CALCULUS

TECH-FREE TEST 1

Writing time: 30 minutes

Structure of test

Number of questions	*Number of questions to be answered*	*Number of marks*
5	5	20

Question 1

Show that if $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$.

2 marks

Question 2

a. Find the derivative of $x\log_e(x^2)$.

2 marks

b. Hence, find the antiderivative of $\log_e(x^2)$.

3 marks

Question 3

The surface area of a sphere increases from 64π to 65π. Find the corresponding approximate increase in radius.

4 marks

Question 4

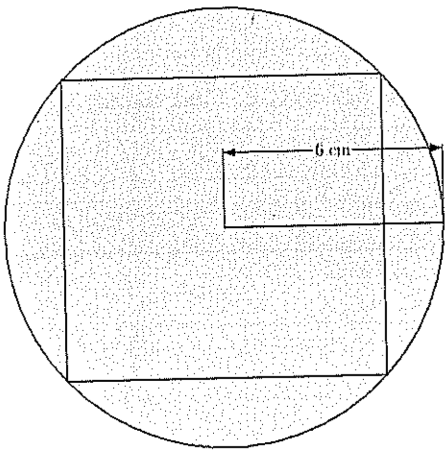
For the following hybrid function, $f(x)$, find the largest subset of $[0, 6\pi]$ for which $f(x)$ is strictly decreasing.

$$f(x) = \begin{cases} \tan\left(-\frac{x}{3}\right), & 0 \leq x \leq 2\pi \\ -\pi x + 2\pi^2 + \sqrt{3}, & x > 2\pi \end{cases}$$

3 marks

Question 5

A cylinder sits exactly in a sphere of radius 6 cm as shown in the cross-sectional diagram below.



The height of the cylinder is h and its radius is r.

a. Show that $r = \dfrac{\sqrt{144 - h^2}}{2}$.

2 marks

b. Hence, state V, the function that describes the volume of the cylinder, in terms of h.

1 marks

c. Use your result from **part b.** to find the maximum volume of the cylinder.

3 marks

END OF TEST

CALCULUS

TECH-FREE TEST 2

Writing time: 30 minutes

Structure of test

Number of questions	*Number of questions to be answered*	*Number of marks*
4	4	20

Question 1

a. Evaluate $\int_0^t \left(e^{-x} + x\right) dx$.

2 marks

b. What does your answer from **part a.** represent?

1 marks

Question 2

Consider the line $y = 2x - 3$ and point A, $(-4, 2)$.

a. State the gradient of the normal to this line.

1 marks

b. Hence, define a normal to $y = 2x - 3$ that passes through a general point, (x_a, y_a), on that line.

2 marks

c. Hence, or otherwise, find the shortest distance between this line and point A.

5 marks

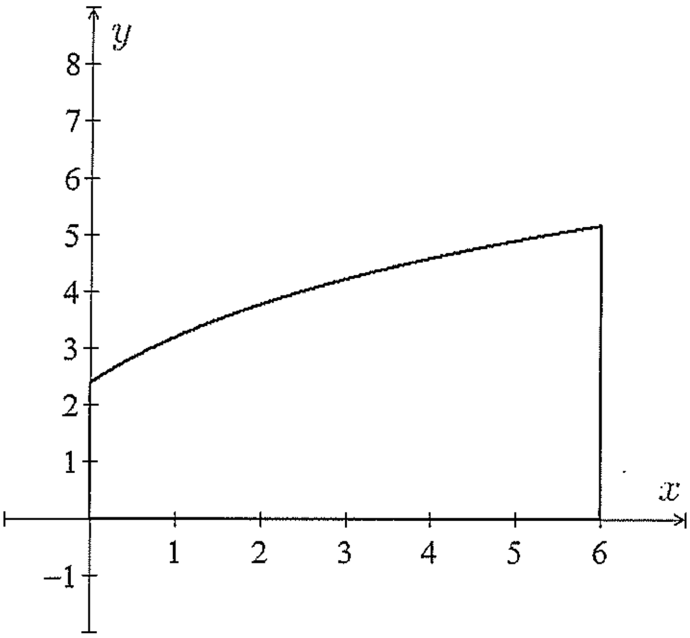
Question 3

Using linear approximation, find an approximate value for $\frac{1}{\sqrt[3]{28}}$.

