St Leonard's College

Mathematical Methods ¾

Calculus – Are you Exam Ready?

Instructions: There are 4 questions in each section:

- Part 1
- Part 2 Multiple Choice
- Part 2 Extended Response

Groups of 3.
The whole group collaboratively does Q1 in each section. Then, Person A does Q2 in each section, Person B does Q3 in each section and Person C does Q4 in each section.

Part 1 – No Calculator Allowed. No reference material.

Question 1

Let $f(x) = \tan\left(\frac{\sqrt{x}}{2}\right)$.

a. If $f(x) = g(h(x))$ write down, the rules for the functions $g(x)$ and $h(x)$.

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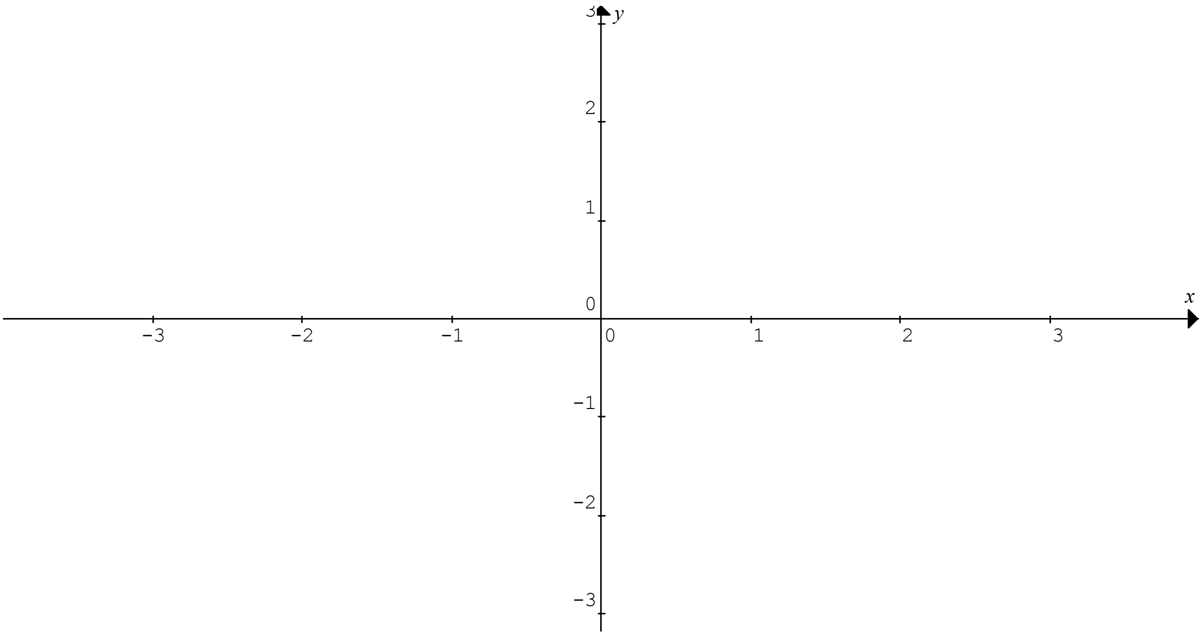
1 mark

b. Evaluate $f'\left(\frac{\pi^2}{9}\right)$.

3 marks

Question 2

a. Sketch the graph of function $y = 1 - \frac{4}{(2x+3)^2}$ on the axes below, clearly indicating all axial cuts and equation of any asymptotes.



2 marks

b. Find the area bounded by the curve $y = 1 - \frac{4}{(2x+3)^2}$, the co-ordinates axes and $x = 1$.

