St Leonard's College

Student Name........ SOLUTIONS / MARK SCHEME

Teacher Name: ______________________________

MATHEMATICAL METHODS UNITS 3 & 4

Term 3 Week 1 TRIAL EXAMINATION 2

2016

Reading Time: 10 minutes
Writing time: 80 minutes

Instructions to students

This exam consists of Section 1 and Section 2.
Section 1 consists of 15 multiple-choice questions, which should be answered on the detachable answer sheet
Section 2 consists of 4 extended-answer questions that should be answered in the spaces provided.
Section 1 is worth 15 marks.
Section 2 is worth 37 marks.
There is a total of 52 marks available.
All questions in Section 1 and Section 2 should be answered.
Unless otherwise stated, diagrams in this exam are not drawn to scale.
Where more than one mark is allocated to a question, appropriate working must be shown.
Exact answers are required unless otherwise specified.
Students may bring one bound reference into the exam.
Students may bring an approved CAS calculator and if desired one scientific calculator into the exam.
A formula sheet will be provided.

SECTION 1

Question 1

Consider the function f with rule $f(x)=1+\sqrt{x+5}$.
The maximal domain of f is

A. $R\setminus\{5\}$
B. R
C. $(1,\infty)$
D. $(-5,\infty)$
E. $[-5,\infty)$

Question 2

The function $f:[a,b]\to R, f(x)=4-2x$ has range $[-2,8]$.
The values of a and b are

A. $a=-12,\quad b=8$
B. $a=-3,\quad b=2$
C. $a=-3,\quad b=3$
D. $a=-2,\quad b=3$
E. $a=0,\quad b=20$

Question 3

The graphs of $y=\frac{a}{x}$ and $y=x+2$ intersect at two distinct points for

A. $a\le -1$
B. $a<-1$
C. $a=-1$
D. $a>-1$
E. $a\ge -1$

Question 4

The transformation $T: R^2 \to R^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0\cdot 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The image of the curve $y = e^x$ under T has the equation

A. $y = e^{2(x-1)} + 1$

B. $y = e^{2(x+1)} + 1$

C. $y = e^{2x-1} + 1$

D. $y = e^{\frac{x}{2}+1} - 1$

E. $y = 2\left(e^{x-1} + 1\right)$

Question 5

The average rate of change of the function $g(x) = \sqrt{\tan(x)}$ between $x = 0$ and $x = \frac{\pi}{4}$ is

A. $\frac{\pi}{4}$

B. $\frac{4}{\pi}$

C. $\frac{2\sqrt{2}}{\pi}$

D. $\frac{4\sqrt{3}}{\pi}$

E. $\frac{4\sqrt{3}}{3\pi}$

Question 6

For the function $f: (-1, \infty) \to R, f(x) = \log_e(x + 1)$ it is false to say that

A. $f(0) = 1$

B. $f'(x) > 0$ for $x < 0$

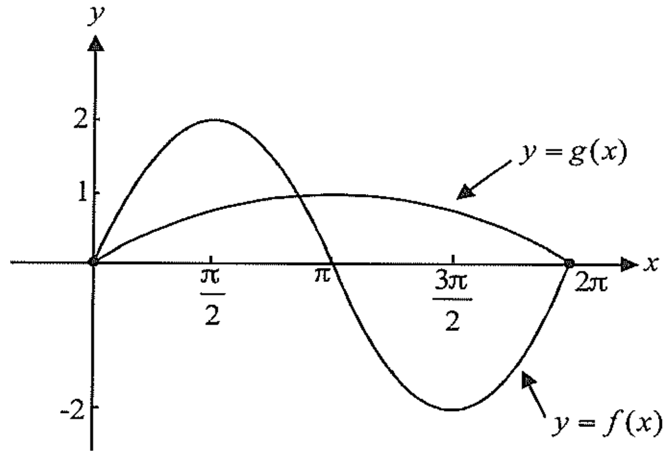
C. the graph of $y = f(x)$ is strictly increasing for $x \in (-1, \infty)$

D. $f'(x) \neq 0$ for $x \in (-1, \infty)$

E. the function f has an inverse function

Question 7

The graph of $y = f(x)$ and part of the graph of $y = g(x)$ are shown below.