3 marks

Question 4

Farmer Bebbington owns a field which is bounded by four roads. Road A is described by the function, $f(x) = 2\log_e(x+2)+1$, Roads B and C are the coordinate axes and Road D is given by $x = 6$.



a.

i) Shade in Farmer Bebbington's field in the diagram above.
ii) Find the inverse of $f(x)$, $f^{-1}(x)$, and sketch it on the above axes.

3 marks

b. Hence, or otherwise, find the area of Farmer Bebbington's field.

3 marks

END OF TEST

CALCULUS
TECH-ACTIVE TEST 1

Writing time: 60 minutes

Structure of test

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	11	11	11
2	2	2	29
			40

SECTION 1 - Multiple-choice questions

Question 1
Let f be a function defined by $f : [0,3] \to \mathbb{R}$, $f(x) = x^2 - 2x - 1$. The maximum and minimum values respectively are

A. $-1, -3$

B. $2, -2$

C. $1, 3$

D. $0, 3$

E. $1, -2$

Question 2
The function f has a derivative function $f'(x) = \frac{1}{2}e^{3x} - \sin(\frac{1}{2}x)$ and passes through the point $(0, 2)$. $f(x)$ is

A. $f(x) = \frac{1}{6}e^{3x} + \cos(\frac{1}{2}x) + \frac{5}{6}$

B. $f(x) = \frac{1}{2}e^{3x} + 2\cos(\frac{1}{2}x) - \frac{1}{2}$

C. $f(x) = \frac{1}{6}e^{3x} + 2\cos(\frac{1}{2}x) - \frac{1}{6}$

D. $f(x) = \frac{1}{2}e^{3x} - \cos(\frac{1}{2}x) + \frac{5}{2}$

E. $f(x) = \frac{1}{6}e^{3x} + \cos(x) + \frac{5}{6}$

Question 3
What is the average value of the function $f(x) = |x^2 - 4|$ over the interval $[-3, 3]$?

A. $\frac{32}{3}$

B. $\frac{23}{9}$

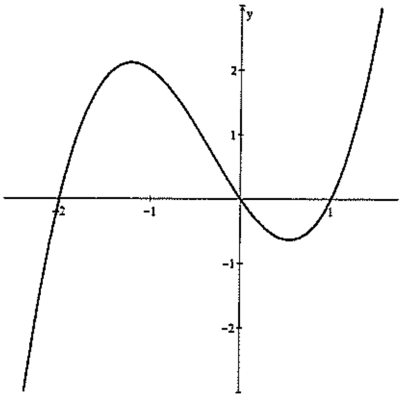
C. $\frac{8}{3}$

D. -6

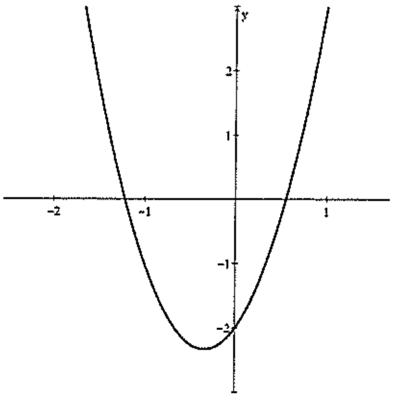
E. 5

Question 4

If a function is given by the following graph, then the derivative could be



A.



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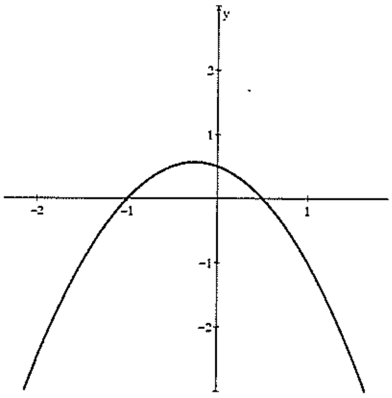
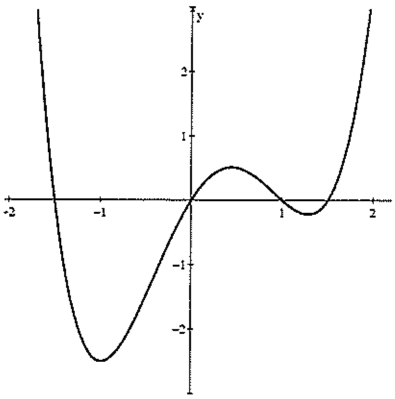
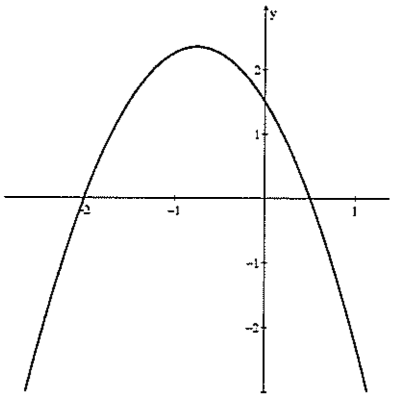


