design for its own sake, and most of the top students will do this. There may be small things that your teachers miss, so it's a good way to make sure that you've covered everything.

The VCAA exams based on the current study design are by far the best practice material for the subject, although we have done our best to create practice exams in the same style. Apart from updating our material in this edition to suit the current study design, we have rewritten major sections of both the question book and the solutions manual in order to more closely match VCAA's style and format. We've also hopefully caught any errors that stuck around, and hopefully not created any new ones, but if you find any then feel free to email us at methods@examproguides.com, and we will update the solutions manual to include any errors that you may have found. Comments and suggestions are also appreciated! We don't claim that every single one of our solutions is the absolute best way to explain a given topic, but we have certainly done our best. Therefore, even a comment such as "the solution to question X is confusing because it doesn't explain Y" is extremely helpful to us in improving the quality of the guide.

There will be plenty more for us to say in the solutions manual, but for now, we wish you the best of luck with your studies.

– Tim Koussas

TABLE OF CONTENTS

MATHEMATICAL METHODS

Functions and graphs
Technology-free test 1

Reading time: 10 minutes
Writing time: 30 minutes

Structure of test

Number of questions	*Number of questions to be answered*	*Number of marks*
5	5	20

Allowed materials
- Pens, pencils, highlighters, erasers, sharpeners and rulers.

Materials supplied
- The two-page formula sheet at the end of this book may be used.

Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

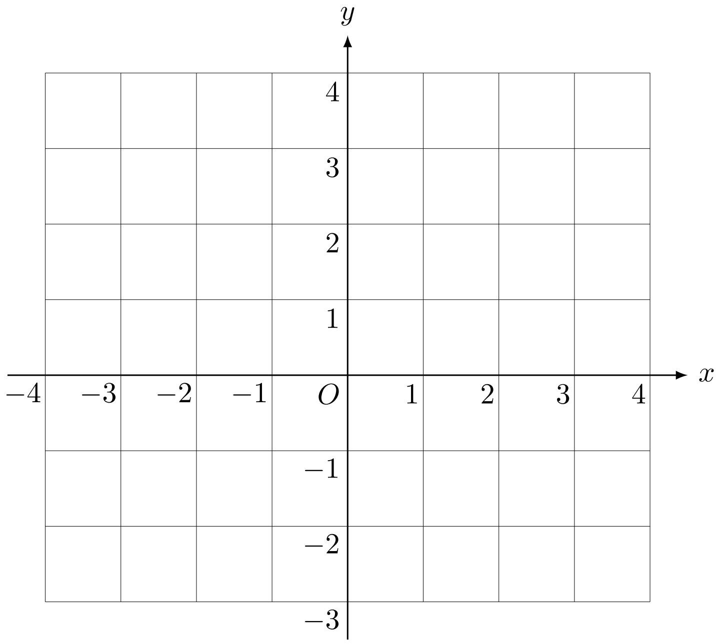
Question 1 (5 marks)

Consider the function $f\colon \mathbb{R} \to \mathbb{R}$, where $f(x) = x^2 - x - 2$.

a. Solve the equation $f(x) = 0$ for $x \in \mathbb{R}$. 2 marks

b. On the set of axes below, sketch the graph of the function f.

Label any turning points with their coordinates. 2 marks



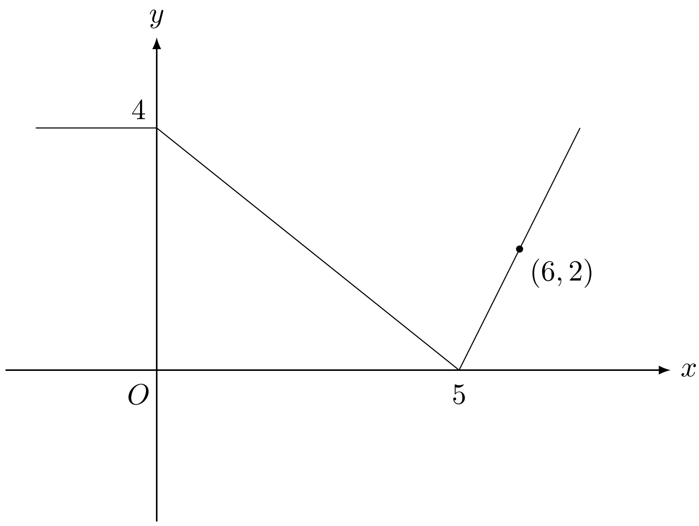
c. Hence, find the set of values of $x \in \mathbb{R}$ for which $f(x) > 0$. 1 mark

Question 2 (2 marks)

The function $h\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) = \begin{cases} 4 & x \leq 0 \\ ax + b & 0 < x < 5 \\ cx + d & x \geq 5 \end{cases}$$

where a, b, c, and d are real constants. Part of the graph of h is shown below.

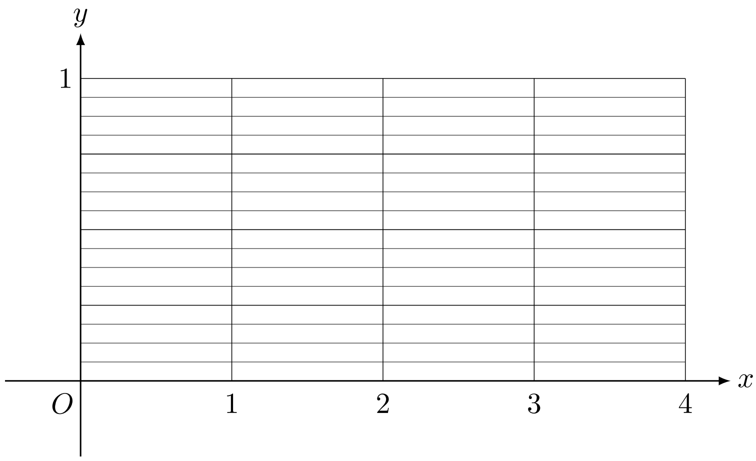


Find the values of a, b, c, and d.

Question 3 (6 marks)

Let $g\colon [0, \infty) \to \mathbb{R}$, where $g(x) = 1 - 4 \cdot 2^{-x-2}$.

a. On the set of axes below, sketch the graph of g. 2 marks



b. State the range of g. 1 mark

c. **i.** Find the rule for the inverse, g^{-1}, of the function g. 2 marks

ii. State the range of g^{-1}. 1 mark

Question 4 (4 marks)

The function f has rule $f(x) = \tan\left(6x - \frac{\pi}{2}\right)$ and is defined over its maximal domain.

a. Find the period of f. 1 mark

b. The line $x = a$ is a vertical asymptote of the graph of f, where $a \in \left[0, \frac{\pi}{2}\right]$.

Find the possible value(s) of a. 3 marks

Question 5 (3 marks)

The transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The image of the curve $y = 3x^2 + 1$ under the transformation T has equation $y = ax^2 + bx + c$.

Find the values of a, b, and c.

END OF TEST

Download the free detailed solutions manual at www.examproguides.com/solutions

Formula sheet

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$
$\frac{d}{dx}\left((ax+b)^n\right) = an(ax+b)^{n-1}$	$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ n \neq -1$
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$	$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$	$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$	

product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A\|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}\big((X-\mu)^2\big) = \mathrm{E}\big(X^2\big) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\, f(x)dx$

Sample proportions

$\widehat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}\big(\widehat{P}\big) = p$
standard deviation	$\mathrm{sd}\big(\widehat{P}\big) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\widehat{p} - z\sqrt{\dfrac{\widehat{p}(1-\widehat{p})}{n}},\ \widehat{p} + z\sqrt{\dfrac{\widehat{p}(1-\widehat{p})}{n}}\right)$

Functions and graphs
Technology-free test 1

Model solutions and marking scheme

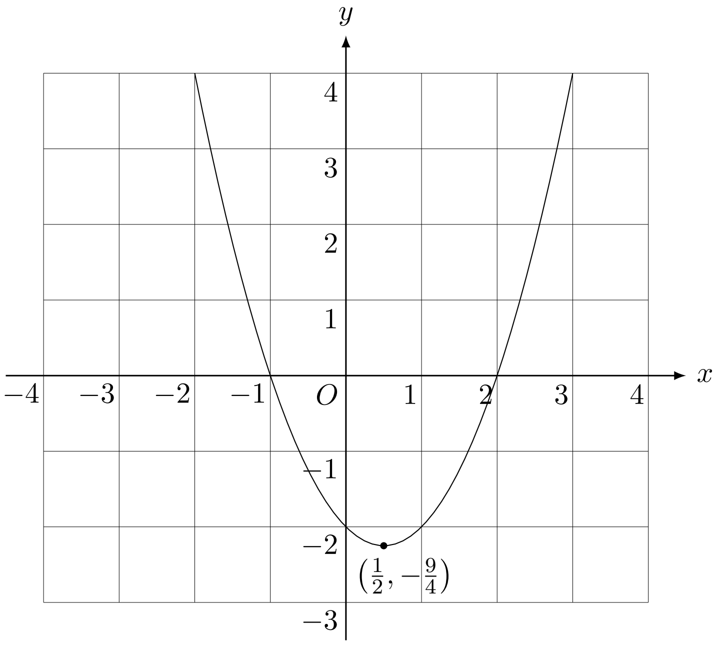
Click to jump: 1a, 1b, 1c, 2, 3a, 3b, 3ci, 3cii, 4a, 4b, 5.

Question 1a. [Go to detailed solutions]

$$\begin{aligned} x^2 - x - 2 = 0 &\implies (x+1)(x-2) = 0 \\ &\implies x = -1 \text{ or } x = 2 \end{aligned}$$

- 1 mark for some valid method of solving the equation, such as factorising $f(x)$ or using the quadratic formula.
- 1 mark for the correct values of x.

Question 1b. [Go to detailed solutions]



- 1 mark for an approximately correct shape, which includes drawing the curve through the correct points on the x- and y-axes, and an appropriate display of the symmetry of the graph about the line $x = \frac{1}{2}$.
- 1 mark for correctly labelling the turning point with its coordinates.

Question 1c. [Go to detailed solutions]

$$(-\infty, -1) \cup (2, \infty)$$

- 1 mark for the correct set of values expressed using correct set notation.

Question 2 [Go to detailed solutions]

$$a = \frac{0-4}{5-0} = -\frac{4}{5}$$

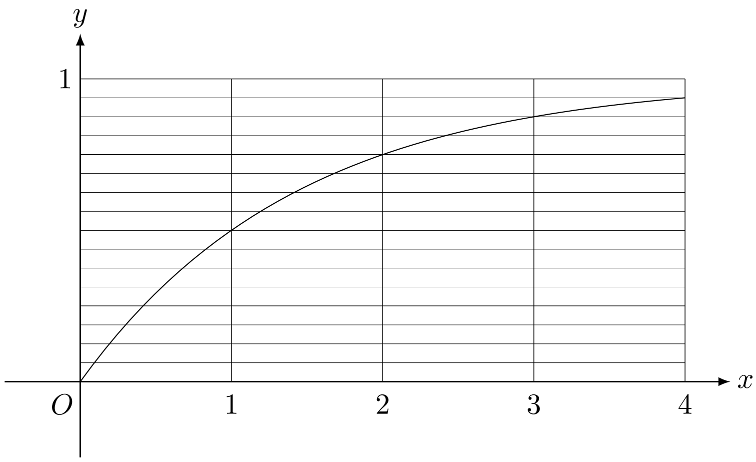
$$b = 4$$

$$c = \frac{2-0}{6-5} = 2$$

$$h(5) = 0 \implies 2 \cdot 5 + d = 0 \implies d = -10$$

- 1 mark for using appropriate methods to calculate a, c, and d, such as calculating gradients or using simultaneous equations.
- 1 mark for the correct values of a, b, c, and d.

Question 3a. [Go to detailed solutions]



- 1 mark for an approximately correct shape.
- 1 mark for drawing the curve with the correct y-coordinates on each of the vertical lines; that is, the graph should pass through $(0,0)$, $\left(1, \frac{1}{2}\right)$, $\left(2, \frac{3}{4}\right)$, $\left(3, \frac{7}{8}\right)$, and $\left(4, \frac{15}{16}\right)$. Slight deviations from these points are acceptable. These points do not need to be labelled.

Question 3b. [Go to detailed solutions]

$$[0, 1)$$

- 1 mark for the correct range.

Question 3ci. [Go to detailed solutions]

Simplifying $g(x)$:

$$g(x) = 1 - 2^2 \cdot 2^{-x-2} = 1 - 2^{-x}.$$

Let $y = g^{-1}(x)$. Then

$$\begin{aligned} g(y) = x &\implies 1 - 2^{-y} = x \\ &\implies 2^{-y} = 1 - x \\ &\implies -y = \log_2(1-x) \\ &\implies y = -\log_2(1-x) \\ &\implies g^{-1}(x) = -\log_2(1-x) \end{aligned}$$

- 1 mark for a valid method for finding inverse functions, such as using the identity $g(g^{-1}(x)) = x$ or solving the equation $g(y) = x$ for y. In the latter case, writing something along the lines of "For inverse, swap x and y" or "Let $y = g^{-1}(x)$" is required.
- 1 mark for a correct expression for $g^{-1}(x)$.

