St Leonard's College

Student Name....SOLUTIONS / MARK SCHEME

MATHEMATICAL METHODS UNITS 3 & 4

TERM 3 TRIAL EXAMINATION 1

2016

Reading Time: 5 minutes
Writing time: 1 hour

Instructions to students

All questions should be answered in the spaces provided. There is a total of 30 marks available. The marks allocated to each of the questions are indicated throughout. Students may **not** bring any calculators or notes into the exam. Where an exact answer is required a decimal approximation will not be accepted. Where more than one mark is allocated to a question, appropriate working must be shown. Diagrams in this trial exam are not drawn to scale.

Question 1

Let $f(x)=\frac{1}{x}+2x$ and $g(x)=x-1$.

Write down the rule for $f(g(x))$.

$f(g(x)) = \frac{1}{x-1} + 2(x-1)$

(1) ans

1 mark

Question 2

Solve the following equations for x.

a. $2e^{(x+1)}=6$.

$e^{x+1} = 3$

$\therefore\ x+1 = \log_e 3$

$\therefore\ x = \log_e 3 - 1$ (1) ans

1 mark

b. $2\log_5(x)-\log_5(3x)=1,\ x>0$

$\log_5 x^2 - \log_5 3x = 1$

$\therefore\ \log_5 \frac{x^2}{3x} = 1$ (1) correct use of any log law

$\therefore\ \frac{x}{3} = 5$

(1) ans

$\therefore\ x = 15$

2 marks

Question 3

a. Let $y = \frac{x^3 - 1}{e^{2x}}$. $\leftarrow u$ $\leftarrow v$

Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{e^{2x}(3x^2) - (x^3-1)2e^{2x}}{(e^{2x})^2}$$

No need to simplify

(1)

1 mark

b. If $h(x) = e^{\tan(x)}$ then evaluate $h'\left(\frac{\pi}{3}\right)$.

$$h'(x) = \frac{1}{\cos^2 x} \times e^{\tan(x)}$$

$$\therefore h'\left(\frac{\pi}{3}\right) = \frac{1}{\left(\cos\frac{\pi}{3}\right)^2} \times e^{\tan\left(\frac{\pi}{3}\right)}$$