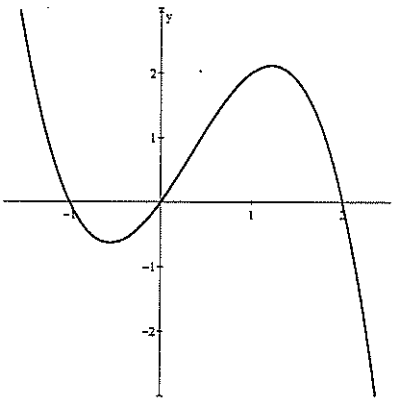
C.



D.

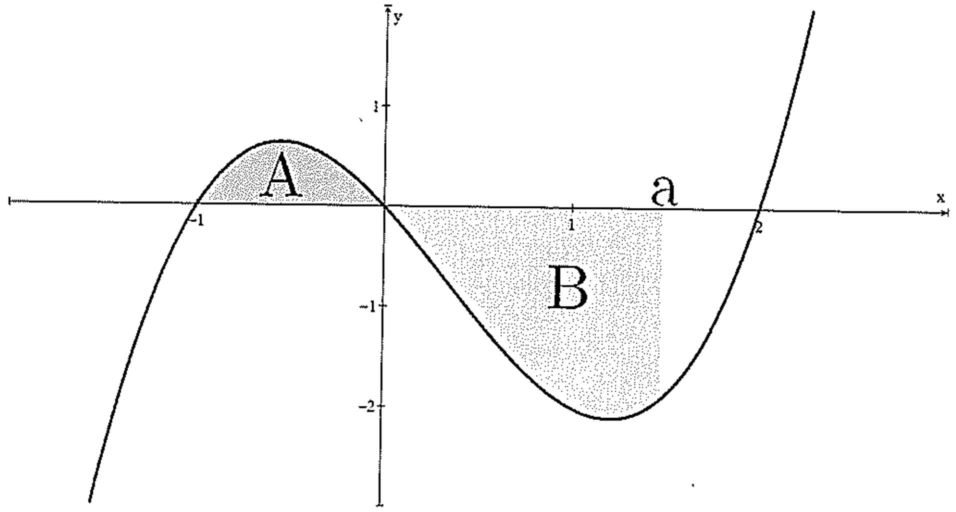


E.



Question 5

If the graph below is of the function $f(x) = x(x-2)(x+1)$ and the area of the region B is three times that of region A, the value of a correct to 4 decimal places is



A. 1.0818

B. 1.2384

C. 1.9802

D. −1.4993

E. 0.8799

Question 6

If f is a function that is defined for $\mathbb{R}$ and differentiable for $\mathbb{R}$, and the following properties hold,

- $f'(x) > 0$ for $x < -1$
- $f'(x) = 0$ for $x = -1$
- $f'(x) > 0$ for $-1 < x < \frac{5}{2}$
- $f'(x) = 0$ for $x = \frac{5}{2}$
- $f'(x) < 0$ for $x > \frac{5}{2}$

Then f has what kinds of stationary points at $x = -1$ and $x = \frac{5}{2}$ respectively?

A. A stationary point of inflection at $x = -1$ and a local minimum at $x = \frac{5}{2}$

B. A local minimum at $x = -1$ and a stationary point of inflection $x = \frac{5}{2}$

C. A local minimum at $x = -1$ and a local maximum at $x = \frac{5}{2}$

D. A stationary point of inflection at $x = -1$ and a local maximum at $x = \frac{5}{2}$

E. A local maximum at $x = -1$ and a local minimum at $x = \frac{5}{2}$

Question 7

The derivative of $y = e^{f(x)}\cos(f(x))$ is

A. $e^{f(x)}f'(x)\cos(f(x))$

B. $e^{f(x)}f'(x)(\cos(f(x)) + \sin(f(x)))$

C. $e^{f(x)}f'(x)(\cos(f(x)) - \sin(f(x)))$

D. $e^{f(x)}f'(x)\sin(f(x))$

E. $e^{f(x)}f(x)(\cos(f(x)) + \sin(f(x)))$

Question 8

The area between the curve $y = e^x$ the y-axis and the line $y = e^2$ is

A. 0

B. e^2

C. $e^2 - 1$

D. $2e^2$

E. $e^2 + 1$

Question 9

A line that is perpendicular to the tangent of the curve of $y = x^2 - 4x + 4$ at the point $(3, 1)$ is