The transformations that the graph of $y = f(x)$ undergoes to become the graph of $y = g(x)$ are

A. a dilation by a scale factor of 2 from the y-axis followed by a translation by one unit down.

B. a dilation by a scale factor of $\frac{1}{2}$ from the y-axis followed by a translation by one unit down.

C. a dilation by a scale factor of $\frac{1}{2}$ from the x and y-axis.

D. a dilation by a scale factor of 2 from the x-axis followed by a dilation by a scale factor of $\frac{1}{2}$ from the y-axis..

E. a dilation by a scale factor of $\frac{1}{2}$ from the x-axis followed by a dilation by a scale factor of 2 from the y-axis

Question 8

The continuous function $f: R \to R$ has the following properties:

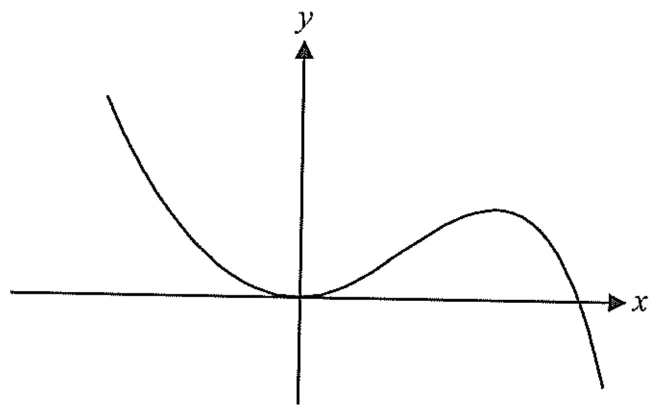
$$f'(x) = 0 \text{ at } x = 1 \text{ and at } x = 3$$
$$f'(x) < 0 \text{ for } x < 1$$
$$f'(x) > 0 \text{ for } 1 < x < 3 \text{ and for } x > 3$$

It is true to say that the graph of f has a

A. maximum turning point at $x = 1$
B. minimum turning point at $x = 3$
C. stationary point of inflection at $x = 1$
D. stationary point of inflection at $x = 3$
E. stationary point of inflection at $x = 1$ and a minimum turning point at $x = 3$

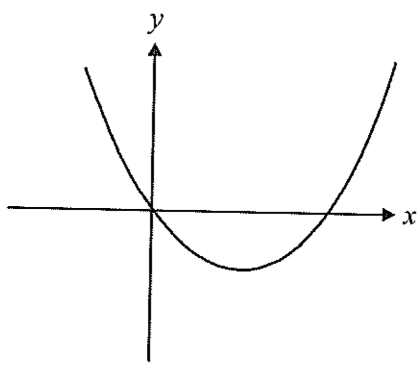
Question 9

The graph of $y = g(x)$ is shown below.

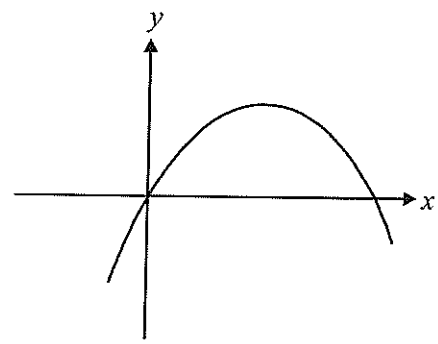


The graph of $y = g'(x)$ could be represented by

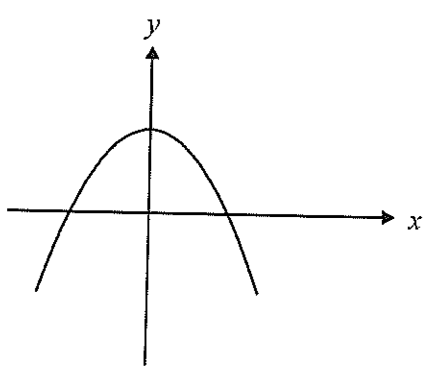
A.