$$= \frac{1}{\left(\frac{1}{2}\right)^2} \times e^{\sqrt{3}}$$

$$= 4e^{\sqrt{3}}$$

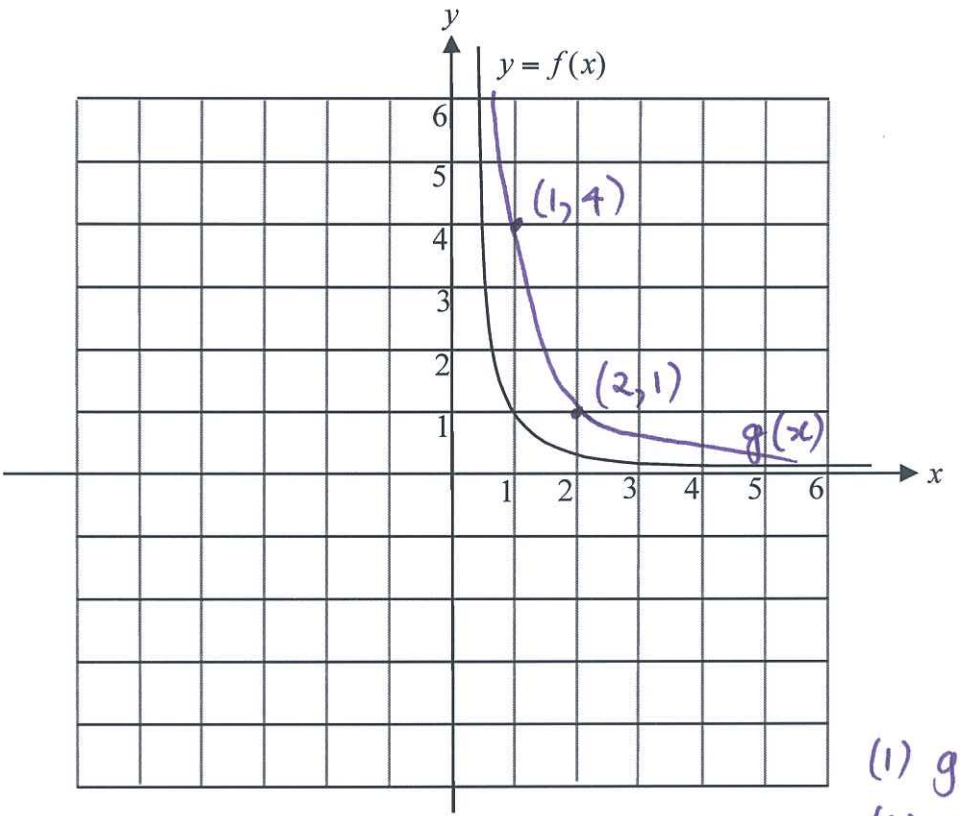
(1) $h'(x)$

(1) ans.

2 marks

Question 4

The graph of $y = f(x)$ where $f : R^+ \to R, f(x) = \frac{1}{x^2}$ is shown below.



(1) graph + 2 pts
(1) rule g(x)

a. This graph of $y = f(x)$ is dilated by a factor of 2 units from the y-axis to become the graph of the function g.

i. On the same set of axes as the graph of $y = f(x)$ above, sketch the graph of $y = g(x)$ labelling clearly two points that lie on the graph.

ii. Write down the rule for g.

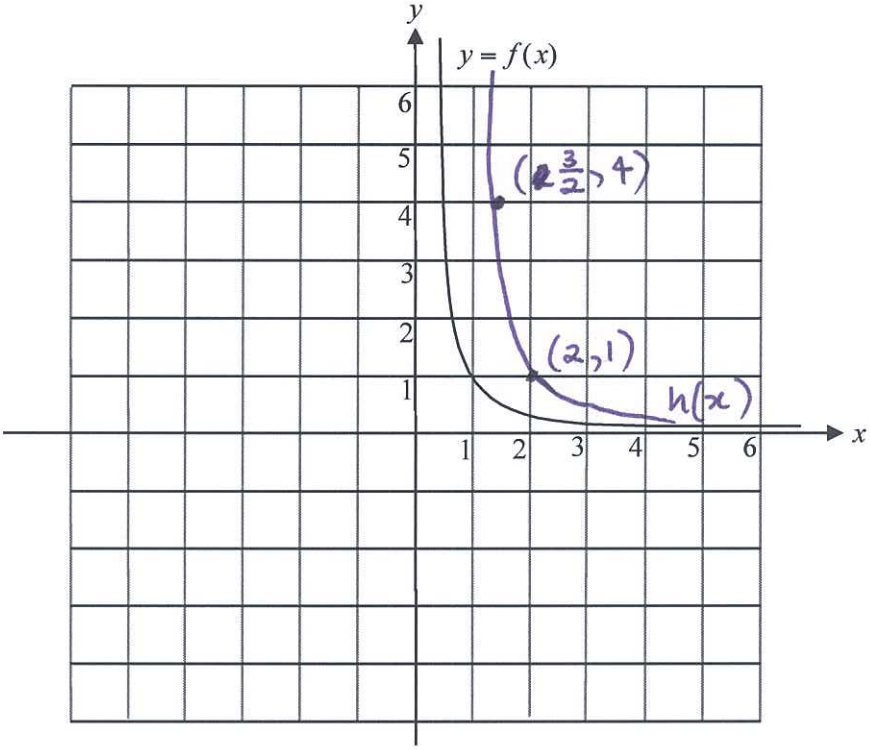
$g(x) = \frac{1}{\left(\frac{x}{2}\right)^2} = \frac{4}{x^2}$

1 + 1 = 2 marks

Question 4 continues on the next page.

Question 4 (continued)

The graph of $y = f(x)$ where $f : R^+ \to R, f(x) = \frac{1}{x^2}$ is shown again below.



b. The graph of $y = f(x)$ is translated one unit to the right to become the graph of the function h.

i. On the same set of axes as the graph of $y = f(x)$ above, sketch the graph of $y = h(x)$ labelling clearly two points that lie on the graph. (1) graph + 2 pts

ii. Write down the equation of the vertical asymptote of the graph of $y = h(x)$.

