Student Name: SOLUTIONS

Home Group: ___________

Teacher's name: (please circle): Ms Nation Ms O'Rielly

Mathematical Methods

Unit 2

Wednesday 8th November 2017

Part I

Total 57 marks

Topics covered:
- Combinatorics
- Circular Functions
- Rates of Change
- Differential Calculus
- Integral Calculus
- Exponential Functions and Logarithms

Complete working must be shown and simplified wherever possible in order to gain full marks.

Reading Time: 15 minutes

Writing Time: 60 minutes

Students are NOT permitted to use any calculators or reference books for this section.

No paper or electronic dictionaries may be used.

Useful formulae:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$${}^{n}C_{r} = \frac{n!}{(n-r)!\,r!} \qquad {}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Newton's Iterative formula for approximating roots of a polynomial:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Any question worth more than 1 mark <u>must have</u> the appropriate working shown to justify the extra marks.

1) A menu offers a choice of five entrees, four mains and three desserts.

   Find the number of meal choices possible

   a) if one of each must be chosen for a 3 course meal.

$$5 \times 4 \times 3 = 60$$

   b) if you have of choice of not having the dessert if you prefer.

3 course or 2 course

$$60 + 5 \times 4 = 80$$

(2 marks)

2) The digits 0, 1, 2, 3 and 4 are used to make a 3-digit number. No digit is repeated.

   a) How many different 3-digit numbers are possible, if 0 cannot be the first digit?

$$4 \times 4 \times 3 = 48$$

   b) If any of the 3-digit numbers in part a is equally likely to have been made, find the probability that number made is greater than or equal to 230.

Number ≥ 300 = $2 \times 4 \times 3 = 24$

+ No. $230 \leq x \leq 300$ = $1 \times 2 \times 3 = 6$ ①

Total $= 30$

$$Pr(\geq 230) = \frac{30}{48} = \frac{5}{8}. \quad ①$$

(1 + 2 = 3 marks)

3) Evaluate ${}^{100}C_2$

$$= \frac{100!}{98!\,2!} = \frac{100 \times 99}{2} = 2\,\overline{)19900} = \boxed{4950}$$

(1 mark)

4) In how many ways can four girls be selected for a table tennis team, if seven girls try out?

$${}^{7}C_4 = \frac{7!}{4!\,3!} = \frac{7 \times 6 \times 5}{3 \times 2} = 35 \quad ①$$

①

(2 mark)

5) Find the exact values of:

a) $\sin 120^0$ $= \sin(180-60^\circ)$ $= \sin 60^\circ$ $= \frac{\sqrt{3}}{2}$	b) $\tan\frac{4\pi}{3}$ $= \tan(\pi + \frac{\pi}{3})$ $= \tan\frac{\pi}{3}$ $= \sqrt{3}$
c) $\sin\left(-\frac{\pi}{6}\right)$ $= -\sin\frac{\pi}{6}$ $= -\frac{1}{2}$	d) $\cos\frac{9\pi}{4}$ $= \cos(\frac{9\pi}{4} - 2\pi)$ $= \cos\frac{\pi}{4}$ $= \frac{\sqrt{2}}{2}$

(4 marks)