Question 3cii. [Go to detailed solutions]

$$[0, \infty)$$

- 1 mark for the correct range.

Question 4a. [Go to detailed solutions]

$$\frac{\pi}{6}$$

- 1 mark for the correct period.

Question 4b. [Go to detailed solutions]

A solution to $\cos\left(6a - \frac{\pi}{2}\right) = 0$ is given by $6a - \frac{\pi}{2} = \frac{\pi}{2}$, so $a = \frac{\pi}{6}$ is one solution. Using the period from **part a.**,

$$a = \frac{\pi}{6} - \frac{\pi}{6},\ \frac{\pi}{6},\ \frac{\pi}{6} + \frac{\pi}{6},\ \frac{\pi}{6} + \frac{2\pi}{6} = 0,\ \frac{\pi}{6},\ \frac{\pi}{3},\ \frac{\pi}{2}$$

- 1 mark for attempting to solve $\cos\left(6a - \frac{\pi}{2}\right) = 0$, or an equivalent equation, for a.
- 1 mark for using an appropriate method to generate all solutions to the equation, such as using a formula for the general solution or using periodicity.
- 1 mark for the correct values of a.

Question 5 [Go to detailed solutions]

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ 3y-1 \end{bmatrix}$$

so T sends $y = 3x^2 + 1$ to $y = 3(3(x-1)^2 + 1) - 1$. Since

$$3(3(x-1)^2 + 1) - 1 = 9(x-1)^2 + 2 = 9x^2 - 18x + 9 + 2 = 9x^2 - 18x + 11,$$

we have

$$a = 9, \quad b = -18, \quad c = 11.$$

- 1 mark for an appropriate method for applying T to the given curve to find the transformed curve, such as decomposing T into simple transformations and applying them in the correct order, defining new coordinate variables and then using algebra, or simply reading off the transformations from the matrices.
- 1 mark for obtaining a correct equation for the transformed curve at some stage, even if not in the correct form; for example, writing down $y = 3(3(x-1)^2 + 1) - 1$.
- 1 mark for the correct values of a, b, and c.

Functions and graphs
Technology-free test 1
Detailed solutions

Click to jump: 1a, 1b, 1c, 2, 3a, 3b, 3ci, 3cii, 4a, 4b, 5.

Question 1a. [Go to model solutions]

The importance of solving quadratic equations in this course cannot be overstated. For this reason, I will give a very lengthy discussion on the topic of quadratic equations, which will hopefully put to rest any uncertainties you have in this area. The solutions to the actual question begin here if you want to skip ahead.

Being able to solve a quadratic equation is an assumed skill from the first two units of Mathematical Methods, but it's never too late to learn how to solve them, unless perhaps your exam starts 10 minutes from reading this. You're always required to solve a quadratic equation or two on the tech-free exam, and since this exam usually consists of about 9-12 questions, a reasonable percentage of the exam marks will be accessible if you can solve quadratic equations. That being said, there's more to it than just being able to solve one that's given to you; sometimes you need to recognise a quadratic equation that's "hidden" (we will have examples of these in later questions), or you might have to find the equation yourself, and then solve it.

Regardless of how they might show up in exams, the first thing to worry about is knowing the precise definition of a quadratic equation. The emphasis on precise mathematical definitions is actually very low in VCE mathematics compared to what (I think) it should be; there are even some past exam questions whose correctness depends on which one of several definitions is used. These are more general comments and don't necessarily relate to quadratic equations in particular, but I really do want to get these ideas across. There is a lot to be gained from knowing exactly what you're dealing with, which is why we don't walk across dark rooms when we can just turn on a light.

With that out of the way, here is the definition. A (real)[1] quadratic equation is an equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where $a, b, c \in \mathbb{R}$ are constants[2], $a \neq 0$, and x is a real variable. The constants a, b, and c are also called the coefficients of the equation $ax^2 + bx + c = 0$. The reason $a = 0$ is excluded is that in this case, the equation reduces to the linear equation $bx + c = 0$, which is a much simpler equation.

It is common to refer to the equation $ax^2 + bx + c = 0$ as a "quadratic equation in x", usually to point out what the variable of the equation is when it may not be clear. An example is the equation $x^4 + 2x^2 + 1 = 0$, which is a quadratic equation in x^2, as $x^4 + 2x^2 + 1 = (x^2)^2 + 2(x^2) + 1$. It is confusing to only refer to $x^4 + 2x^2 + 1 = 0$ as a quadratic equation, as it is not a quadratic equation in x, but the less-obvious variable x^2. This line of thinking is the basis of some of the more difficult questions involving quadratics, but for now we will return to focus on the basic solution processes.

We are first going to explore the factorisation method for solving quadratics – which you are likely to have learned before, but perhaps have never considered the topic in this level of detail. We will then discuss a more comprehensive method for solving quadratics, which is the quadratic formula.

In brief, the factorisation method is to first write $ax^2 + bx + c$ as a product of two expressions that are linear in x, apply the null factor law[3] to obtain two (possibly identical) linear equations in x, and then

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[1]There are versions of the quadratic equation that use complex numbers rather than just real numbers. Complex numbers are dealt with in Specialist Mathematics, but not Mathematical Methods.

[2]Note that we use the symbol $\mathbb{R}$ for the set of real numbers. VCAA tends to use a plain unbolded R, which is a questionable choice.

[3]The null factor law is: for real numbers a, b, if $ab = 0$, then $a = 0$ or $b = 0$. Intuitively, this says that multiplying two non-zero real numbers can never give 0 as a result.

solve these individually. This description might sound esoteric at this point, but all will be explained. The hardest part of this method is the factorisation itself, since it's relatively easy to solve the equation $ax^2 + bx + c = 0$ once the expression $ax^2 + bx + c$ is factorised. Consider, for example, the quadratic equation $(5x - 2)(x + 1) = 0$. The null factor law implies that either $5x - 2 = 0$ or $x + 1 = 0$. We then individually solve these equations, obtaining $x = \frac{2}{5}$ and $x = -1$ respectively, which gives us the two solutions to the original equation $(5x - 2)(x + 1) = 0$. In symbols,

$$(5x-2)(x+1) = 0 \implies 5x - 2 = 0 \text{ or } x + 1 = 0 \implies x = \frac{2}{5} \text{ or } x = -1,$$

which is not much work at all.

Having given a small refresher on why factorisation is useful, we should now think about how to obtain a factorised expression in the first place. An important thing to note is that this method won't always work, in that you won't always get factored expressions such as $(5x - 2)(x + 1)$ that don't involve any moderately complicated irrational numbers; the reason for this, of course, is that some quadratic equations have moderately complicated irrational numbers as solutions.

To illustrate this point, consider the pleasant-looking equation $x^2 - 4x + 2 = 0$, which has solutions $x = 2 + \sqrt{2}$ and $x = 2 - \sqrt{2}$ (never mind why for the moment). To see how this relates to the factored form, the expression $x^2 - 4x + 2$ factorises to $\left(x - 2 - \sqrt{2}\right)\left(x - 2 + \sqrt{2}\right)$. The usual processes of finding factors won't be able to find relatively complicated factors like these (as you shouldn't bother looking for possible factors after a certain point), so an attempt to factorise $x^2 - 4x + 2$ the usual way won't get you anywhere.

Apart from not having nice solutions, it's also possible for a quadratic equation to have no solutions at all, such as the equation $x^2 + 2x + 2 = 0$. This can be written as $(x + 1)^2 = -1$, which has no real solutions, since $(x + 1)^2$ is never negative. As such, the expression $x^2 + 2x + 2$ can't be factorised in any way using only real numbers; if it could, then it would have real solutions, but it doesn't.

Now, how do we look for factors, and how do we know when to give up? We will start with a simplified version of the quadratic equation, $x^2+bx+c = 0$. We want to write x^2+bx+c in the form $(x+h)(x+k)$, so we just need to figure out what h and k are. This is done by expanding $(x+h)(x+k)$, which makes it easier to compare it to $x^2 + bx + c$. Expanding $(x + h)(x + k)$, we get

$$(x + h)(x + k) = x^2 + hx + kx + hk = x^2 + (h + k)x + hk.$$

Therefore, we want to find h and k such that

$$x^2 + (h + k)x + hk = x^2 + bx + c.$$

By comparing these expressions, we see that we require $h + k = b$ and $hk = c$. In other words, to be able to write $x^2 + bx + c$ in the form $(x + h)(x + k)$, we need to find two numbers that add up to b and multiply to give c.

We still need to address the more general case of factoring $ax^2 + bx + c$, but we will give an example of the simpler case first. Consider the expression $x^2 - 3x - 10$. To write this in the form $(x + h)(x + k)$, we need to find h and k such that $h + k = -3$ and $hk = -10$.

Now, we've come to an important point in the factorisation method, which is how we go about finding h and k. Generally, we will only try to find integer solutions, since any more than that will take too long. At some point it becomes quicker to use the quadratic formula, and the line is generally drawn at looking for integer solutions, after which we stop trying to factorise.

In a sense, factorisation could theoretically work every time (that solutions exist) if you just keep trying numbers, but it is ridiculous to attempt this knowing how complicated some solutions can be. Imagine how long it would take to see that $x^2 - 4x + 2$ factorises to $\left(x - 2 - \sqrt{2}\right)\left(x - 2 + \sqrt{2}\right)$ just by trying values of h and k.

Now, even if we restrict ourselves to looking for integer solutions to $h+k = -3$, there are still infinitely many options for h and k; that is, there are infinitely many pairs of integers that add to -3. For this reason, we first consider $hk = -10$, which only has finitely many integer solutions (because -10 only has finitely many divisors). This will generally be the case when trying to factorise $x^2 + bx + c$; there are only finitely many ways to write an integer c as a product of two integers.

The possible ways of writing 10 as a product of two positive integers are 1×10 and 2×5, disregarding the order in which the factors are written. We want two integers that multiply to -10, so we can extend this by adding negative signs in: the options are $(-1) \times 10$, $1 \times (-10)$, $(-2) \times 5$, and $2 \times (-5)$. The next step is to see if any of these pairs add up to -3 (recall that we want numbers that multiply to -10 and add to -3). Our winner is the pair $2, -5$, so we can choose $h = 2$, $k = -5$, and hence obtain the factorisation $x^2 - 3x - 10 = (x+2)(x-5)$. Of course I chose an example where this method would get us somewhere, but if none of the four pairs we found added up to -3, it would then be best to use the quadratic formula. You may be wondering if it matters which number we call h and which one we call k. Swapping h and k just swaps the factors of $(x+h)(x+k)$, so in fact it doesn't matter.
Now, how do we factorise the more general expression $ax^2 + bx + c$? As we've already alluded to, there's no point in trying to factorise this expression unless a, b, and c are integers, since if they aren't then the quadratic formula is likely to be more time-efficient. If a, b, and c are at least rational, then the equation $ax^2 + bx + c = 0$ can be modified to have integer coefficients. For example, the equation $\frac{1}{3}x^2 + 2x + 4 = 0$ can be multiplied by 3 to give $x^2 + 6x + 12 = 0$, which has the same solutions but is easier to work with.
It's worth mentioning that, if you can take a out as a common (integer) factor, then it will probably make life easier. An example is the quadratic expression $3x^2 + 6x + 3$. Here we can take out 3 as a common factor to obtain the expression $3\big(x^2 + 2x + 1\big)$. This reduces the situation to the one we've already discussed, which is factorising an expression of the form $x^2 + bx + c$. Even if a doesn't come out as a common factor, the three coefficients a, b, and c might have some smaller common factor other than 1 (such as in $4x^2 + 6x + 2$), and taking out this factor will still help, since it makes the coefficients smaller and thus reduces the number of possible factors to try.
After making as many simplifications as possible and still arriving at an equation of the form $ax^2 + bx + c = 0$ (with $a \neq 1$), we attempt to factorise $ax^2 + bx + c$ in the form $(mx+h)(nx+k)$. As before, we expand this and relate our unknowns h, k, m, and n to a, b, and c. We have

$$ax^2 + bx + c = (mx+h)(nx+k) = mnx^2 + nhx + mkx + hk = mnx^2 + (nh + mk)x + hk.$$

As you can probably tell, having the a in the equation $ax^2 + bx + c = 0$ complicates matters quite a bit, which is why efforts should be made to get rid of it or at least make it smaller. Still, there are some similar features to the method of solving the simplified version $x^2 + bx + c = 0$. For starters, we can see that $mn = a$ and $hk = c$, so that we end up having only finitely many options for (integer) h, k, m, and n.
What makes this harder is having to try two pairs of integers at once, which usually means we end up with many more possibilities that need to be tried; for each choice of h, k, m, and n such that $mn = a$ and $hk = c$, we need to check that $nh + mk = b$. Depending on how many factors a and c have, it may still be more efficient to go straight to using the quadratic formula.
An example is probably in order. We will solve the equation $4x^2 - 30x + 14 = 0$ by factorisation. First, divide by 2 to get $2x^2 - 15x + 7 = 0$. Both 2 and 7 are prime numbers, so we only have a few possible factorisations of each number. The number 2 can be written as 1×2 or $(-1) \times (-2)$, and similarly 7 can be written as 1×7 or $(-1) \times (-7)$.
Now, here the order in which we choose h and k (or m and n) matters, since interchanging h and k in $(mx+h)(nx+k)$ can result in a genuinely different expression. However, we can swap h and k if we also swap m and n, because this amounts to writing the two linear factors in the opposite order.
Before trying out possibilities, the following observations will always come in handy while factorising. First, note that you can always make the coefficient of x^2 positive, because you can multiply the equation $ax^2 + bx + c = 0$ by -1. Recall that to have $ax^2 + bx + c = (mx+h)(nx+k)$ we at least need $mn = a$, so if we ensure that a is positive we can also ensure that m and n have the same sign. Finally, because of the fact that

$$(mx+h)(nx+k) = (-mx-h)(-nx-k),$$

we can ensure that m and n are both positive. For instance, if we tried $m = -1$ and $n = -2$, we could just as well try $m = 1$ and $n = 2$ if we also swap the signs of h and k, because $(mx+h)(nx+k)$ turns out the same either way.