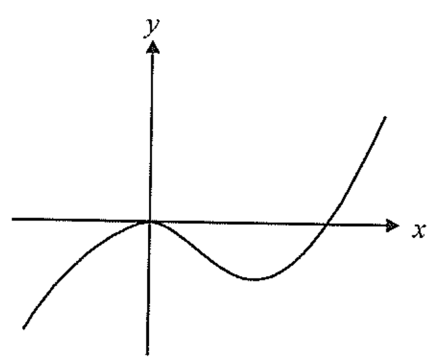
B.



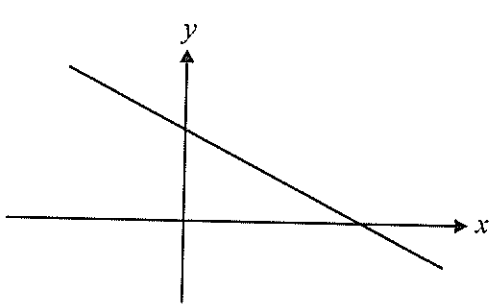
C.



D.



E.



Question 10

If $f(x) = h(x(x+2))$ then $f'(x)$ is equal to

A. $h'(2(x+1))$
B. $h'(x(x+2))$
C. $2(x+1)h'(x(x+2))$
D. $2(x+1)h'(2(x+1))$
E. $2(x+1)h(x(x+2)) + 2(x+1)h'(x(x+2))$

Question 11

The simultaneous linear equations

$$mx + 8y = 6$$
$$2x + my = m$$

have no solution for

A. $m = -4$
B. $m = 4$
C. $m = \pm 4$
D. $m \in R \setminus [4]$
E. $m \in R \setminus [-4,4]$

Question 12

$$f(x)=\begin{cases}2\cos(2x) & \text{if} \quad 0<x<\dfrac{\pi}{4}\\ 0 & \text{elsewhere}\end{cases}$$

The value of a such that $\int_0^a f(x)\ dx = 0.4$ is closest to

A. -0.2
B. 0.1
C. 0.2
D. 0.4
E. 11.8

Question 13

If $\sin^2(x)-\frac{1}{2}\sin(x)=0$ and $x\in\left[0,\frac{\pi}{2}\right]$ then

A. $x=0$ only

B. $x=\frac{\pi}{6}$ only

C. $x=0$ or $x=\frac{\pi}{6}$

D. $x=0,\ x=\frac{\pi}{6}$ or $x=\frac{\pi}{2}$

E. $x=0,\ x=\frac{\pi}{6}$ or $x=\frac{5\pi}{6}$

Question 14

If $\int_a^b g(x)dx=4$ then $\int_b^a (3g(x)+5)dx$ is equal to

A. -7
B. 17
C. $5(a-b)-42$
D. $5(a-b)-12$
E. $3(a-b)-12$

Question 15

The average value of the function $y = \frac{1}{x+2} - 3$ over the interval $[1, 3]$ is