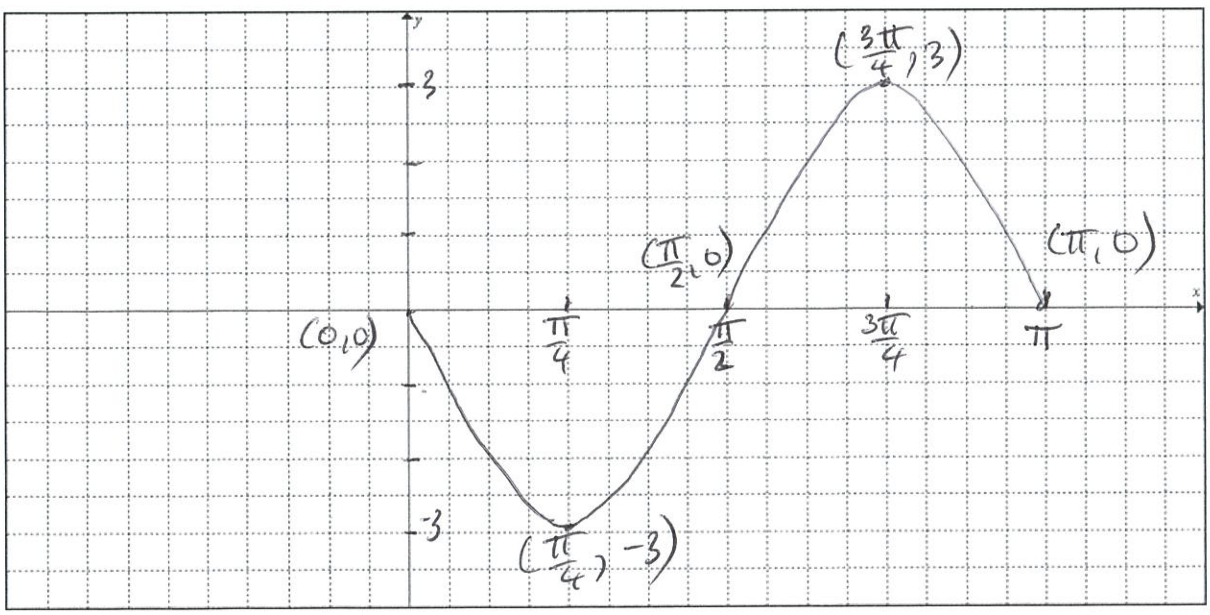
6) a) What is the period and the amplitude of the graph of

$y = -3\sin 2x$?

period $= \frac{2\pi}{2} = \pi$ amp : 3

*reflected

b) Sketch the graph, showing one complete cycle. Clearly label key points.



shape ①
key points labelled ①
1 cycle ①

(2 + 3 = 5 marks)

7) Solve the following equation, giving your answer(s) as exact values:

$$\sqrt{2}\sin x + 1 = 0, \quad \underline{0 \le x \le 4\pi}$$

$\sqrt{2}\sin x = -1$

$\sin x = \frac{-1}{\sqrt{2}}$

$= \frac{-\sqrt{2}}{2}$

reference angle (Q1) $\theta = \frac{\pi}{4}$ 1

in Q3 : $x = \pi + \theta$

$= \frac{5\pi}{4}$ 1

in Q4 : $x = 2\pi - \theta$

$= 2\pi - \frac{\pi}{4}$

$= \frac{7\pi}{4}$

$x = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{5\pi}{4} + 2\pi, \frac{7\pi}{4} + 2\pi$

$x = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$ 1

(3 marks)

8) If sin θ = 0.66, cos θ= 0.75 and tan θ= 0.87, write down the value of:

sin (2π - θ) $= -\sin\theta$ $= -0.66$	tan (π + θ) $= \tan\theta$ $= 0.87$
cos (-θ) $= \cos\theta$ $= 0.75$	cos (π - θ) $= -\cos\theta$ $= -0.75$

(4 marks)

9) If Evie drives at 60 km/h for 2 hours and 110 km/h for 3 hours, what is her average speed for the entire journey?

Total distance $= 60\times 2 + 110\times 3$

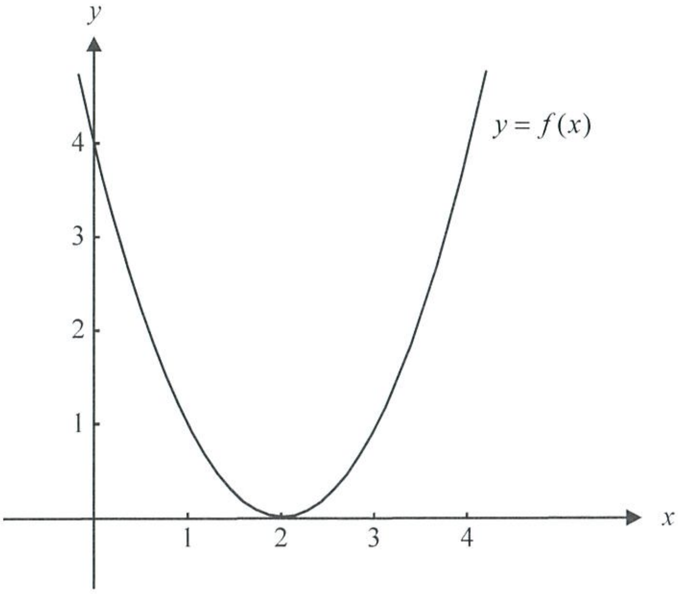
$= 120 + 330$

$= 450$ km

Av. speed $= \frac{450}{5} = 90$ km/hr

(1 mark)

10) The graph of the function $f : R \to R,\ f(x) = (x-2)^2$ is shown below.



a) Find the average rate of change of $y = f(x)$ with respect to x, between x = 1 and x = 4.