3 marks

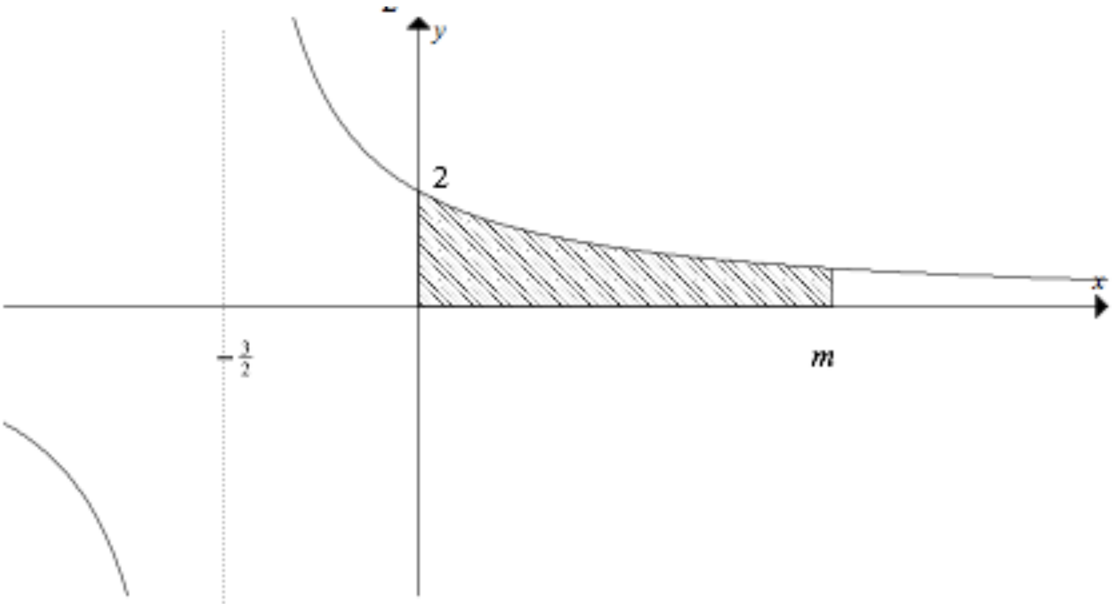
Question 3

Differentiate $x e^{-3x}$ and, hence, find $\int x e^{-3x} \, dx$.

3 marks

Question 4

Consider the graph of the function $f : R \setminus \{-\frac{3}{2}\} \to R,\ f(x) = \dfrac{b}{2x + a}$. The graph has a vertical asymptote at $x = -\dfrac{3}{2}$ and crosses the y-axis at $y = 2$, as shown below.



The shaded area is the area bounded by the graph of $y = \dfrac{b}{2x + a}$, the coordinate axes and the line $x = m$. If the shaded area is equal to $\log_e(27)$ square units, find the values of a, b and m.

4 marks

Part 2 – Calculator Allowed. Reference Material Allowed.

Question 1

If $f(x)$ and $g(x)$ are two differentiable functions with $f'(x)=\frac{d}{dx}(f(x))$ and $g'(x)=\frac{d}{dx}(g(x))$, then $\frac{d}{dx}(f(g(x)))$ is equal to

A. $f'(g(x))$

B. $f'(g'(x))$

C. $f'(g(x))+f(g'(x))$

D. $g'(x)f'(g(x))$

E. $f'(x)g'(x)$

Question 2

The average value of the function with the rule $f(x)=x^3+e^{2x}$ between $x=0$ and $x=2$ is equal to

A. $\frac{7+e^4}{2}$

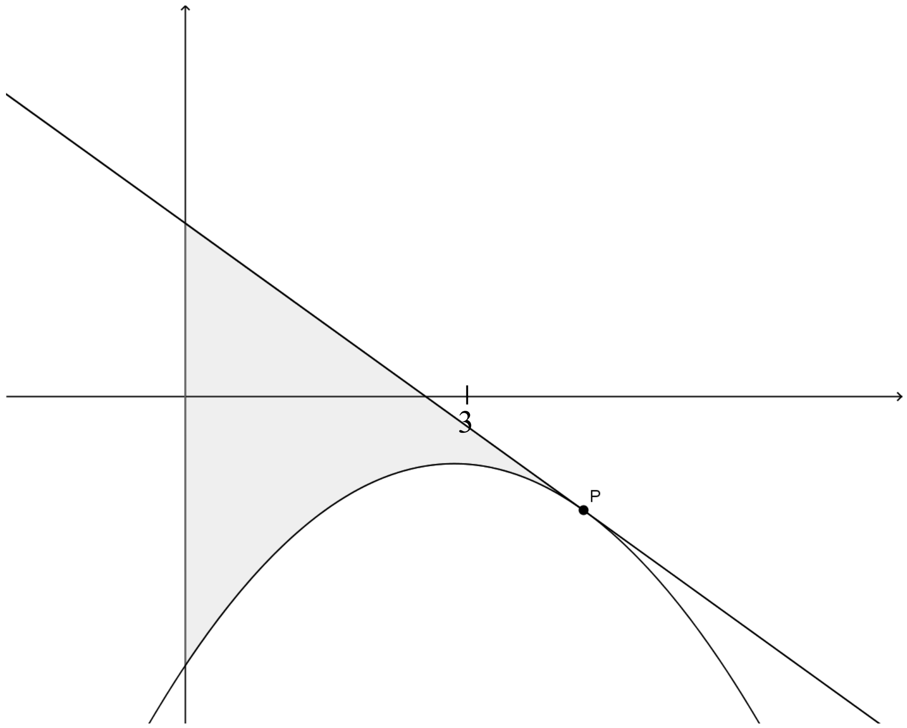
B. $\frac{7+e^4}{4}$

C. $12+2e^4$

D. $5+e^4$

E. $\frac{11+2e^4}{2}$

Question 3



The diagram above show the graph of $y=-x^2+3x-3$ and the tangent to the graph at the point P, where $x=3$. The tangent has the equation $y=mx+c$. The shaded area A is the area between the graph, the tangent and the y-axis. Which of the following is **FALSE**?