A ~~F.~~ $-\frac{1}{2}\log_e\left(\frac{3}{2}\right)+3$

B ~~G.~~ $-\frac{1}{2}\log_e(3)+5$

C ~~H.~~ $\frac{1}{2}\log_e\left(\frac{5}{3}\right)-3$

D ~~I.~~ $\frac{1}{2}\log_e(3)-1$

E ~~J.~~ $\frac{1}{2}\log_e(5)+1$

$$\frac{1}{2}\int_1^3 \frac{1}{x+2} - 3\,dx$$

$$= \frac{1}{2}\left[\log_e(x+2) - 3x\right]_1^3$$

$$= \frac{1}{2}\left[(\log_e 5 - 9) - (\log_e 3 - 3)\right]$$

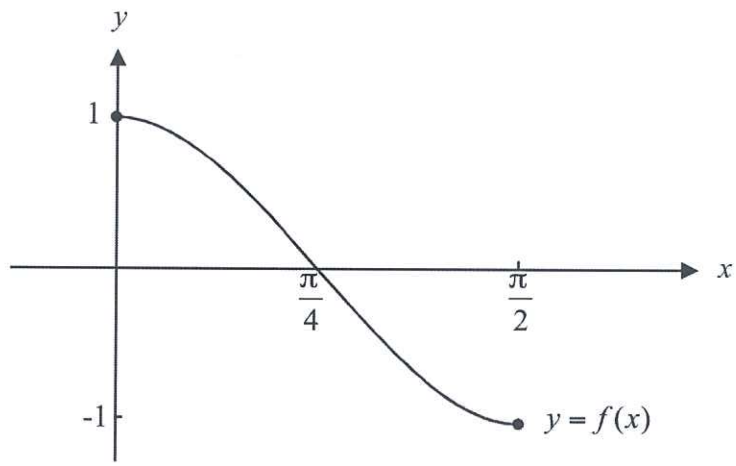
$$= \frac{1}{2}\left(\log_e \frac{5}{3} - 6\right)$$

SECTION 2

Answer all questions in this section.

Question 3

The graph of the function $f:\left[0,\frac{\pi}{2}\right] \to R, f(x) = \cos(2x)$ is shown below.



a. Find $f'(x)$.

$f'(x) = -2\sin(2x)$

(1) ans.

1 mark

b. Find the coordinates of the point(s) where the gradient of the tangent to the graph of $y = f(x)$ is -1. ~~Express the coordinates as exact values.~~

$-2\sin(2x) = -1$

$x = \frac{\pi}{12}, \frac{5\pi}{12}$

Co-ords. $\left(\frac{\pi}{12}, \frac{\sqrt{3}}{2}\right)$ $\left(\frac{5\pi}{12}, -\frac{\sqrt{3}}{2}\right)$

(1) ans (1) ans.

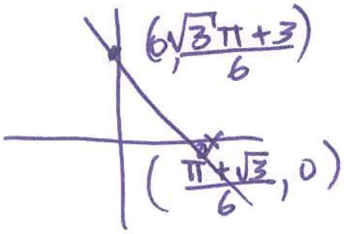
2 marks

A tangent to the graph of $y = f(x)$ has an x-intercept of $\frac{\pi + \sqrt{3}}{6}$ and a y-intercept of $\frac{\sqrt{3}\pi + 3}{6}$.

c. i. Show that the gradient of this tangent is $-\sqrt{3}$.

(1) correct formulatⁿ grad.

(1) rationalising denom.



$$m_{tangent} = \frac{\frac{\sqrt{3}\pi+3}{6} - 0}{0 - \left(\frac{\pi+\sqrt{3}}{6}\right)}$$

$$= \frac{\sqrt{3}\pi + 3}{6} \times \frac{-6}{\pi+\sqrt{3}}$$

$$= -\frac{(\sqrt{3}\pi + 3)}{\pi + \sqrt{3}} = -\frac{(\sqrt{3}\pi + 3)}{\pi+\sqrt{3}} \times \frac{\pi-\sqrt{3}}{\pi-\sqrt{3}}$$

$$= -\frac{(\sqrt{3}\pi^2 - 3\sqrt{3})}{\pi^2 - 3}$$

$$= -\frac{\sqrt{3}(\pi^2-3)}{\pi^2-3}$$

$$= -\sqrt{3}.$$

ii. Using your answer to part **a.** find the values of x where the function f has a gradient of $-\sqrt{3}$.

Solving $f'(x) = -\sqrt{3}$ (1)

gives $x = \frac{\pi}{6}, \frac{\pi}{3}$ (1)

iii. Hence find the coordinates of the point of tangency.

✓✓ Tangent at $x = \frac{\pi}{6}$ $y = \frac{\pi\sqrt{3}+3}{6} - \sqrt{3}x$

$x = \frac{\pi}{3}$ $y = \frac{2\pi\sqrt{3}+3}{6} - \sqrt{3}x$

Pt. of tangency $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ (1) ans.

2 1 5

~~1~~ + 2 + ~~2~~ = ~~5~~ marks

Let $g: R \to R, g(x) = \cos(2x)$.

d. Find the general solution for x of the equation $g(x) = 0 \cdot 5$.