A. $y = 2x - 1$

B. $x = 1$

C. $y = 3 - x$

D. $y = -\frac{1}{2}$

E. $y = -\frac{1}{2}x + 2$

Question 10

The acute angle between the tangents to the curves $f(x) = x^3 - x^2 - 2x + 3$ and $g(x) = (x-2)^2 + 3$ at the intersection of the two curves correct to 2 decimal places is

A. 9.46°

B. 45.00°

C. 9.46°

D. 32.42°

E. 80.54°

Question 11

The function $f(x) = ax^3 + 2ax^2 + 2x - 1$ has two turning points for

A. $a \in \mathbb{R} \setminus [0, \frac{3}{2}]$

B. $a \in (0, \frac{3}{2})$

C. $a \in \mathbb{R}$

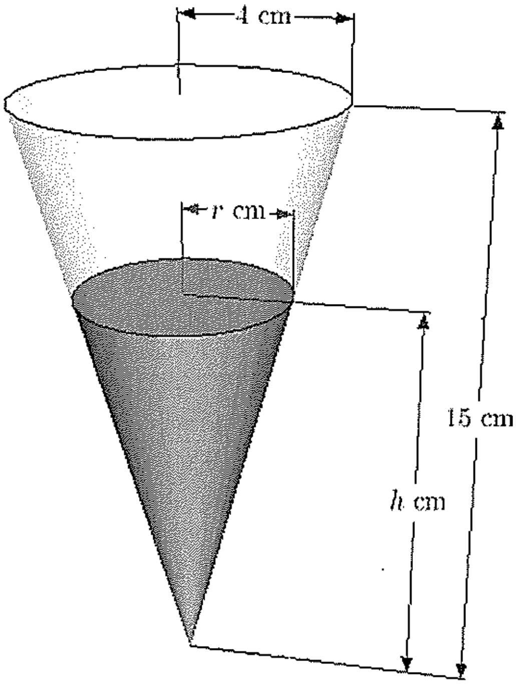
D. $a \in \mathbb{R} \setminus (0, \frac{3}{2})$

E. $a \in [0, \frac{3}{2}]$

SECTION 2 - Extended response questions

Question 1

An ice-cream cone is modeled by an inverted cone with radius 4 cm and height 15 cm. The empty cone is filled with ice-cream. Let r represent the radius of the surface of that top of the ice-cream in the cone and h represent the height of the ice-cream in the cone.



a. Find r in terms of h.

1 mark

b. Find an expression for the Volume, V cm^3 in terms of h.

1 mark

c. Find the height of the cone (in cm) that is filled with ice-cream when the cone is half filled to 2 decimal places.

3 marks

d. If the volume of ice-cream in the cone is increasing at $20\ \text{cm}^3/\text{s}$, find the rate at which the height of the ice-cream in the cone is increasing when $h = 5$ cm.

3 marks

e. If ice-cream is added at a rate of $20t\,\text{cm}^3/\text{s}$ where t is the time in seconds, find the total time taken to fill the cone, given that the cone was initially empty.

3 marks

Question 2

A small mountain range is modeled by the function $f : [0, 5] \to R,\ f(x) = ax^6 + bx^5 - \frac{705}{8}x^4 + \frac{1545}{4}x^3 - \frac{1215}{2}x^2 + 500$ where x is the distance in kilometres horizontally from the peak of the larger hill and $f(x)$ is the height of the ground in metres. The main peak of the hill is at $x = 0$ km. Assume that the peaks are maximums when graphed.