In this scenario, we can assume that $m = 1$ and $n = 2$, because the only way to write 2 as a product of two positive integers is 1×2 (disregarding order, as usual). Why not $m = 2$ and $n = 1$? Because the order of the factors in $(mx + h)(nx + k)$ doesn't matter. Thus, we effectively end up with only one possibility for m and n, so we just need to try different possibilities for h and k.

Another fortunate feature of this quadratic equation is that the coefficient of x (namely -15) is negative. This is useful because the requirement that $nh + mk = -15$ implies that h, k, m, and n can't all be positive. We've already chosen m and n to be positive, so we just need to test h and k with $hk = 7$, where h and k aren't both positive. This leaves just two possibilities, $h = -1$ and $k = -7$, or $h = -7$ and $k = -1$. You won't always be able to make this simplification, but it's definitely worth looking out for.

A common way of testing out possibilities is to write h, k, m, and n in a square like so:

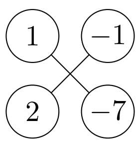
$$\begin{array}{cc} m & h \\ n & k \end{array}$$

The case $m = 1$, $n = 2$, $h = -7$ and $k = -1$ would be written as

$$\begin{array}{cc} 1 & -1 \\ 2 & -7 \end{array}$$

This actually avoids having to introduce new pronumerals such as h, k, m, and n, since you can jump straight to writing down squares like these once you know what equation you want to solve ($2x^2 - 15x + 7 = 0$ in the present case). Thinking up factors of 2 and 7 can be done in your head, as well as simplifications such as the fact that the left-hand column is effectively the only thing you can write down. Because this is really a trial-and-error process, though, you may need to keep rubbing out parts of the diagram if there isn't enough space on the exam or test paper to keep drawing new squares.

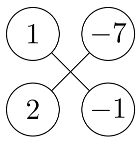
How does this diagram get us anywhere, though? Each row can be thought of as a possible factor of $2x^2 - 15x + 7$. The above corresponds to testing the factors $x - 1$ (from the top row) and $2x - 7$ (from the bottom row). The way of testing that you have the right factors is to calculate $nh + mk$, which is made easy by writing the four numbers in a square. It amounts to multiplying the diagonallly opposite numbers with each other and then adding the two products.



$$\longrightarrow \qquad 2 \cdot (-1) + 1 \cdot (-7) = -9$$

Due to this way of calculating $nh + mk$, this particular way of finding factors is often referred to as the "cross method".

Here we end up with $nh+mk = -9$, but we want $nh+mk = -15$, so we need to make another guess. We rub out -1 and -7 and then write them in the opposite order.



$$\longrightarrow \qquad 2 \cdot (-7) + 1 \cdot (-1) = -15$$

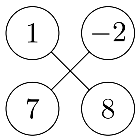
Success! Now we read off the factors from the square:

$$\begin{array}{cc} 1 & -7 \\ 2 & -1 \end{array} \longrightarrow \begin{array}{c} (1x - 7) \\ \times \\ (2x - 1) \end{array}$$

Hence, we have $2x^2 - 15x + 7 = (x - 7)(2x - 1)$. The original equation $4x^2 - 30x + 14 = 0$ is therefore equivalent to the equation $(x - 7)(2x - 1) = 0$, which has solutions $x = 7$ and $x = \frac{1}{2}$.
To give another example, we will solve $-7x^2 + 6x + 16 = 0$. Making the coefficient of x^2 positive, we multiply by -1 to give $7x^2 - 6x - 16 = 0$. The coefficients have no common factor other than 1, so, like in the last example, we proceed to looking for a factorisation of the form $(x + h)(7x + k)$ (since 7 is prime, we only really have one choice for the coefficients of x in the linear factors, given the observations made in the previous example). We can start writing down a square of numbers.

$$\begin{matrix} 1 & \_\_ \\ 7 & \_\_ \end{matrix}$$

There is no obvious way to eliminate any possibilities for h and k, and trying to think of something clever might waste too much time. There are quite a few ways to factorise 16 into a product of two positive integers, so taking signs into account, and the fact that h and k aren't interchangeable this time, we end up with about 10 different cases to consider. At this point it would be reasonable to stop trying to factorise and instead use the quadratic formula, although it's worth checking just a few cases since you might get lucky and find the factors on the first or second attempt. Potentially, though, it could be the last one you try, so it wouldn't be a good idea to try all of them if you can see there are a lot.
With enough practice, you should be able to check cases fairly quickly. It's probably a good idea to write possibilities down lightly in pencil until you get the right one. In this case, it turns out that the following square does the trick:



$$\longrightarrow \quad 7 \cdot (-2) + 1 \cdot 8 = -6$$

From this we obtain the factorisation $7x^2 - 6x - 16 = (x - 2)(7x + 8)$. If you could see this immediately, then you're pretty much set for life, but otherwise you can always fall back on the quadratic formula.
One more strategy that could help you find factors is to guess a solution to the equation you're trying to solve, since that will actually tell you one of the factors. For example, if you saw that $x = 2$ was a solution to $7x^2 - 6x - 16 = 0$ (probably not that likely), that would tell you that $x - 2$ is a factor of $7x^2 - 6x - 16$, from which you can figure out the other factor with ease. Of course, there may not be any obvious solutions, so I wouldn't suggest spending too much time on that.
There's not much more to say on the factorisation method, except that it should be practised thoroughly unless you prefer to rely on the quadratic formula every time (which is fine, but may be less appealing to some). Whether you should use the factorisation method over the quadratic formula depends on the equation, as well as how comfortable you are with each method. The efficiency with which you perform each method can also change over time.
We will now discuss the quadratic formula. Given a quadratic equation $ax^2 + bx + c = 0$, the quadratic formula tells us the solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This needs to be memorised for the tech-free exam, because you can't bank on being able to factorise every quadratic that shows up. It also doesn't appear on the exam formula sheet; you may have noticed that our formula sheet doesn't contain it, and that's because ours contains exactly the same amount of information as the official exam formula sheet.
In the end, the quadratic formula is just that – a formula – and you can plug in the values of a, b, and c to get your solutions. There's no other working you have to show in exams, as you're expected to apply the quadratic formula wherever it's appropriate. However, I do want to show how the formula comes about, as it brings up a lot of the important features of quadratics.

To lead into the derivation of the formula, we will first discuss the method of completing the square. Recall the perfect square expansion:

$$(x+p)^2 = x^2 + 2px + p^2$$

This can sometimes be used backwards to factorise quadratic espressions. Not all quadratics are perfect squares, but fortunately it is easy to tell if a quadratic x^2+bx+c is a perfect square. We want

$$x^2 + bx + c = (x+p)^2 = x^2 + 2px + p^2,$$

and this gives us a way to find a relationship between b and c that can be used as a test. From the above equalities, we have $b = 2p$ and $c = p^2$, which allows us to relate b and c as follows: given b, we can halve it and then square it to obtain c. To show this symbolically,

$$\left(\frac{b}{2}\right)^2 = \left(\frac{2p}{2}\right)^2 = p^2 = c,$$

so the quadratic $x^2 + bx + c$ is a perfect square precisely when

$$c = \left(\frac{b}{2}\right)^2.$$

I wouldn't suggest memorising this, because there's an easier way to derive it that we will see later. But for now, let's try this out on $x^2 + 4x + 4$, where $b = c = 4$. We have

$$\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 2^2 = 4 = c,$$

and hence $x^2 + 4x + 4$ is a perfect square and can therefore be written in the form $(x+p)^2$. Finding p is easy given that we've already shown that $b = 2p$. This says that p is half of b, so $p = 2$ in this case. Hence,

$$x^2 + 4x + 4 = (x+2)^2,$$

as you can check by expansion.
Now, the coefficient of x^2 could always be something other than 1, but you can still apply the above test by taking out the coefficient of x^2 as a common factor. Consider for example the quadratic $4x^2 + 4x + 1$, which can be written as

$$4x^2 + 4x + 1 = 4\left(x^2 + x + \frac{1}{4}\right).$$

You can then apply the test $c = \left(\frac{b}{2}\right)^2$ to the quadratic $x^2 + x + \frac{1}{4}$. Here it turns out that the equation $c = \left(\frac{b}{2}\right)^2$ holds, and so we can write

$$4x^2 + 4x + 1 = 4\left(x^2 + x + \frac{1}{4}\right) = 4\left(x + \frac{1}{2}\right)^2,$$

which, for the sake of interest, can be simplified as

$$4\left(x + \frac{1}{2}\right)^2 = \left(2\left(x + \frac{1}{2}\right)\right)^2 = (2x+1)^2.$$

Although not every quadratic is a perfect square, it is still possible to write a quadratic expression in the turning-point form $a(x+h)^2+k$, which is as close to a perfect square expression as you can get. This way of expressing a quadratic is arguably the most useful in terms of solving $a(x+h)^2+k=0$ as well as sketching the curve $y = a(x+h)^2+k$. This is how easy it is to solve the equation $a(x+h)^2+k=0$ for x:

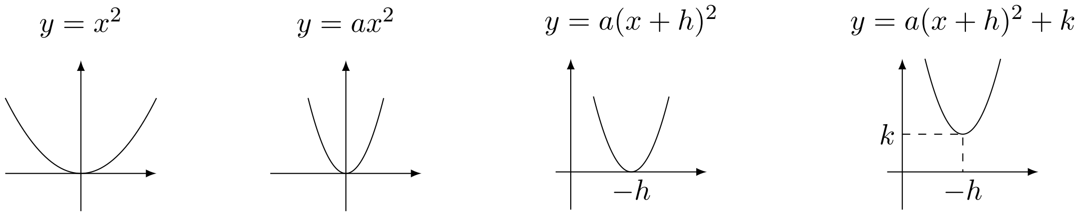
$$a(x+h)^2 + k = 0 \implies (x+h)^2 = -\frac{k}{a} \implies x + h = \pm\sqrt{-\frac{k}{a}} \implies x = -h \pm \sqrt{-\frac{k}{a}}$$

There is a problem in that $-\frac{k}{a}$ will be negative if a and k have the same sign, which implies that we can't take its square roots, but on the bright side, this observation makes it easy to tell when a quadratic equation (with the quadratic written in turning-point form) has no solutions: we require either that $k = 0$, or that a and k have opposite sign. For example, the equation $-2(x+1)^2 + 5 = 0$ will have solutions, while the equation $2(x-2)^2 + 1 = 0$ has no solutions.

As for how the turning-point form helps to sketch the graph, first observe the transformations required to obtain the curve $y = a(x+h)^2 + k$ from the curve $y = x^2$ (doing this is desirable because the shape of $y = x^2$ is familiar and relatively easy to understand). We can get to $y = a(x+h)^2 + k$ from $y = x^2$ as follows:

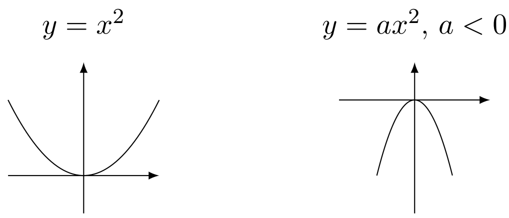
$$y = x^2 \quad \to \quad y = ax^2 \quad \to \quad y = a(x+h)^2 \quad \to \quad y = a(x+h)^2 + k$$

Hence, dilating[4] by a factor of a parallel to the y-axis, translating by $-h$ units in the x-direction, and translating by k units in the y-direction (in that order) sends the curve $y = x^2$ to the curve $y = a(x+h)^2 + k$.



In applying these three basic transformations, what happens to the turning point of $y = x^2$? It starts out at $(0,0)$. Dilating doesn't move the origin anywhere. Translating by $-h$ horizontally sends $(0,0)$ to $(-h,0)$, and translating by k vertically sends $(-h,0)$ to $(-h,k)$, so the overall process sends $(0,0)$ to $(-h,k)$. The turning point of $y = a(x+h)^2 + k$ is therefore located at $(-h,k)$. If you aren't particularly comfortable with transformations and can't see how this works, then I suggest reading the detailed solution to question 5 of this test (which begins here), where we cover transformations in depth.