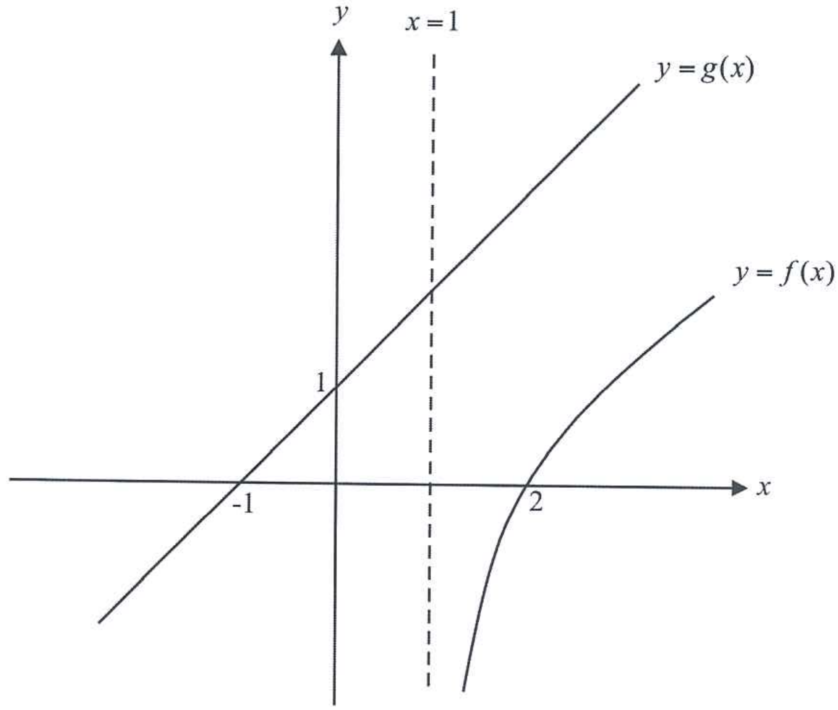
$x = \frac{(6n \pm 1)\pi}{6}$ $n \in Z$

(1) (1)

2 marks

Question 1

The graphs of the functions $f:(1,\infty)\to R,\ f(x)=2\log_e(x-1)$ and $g:R\to R,\ g(x)=x+1$ are shown below.



a. Find the rule and the domain of the inverse function f^{-1}.

$f(x)=y=2\log_e(x-1)$

$f^{-1}(x)=1+e^{\frac{x}{2}}$

dom f^{-1} = ran f = R (1) rule

(1) dom

2 marks

Consider the function h where $h(x)=g(x)-f(x)$.

b. Write down the domain of h.

dom h = dom g $\cap$ dom f (1) ans.

$=(1,\infty)$

1 mark

c. Find the value of x for which $h(x)$ is a minimum. (You are not required to justify that the stationary point is a minimum).

$h(x) = x + 1 - 2\log_e(x-1)$

$h'(x) = 1 - 2 \times \frac{1}{x-1}$ (1) method

$h'(x) = 0$ for max/min

$x = 3$ (1) ans.

2 marks

d. Hence find the minimum vertical distance between the graphs of $y = g(x)$ and $y = f(x)$.

$h(3) = 4 - 2\log_e(2)$ units

(1) ans

1 mark

Total 6 marks

Question 2

Consider the function $f: R \rightarrow R, f(x) = e^{ax}(x^2 - bx)$ where a and b are constants.