Using rule $f(1) = (1-2)^2 = 1$

$f(4) = (4-2)^2 = 4$

av. rate of change $= \dfrac{f(4) - f(1)}{4 - 1} = \dfrac{4-1}{4-1} = 1$

b) Find the instantaneous rate of change of $y = f(x)$ with respect to x at the point where

i) $x = 2$	ii) $x = 4$
$f'(x) = 0$ (tp of graph)	$f'(x) = 2x - 4$ 1 $f'(4) = 2(4) - 4$ $= 4$ 1

(1 + 1 + 1 = 3 marks)

11) If $f(x) = (x+2)(x+3)$, find $f'(3)$.

$f(x) = x^2 + 3x + 2x + 6$

$= x^2 + 5x + 6$ 1

$\Rightarrow f'(x) = 2x + 5$

$f'(3) = 2(3) + 5$

$= 6 + 5$

$= 11$ 1

(2 marks)

12) Find, using first principles, the derivative of

$$f(x) = 3x^2 + x - 2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x+h)^2 + (x+h) - 2 - (3x^2 + x - 2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + h^2 + \cancel{x} + h - \cancel{2} - \cancel{3x^2} - \cancel{x} + \cancel{2}}{h}$$

$$= \lim_{h \to 0} \frac{6xh + h^2 + h}{h} \quad (1)$$

$$= \lim_{h \to 0} \frac{\cancel{h}(6x + h + 1)}{\cancel{h}}$$

$$= \lim_{h \to 0} 6x + h + 1$$

$$= 6x + 1 \quad (1)$$

(1) correct, notation throughout.

(3 marks)

13) Find $\lim_{x \to 3} \frac{x^2 - x - 12}{x^2 - 16}$ $= \lim_{x \to 3} \frac{(x-4)(x+3)}{(x+4)(x-4)}$

$$= \lim_{x \to 3} \frac{x+3}{x+4} \quad (1)$$

$$= \frac{3+3}{3+4} \quad \boxed{= \frac{6}{7}} \quad (1)$$

(2 marks)

14) Evaluate:

a) $\int (4x^3 - x^2 + 9)dx$	b) $\int \frac{3x^4 + 5x^3}{2x} dx$
$= x^4 - \frac{1}{3}x^3 + 9x + c$	$= \int \frac{3x^4}{2x} dx + \int \frac{5x^3}{2x} dx$ (1) $= \int \frac{3}{2}x^3 dx + \int \frac{5}{2}x^2 dx$ $= \frac{3}{8}x^4 + \frac{5}{6}x^3 + c$ (1)

(1 + 2 = 3 marks)

15) A particle moves in a straight line with velocity of $v(t) = 6t^2 - 4t \ (m/s)$ at time t seconds (t ≥ 0). The particle has an initial position $x(t)$ of 3m left of the origin, O.

a) Find the equation of the position of the particle, $x(t)$

$x(t) = \int v(t)\,dt$

$= \int 6t^2 - 4t \, dt$

$= 2t^3 - 2t^2 + c$ ①

Initial $pos^n = -3$ m $(0, -3)$

$\Rightarrow -3 = 2(0)^3 - 2(0)^2 + c$

$c = -3$

$x(t) = 2t^3 - 2t^2 - 3$ ①

b) Find the acceleration of the particle at $t = 2$ seconds

$a(t) = \frac{dv}{dt}$

$= 12t - 4$ ①

at $t = 2$ $a(2) = 12(2) - 4$

$= 20\ ms^{-2}$ ①

(2 + 2 = 4 marks)