Incidentally, the sign of a tells us whether the turning point is a maximum or a minumum. That is, it will be a minimum if a is positive and a maximum if a is negative. This is solely due to the first transformation multiplying the y-coordinates by a, because translations don't change the nature of stationary points. If $a > 0$ then $y = ax^2$ will still have a minimum. If $a < 0$, going from $y = x^2$ to $y = ax^2$ will reflect the curve in the x-axis as well as dilating vertically, which turns the minimum into a maximum.



Using the two examples from earlier, the turning point of $y = -2(x+1)^2 + 5$ is a maximum and the turning point of $y = 2(x-2)^2 + 1$ is a minimum.

Since the curve $y = a(x+h)^2 + k$ has its turning point at $(-h,k)$, we can see that it is easy to find the turning point of a quadratic curve when expressed in turning-point form – hence the name. Using our two examples again, the curve $y = -2(x+1)^2 + 5$ has its turning point at $(-1,5)$, and the curve $y = 2(x-2)^2 + 1$ has its turning point at $(2,1)$.

This is all well and good, but we still need to know how to find the turning-point form of a quadratic $ax^2 + bx + c$. Unlike the factorisation method we've discussed, this involves no trial and error. First, we write

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c.$$

[4] To be precise, if $a < 0$ then we are really dilating by $-a$ and then reflecting in the x-axis.

Here we've just taken a out as a common (real-number) factor of ax^2 and bx, for reasons that will hopefully become clear. Let's concentrate on the part inside the brackets, which is

$$x^2 + \frac{b}{a}x.$$

We know the form of a perfect square: $x^2 + 2px + p^2$. As we've discussed, this tells us what the constant term should look like in terms of the coefficient of x. Taking $\frac{b}{a}$, halving it, and then squaring it, we get $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$; hence, the following is a perfect square:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$$

Now, you might be thinking that we can't just add whatever we want. But we can if we also subtract it. That is,

$$x^2 + \frac{b}{a}x = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}.$$

We've just added and subtracted the same number, but remarkably this gets us most of the way there! The reason is that $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$ is a perfect square, as we've already mentioned, and so

$$x^2 + \frac{b}{a}x = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}.$$

We can now replace $x^2 + \frac{b}{a}x$ in the expression $a\left(x^2 + \frac{b}{a}x\right) + c$ and get

$$a\left(x^2 + \frac{b}{a}x\right) + c = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c.$$

We're almost there; we just need to do a small amount of algebra.

$$\begin{aligned} a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c &= a\left(x + \frac{b}{2a}\right)^2 - a \cdot \frac{b^2}{4a^2} + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}. \end{aligned}$$

Hence, we've shown that

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.$$

This indeed has the form $a(x+h)^2 + k$, with $h = \frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$.
In practice, if you want to convert a quadratic to turning-point form, then it's best to remember the general procedure rather than the formula $ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}$. There's nothing stopping you from memorising it, but it would be more productive to use your brain space on other things, like the quadratic formula. Remembering the general procedure might even help you memorise the formula. We will demonstrate the general procedure on the expression $4x^2 + 4x + 2$. First write

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2,$$

then add and subtract the appropriate constant inside the bracket, which in this case is $\frac{1}{4} = \left(\frac{1}{2}\right)^2$ (the square of half of the coefficient of x in $x^2 + x$, which is 1). Following this by some algebra, we arrive at the turning-point form without too much effort.

$$\begin{aligned} 4(x^2 + x) + 2 &= 4\left(x^2 + x + \frac{1}{4} - \frac{1}{4}\right) + 2 \\ &= 4\left(x^2 + x + \frac{1}{4}\right) - 1 + 2 \\ &= 4\left(x + \frac{1}{2}\right)^2 + 1. \end{aligned}$$

Now that we can write any quadratic expression in turning-point form, we can derive the quadratic formula without much more effort. Remember that the quadratic formula just gives the solutions to the equation $ax^2 + bx + c = 0$, when they exist. The way to derive the quadratic formula, then, is to solve the equation $ax^2 + bx + c = 0$ algebraically. Expressing $ax^2 + bx + c$ in turning-point form was most of the work – we've already seen how to solve quadratic equations using the turning-point form, so we just need to repeat this process.

$$\begin{aligned} ax^2 + bx + c = 0 &\implies a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0 \\ &\implies a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c = \frac{b^2 - 4ac}{4a} \\ &\implies \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \end{aligned}$$

If the right-hand side of the last equation is negative, there will be no solutions. Otherwise, we can proceed to solve:

$$\begin{aligned} ax^2 + bx + c = 0 &\implies \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \\ &\implies x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\ &\implies x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &\implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

This is the quadratic formula, so we are done. There is more to be said, however. The quantity $b^2 - 4ac$ appearing under the square root sign plays an important role in deciding the number of solutions to the equation $ax^2 + bx + c = 0$. Due to this, the number $b^2 - 4ac$ is referred to as the discriminant of the equation $ax^2 + bx + c = 0$. To explain this in more detail, recall that $ax^2 + bx + c = 0$ is equivalent to

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Noting that $4a^2$ is positive, we see that the sign of the right-hand side is the same sign as the discriminant $b^2 - 4ac$, which appears as the numerator. If $b^2 - 4ac < 0$, then the equation $ax^2 + bx + c = 0$ has no solutions. If $b^2 - 4ac = 0$, then there is one solution:

$$\left(x + \frac{b}{2a}\right)^2 = 0 \implies x = -\frac{b}{2a}.$$

This can also be obtained from the quadratic formula. Finally, if $b^2 - 4ac > 0$, then $\sqrt{b^2 - 4ac}$ is non-zero, leading to two solutions to the equation $ax^2 + bx + c = 0$ as given by the quadratic formula. Another useful feature of the discriminant is that it enables us to tell if a quadratic $ax^2 + bx + c$ can be written as a perfect square $a(x + h)^2$. To see this, consider the turning-point form as found before:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

For $ax^2 + bx + c$ to be a perfect square, then, we require that

$$c - \frac{b^2}{4a} = 0 \implies \frac{b^2}{4a} = c \implies b^2 = 4ac \implies b^2 - 4ac = 0.$$

That is, the $ax^2 + bx + c$ is a perfect square exactly when $b^2 - 4ac = 0$, in which case

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2.$$

How does this relate to $c = \left(\frac{b}{2}\right)^2$ from earlier? In that situation we had $a = 1$, so $b^2 - 4ac = 0$ becomes $b^2 - 4c = 0$, and you can show with a small amount of manipulation that this is the same as $c = \left(\frac{b}{2}\right)^2$.
As a final example before giving solutions to the actual question, we'll demonstrate the use of the formula to solve the equation $16x^2 + 6x - 7 = 0$. We have $a = 16$, $b = 6$, and $c = -7$, so

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 16 \cdot (-7)}}{2 \cdot 16}.$$

This is about as tricky as the arithmetic can possibly get in a tech-free Methods exam (and this is actually an equation from a past exam). Multiplying out $4 \cdot 16 \cdot 7$ is a little difficult, more so in a high-stakes exam. But you actually don't need to work out what it is; there are smarter ways to go about it. Let's concentrate on the part under the square root sign (the discriminant):

$$6^2 - 4 \cdot 16 \cdot (-7) = 36 + 4 \cdot 16 \cdot 7 = 4 \cdot (9 + 16 \cdot 7)$$

That's a little better. Since 6 is even, 6^2 is divisible by 4, so 4 comes out as a common factor due to the $4ac$ term. We've reduced the problem to working out $9 + 16 \cdot 7$, which turns out to be equal to 121. Having broken things down a little, we can calculate the solutions:

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{6^2 - 4 \cdot 16 \cdot (-7)}}{2 \cdot 16} \\ &= \frac{-6 \pm \sqrt{4 \cdot (9 + 16 \cdot 7)}}{2 \cdot 16} \\ &= \frac{-6 \pm \sqrt{4 \cdot 121}}{2 \cdot 16} \\ &= \frac{-6 \pm 2 \cdot 11}{2 \cdot 16} \\ &= \frac{-3 \pm 11}{16} \\ &= \frac{-14}{16}, \frac{8}{16} \\ &= -\frac{7}{8}, \frac{1}{2} \end{aligned}$$

The overall point of this example is that you don't actually have to compute every product you see, since it might be easier to cancel things out first. You still can, of course, but you should choose methods that minimise the chances of you making arithmetic errors, for obvious reasons.
Now, to the actual question, which is to solve the equation $x^2 - x - 2 = 0$. We will solve this in three ways: factorisation, completing the square, and using the quadratic formula.
To factorise, we look for two integers that multiply to -2 and add to -1, and it's pretty easy to see that 1 and -2 do the job, so we can write $x^2 - x - 2 = (x + 1)(x - 2)$. Hence, the solutions to the equation $x^2 - x - 2 = 0$ are $x = -1$ and $x = 2$. This is an example where factorisation is definitely the quickest method, and can even be completed in reading time in your head.
To complete the square, we take the coefficient of x (which is -1), halve it, then square it, to obtain $\frac{1}{4}$. We then add and subtract the resulting number and then factor the perfect square.

$$x^2 - x - 2 = x^2 - x + \frac{1}{4} - \frac{1}{4} - 2 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - 2 = \left(x - \frac{1}{2}\right)^2 - \frac{9}{4}.$$

Solving $x^2 - x - 2 = 0$ for x, we use the turning-point form.

$$\left(x - \frac{1}{2}\right)^2 - \frac{9}{4} = 0 \implies \left(x - \frac{1}{2}\right)^2 = \frac{9}{4} \implies x - \frac{1}{2} = \pm\frac{3}{2} \implies x = \frac{1}{2} \pm \frac{3}{2}$$

The solutions are therefore

$$x = \frac{1}{2} + \frac{3}{2} = \frac{4}{2} = 2$$

and

$$x = \frac{1}{2} - \frac{3}{2} = \frac{-2}{2} = -1.$$

The same as we got from factorisation, as you would hope.
Finally, we apply the quadratic formula to $x^2 - x - 2 = 0$.

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1} \\ &= \frac{1 \pm \sqrt{9}}{2} \\ &= \frac{1 \pm 3}{2} \\ &= \frac{-2}{2},\ \frac{4}{2} \\ &= -1,\ 2 \end{aligned}$$

To re-iterate, the factorisation method is the way to go in this question, since there are just two ways to write -2 as a product of two integers: $(-1) \times 2$ and $1 \times (-2)$. Before even attempting the question, you should be able to see (with enough experience) that the worst-case scenario with the factorisation method is that you try these two pairs of integers and neither of them work, in which case you would use the quadratic formula. In more complicated situations, like with $16x^2 + 6x - 7 = 0$, it's reasonable to jump straight to using the quadratic formula, since you'll have to check factors of 16 and -7, and there are a lot of these.
This should cover everything you could possibly want to know about solving quadratic equations. There's still the issue of recognising them, but 11 pages is probably enough for now, so we will address this issue in later questions.

Question 1b. [Go to model solutions]

In sketching the graph of a quadratic function, one can almost always take advantage of the highly symmetrical nature of their graphs, which are commonly referred to as parabolas[5].

First, consider the parabola $y = x^2$. The symmetrical properties of the curve $y = x^2$ are easy to understand, because the equation is so simple. The fact that $(-x)^2 = x^2$ implies that the curve $y = x^2$ is symmetrical about the y-axis, for it is unchanged upon reflection in this axis. In this sense, the y-axis may be referred to as an axis of symmetry of the curve $y = x^2$. This function also has a turning point at the origin, which we really need calculus to define properly, but it is easy to at least observe that the origin $(0, 0)$ corresponds to a global minimum of the function $f(x) = x^2$, because it just follows from the fact that $x^2 \geq 0$ for all real x.

Recall from the solution to the previous question that a general quadratic curve $y = ax^2 + bx + c$ can be written in turning-point form $y = a(x+h)^2 + k$. This is obtained from $y = x^2$ by a vertical dilation and possibly a reflection in the x-axis, followed by vertical and horizontal translations. Which of these transformations affect the axis of symmetry of $y = ax^2$? Since $a(-x)^2 = ax^2$, the curve $y = ax^2$ is still symmetric about the y-axis. Reflection in the x-axis also preserves this axis of symmetry. Horizontal translation will move the axis of symmetry, but importantly the resulting shape will still be symmetric about a vertical line. Finally, vertical translation leaves the axis of symmetry unchanged (since it is a vertical line).

To summarise, of the basic transformations taking $y = x^2$ to $y = a(x + h)^2 + k$, the only one that changes the axis of symmetry is the horizontal translation by $-h$, and so the curve $y = a(x + h)^2 + k$ is symmetric about the line $y = -h$.

Knowing the location of the axis of symmetry helps to both sketch the parabola and find the turning point, because the turning point of the graph of a quadratic function always lies on the axis of symmetry. There is an easy and general method to finding the axis of symmetry: if we know two points on the graph of a quadratic function with the same y-coordinate (for example, two x-intercepts, as they both have y-coordinate 0), then we can deduce that the axis of symmetry is halfway between them. In the present situation, we know from part **a.** that the quadratic curve we need to sketch has x-intercepts at $(-1, 0)$ and $(2, 0)$, so the graph of f is symmetric about the vertical line between these two points.