A. $c=6$

B. $m=-3$

C. $-3m+6=-3$

D. $m<0$ and $c>0$

E. $A=\int_0^3 \left(x^2+(m-3)x+(c+3)\right)dx$

Question 4

If $f(x)=f(-x)$ and $\int_{-6}^{6} f(x)\,dx=10$, then $\int_0^6 \left(2f(x)-1\right)dx$ is equal to

A. 4

B. 14

C. 7

D. $10-x$

E. $20-x$

Extended Response Questions
Question 1

Consider the function $f: R \rightarrow R$, $f(x) = x^3 - 3x^2 + cx + d$, where c and d are real numbers.

a. The coordinates of the turning point on the graph of $y = f(x)$ are $(-1, 5)$ and (u, v).

i. Show that in this case $c = -9$, $d = 0$ $v = -27$ and determine the value of u.

2 marks

ii. For what values of d does the graph of $y = x^3 - 3x^2 - 9x + d$ cross the x-axis at three distinct points?

2 marks

b. For what values of c and d does the graph of $y = x^3 - 3x^2 + cx + d$ have two distinct turning points?

2 marks

c. If the graph of $y = f(x)$ is translated p units to the left away from the y-axis, it becomes the graph of $y = x^3$. Find the values of p, c and d in this case.

2 marks

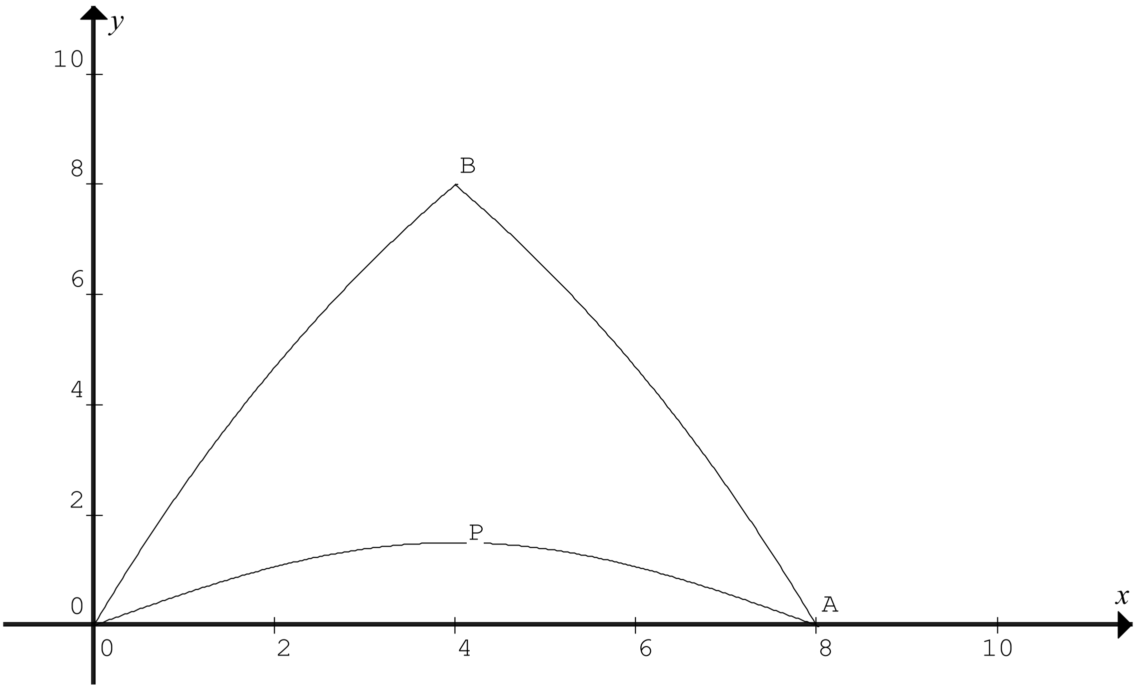
d. Let A be the area bounded by the graph of $y = x^3 - 3x^2 + cx + d$, the coordinate axes and $x = 2$, and that $y \geq 0$ for $0 \leq x \leq 2$. If this area is approximated by four equally spaced left rectangles, the area is 10 square units. If this area is approximated by four equally spaced right rectangles, the area is 6 square units. The exact area bounded by the graph of $y = x^3 - 3x^2 + cx + d$, the coordinate axes and $x = 2$ is 8 square units. Determine the values of c and d.