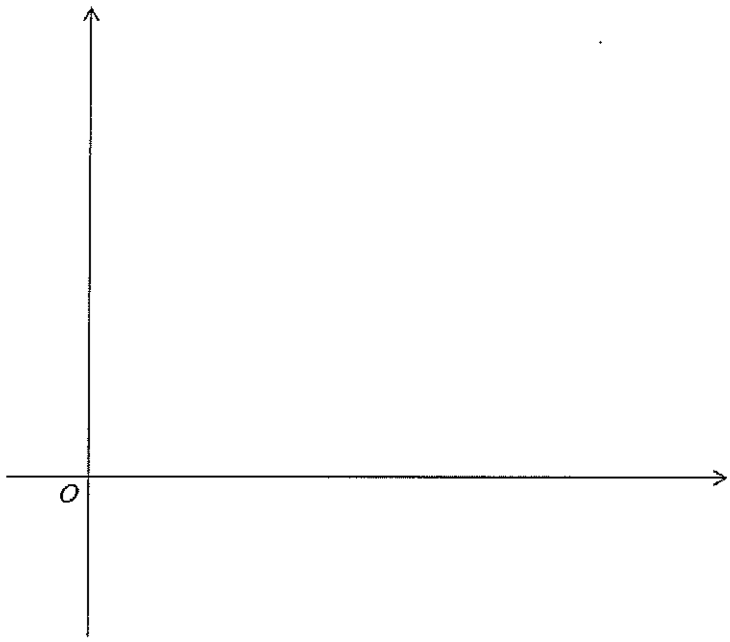
a. If the minor peak is at $(4, 300)$, find the values of a and b.

3 marks

b. i. Find the x and y intercepts of the function.

ii. Find the turning points of the function.

iii. Sketch the mountain range labeling the turning points and axis intercepts.



2 + 2 + 3 = 7 marks

c. Under heavy rains the gap between the two peaks may fill up with water, forming a temporary dam. On a particular day the water fills up to a height of 100 m. If the mountain range is 1000 m long (assuming a constant cross-sectional area), find the volume of water in m^3 that is in the temporary dam correct to the nearest 1000 m^3.

3 marks

d. If the height of the water remains at 100 m, and Nina slides down the slope from the minor peak into the water in the dam, what is the acute angle that she will hit the water at, in degrees correct to the nearest degree?

2 marks

e. Find the minimum volume of water in m^3 needed to overfill the temporary dam correct to the nearest $1000\ m^3$.

3 marks

END OF TEST

CALCULUS

TECH-ACTIVE TEST 2

Writing time: 60 minutes

Structure of test

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	11	11	11
2	3	3	29
			40

SECTION 1 - Multiple-choice questions

Question 1

The derivative of $y = \log_e(f(3x+1))\,x^2$ is

A. $6x\log_e(f(3x+1)) + 3x^2\dfrac{f'(3x+1)}{f(3x+1)}$

B. $2x\log_e(f(3x+1)) + 3x^2\dfrac{f'(3x+1)}{f(3x+1)}$

C. $2x\log_e(3x+1) + \dfrac{x^2}{f(3x+1)}$

D. $2x\log_e(f(3x+1)) + x^2\dfrac{f'(3x+1)}{f(3x+1)}$

E. $6x\log_e(f(3x+1)) + \dfrac{f'(3x+1)}{f(3x+1)}$

Question 2

The tangent of $f(x) = -2\log_e(x-2)$ at $x = 4$ is parallel to the tangent $g(x) = -(x-a)^2$ at $x = 4$. The value of a is

