Answers

Past VCAA exam questions on continuous random variables and the normal distribution.

2011 Exam 2

Question 6

For the continuous random variable X with probability density function

$$f(x) = \begin{cases} \log_e(x) & 1 \le x \le e \\ 0 & elsewhere \end{cases}$$

the expected value of X, $E(X)$, is closest to

A. 0.358

B. 0.5

C. 1

D. 1.859

E. 2.097

$\int_1^e x\log_e(x)\,dx$

$= 2.097$

2011 Exam 2

Question 12

The continuous random variable X has a normal distribution with mean 30 and standard deviation 5. For a given number a, $\Pr(X > a) = 0.20$.

Correct to two decimal places, a is equal to

A. 23.59

B. 24.00

C. 25.79

D. 34.21

E. 36.41

Inverse Normal

Left tail

2011 Exam 2

Question 13

In an orchard of 2000 apple trees it is found that 1735 trees have a height greater than 2.8 metres. The heights are distributed normally with a mean μ and standard deviation 0.2 metres.

The value of μ is closest to

A. 3.023

B. 2.577

C. 2.230

D. 1.115

E. 0.223

$\Pr(x > 2.8) = \frac{1735}{2000}$

solve (normCdf(-∞, 2.8, 0.2, x) = $\frac{1735}{2000}$

$x = 2.577$

This should be A, I found the wrong area as you can see by my solve sentence.

Question 2

In a chocolate factory the material for making each chocolate is sent to one of two machines, machine A or machine B.

The time, X seconds, taken to produce a chocolate by machine A, is normally distributed with mean 3 and standard deviation 0.8.

The time, Y seconds, taken to produce a chocolate by machine B, has the following probability density function.

$$f(y) = \begin{cases} 0 & y < 0 \\ \dfrac{y}{16} & 0 \le y \le 4 \\ 0.25e^{-0.5(y-4)} & y > 4 \end{cases}$$

a. Find correct to four decimal places

i. $\Pr(3 \le X \le 5)$

$X \sim N(3, 0.8^2)$ $\quad \Pr(3 \le x \le 5)$

≈ 0.4938

ii. $\Pr(3 \le Y \le 5)$

$\Pr(3 \le Y \le 5) = \int_3^4 \frac{1}{16} y \, dy + \int_4^5 0.25e^{-\frac{1}{2}(y-4)} dy$

≈ 0.4155

1 + 3 = 4 marks

b. Find the mean of Y, correct to three decimal places.

$E(Y) = \int_0^4 (y \cdot \frac{y}{16}) \, dy + \int_4^\infty y \cdot 0.25e^{-\frac{1}{2}(y-4)} dy$

≈ 4.333

3 marks

c. i. Find the median of Y.

$\int_0^4 \frac{y}{16} dy = 0.5$

So median of Y is 4

ii. Find the value of a, correct to two decimal places, such that $\Pr(Y \le a) = 0.7$.

$\Pr(Y \le a) = 0.7$ but $\Pr(Y \le 4) = 0.5$

Solve, $\int_4^a 0.25e^{-\frac{1}{2}(y-4)} dy = 0.2$

$a = 5.02$

1 + 2 = 3 marks

d. It can be shown that $\Pr(Y \le 3) = \frac{9}{32}$. A random sample of 10 chocolates produced by machine B is chosen. Find the probability, correct to four decimal places, that exactly 4 of these 10 chocolates took 3 or less seconds to produce.

Binomial $n = 10$, $\frac{9}{32}$

Pr(4 chocs take $\le$ 3s to produce)

≈ 0.1812

2 marks

All of the chocolates produced by machine A and machine B are stored in a large bin. There is an equal number of chocolates from each machine in the bin.

It is found that if a chocolate, produced by either machine, takes longer than 3 seconds to produce then it can easily be identified by its darker colour.

e. A chocolate is selected at random from the bin. It is found to have taken longer than 3 seconds to produce. Find, correct to four decimal places, the probability that it was produced by machine A.