There is another method to finding the turning point, which follows from the work we did in the previous part in converting the expression $ax^2 + bx + c$ to turning-point form. We found that

$$ax^2 + bx + c = a\Big(x + \frac{b}{2a}\Big)^2 + c - \frac{b^2}{4a},$$

which, by the above discussion, tells us that the axis of symmetry of the curve $y = ax^2 + bx + c$ is the line

$$x = -\frac{b}{2a}.$$

This is most useful when your quadratic expression is in the form $ax^2 + bx + c$, because you can read off the coefficients. You may get a quadratic expression in factorised form, in which case it is easier to read off the x-intercepts, and then calculate the axis of symmetry as being halfway between these intercepts (if there is only one intercept, then that's where the axis of symmetry is).

If you completed part **a.**, then these two methods would be more or less equal in efficiency. In the form $f(x) = x^2 - x - 2$, we can calculate that the axis of symmetry is

$$x = -\frac{b}{2a} = -\frac{(-1)}{2} = \frac{1}{2}.$$

Alternatively, we can use the x-intercepts $x = -1, 2$ and find their midpoint by taking their average:

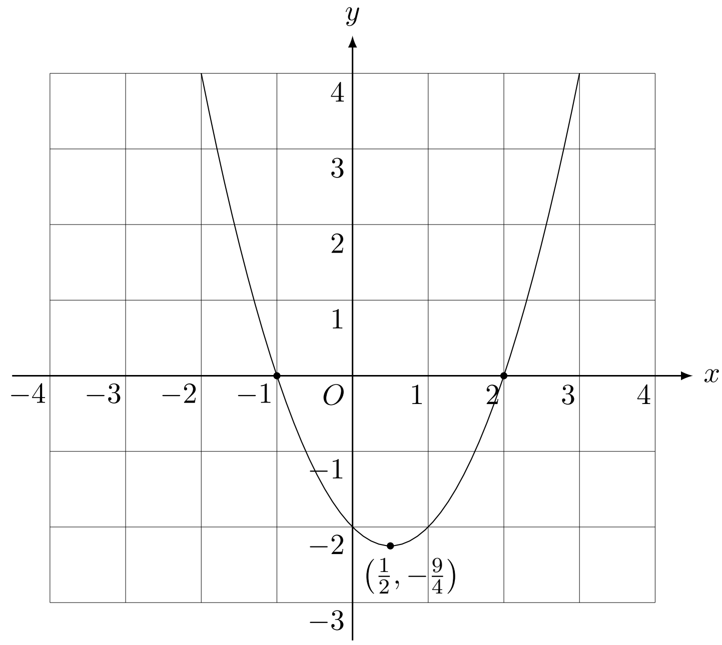
$$\frac{-1 + 2}{2} = \frac{1}{2}.$$

[5] A parabola is what you get when you cut a cone at a certain angle, and this happens to be the shape of the graph of every quadratic function. Parabolas were studied long before quadratic functions, and even before algebra.

We've calculate in two different ways that the graph of f has the line $x = \frac{1}{2}$ as its axis of symmetry. The turning point of the graph lies on this line, so we can use this x-value to find the y-coordinate of the turning point:

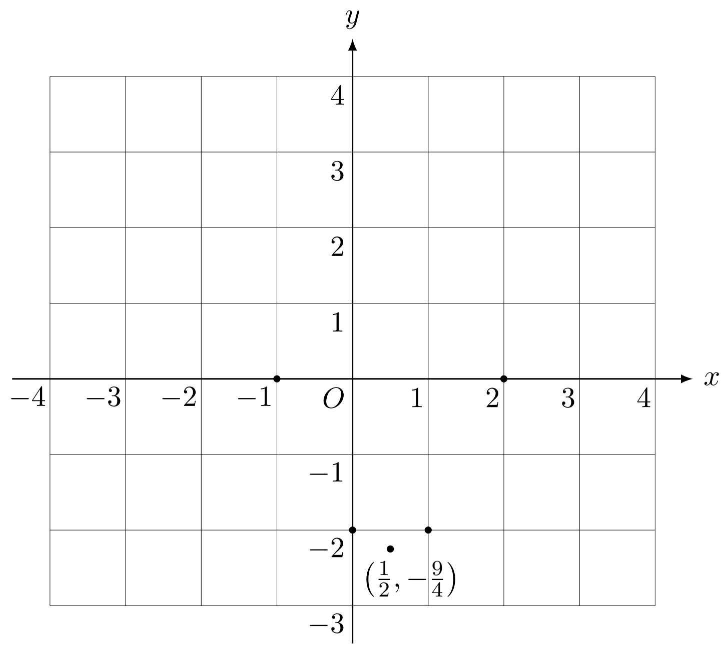
$$f\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4}$$

Hence, the turning point occurs at $\left(\frac{1}{2}, -\frac{9}{4}\right)$. This is a good thing to mark on the provided axes before attempting to draw the curve itself. We know the x-intercepts, so we can mark these in too.

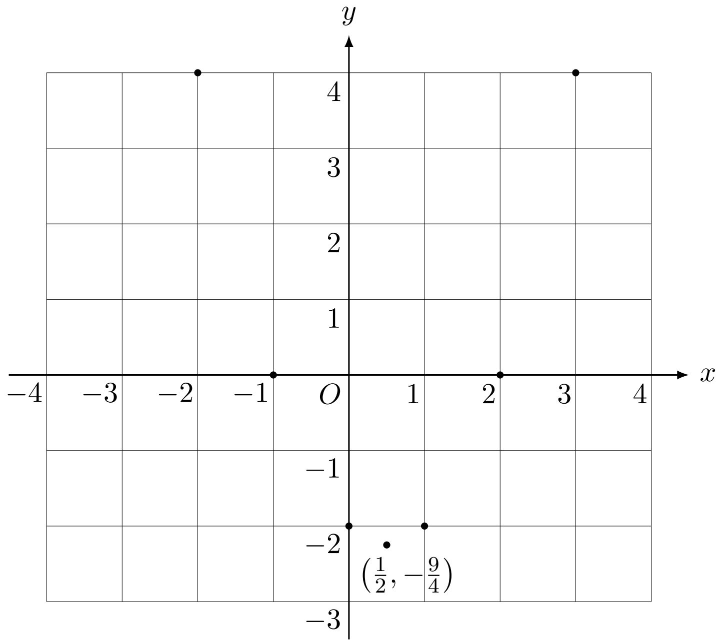


Drawing a nice, symmetric, and smooth curve can be tough, especially when you're under pressure. For this reason, you should try to calculate as many points as possible without wasting too much time. There is some balance to be struck, because you want to get the curve right the first time, but you don't want to be calculating coordinates for ten minutes.

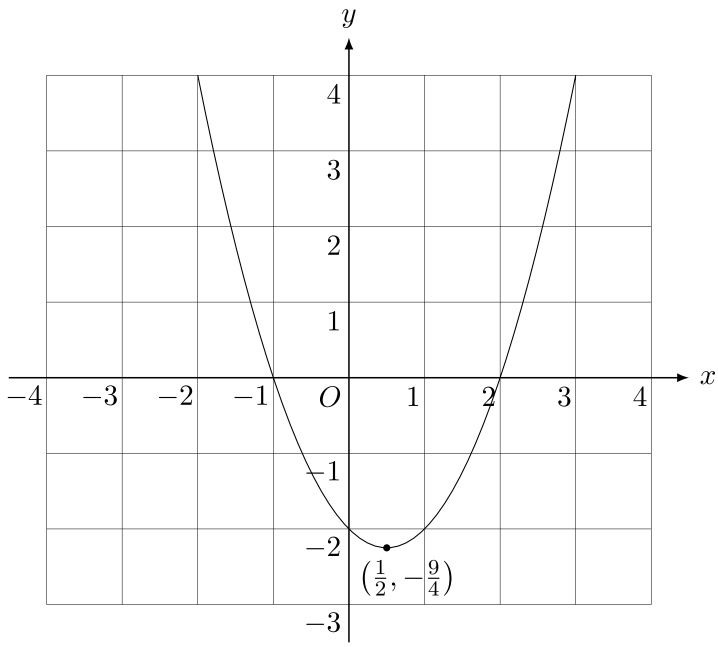
Once you've calculated the coordinates that are absolutely required for marks, you can look for some low-hanging fruit. In this case (and most others), the y-intercept is easy to calculate: $f(0) = -2$. Using symmetry about the line $x = \frac{1}{2}$, we also have $f(1) = -2$ (which can also be calculated directly without difficulty). We can mark these points on the grid. Labelling them isn't required, so it would waste precious time to do so.



Things are starting to take shape. This is probably enough to draw a reasonable curve, but it wouldn't hurt to calculate some more points for integer x-values. We have $f(3) = 9 - 3 - 2 = 4$, and $f(-2)$ should have the same value by symmetry, so we have two more points.



This is about as good as it's going to get. Trying any more integer x-values will take us outside of the grid, so we'd be forced to substitute in fractions to obtain more y-coordinates, and that would be more trouble than it's worth. The seven points above give a decent picture of what the graph will look like, so now would be an appropriate time to draw the curve.



Before drawing anything, remember to make the curve flat near the turning point; there shouldn't be any "sharp" bits on the curve. Because of the symmetry, you could draw half of the curve first (for example, the part of the graph where $x \leq \frac{1}{2}$), and then draw the other half by mirroring the half you've already drawn.

Always double check which points need to be labelled; losing marks for not precisely answering the question can cause a bit of heartbreak, especially when you've worked everything out perfectly.

Question 1c. [Go to model solutions]

Given the first two parts, this is now relatively straightforward. The main issue is probably writing the set down correctly. From the graph, we can see that $f(x) > 0$ when either $x < -1$ or $x > 2$. The phrase "$x < -1$ or $x > 2$" on its own wouldn't be a valid answer, since this isn't a set, but a statement. Writing "$x < -1$ and $x > 2$" is wrong for another reason, since no values of x satisfy both inequalities simultaneously. Admittedly, there is room for interpretation, but that really means it's not a good way of expressing the answer.

The answer could be written as $\{x \in \mathbb{R} : x < -1 \text{ or } x > 2\}$, since this is a set. This can also be written as $(-\infty, -1) \cup (2, \infty)$, or even $\mathbb{R}\backslash[-1, 2]$. Any correct use of set notation will do.

If you aren't familiar with the notation $\{x \in \mathbb{R} : x < -1 \text{ or } x > 2\}$, it is read as "the set of all real numbers x such that $x < -1$ or $x > 2$". More generally, we use this notation to form sets using sets we've already defined. That is, if S is a set, then we can form a new set $\{x \in S : P(x)\}$, where $P(x)$ is some statement about x (such as "$x < -1$ or $x > 2$"). The set $\{x \in S : P(x)\}$ is defined to be the set of elements of S for which the statement $P(x)$ is true.

This notation is a way of selecting members of a larger set based on some property; in this case, we are selecting real numbers with the property that $x < -1$ and $x > 2$. The reason for using this notation is the same reason we use any mathematical notation: it's succinct, and allows for more precise statements.

With that in mind, the set $\{x \in \mathbb{R} : f(x) > 0\}$ is technically an answer to the question, because it is literally what the question asks for: the set of values of x for which $f(x) > 0$. However, VCAA generally requires concrete answers to questions, so you need to express the set in one of the ways we've shown in the above paragraphs. Answering with the set $\{x \in \mathbb{R} : f(x) > 0\}$ is like answering "State the value of p" with "The value of p".

You may be wondering if this question can be done without reference to a graph. The inequality $f(x) > 0$ is really a statement about numbers, so it feels like you should be able to do it using algebra. You would not need to do it this way in the exam, but it's good to know how to do it, especially if you're continuing with maths after VCE.

First, write $f(x)$ in factorised form, $f(x) = (x+1)(x-2)$. We want to know when $(x+1)(x-2) > 0$. The key observation is this: if a product of two numbers is positive, then the two numbers are either both positive or both negative. To solve this inequality, these two cases need to be considered separately.

If both factors are positive, we have $x + 1 > 0$ and $x - 2 > 0$, so $x > -1$ and $x > 2$. Saying "$x > -1$ and $x > 2$" is actually the same as just saying that $x > 2$, because if $x > 2$, then $x > -1$ anyway. Thus, if both factors are positive, we have $x > 2$.

If both factors are negative, we have $x + 1 < 0$ and $x - 2 < 0$, so $x < -1$ and $x < 2$. Using similar logic to the previous case, this is the same as saying that $x < -1$.

So, in the first case, we get $x > 2$, and in the second case, we get $x < -1$. We therefore have $f(x) > 0$ when $x < -1$ or $x > 2$. This can also be written without using so many words:

$$\begin{aligned}(x+1)(x-2) > 0 &\implies \Big(x + 1 > 0 \text{ and } x - 2 > 0\Big) \text{ or } \Big(x + 1 < 0 \text{ and } x - 2 < 0\Big) \\ &\implies \Big(x > -1 \text{ and } x > 2\Big) \text{ or } \Big(x < -1 \text{ and } x < 2\Big) \\ &\implies x > 2 \text{ or } x < -1\end{aligned}$$

Note that the first line is a compact way of saying that the factors are both positive or both negative. Again, you won't need to solve quadratic inequalities this way in VCE, but it's nice to know how to do it.