A. -4

B. $\frac{7}{2}$

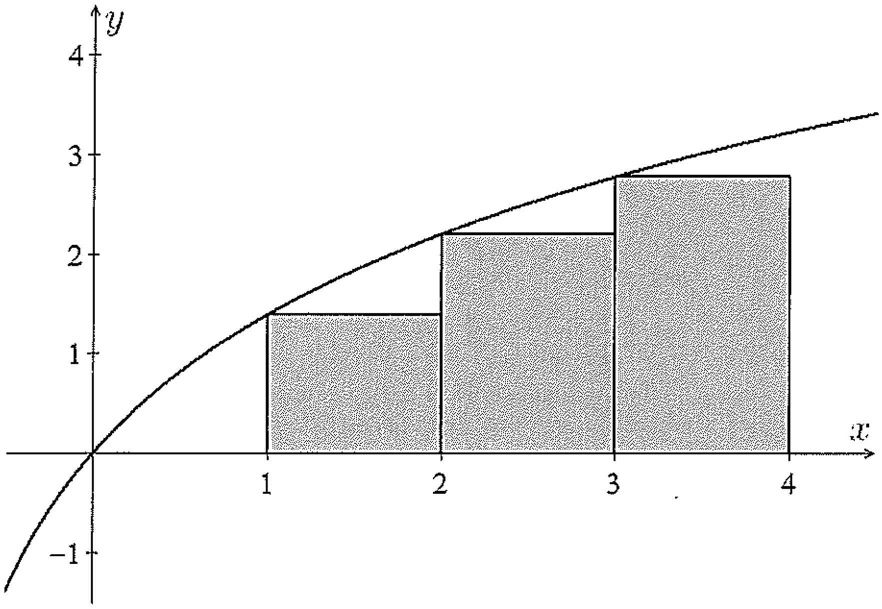
C. 3

D. $\frac{5}{2}$

E. -2

Question 3

The area under the curve $y = 2\log_e(x+1)$ from $x = 1$ to $x = 4$ can be approximated by the following rectangles:



The area of this approximation is

A. $\log_e(2) + \log_2(3) + \log_e(4)$

B. $\log_e(576)$

C. $2(5\log_e(5) - 2\log_e(2) - 3)$

D. $5\log_e(5) - 2\log_e(2) - 3$

E. $\log_e(29)$

Question 4

The area bound by the curve $f(x) = \dfrac{6}{4x+1} - \dfrac{1}{2}$, the x-axis and the y-axis may be found by evaluating

A. $\left[\dfrac{3}{2}\log_e(|4x+1|) - \dfrac{1}{2}x\right]_0^{\frac{11}{4}}$

B. $\left[\dfrac{3}{2}\log_e(|4x+1|) - \dfrac{1}{2}x\right]_0^{\frac{11}{2}}$

C. $\left[3\log_e(|x+1|) - \dfrac{1}{2}x\right]_0^{\frac{11}{4}}$

D. $\left[3\log_e(x+1) - \dfrac{1}{2}x\right]_0^{2}$

E. $\left[\dfrac{3}{2}\log_e(-4x-1) - \dfrac{1}{2}x\right]_0^{\frac{11}{4}}$

Question 5

The cubic $f(x) = ax^3 + 2ax^2 - bx + 2$ has 1 stationary point for

A. $a = 0$

B. $a = 0$ and $a = -\frac{3b}{4}$

C. $a = \frac{1}{2}b$ and $a = -\frac{3b}{4}$

D. $a = \frac{1}{2}b$ and $b \neq 0$

E. $a = -\frac{3b}{4}$ and $b \neq 0$

Question 6

If the area bounded by the x-axis, y-axis, the function $f(x) = \cos(x)$ and the line $x = a$ is equal to the area bounded by the x-axis, the function $g(x) = \sin(2x)$ and $x = a$, then given that $0 < a < \pi$, the value of a is

A. $\frac{\pi}{8}$

B. $\frac{\pi}{4}$

C. $\frac{3\pi}{8}$

D. $\frac{\pi}{2}$

E. $\frac{5\pi}{8}$

Question 7

The average value of the function $f(x) = |x\cos(x)|$ over the interval $\left[0, \frac{3\pi}{2}\right]$ can be found by evaluating which of the following expressions;

A. $\int_0^{\frac{3\pi}{2}} |x\cos(x)|dx$

B. $\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x\cos(x)dx - \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\cos(x)dx$

C. $\frac{2}{3\pi}\left(\int_0^{\frac{\pi}{2}} x\cos(x)dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x\cos(x)dx\right)$

D. $\frac{2}{3\pi}\left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x\cos(x)dx - \int_0^{\frac{\pi}{2}} x\cos(x)dx\right)$

E. $\frac{2}{3\pi}\left(\int_0^{\frac{3\pi}{2}} x\cos(x)dx\right)$

Question 8

If $\int_0^{10} f(x)dx = 2a$ then $\int_0^{10} 6f(x) - 2dx$ is equal to

A. $6a - 10$

B. $12a$

C. $12a - 20$

D. $6a - 20$

E. $2a - 20$

Question 9

If a function is given by

$$f(x)=\begin{cases}(x-1)^2-1 & x\leq 2\\ 2x-4 & 2<x<4\\ 4 & x\geq 4\end{cases}$$