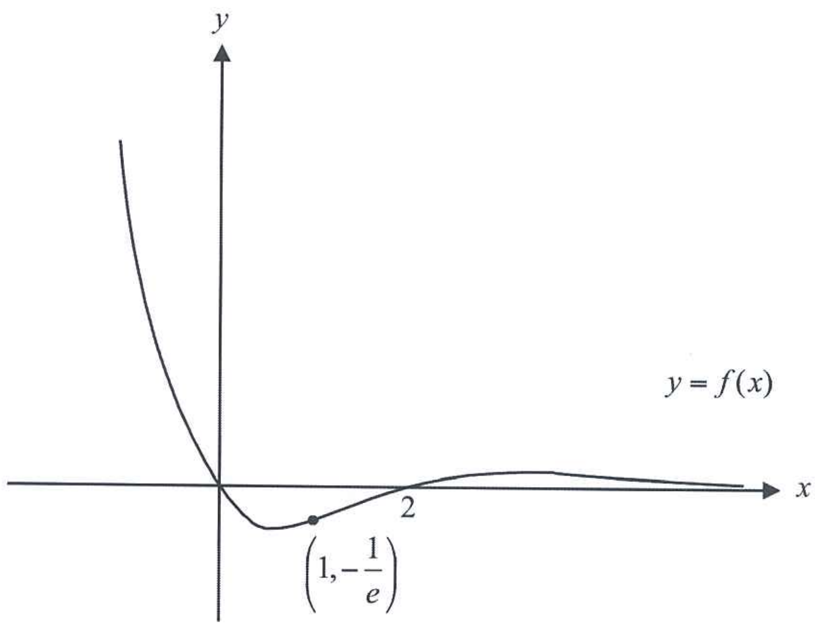
a. Find the value(s) of x for which $f(x) = 0$.

$x = 0, b$

(1) ans.

1 mark

The graph of $y = f(x)$ is shown below.



b. Explain why $a = -1$ and $b = 2$.

From graph x-ints at 0, 2 $\therefore$ From part (a) $b = 2$

Pt. $(1, -\frac{1}{e})$

$\therefore -\frac{1}{e} = e^a(1-2)$ (1) a

$\therefore -\frac{1}{e} = -e^a$ (1) b

$\therefore a = -1$ shown

2 marks

c. Find the range of f. Express values correct to 2 decimal places where appropriate.

$\text{ran} f = [-0.46, \infty)$

(1) notation
(1) values

2 marks

d. Find the maximum value of f for $x \in [-1,3]$.

Max value is $3e$.

(1) ans.

1 mark

e. Show that at the point on the graph of $y = f(x)$ where $x = 4$, the gradient is $-2e^{-4}$.

$f'(x) = e^{-x}(-x^2 + 4x - 2)$

$f'(4) = e^{-4}(-16 + 16 - 2)$

$= -2e^{-4}$ shown

(1)

1 mark

f. Find the x-coordinates of the other points on the graph of $y = f(x)$ where the gradient is $-2e^{-4}$. Express your answers correct to 2 decimal places.

Solving $f'(x) = -2e^{-4}$

gives $x = 4, 0.56, 5.87$

$\therefore$ Other pts. have x values 0.56 & 5.87 (1) ans.

1 mark

g. Find the y-intercept of the tangent to the graph of $y = f(x)$ at the point where $x = 4$.

Eqtn. of tangent is

$y - 8e^{-4} = -2e^{-4}(x-4)$

(or equivalent)

When $x = 0$, $y = 16e^{-4}$

$\therefore$ y-int is $16e^{-4}$ or $(0, 16e^{-4})$

2 marks

Total 10 marks

(1) eqtn.

(1) ans.

Question 4

The cross-sectional view of an earth moving machine at a mine is shown in Diagram 1 below.