Question 2 [Go to model solutions]

The thrust of this question is using points to calculate gradients and intercepts of straight lines. The function h is made of components, and this question should be approached by looking at each piece separately.

For $0 < x < 5$, the function h has the rule $h(x) = ax + b$, so this part of the graph of h is a straight line (which we can also see from the provided graph). To find a and b, then, we apply the usual techniques of finding a line passing between two points, which in this case are $(0, 4)$ and $(5, 0)$. We can immediately see that $b = 4$, since this is the y-intercept[6]. The value of a is the gradient of the line through these two points, so we can use the standard "rise over run" formula. The gradient of the line passing through the two points (x_1, y_1) and (x_2, y_2) (when $x_1 \neq x_2$) is

$$\frac{y_2 - y_1}{x_2 - x_1},$$

so to calculate a, we use the points $(0, 4)$ and $(5, 0)$, giving

$$a = \frac{0 - 4}{5 - 0} = -\frac{4}{5}.$$

The same formula can be used with the points $(5, 0)$ and $(6, 2)$ to find that $c = 2$. Finding d is only slightly less straightforward, but if we know c it's not too bad. We can see that $h(5) = 0$ from the given graph, so if we use the actual rule for h, we get $h(5) = 5c + d = 0$. Knowing the value of c, we get $d = -5c = -10$.
This is more of a revision question than an exam-style question, but you will need to know straight lines pretty well, since many of the more complicated questions are based on this knowledge.

Question 3a. [Go to model solutions]
This is a bit of a trick question, since the expression for $g(x)$ isn't given in the simplest form possible. It's common for VCAA to test manipulation of powers of small numbers. Here, you need to spot that rewriting 4 as 2^2 will allow you to combine powers and simplify the expression for $g(x)$; that is,

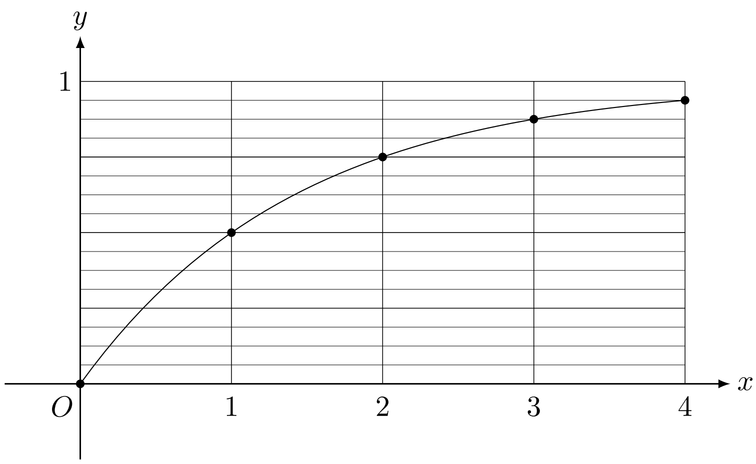
$$g(x) = 1 - 2^2 \cdot 2^{-x-2} = 1 - 2^{2-x-2} = 1 - 2^{-x}.$$

(Here we used the index law $a^x a^y = a^{x+y}$ to combine powers.) This is a less cumbersome expression, and it's easier to see what the graph will look like. Of course, you're not required to simplify it to get marks, but it will make your life easier, considering the lack of calculator access. The 2^{-x} term will become arbitrarily small with increasing x, so the graph of g will approach the line $y = 1$. With the set of axes we are given, you would hope that $g(x)$ never exceeds 1, and this is so: the quantity 2^{-x} is always positive, so $g(x) = 1 - 2^{-x}$ will always be less than 1. The graph of g must therefore approach the line $y = 1$ from below. Another thing to note is that, since 2^{-x} gets smaller as x increases, the value of $g(x)$ will increase with increasing x.
This should give some idea of what the graph of g looks like, but still doesn't really help you nail down the exact shape. An obvious thing to do is find the y-intercept. Using the simplified expression for $g(x)$, we have $g(0) = 1 - 2^0 = 1 - 1 = 0$, so the graph of g passes through the origin. The simplified expression found above makes it much easier to calculate $g(x)$ for any given x-value, and once again we are reaping the benefits of finding it. The vertical lines on the grid occur at integer x-values, so finding the corresponding y-values would help quite a bit in drawing the graph.
We have

$$g(1) = 1 - 2^{-1} = 1 - \frac{1}{2}, \quad g(2) = 1 - \frac{1}{4}, \quad g(3) = 1 - \frac{1}{8}, \quad g(4) = 1 - \frac{1}{16}.$$

Here we used another index law: $a^{-x} = \frac{1}{a^x}$.
As we mentioned in the first question, you don't need to fully simplify these values unless they're explicitly required as an answer to a question. The reason for doing this isn't that simplifying the fractions is too difficult; it's more that leaving the values as they are above gives you more of a feel for what's going on. It's also nice and easy to work with in this situation; the value of $g(1)$ is 1/2 of a unit from the top line, the value of $g(2)$ is 1/4 of a unit from the top line, and so on. These points should be marked in before you start drawing the curve, as it makes the task much easier and allows for more precision on your part. Adding in extra key points to your free-hand graphs in exams will ensure you never lose marks for graph shape in these questions (if done correctly).

[6] Recall that, for a line with equation $y = mx + c$, the y-intercept occurs at $(0, c)$, while the gradient of the line is m.



You don't need to label any points in this case, but always be sure of what points you're asked to label. You're expected to use the gridlines (if any are given to you) to help get the shape right, so you may lose marks if you just draw the rough shape. This particular curve should definitely have an endpoint at the origin. Marks may be lost if the right-hand endpoint was drawn at $(4, 1)$, for example.

Question 3b. [Go to model solutions]

An obvious trap here lies in not reading the question carefully. We are given that the domain of g is $[0, \infty)$, but this is not the largest domain that g could have with its rule. From the graph, we know that $g(0) = 0$, and $g(x)$ increases from there, so 0 is the smallest value in the range of g. We also know that $g(x) < 1$, but $g(x)$ will get arbitrarily close to 1, so the range of g is $[0, 1)$.

Question 3ci. [Go to model solutions]

As in part **a.**, simplyfing $g(x)$ as $g(x) = 1 - 2^{-x}$ will help here, as it will reduce the number of steps in finding the inverse function.

As far as finding $g^{-1}(x)$ goes, there are a few approaches, but they are all essentially the same. Graphically, the inverse of a one-to-one[7] function is obtained by reflecting its graph in the line $y = x$. Algebraically, this means y and x are swapped, and doing this is a common approach to finding inverses.

The graph of g is the curve $y = g(x)$, while the graph of g^{-1} is the curve $y = g^{-1}(x)$. What does this have to do with reflecting in the line $y = x$, then? Well, reflecting the curve $y = g(x)$ (the graph of g) in the line $y = x$ gives the curve $x = g(y)$. The equation of the graph of g^{-1}, however, is $y = g^{-1}(x)$. Are these the same equation?

The fact that the equations $y = g^{-1}(x)$ and $x = g(y)$ are the same is intuitively obvious if you think about how you would solve them. Solving $y = g^{-1}(x)$ for x gives $x = g(y)$, and solving $x = g(y)$ for y gives $y = g^{-1}(x)$. This shows how the equations $y = g^{-1}(x)$ and $x = g(y)$ are equivalent.

This can be shown more rigorously if one uses the proper (Methods) definition of the inverse of a one-to-one function g, which is the function $g^{-1}\colon \operatorname{ran}(g) \to \mathbb{R}$ satisfying $g\big(g^{-1}(x)\big) = x$ for all $x \in \operatorname{dom}\left(g^{-1}\right) = \operatorname{ran}(g)$, and also $g^{-1}(g(x)) = x$ for all $x \in \operatorname{dom}(g)$. Using these facts, we see that

$$y = g^{-1}(x) \implies g(y) = g\big(g^{-1}(x)\big) \implies g(y) = x,$$

and

$$g(y) = x \implies g^{-1}(g(y)) = g^{-1}(x) \implies y = g^{-1}(x)$$

which again shows how the equations $y = g^{-1}(x)$ and $x = g(y)$ are equivalent. This really just gave some formalism to the idea of solving the equations $y = g^{-1}(x)$ and $x = g(y)$ for x and y, respectively.

[7] Recall that a function g is one-to-one if $g(a) = g(b)$ implies $a = b$ for all a, b in the domain of g. More intuitively, this says that g doesn't take the same y-value at two different x-values, or alternatively that no horizontal line intersects the graph of g at more than one point (this is the so-called "horizontal line test").

To find $g^{-1}(x)$, we let $y = g^{-1}(x)$. This implies that $g(y) = x$, so we solve this equation for y.

$$\begin{aligned} g(y) = x &\implies 1 - 2^{-y} = x \\ &\implies 2^{-y} = 1 - x \\ &\implies -y = \log_2(1-x) \\ &\implies y = -\log_2(1-x) \\ &\implies g^{-1}(x) = -\log_2(1-x) \end{aligned}$$

The idea behind this approach is that we know what g is already, so it can be applied to y. In case you missed it, in going from the second line to the third, we used the fundamental relationship between exponential functions and logarithms, which is that, for $a > 1$ and $y > 0$, we have

$$a^x = y \iff x = \log_a(y)$$

In other words, the equations $a^x = y$ and $x = \log_a(y)$ are logically equivalent, so you can use them to convert between exponential functions and logarithms. In more detail, solving $a^x = y$ for x gives $x = \log_a(y)$, while solving $x = \log_a(y)$ for y gives $a^x = y$. This may resemble the discussion of how the equations $y = g^{-1}(x)$ and $x = g(y)$ are equivalent, and that is because $\log_a(x)$ is the inverse of the function a^x, which is actually the definition of $\log_a(x)$.
These questions (where you find the inverse of a given function) will always involve manipulation of power functions, exponential functions, or logarithms (inverses of circular functions are not part of the Methods course), so you can prepare pretty well for them.

Question 3cii. [Go to model solutions]

This is a straightforward test on the relationship between the domain and range of an invertible function and the domain and range of its inverse. In effect, the domain and range of a function are swapped when you invert the function. This can be understood graphically; the x- and y-values are swapped in going from the graph of a function to the graph of its inverse, as we showed in the previous part.
Based on this observation, the range of g^{-1} is the domain of g, which is $[0, \infty)$; this is given to you explicitly as the domain of g. Again, it would be easy to miss this and assume g is defined everywhere, which would imply that the range of g^{-1} is all of $\mathbb{R}$ (but it isn't).

Question 4a. [Go to model solutions]

The period of a function f is the smallest positive number p such that $f(x+p) = f(x)$ for all x in the domain of f, if such a number exists. The period of f is then the smallest (non-zero) amount by which you can translate the graph of f horizontally and end up with exactly the same curve. Given a function f, it is useful to know what simple transformations of f will affect the period of f, since in Methods you will only be asked to find the period of a function obtained as a simple transformation of sin, cos, or tan. In the following, it will be useful to think of f as one of these three functions.
Before getting started, click here and have a read if you aren't particularly comfortable with transformations, since most of what follows will be based on transformations.
Suppose that $g(x) = af(x)$, where $a \neq 0$. Then g must have the same period as f; the values of f have been uniformly scaled (with an order reversal if $a < 0$), so the cyclic pattern will remain the same. Think of this graphically; vertical scaling or reflecting in the x-axis will not affect the period. Similarly, if $g(x) = f(x) + b$, then g will have the same period as f, since this just moves the graph of f vertically. Thus, applying transformations in the y-direction of the graph of f will not affect its period.
What about translating in the x-direction? If $g(x) = f(x+c)$, then g still has the same period as f. If you change the position of the hands of an analogue clock, it will still repeat every 12 hours.
For reflecting in the y-axis, we consider $g(x) = f(-x)$. Again, the period of g is the same as f. If a clock runs backwards at its normal speed, it will still repeat every 12 hours.