5 marks

Total 13 marks

Question 2

The diagram below shows a "triangular" shade cloth, which is designed to block the direct sunlight onto a children's playground. The cloth lies in a horizontal plane and has vertical posts erected at points O, A and B. The point O is the origin and the co-ordinates of the points A , P and B are $(8,0)$, $(4,1.5)$ and $(4,8)$ respectively. The axes are shown on the drawing and the units are in metres.



The curve OPA has the form $y = a\sin(nx)$.

a. Explain why $a = 1.5$ and $n = \dfrac{\pi}{8}$.

1 mark

The curve OB has the form $y = 16\left(1 - e^{-kx}\right)$.

b. Find the value of k.

1 mark

c. The curve BA is the reflection of the curve OB in the line $x = 4$.

i. Write down **two** transformations, which take the curve OB into the curve BA.

1 mark

ii. Hence write down a **function** in terms of k, which describes the curve BA.

2 marks

d. i. Write down a definite integral, in terms of k, which gives the total area of the shade cloth.

1 mark

ii. If the area of the shade cloth can be expressed in the form $p + \frac{q}{k} + \frac{r}{\pi}$, find the values of p, q and r.

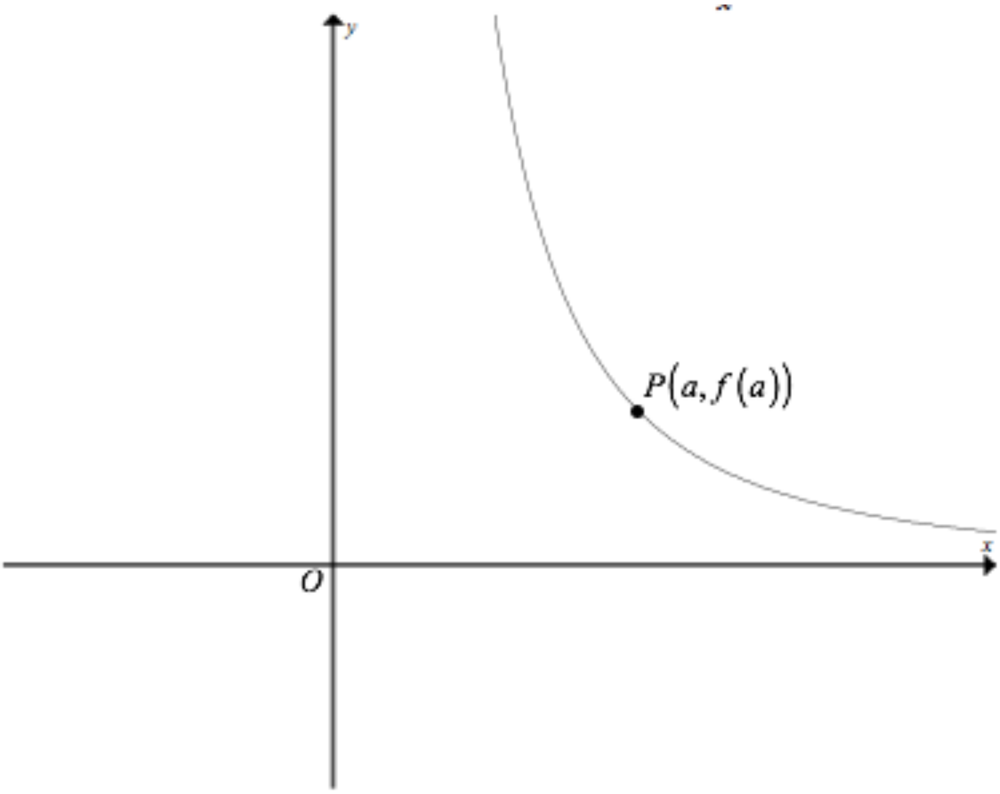
4 marks

Total 10 marks

Question 3

The diagram below shows part of the graph of the function $f:(0,\infty)\to R$ where $f(x)=\dfrac{4}{x^2}$.

Let $P(a,f(a))$ where $a>0$ be a point on the graph of $y=\dfrac{4}{x^2}$.



a. Show that the distance s from the origin O to the point P is given by $s=\dfrac{\sqrt{16+a^6}}{a^2}$.