$x = 1$

iii. Write down the equation of the horizontal asymptote of the graph of $y = h(x)$.

$y = 0$

iv. Write down the rule for h.

$h(x) = \frac{1}{(x-1)^2}$

$1 + 1 + 1 + 1 = 4$ marks

Dom $x \in [0,20]$
$\therefore \frac{\pi x}{8} \in [0, \frac{5\pi}{2}]$

Question 5

Consider the function $g: R \to R, g(x) = \cos\left(\frac{\pi x}{8}\right)$.

a. Find the solution(s) to the equation $\cos\left(\frac{\pi x}{8}\right) - \frac{1}{\sqrt{2}} = 0$ for $x \in [0,20]$.

$\cos\left(\frac{\pi x}{8}\right) = \frac{1}{\sqrt{2}}$

BA $= \frac{\pi}{4}$

$\frac{\pi x}{8} = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}$

$\therefore \pi x = 2\pi, 14\pi, 18\pi$

$\therefore x = 2, 14, 18$

(1) $\cos(\ ) = \frac{1}{\sqrt{2}}$ + BA

(1) ans.

2 marks

Consider the function $h: R \to R, h(x) = -\cos\left(\frac{\pi}{8}(x+4)\right) - 1$.

b. Find the value(s) of x for which $h(x)$ is a minimum for $x \in [0,20]$.

Minimum value $h(x) = -2$

$\therefore -\cos\left(\frac{\pi}{8}(x+4)\right) - 1 = -2$

$\therefore \cos\left(\frac{\pi}{8}(x+4)\right) = 1$

$\therefore \frac{\pi}{8}(x+4) = 0, 2\pi, 4\pi, \ldots$

$\therefore x + 4 = 0, 16, 32, \ldots$

$\therefore x = -4, 12, 28, \ldots$

2 marks

For $x \in [0, 20]$ min^m occurs at $x = 12$.

(1) Understanding shown of what 'minimum' means and an attempt to find x

(1) ans.

Question 6

Find the equation of the tangent to the graph of $y = \cos(2x)$ at the point where $x = \frac{\pi}{6}$.

$m = \frac{dy}{dx}$ at $\frac{\pi}{6}$

$\frac{dy}{dx} = -2\cancel{\cos}\sin\left(\frac{\pi}{3}\right) = -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3}$

(x_1, y_1) $\quad y_1 = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

Tangent: $\quad y - \frac{1}{2} = -\sqrt{3}\left(x - \frac{\pi}{6}\right)$

(1) m + tangent

(1) eqtn.

2 marks

Question 7

a. Find the derivative of $\log_e(\sin(x))$.

deriv $= \frac{\cos x}{\sin x}$

(1) ans

1 mark

b. Hence evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{3\cos(x)}{\sin(x)}\,dx$.

$\log_e(\sin(x)) \xrightarrow{\text{diff}} \frac{\cos x}{\sin x}$

$\therefore \int \frac{\cos x}{\sin x}\,dx = \log_e(\sin(x))$

Hence $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{3\cos(x)}{\sin(x)}\,dx = \left[3\log_e(\sin(x))\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= 3\log_e\left(\sin\left(\frac{\pi}{2}\right)\right) - 3\log_e\left(\sin\frac{\pi}{6}\right)$

(1) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\frac{\cos(x)}{\sin(x)}\,dx = \left[3\log_e(\sin(x))\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

2 marks

(1) ans.