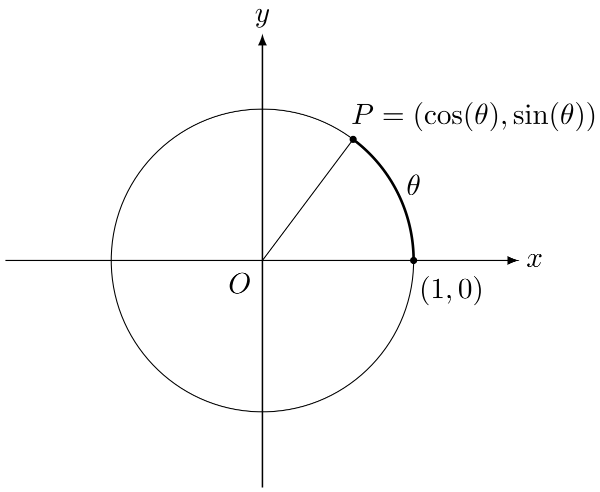
All that's left is dilation in the x-direction, and this is the only thing that affects the period. The graph of f will repeat itself every p units in the x-direction, so if we dilate the graph of f by a factor of a in the x-direction (where $a > 0$), it will repeat itself every ap units; that is, the new period is ap. Now, dilating by a factor of a corresponds to the transformed function $g(x) = f\left(\frac{x}{a}\right)$; always remember the reversal that happens when you transform the x-coordinates. If we instead have $g(x) = f(ax)$, then, the period of g will be $\frac{p}{a}$, because the dilation taking $y = f(x)$ to $y = f(ax)$ is actually a dilation by $\frac{1}{a}$. In terms of clocks... slowing the speed at which a clock runs will change the amount of time it takes to repeat by the same factor. For example, if a clock runs at half speed, it will repeat every 24 hours rather than every 12 hours (i.e., halving the speed doubles the period).
So how does this help with the task at hand? We are concerned with the function

$$f(x) = \tan\left(6x - \frac{\pi}{2}\right)$$

As we've discussed, we can ignore horizontal translations; the period of f is the same as the period of $\tan(6x)$. (There aren't any transformations applied in the y-direction, but in a situation where there are any, you can ignore them as far as the period is concerned.) If p is the period of tan, then the period of f will be $\frac{p}{6}$, so we will have our answer. Thus, we only really need to know what the period of tan is.
Understanding tan really comes down to understanding sin and cos, since tan is defined in terms of these functions. The definition of tan is

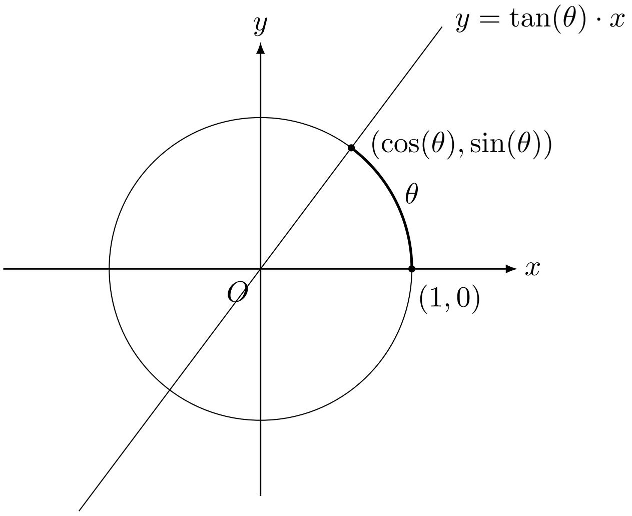
$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

The maximal domain of tan is the set $\{x \in \mathbb{R} : \cos(x) \neq 0\}$.
Now, we need to know what sin and cos are. Consider the unit circle. Starting at the point $(1, 0)$, we can travel as far as we want along the circle in either direction (i.e., clockwise or anticlockwise). Travelling 2π units along the circle will get us back to where we started, since 2π is the circumference of the unit circle. This would also happen if we travel 4π units. In principle, we can travel by any amount. Conventionally, we allow ourselves to travel both positive and negative distances; travelling by a positive amount means travelling anticlockwise, and travelling by a negative amount means travelling clockwise. Travelling by -2π around the unit circle from $(1, 0)$, for example, means travelling around the circle once in the clockwise direction.
So, given any $\theta \in \mathbb{R}$, we can travel θ units along the unit circle (starting at $(1, 0)$). Whatever θ is, we will end up at some point P on the unit circle. **By definition**, $\cos(\theta)$ is the x-coordinate of P, and $\sin(\theta)$ is the y-coordinate of P.

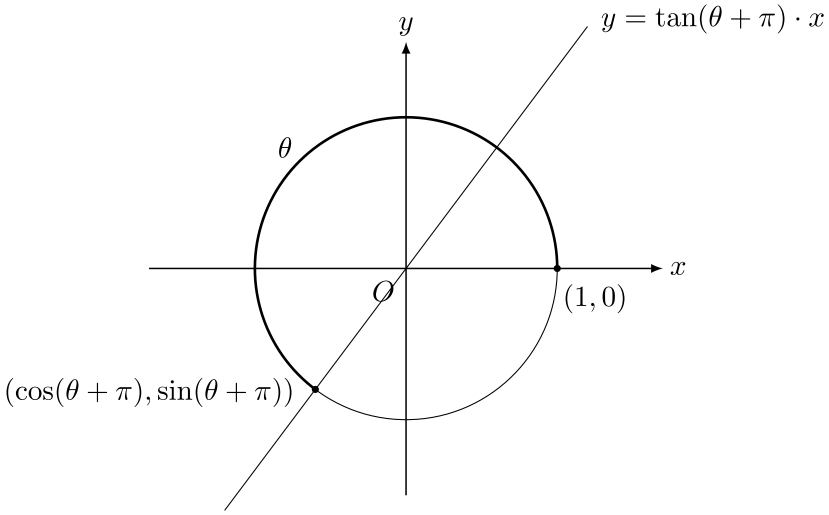


Commonly, θ is referred to as the radian angle between the line segment OP and the x-axis, although there are some features of this measure of angle that are a little odd as far as the usual understanding of angles goes. It's best to think of radian measure as a little more than an angle; it's really how far you've travelled around the circle. The functions cos and sin tell you the coordinates of the point you'll reach when you travel a certain distance.

There are lot of things we can take from this definition, but for now we should probably return to the original question of finding the period of tan. To this end, it would be useful to interpret the tan function geometrically. When it is defined, $\tan(\theta)$ is the gradient of the line through the origin and the point $(\cos(\theta), \sin(\theta))$; this follows from a simple application of the gradient formula $\frac{y_2-y_1}{x_2-x_1}$.



If you travel around the circle by a further π units (and no less!), you'll end up on the other side of the circle (remember that the circumference of the unit circle is π), but the line through the origin and the point you reach will be exactly the same. In particular, it will have the same gradient.



The two gradients are equal, so $\tan(\theta) = \tan(\theta + \pi)$. From this simple observation, we can see that the period of tan is π; hence, the period of f is $\frac{\pi}{6}$. More generally, the period of a function of the form $f(x) = \tan(ax)$, with $a > 0$, is $\frac{\pi}{a}$.

Question 4b. [Go to model solutions]

Here, we will assume some knowledge of the graph of tan. It's good to see how the graph connects with the geometric interpretation above. As θ increases from 0 to $\frac{\pi}{2}$, the gradient of the line through $(\cos(\theta), \sin(\theta))$ and the origin will tend towards infinity. Since $\tan(\theta)$ is the gradient of this line, we see this behaviour in the graph of tan.

The graph of tan will have vertical asymptotes at every value of θ such that the line through $(\cos(\theta), \sin(\theta))$ and the origin is vertical. This happens when the point $(\cos(\theta), \sin(\theta))$ is either at the very top or the very bottom of the unit circle; for example, when $\theta = \frac{\pi}{2}$. Another asymptote is reached whenever θ increases or decreases by π.

The point is that we can find asymptotes by using the period, provided we know where one asymptote is. We can do this for any simple transformation of the tan function, such as the function in this question. Unlike the period, the location of asymptotes will usually be affected by horizontal translations, but the distance between the asymptotes remains the same. The distance between adjacent asymptotes of tan is the same as its period, and this is true of any simple transformations of tan.

To algebraically find the location of an asymptote of a transformed tan function, it's best not to work with ∞ or $-\infty$ as if it were a number; avoid writing things like "Let $f(x) = \infty$". Apart from the fact that it probably won't help, it's nonsensical in a couple of ways. The asymptotes of $y = \tan(x)$ occur when $\cos(x) = 0$, so this is really the equation you want to solve. In this case, we have

$$f(x) = \tan\left(6x - \frac{\pi}{2}\right) = \frac{\sin\left(6x - \frac{\pi}{2}\right)}{\cos\left(6x - \frac{\pi}{2}\right)}$$

We know $x = a$ is an asymptote of the graph of f, so we solve $\cos\left(6a - \frac{\pi}{2}\right) = 0$ to find a. Since $\cos\left(\frac{\pi}{2}\right) = 0$, a possible solution is $6a - \frac{\pi}{2} = \frac{\pi}{2}$; solving this for a gives $a = \frac{\pi}{6}$. We can find the locations of other asymptotes by adding and subtracting the period of f. We are asked for a within the interval $\left[0, \frac{\pi}{2}\right]$, so there are only finitely many values of a; a general solution is not required (general solutions have never been required in tech-free exams, at least in the last 11 years).

Going backwards from $a = \frac{\pi}{6}$, subtracting the period of f (which is also $\frac{\pi}{6}$) gives 0, and subtracting any more will take us out of the required interval. If we add the period, we get $a = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$, and adding once more gives $a = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$, which brings us to the other end of the interval. The possible values of a are

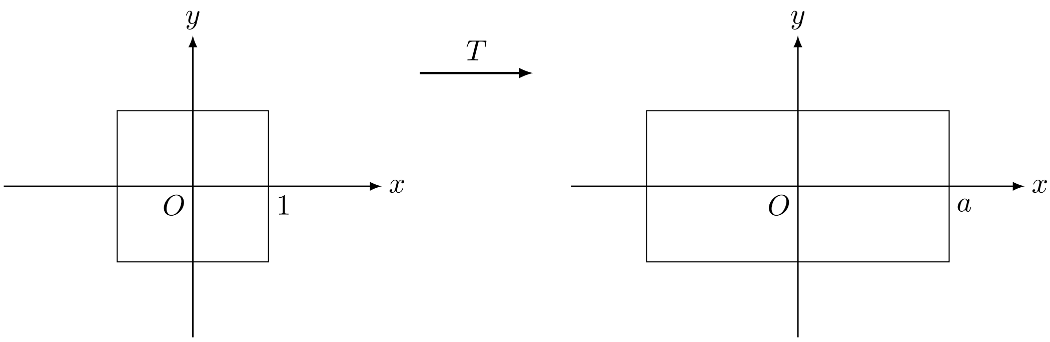
$$a = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$$

Question 5 [Go to model solutions]

This is probably a good place to carefully cover the theory behind transformations of the plane in order to address any subtleties you might have missed. The solutions to the actual question, if you want to skip ahead, start here.

This area is often taught poorly, but it is genuinely hard to teach, mainly due to fact that the Methods course lacks a lot of important concepts that would make it easier to explain. We won't introduce concepts outside of the course, though, so we'll explain transformations using only Methods knowledge.

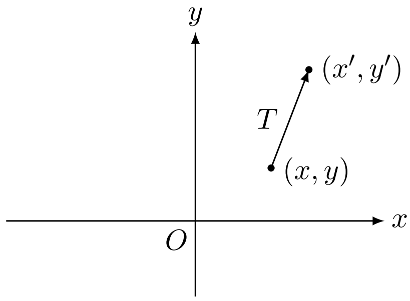
First, we'll define dilations. For $a > 0$, a dilation by a factor of a in the x-direction (also called a dilation by a factor of a from the y-axis) is the function[8] $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (ax, y)$. Essentially, T multiplies the x-coordinate of every point in the plane by a. (I'm going to use the symbol T many times, so you'll have to keep track of what T is at any given time.)



Similarly, a dilation by a factor of a in the y-direction (or a dilation by a factor of a from the x-axis) is defined as the function $T(x, y) = (x, ay)$. Geometrically, this is a vertical stretching rather than a horizontal stretching.

[8] You may be used to thinking of a function as having subsets of $\mathbb{R}$ as its domain and range, but a function can map between any two sets. The notation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ shows up in Methods exams but not in a big way.

It is usually of interest (in this course) to know how dilations affect the graphs of functions. Let $f\colon D \to \mathbb{R}$, where D is some subset of $\mathbb{R}$. The graph of f is the set of points (x, y) such that $y = f(x)$. Under a dilation, these points may be mapped to new points, which will result in a new curve.
To carefully verify results about transformations, it is useful to think of a function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ as sending a point (x, y) to a new point (x', y'). Giving names to the coordinates of the new point gives you more to work with (in a helpful way, as we will see).