Then the derivative is given by

A. $f'(x)=\begin{cases}2(x-1) & x<2\\ 2 & 2<x<4\\ 0 & x>4\end{cases}$

B. $f'(x)=\begin{cases}2(x-1) & x<2\\ 2 & 2<x<4\\ 4 & x\geq 4\end{cases}$

C. $f'(x)=\begin{cases}2(x-1) & x<2\\ 2 & 2<x<4\\ 4 & x>4\end{cases}$

D. $f'(x)=\begin{cases}2(x-1) & x\leq 2\\ 2 & 2<x<4\\ 4 & x\geq 4\end{cases}$

E. $f'(x)=\begin{cases}2(x-1) & x\leq 2\\ 2 & 2<x<4\\ 0 & x>4\end{cases}$

Question 10

If a function is defined by $f(x)$ and

- $f'(x)>0$ for $x<-1$
- $f'(x)<0$ for $-1<x<1$
- $f'(x)<0$ for $1<x<3$
- $f'(x)>0$ for $x>3$
- $f'(x)=0$ for $x=\{-1,1,3\}$

Then the number of stationary points of inflections, local maxima and local minima respectively are

A. $1,1,1$
B. $1,2,0$
C. $1,0,2$
D. $0,2,1$
E. $0,3,0$

Question 11

If $f'(x)=4x^3+\dfrac{1}{x^2}-\dfrac{1}{x}$ and $f(x)$ passes through the point $(1,2)$ then the rule for $f(x)$ is given by

A. $f(x)=x^4-\dfrac{1}{x}-\ln|x|$

B. $f(x)=\dfrac{1}{4}x^4-\dfrac{1}{x}-\ln|x|+2$

C. $f(x)=x^4-\dfrac{1}{x}-\ln|x|+2$

D. $f(x)=\dfrac{1}{4}x^4+\dfrac{1}{x}+\ln|x|+2$

E. $f(x)=x^4+\ln|x|$

SECTION 2 - Extended response questions

Question 1

The depth of a river is modelled by the function $d : [-25, 25] \to R$, $d(x) = \frac{1}{50}\left(x^2 - 625\right)$ where x is the horizontal distance from the lowest point of the bank in m and $d(x)$ is the depth of the river in m. The river carries flowing water through a rainforest up until it goes over the edge of a waterfall.

a. Find the cross-sectional area of the river in m^2

2 marks

Over time, sediment builds up on the bottom of the river. The layer of sediment can be modelled by the function $s : [-a, a] \to R$, $s(x) = -\sqrt{\frac{3}{25}\left(625 - x^2\right)}$.

b. Find the value of a.

1 mark

c. Find the cross-sectional area of the sediment on the bottom of the river correct to the nearest square metre.

2 marks

The river winds its way around the rainforest and follows a path of the function $g : [-3, 10] \to R, g(x) = -\frac{3}{x+a} - bx + 5$. All units are in kilometers. A Nuclear Power Plant needs to draw water from the river to cool the reactor, which is situated at the origin, $(0, 0)$.

d. Given that $g(x)$ has a turning point at $\left(\sqrt{6} - 3, \frac{13}{2} - \sqrt{6}\right)$, find the values of a and b, where $a > 2$ and $b < 5$.

3 marks

A small canal redirects some of the flow of the river, bypassing the stationary point of the function. This canal follows the function $h : \left[-\frac{12}{5}, 3\right] \to R, h(x) = \frac{1}{3}x + 2$. Another small canal is to be dug from the original canal to provide cooling water to the power plant.

e. Find the minimum distance between the original canal and the Nuclear Power Plant, and the point on the canal at which this occurs.

4 marks

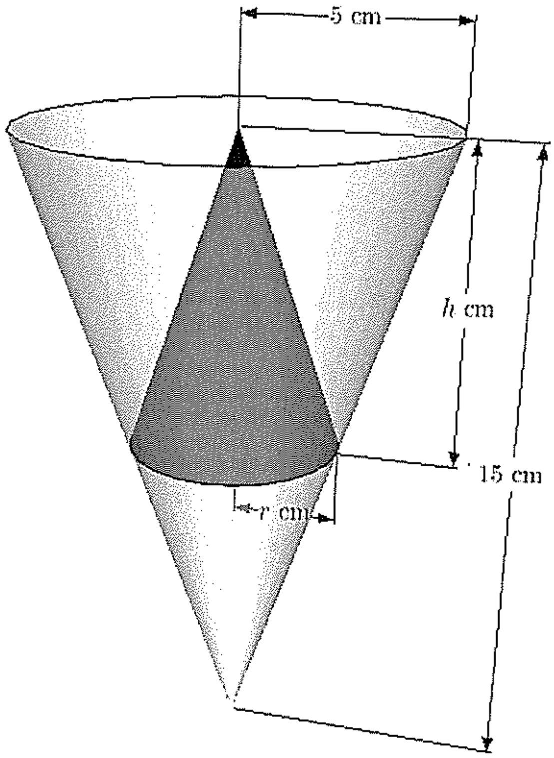
The water from the canal fills up a tank inside the Nuclear Power Plant. Over time water is taken from the tank and fed into and out of the reactor to control the temperature. The volume flow rate of water into the reactor is given by $V' : [0, 180] \to R$, $V'(t) = e^{-\frac{1}{15}(t-30)} + \cos(\frac{1}{15}t)$ m^3/s and t is the time in seconds.