$= -3\log_e\left(\frac{1}{2}\right)$

OR $\quad 3\log_e 2$

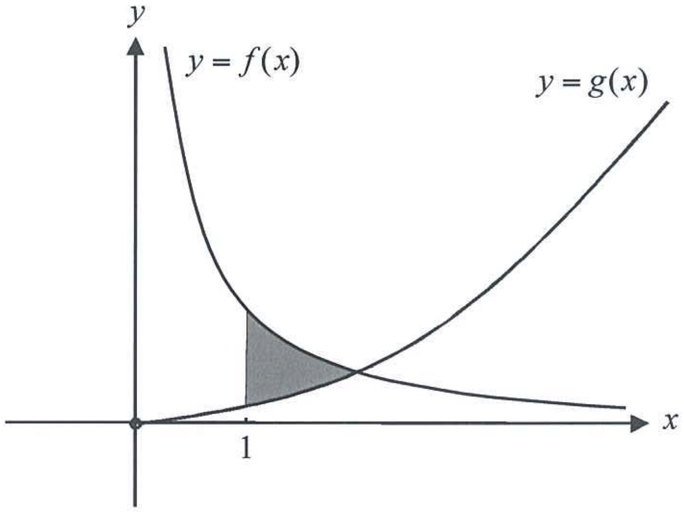
Question 8

The graphs of the functions

$$f : R^+ \to R,\ f(x) = \frac{1}{x}$$

$$\text{and} \quad g : R^+ \to R,\ g(x) = \frac{x^2}{8}$$

are shown below.



The region enclosed by the graphs of $y = f(x)$, $y = g(x)$ and the line $x = 1$ is shaded in the diagram. Find the area of this shaded region.

Pt. of Int^n $\quad \frac{1}{x} = \frac{x^2}{8}$

$\therefore\ x^3 = 8$

$\therefore\ x = 2$

Area $= \int_1^2 (f(x) - g(x))\,dx = \int_1^2 \frac{1}{x} - \frac{x^2}{8}\,dx$

$= \left[\log_e x - \frac{x^3}{24}\right]_1^2$

$= \left(\log_e 2 - \frac{8}{24}\right) - \left(\log_e 1 - \frac{1}{24}\right)$

$= \log_e 2 - \frac{7}{24}$ sq. units.

(1) x value pt. of int^n

(1) Correct integral

(1) Correct integration

(1) ans.

4 marks

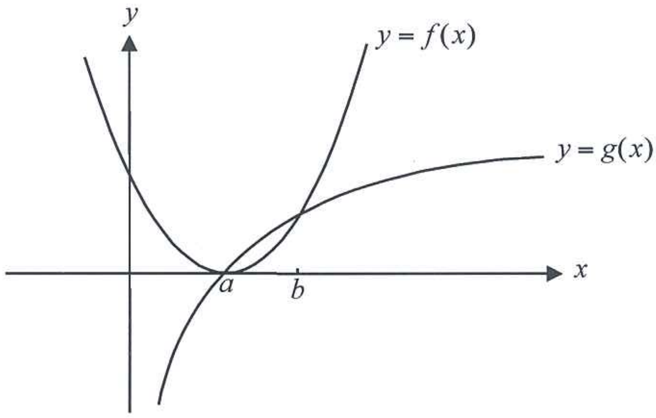
Question 9

The graphs of the functions

$$f: R \to R,\ f(x) = (x-1)^2$$

$$\text{and } g: R^+ \to R,\ g(x) = \log_e(x)$$

are shown below.



The graphs intersect at the points where $x = a$ and $x = b$, $a < b$.

a. Show that $a = 1$.

$f(x)$ & $g(x)$ intersect at $x = a$.

$f(x) = (x-1)^2 \therefore$ Intersects x axis at $x = 1$

$g(x) = \log_e(x)$ So $g(x)$ also has x-int of 1

$0 = \log_e x$ $\quad \therefore a = 1$

$\therefore 1 = x$

1 mark

(1) something sensible

b. Find the value of x for which the difference between the value of f and the value of g is a maximum for $x \in [1, b]$.

Deriv
(1) $h'(x) = 0$
(1) ~~Attempt~~ Attempt to solve quadratic
(1) ans.

For $x \in [1, b)$, $g(x) > f(x)$

$g(x) - f(x) = \log_e(x) - (x-1)^2$

Deriv $= \frac{1}{x} - 2(x-1)$

For max/min $0 = \frac{1}{x} - 2(x-1)$

$\therefore \frac{1}{x} = 2x - 2$

$\therefore 1 = 2x^2 - 2x$

$\therefore 0 = 2x^2 - 2x - 1$

$\therefore x = \frac{2 \pm \sqrt{12}}{4}$

$= \frac{1 \pm \sqrt{3}}{2}$

3 marks

~~From graph above~~

$x \in [1, b)$ so min^m occurs at $x = \frac{1 + \sqrt{3}}{2}$