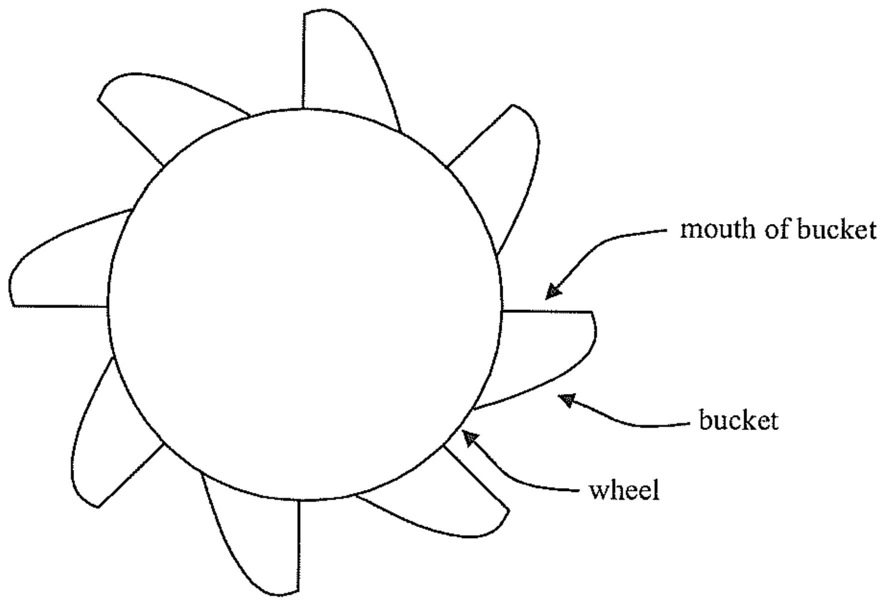


Diagram 1

Eight identically sized buckets; in which the earth is collected, are attached evenly around a wheel that turns anticlockwise. The mouth of the bucket is a straight line of length 1 metre.
Diagram 2 below shows the wheel, with centre located at $O(0,0)$ on the Cartesian plane, and two of the buckets. The mouth of both of these buckets is running vertically; that is, along the y-axis.

$f(a) = w(a)$

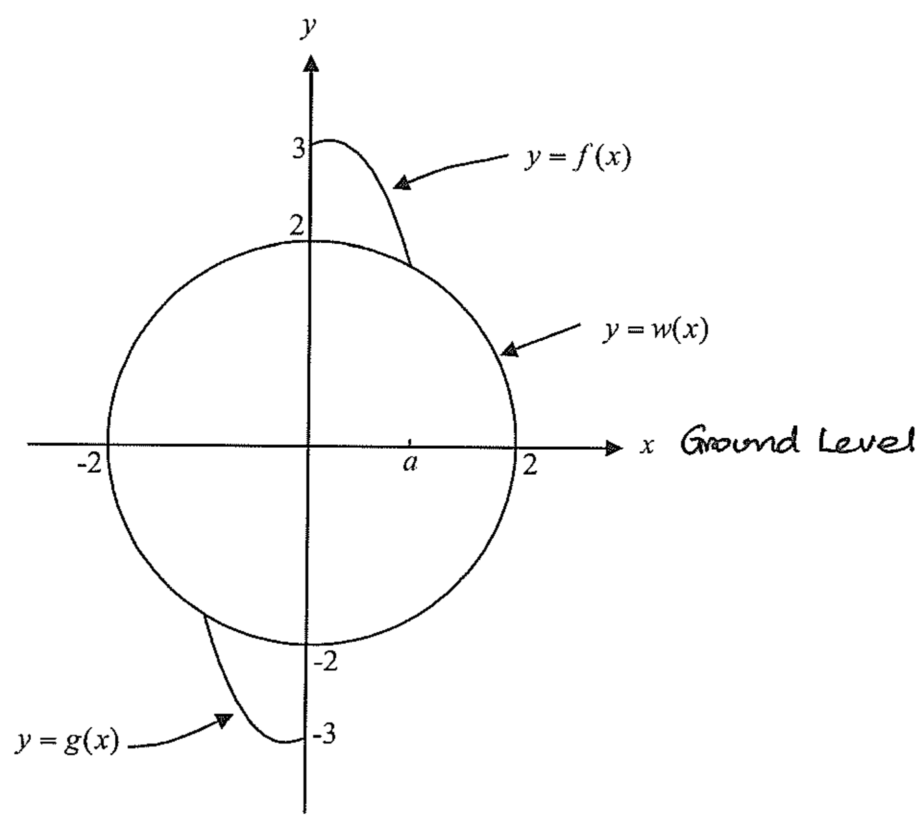


Diagram 2

The right hand edge of the top bucket can be defined by the function

$$f:[0,a]\to R, f(x)=-2x^2+(\sqrt{3}-1)x+3.$$

The edge of the top half of the wheel can be defined by the function

$$w:[-2,2]\to R, w(x)=\sqrt{4-x^2}.$$

a. Find the value of a.