f. If, initially, the volume of water in the tank is $20\,\text{m}^3$, find the function for the volume of water left in the reactor in m^3 at time t.

3 marks

Question 2

An inverted cone of radius r and heigth h is inscribed inside a larger cone of radius 5 cm and heigth 15 cm as in the diagram below.



a.

i. Show that $r = 5 - \frac{1}{3}h$

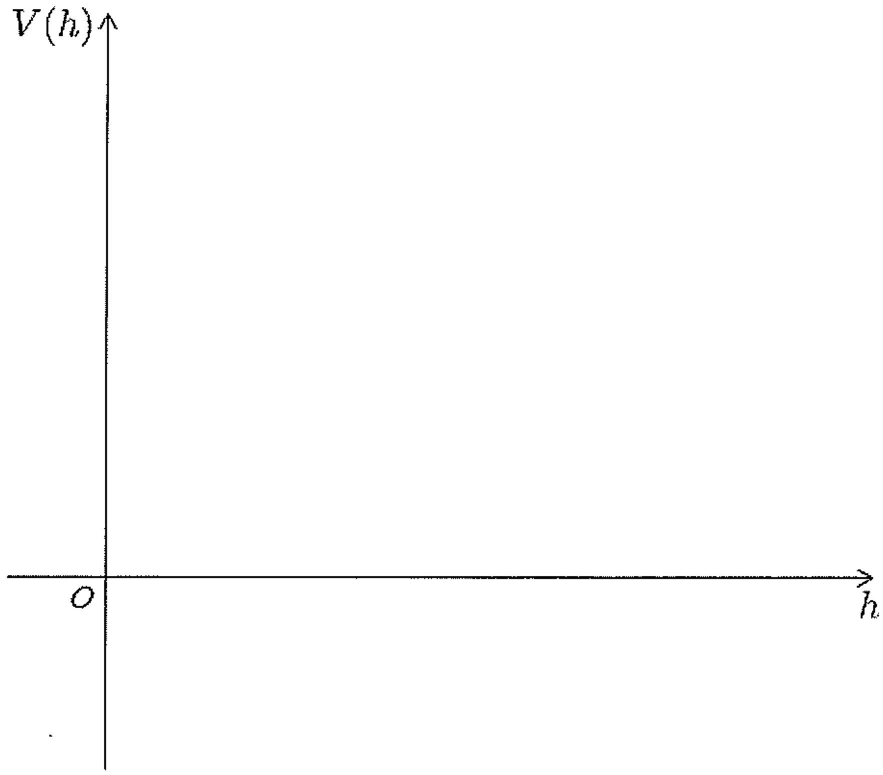
ii. Find a function for the volume V of the smaller cone in cm^3, in terms of h only

1 + 2 = 3 marks

b. Find the maximum volume of the smaller cone in cm^3 correct to 2 decimal places and the value of h that it occurs at.

3 marks

c. Graph the function $V(h)$ for the appropriate domain, labelling the turning point.



2 marks

Question 3

a. Given that $a > 0$, find the values of a for which $f(x) = 2\sin(ax)$ has only 5 stationary points over the domain $(0, 2\pi)$

2 marks

b. If $f(x) = ax^3 + 2ax^2 + 3ax + 4$ for $a \in R \backslash \{0\}$ and f has n stationary points, then what possible values can n have?

2 marks

c. The average value of the function $f(x) = 2\cos\left(\frac{1}{2}x\right) + 1$ over the interval $[0, b]$ is $\frac{5}{2}$. If $b > 0$, find the value of b correct to 2 decimal places.

2 marks

END OF TEST