16) Simplify these expressions using appropriate index or logarithm laws:

a) $\frac{25^{x+3} \times 5^{6x}}{125^{2x-1}}$	b) $\frac{(2x^4y^{-3})^3}{2(x^{-3}y^2)^2}$	c) $2\log_{10} 5 + \log_{10} 4$
$= \frac{(5^2)^{x+3} \times 5^{6x}}{(5^3)^{2x-1}}$ ①	$= \frac{2^3 x^{12} y^{-9}}{2x^{-6}y^4}$ ①	$= \log_{10} 5^2 + \log_{10} 4$
$= \frac{5^{2x+6} \times 5^{6x}}{5^{6x-3}}$	$= 4x^{12--6}y^{-9-4}$	$= \log_{10} 25 \times 4$ ①
$= 5^{2x+6+6x-(6x-3)}$	$= 4x^{18}y^{-13}$	$= \log_{10} 100$
$= 5^{2x+9}$ ①	$= \frac{4x^{18}}{y^{13}}$ ①	$= \log_{10} 10^2$
		$= 2\log_{10} 10$
		$= 2$ ①

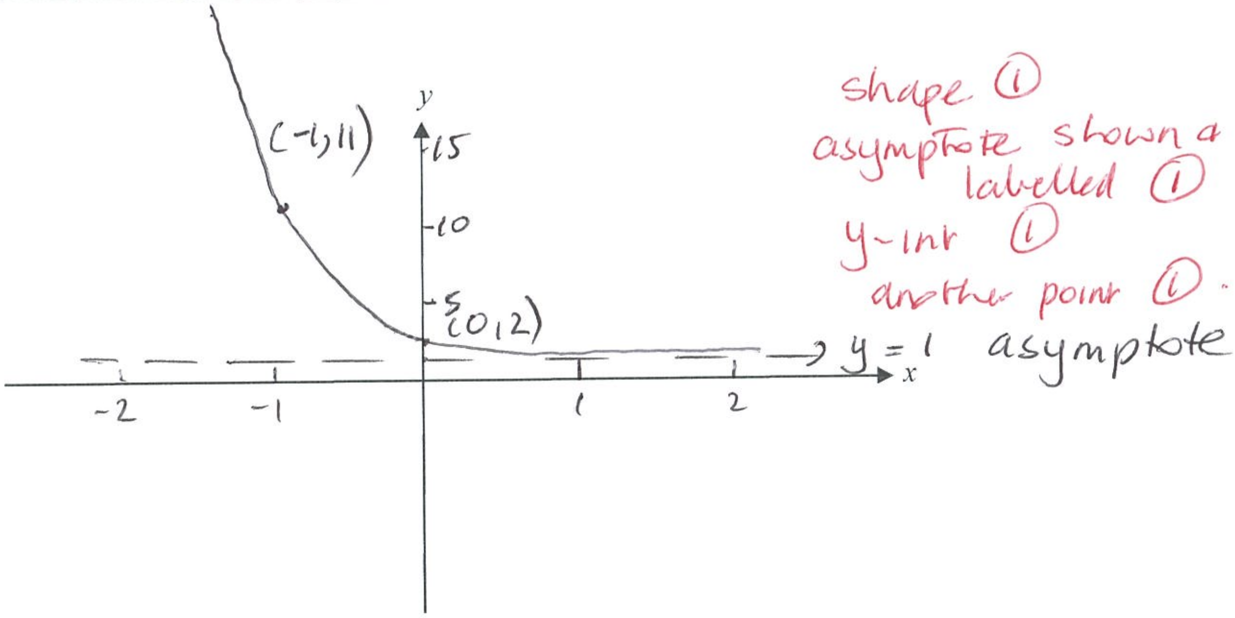
(3 x 2 = 6 marks)

17) Solve the following equations for x:

a) $9^{2x} = 27^{2x-4}$	b) $\log_2(3x - 5) = 4$
$(3^2)^{2x} = (3^3)^{2x-4}$ $3^{4x} = 3^{6x-12}$ (1) $4x = 6x - 12$ $-2x = -12$ $x = 6$ (1)	$2^4 = 3x - 5$ (1) $16 = 3x - 5$ $21 = 3x$ $7 = x$ (1)

(2 x 2 = 4 marks)

18) a) Sketch the graph of the function $y = 10^{-x} + 1,\ x \in R$ on the set of axes below. Indicate clearly on the graph any intercepts or asymptotes.



shape (1)
asymptote shown & labelled (1)
y-int (1)
another point (1).

b) What is the range of this function?

$(1, \infty)$

(4 + 1 = 5 marks)

END OF EXAM