Let's consider again a dilation by a factor of a in the x-direction, which is the function $T(x, y) = (ax, y)$. The point (x, y) gets sent to the point (ax, y), so we let $(x', y') = (ax, y)$.
To understand what happens to the graph of f, we ask: if (x, y) satisfies the equation $y = f(x)$, what equation does the point (x', y') satisfy? This is a slightly ill-defined problem, but the aim is to get an equation kind of like $y = f(x)$.
We defined $(x', y') = (ax, y)$, so we have a relationship between the new and old coordinates. More specifically, we have $x' = ax$ and $y' = y$. Rearranging the former gives $x = \frac{x'}{a}$; there is not much we can do with the latter.
Since $y = y'$ and $x = \frac{x'}{a}$, the equation $y = f(x)$ is the same as the equation $y' = f\left(\frac{x'}{a}\right)$. This says that the curve $y = f(x)$ gets mapped to the curve $y = f\left(\frac{x}{a}\right)$. The reason we can change the variable names back to normal is always tricky to explain, but it really comes down to the fact that the graph of a function is technically a set of points, not an equation. The graph of f is really the set

$$\{(x, y) : y = f(x)\}$$

As we have seen, under T, this set becomes

$$\left\{(x', y') : y' = f\left(\frac{x'}{a}\right)\right\}$$

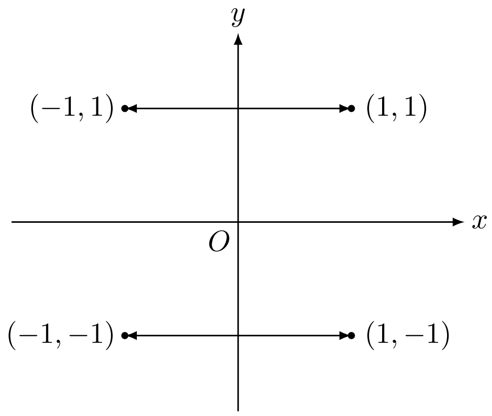
Since this is a set, it doesn't really matter what you called the variables, so it is the same as the set

$$\left\{(x, y) : y = f\left(\frac{x}{a}\right)\right\}.$$

This can be done even more carefully and convincingly by using more sophisticated set-theoretic notions, but this would take us outside of the course. In a nutshell, dilating by a factor of a in the x-direction sends the curve $y = f(x)$ to the curve $y = f\left(\frac{x}{a}\right)$. This is all you really need to know, but it's good to know why dilating in the x-direction has this effect.
We can apply the same approach to see what happens when we dilate in the y-direction via the function $T(x, y) = (x, ay)$. Again, we write (x', y') for the new coordinates, so $x' = x$ and $y' = ay$, the latter being equivalent to $y = \frac{y'}{a}$. By replacing old variables with the new ones, the equation $y = f(x)$ is equivalent to $\frac{y'}{a} = f(x')$, which in turn is equivalent to $y' = af(x')$. The equation of the new curve is therefore $y = af(x)$.
Hopefully you can see the differences between the two types of dilations – particularly, why one involves division by a and the other doesn't. The answer is basically that both involve division by a, but we generally rearrange equations to express things in terms of y.
In hindsight, we could have allowed $a < 0$, since it causes no problems with the algebra[9], but it's a little trickier to interpret this geometrically. For this reason, we usually look at the specific case $a = -1$.

[9] The case $a = 0$ does cause some problems, particularly in transforming graphs, since we divided by a. The corresponding transformations are perfectly sensible – they just act a little extremely, namely by squashing the entire plane onto an axis. This type of transformation won't come up in VCE.

The function $T(x, y) = (-x, y)$ takes the negative of each x-coordinate – the points (x, y) and $(-x, y)$ are swapped by T, so the entire plane is reflected in the y-axis.



Similarly, the function $T(x, y) = (x, -y)$ swaps the points (x, y) and $(x, -y)$, reflecting the plane in the x-axis. We've actually already worked out what these transformations do to the curve $y = f(x)$; simply substitute $a = -1$ into what we've already obtained. The reflection $T(x, y) = (-x, y)$ sends $y = f(x)$ to $y = f\left(\frac{x}{-1}\right)$; i.e., $y = f(-x)$. The reflection $T(x, y) = (x, -y)$ sends the curve $y = f(x)$ to $y = -f(x)$.

So, for $a < 0$, we can more easily describe the transformation $T(x, y) = (ax, y)$ as follows. First, note that $a = -b$, where b is a positive number (well, $b = -a$, to be precise). This means $T(x, y) = (-bx, y)$, so the x-values are dilated by b, and the plane is also reflected in the y-axis (the order in which these two transformations are carried out doesn't matter, just because the order in which you multiply x by -1 and b doesn't matter, because you end up with $-bx$ either way).

Decomposing T in this way is good for interpreting T geometrically, but it complicates the algebra. We could simply say that T multiplies the x-coordinate of every point by a (whether a is positive or negative), but this gives a little less intuition for what is happening[10]. In describing transformations in words, VCAA expects geometric words like "dilation" and "reflection" rather than "multiplication".

Since we can dilate along both coordinate axes separately, we can try to do both at once. Let $a, b \in \mathbb{R}$, let $T_1(x, y) = (ax, y)$, and let $T_2(x, y) = (x, by)$. These types of transformations have already been discussed, but we need to give them different names in order to combine them. We can define a new transformation T by $T(x, y) = T_1(T_2(x, y))$; this is one transformation followed by another, resulting in a new transformation. We calculate a more explicit rule for T:

$$T(x, y) = T_1(T_2(x, y)) = T_1(x, by) = (ax, by).$$

We get the same thing if we perform T_1 and T_2 in the other order, which hopefully makes sense geometrically – multiplying the x-coordinates doesn't affect the y-coordinates, so it doesn't matter if you multiply the y-coordinates before or after the x-coordinates. Basically, we can do them both at once, which is what T does.

We defined T as the function "do T_2, then do T_1", so we can almost immediately say what T does to the curve $y = f(x)$. Applying T_2, we get the curve $y = f\left(\frac{x}{a}\right)$, and applying T_1 to this curve gives $y = bf\left(\frac{x}{a}\right)$, so T sends the curve $y = f(x)$ to the curve $y = bf\left(\frac{x}{a}\right)$.

The function $T(x, y) = (ax, by)$ encapsulates all dilations and reflections; by choosing appropriate values of a and b, you can perform any sequence of dilations and reflections by using the single transformation T.

Let's see how this works with an example. Consider the sequence of transformations:

- Reflection in the x-axis
- Dilation by a factor of 2 from the x-axis

[10]This is a common trade-off in mathematics; concepts can be described more compactly at the cost of an intuitive understanding. This problem can be tackled by trying to understand each concept in a number of different ways.

- Reflection in the y-axis
- Dilation by a factor of 3 from the y-axis
- Reflection in the y-axis

We want to write all of this as a single function. To do this, we look at what this sequence of transformations does to the point (x, y).
Reflecting in the x-axis means multiplying y-coordinates by -1, so (x, y) gets sent to $(x, -y)$.
Dilating by a factor of 2 from the x-axis means multiplying y-coordinates by 2, so $(x, -y)$ gets sent to $(x, -2y)$.
Reflecting in the y-axis means multiplying x-coordinates by -1, so $(x, -2y)$ gets sent to $(-x, -2y)$.
Dilating by a factor of 3 from the y-axis means multiplying x-coordinates by 3, so $(-x, -2y)$ gets sent to $(-3x, -2y)$.
The last reflection sends $(-3x, -2y)$ to $(3x, -2y)$.
This sequence of 5 transformations can be written as a single function $T(x, y) = (3x, -2y)$. The order of this particular sequence doesn't matter; it only matters when translations come into play. We will now define translations.
Let $c, d \in \mathbb{R}$. We define new transformations $T_1(x, y) = (x + c, y)$ and $T_2(x, y) = (x, y + d)$. The transformation T_1 adds c to every x-coordinate, which has the effect of shifting the plane horizontally. The plane will shift to the right if c is positive and to the left if c is negative; if $c = 0$, nothing happens. Similarly, T_2 will shift the plane upwards if d is positive and downwards if d is negative.
As with dilations, we can combine translations in the x- and y-directions into a single function that performs both operations. Combining T_1 and T_2 (in either order) gives a new transformation $T(x, y) = (x + c, y + d)$. Any sequence of translations can be combined into a function of this form, in more or less the same way that a sequence of dilations and reflections can be combined into a single function.
Let's see what $T(x, y) = (x + c, y + d)$ does to the curve $y = f(x)$. Let $(x', y') = (x + c, y + d)$, so $x' = x + c$ and $y' = y + d$. We can rewrite these equations as $x = x' - c$ and $y = y' - d$, so $y = f(x)$ is the same as $y' - d = f(x' - c)$, which in turn is the same as $y' = f(x' - c) + d$. The transformed curve is therefore $y = f(x - c) + d$.
We can further combine dilations, reflections and translations into a single function. Let $T_1(x, y) = (ax, by)$ and $T_2(x, y) = (x + c, y + d)$. If we perform T_1 followed by T_2, we get

$$T_2(T_1(x, y)) = T_2(ax, by) = (ax + c, by + d).$$

You might have noticed that I didn't say what a transformation is yet, but the Methods definition is a function of the form $T(x, y) = (ax + c, by + d)$; that is, a composition of dilations, reflections and translations.
Now that we have the most general form of transformation that you'll encounter, we should look at how it changes the graph of a function. We already know what T_1 and T_2 do individually, so it won't be much work. If we apply T_1, the curve $y = f(x)$ is sent to the curve $y = bf\left(\frac{x}{a}\right)$, and then T_2 sends this to the curve $y = bf\left(\frac{x-c}{a}\right) + d$. We can summarise the last three pages or so by saying that

$$T(x, y) = (ax + c, by + d) \text{ sends the curve } y = f(x) \text{ to the curve } y = bf\left(\frac{x - c}{a}\right) + d.$$

It's important to look at what happens if we compose T_1 and T_2 the other way.

$$T_1(T_2(x, y)) = T_1(x + c, y + d) = (ax + ac, by + bd).$$

Generally, this will be a different from following T_1 by T_2, but the form is the same: if we let $c' = ac$ and $d' = bd$, we have

$$T_1(T_2(x, y)) = (ax + c', by + d').$$

No matter what sequence of translations, dilations, and reflections you apply to the plane, you will be able to express the resulting transformation in the form $T(x, y) = (ax+c, by+d)$, and we know exactly what this does to the graph of a function. Moreover, given such a function, you can express it as a sequence of dilations, reflections, and translations. As an example, take $T(x, y) = (-2x - 1, 5y + 3)$. To decompose T, you just need to think of how to build up T using basic transformations. In my head, I would think of something like this:

$$(x, y) \mapsto (2x, y) \mapsto (2x, 5y) \mapsto (-2x, 5y) \mapsto (-2x - 1, 5y) \mapsto (-2x - 1, 5y + 3)$$

At each stage, only one dilation, reflection or translation is applied. This shows that T is equivalent to the following sequence of transformations:

- Dilation by a factor of 2 from the y-axis
- Dilation by a factor of 5 from the x-axis
- Reflection in the y-axis
- Translation by 1 unit in the negative direction of the x-axis (i.e., to the left)
- Translation by 3 units in the positive direction of the y-axis (i.e., upwards)

If you are given such a list, you can find the resulting transformation by applying each transformation in the list to the point (x, y). We can apply the above list of transformations to the point (x, y):

$$(x, y) \mapsto (2x, y) \mapsto (2x, 5y) \mapsto (-2x, 5y) \mapsto (-2x - 1, 5y) \mapsto (-2x - 1, 5y + 3)$$

This gives back the original function $T(x, y) = (-2x - 1, 5y + 3)$. So a transformation can be written either as a function $T(x, y) = (ax + c, by + d)$ or as a list of dilations, reflections, and translations. If you prefer working with one form, you should learn how to go from the form you don't like to the form you do like, since they can be given to you in either form. It's also possible that you will be asked to write a transformation in a certain form, but this is not very common. To be on the safe side, you should be comfortable with converting a list to a function and vice versa.

I haven't yet addressed matrix notation, but it's really just another way of writing out the function $T(x, y) = (ax + c, by + d)$. A point (x, y) can alternatively be written in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

Of course, the information that it carries is the same, but now it's vertical. So far so good. Now, if we have a function $T(x, y) = (ax + c, by + d)$, we can write the pairs as matrices:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax + c \\ by + d \end{bmatrix}$$

Again, the pairs have just been written vertically. The next part uses some matrix multiplication and addition. Observe that

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ax + c \\ by + d \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

So, finally, we have

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$$

In this course, transformations will usually be written in matrix form, rather than the form $T(x, y) = (ax + c, by + d)$. You may be wondering whether there is an advantage of writing T in matrix form, since it is a lot less compact than the alternative. As far as this course is concerned, there is actually no advantage whatsoever, and I find VCAA's choice to put it in the course a little confusing. It's just

something you'll have to cope with. That's not to say matrices aren't useful – quite the opposite is true – it's just that you don't really need to use them for transformations as simple as these.
We now have enough theory to be able to answer the original question (as well as every question about transformations). We are given

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and we want to see how this transforms the curve $y = 3x^2 + 1$. This isn't exactly how the question is phrased, but that is really what it's asking. It's asked in a way that forces you to write the answer in a certain form (this really just makes it easier to mark).
First, we can rewrite T more compactly as $T(x, y) = (x + 1, 3y - 1)$. We now apply the result

$$T(x, y) = (ax + c, by + d) \text{ sends the curve } y = f(x) \text{ to the curve } y = bf\left(\frac{x - c}{a}\right) + d,$$

In this specific case, we have $a = 1$, $b = 3$, $c = 1$, $d = -1$, and $f(x) = 3x^2 + 1$, so we get

$$T(x, y) = (x + 1, 3y - 1) \text{ sends the curve } y = 3x^2 + 1 \text{ to the curve } y = 3(3(x - 1)^2 + 1) - 1.$$

All that's left is to simplify the right-hand side of $y = 3(3(x - 1)^2 + 1) - 1$.

$$\begin{aligned} y &= 3(3(x - 1)^2 + 1) - 1 \\ &= 9(x - 1)^2 + 3 - 1 \\ &= 9(x^2 - 2x + 1) + 2 \\ &= 9x^2 - 18x + 9 + 2 \\ &= 9x^2 - 18x + 11. \end{aligned}$$

This is the form required by the question; you just need to finish up by saying what a, b, and c are[11].
In terms of writing out the answer, be sure to refer to the model solutions.

[11] These aren't the same a, b, c that appear in the equation $y = bf\left(\frac{x-c}{a}\right) + d$. You should use different names if you want to write out the general equation in your answer, but you don't really need to write out the general equation.