Pt. of int^n of $f(x)$ & $w(x)$ is $x=a$.
Solving $f(x)=w(x)$ gives $x=1$
$\therefore a=1$

(1)

1 mark

The left hand edge of the bottom bucket can be defined by the function g.

b. **i.** Write down a sequence of two transformations (excluding rotations) that map the graph of $y=f(x)$ on to the graph of $y=g(x)$.

Ref^n in x axis followed by
Ref^n in y axis
or vice versa

(1) for each correct transf^n.

ii. Hence find the rule for g.

$g(x)=-f(-x)$ (1) method
$\therefore g(x)=2x^2+(\sqrt{3}-1)x-3$ (1) ans.

iii. Write down the domain of g.

dom $g=[-1,0]$ (1) ans

2 + 2 + 1 = 5 marks

~~The centre of the wheel is 5m above the ground and the machine has its buckets in the~~ **position** ~~**indicated in Diagram**~~ **2**. Also, the unit of measurement in Diagram 2 is the metre.

c. **i.** Find the value of x for which the height of the ~~machine~~ bucket defined by $f(x)$ above the ground is a maximum.

$f'(x) = 0$ (1) deriv = 0

Solving gives $x = \frac{\sqrt{3}-1}{4}$ (1) ans.

ii. Hence find the maximum height of the machine above the ground. Express your answer in metres correct to 2 decimal places.

Max. height $= f\left(\frac{\sqrt{3}-1}{4}\right) + 5$

$\simeq 3 \cdot 07 \cancel{+5}$ (1) ans.

$= \cancel{8 \cdot 07 m}$

2+1 = 3 marks

d. Hence write down the range of the function f.

ran $f = [\sqrt{3}, 3 \cdot 07]$

(1) notation

(1) values

2 marks

The functions f and w are shown in Diagram 3 below.
The cross-sectional area of the top bucket is bounded by the y-axis and the graphs of $y = f(x)$ and $y = w(x)$.

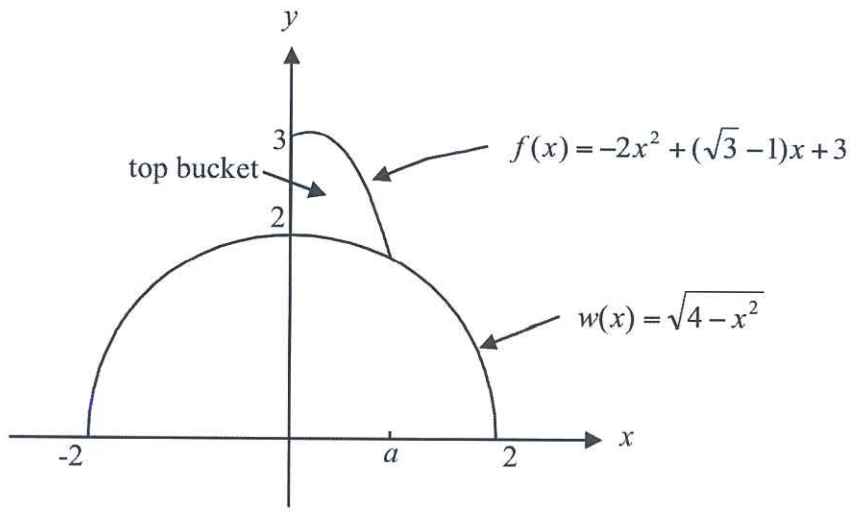


Diagram 3

e. **i.** Write down a definite integral that gives the cross-sectional area of the top bucket shown.

$\int_0^1 (f(x) - w(x))\, dx$

or $\int_0^1 \left(-2x^2 + (\sqrt{3}-1)x + 3 - \sqrt{4-x^2}\right) dx$

(1) terminals
(1) integrand

ii. Hence find the cross-sectional area of the top bucket.

Area $= \int_0^1 (f(x) - w(x))\, dx$

$= \left(\frac{11}{6} - \frac{\pi}{3}\right)$ sq. units (1) ans.

2+1=3 marks
Total 14 marks

Multiple Choice Answer Sheet

NAME : ______________________________

Question No.					
1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E