2 marks

b i. Find, the exact value of a, for which the distance s is a minimum.
Verify that it is a minimum.

3 marks

ii. Find the minimum distance, give an exact answer.

1 mark

c.i. Find in terms of a, the equation of the normal to the curve $y = f(x)$ at the point P.

2 marks

ii. Find the value of a, for which the normal passes through the origin.

1 mark

Total 9 marks

Question 4

Given the function $g : R \to R$, $g(x) = \sqrt{3}\sin(2x) + \cos(2x)$

a. Find the general solution of $g(x) = 0$.

1 mark

b. State the smallest positive value of T for which $g(x+T) = g(x)$.

1 mark

Given the function $f:[0,2\pi]\to R$, $f(x)=\sqrt{3}\sin(2x)+\cos(2x)$.

c. Find $\{x: f(x)=0\}$.

1 mark

d.i. Find $\{x: f'(x)=0\}$.

2 marks

ii. Find the exact coordinates of the maximum and minimum turning points on the graph of $y = f(x)$.

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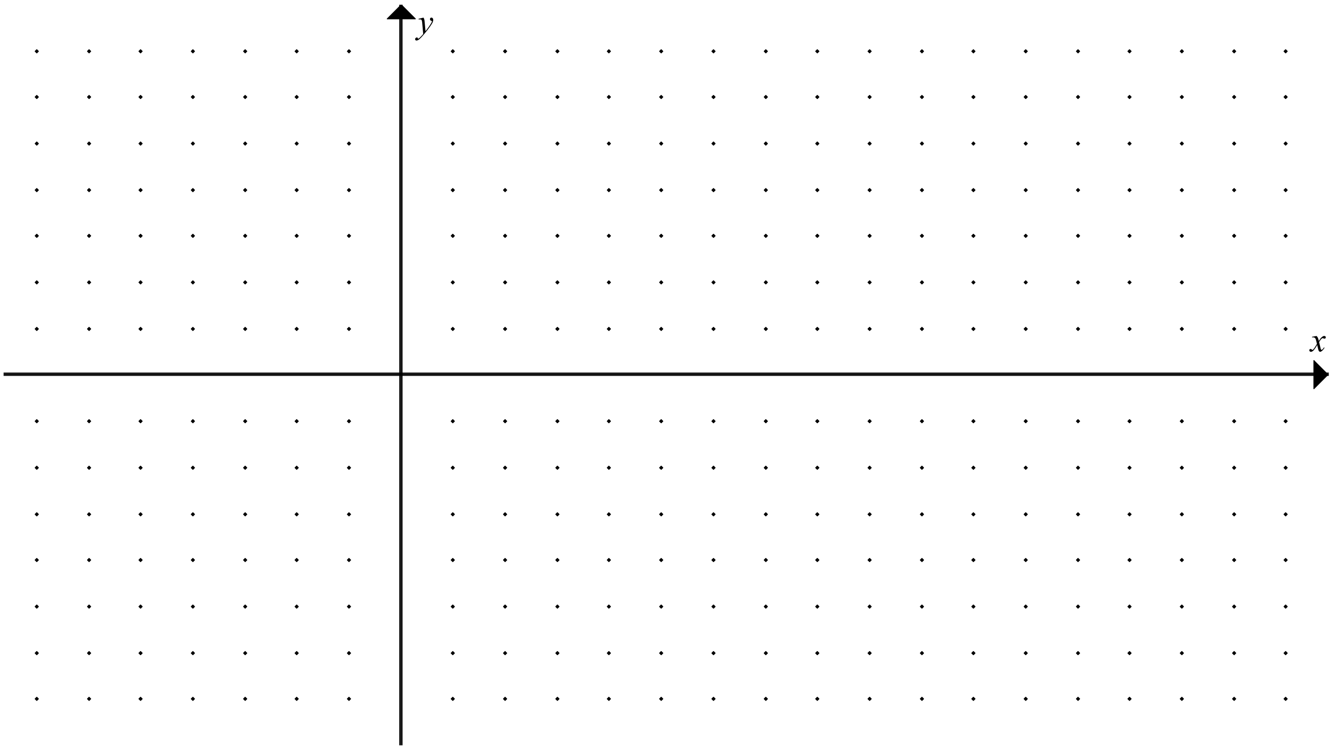
1 mark

e. Sketch the graph of $y = f(x)$ on the axes below, clearly labelling the scale.



2 marks

f. If $f(x) = A\sin(2(x+\alpha))$ state the values of A and α.

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2 marks

Total 10 marks