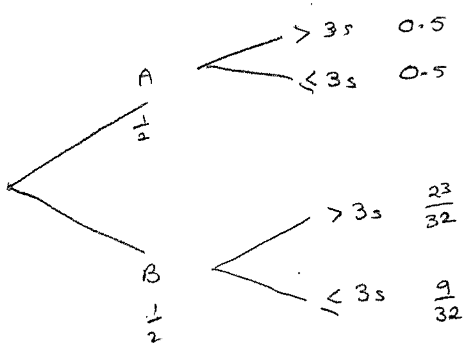
$\Pr(\text{want} \mid \text{know}) = \Pr(\text{Machine A} \mid > 3s)$

$= \dfrac{\Pr(\text{Machine A and} > 3s)}{\Pr(> 3s \text{ from any machine})}$

$= \dfrac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{23}{32}}$

≈ 0.4103

3 marks

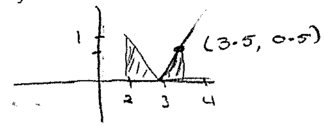


$A \sim N(3, 0.8)$

Question 5

The probability distribution function for the continuous random variable X is given by

$$f(x) = \begin{cases} |3-x| & \text{if } \ 2 \le x \le 4 \\ 0 & \text{elsewhere} \end{cases}$$



a. Find $\Pr(X < 3.5)$.

$= \int_2^{3.5} |3-x| \, dx$

$= \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$ $(= 0.625)$

2 marks

b. Find $\Pr(X < 2.5 \mid X < 3.5)$.

$= \dfrac{\Pr(x < 2.5)}{\Pr(x < 3.5)}$

$\Pr(x < 2.5) =$ unshaded area above $= 1 - \frac{5}{8} = \frac{3}{8}$

$= \dfrac{3/8}{5/8}$

$= 0.6$

2 marks

2010 Exam 2

Question 11

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \cos(2x) & \text{if } \ \dfrac{3\pi}{4} < x < \dfrac{5\pi}{4} \\ 0 & \text{elsewhere} \end{cases}$$

The value of a such that $\Pr(X < a) = 0.25$ is closest to

A. 2.25
B. 2.75
C. 2.88
D. 3.06
E. 3.41

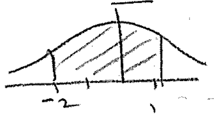
$\int_{\frac{3\pi}{4}}^{a} \cos(2x) \, dx = 0.25$

2010 Exam 2

Question 13

The continuous random variable X has a normal distribution with mean 20 and standard deviation 6. The continuous random variable Z has the standard normal distribution.

The probability that Z is between −2 and 1 is equal to



A. $\Pr(18 < X < 21)$ $= \Pr\left(-\frac{1}{3} < z < \frac{1}{6}\right)$ $\Pr(-2 < z < 1)$

B. $\Pr(14 < X < 32)$ $= \Pr(-1 < z < 2)$

C. $\Pr(14 < X < 26)$

D. $\Pr(8 < X < 32)$

E. $\Pr(X > 14) + \Pr(X < 26)$

$z = \frac{x - \mu}{\sigma}$

2010 Exam 1

Question 5

Let X be a normally distributed random variable with mean 5 and variance 9 and let Z be the random variable with the standard normal distribution.

$\sigma = 3$

a. Find $\Pr(X > 5)$.

$\Pr(x > 5) = \Pr\left(z > \frac{5-5}{3}\right)$

$= \Pr(z > 0) = 0.5$

1 mark

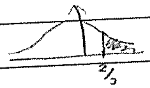
b. Find b such that $\Pr(X > 7) = \Pr(Z < b)$.

$\Pr(x > 7) = \Pr\left(z > \frac{7-5}{3}\right)$

$= \Pr\left(z > \frac{2}{3}\right)$

$= \Pr\left(z < -\frac{2}{3}\right)$

$\therefore\ b = -\frac{2}{3}$



2 marks

2010 Exam 1

2010 Exam 1

Question 7

The continuous random variable X has a distribution with probability density function given by

$$f(x) = \begin{cases} ax(5-x) & \text{if } 0 \le x \le 5 \\ 0 & \text{if } x < 0 \text{ or if } x > 5 \end{cases}$$

where a is a positive constant.

a. Find the value of a.

$\int_0^5 ax(5-x)\,dx = 1$

$a \int_0^5 (5x - x^2)\,dx = 1$

$a \left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_0^5 = 1$

$a\left[\left(\frac{5}{2} \times 25 - \frac{1}{3} \times 125\right) - 0\right] = 1.$

$a\left(\frac{125}{2} - \frac{125}{3}\right) = 1$

$a\left(\frac{3 \times 125 - 2 \times 125}{6}\right) = 1.$

$a\left(\frac{375 - 250}{6}\right) = 1$

$a\left(\frac{125}{6}\right) = 1$

$a = \frac{6}{125}$

3 marks

b. Express $\Pr(X < 3)$ as a definite integral. (Do not evaluate the definite integral.)

$\Pr(x < 3) = \int_0^3 \frac{6}{125} x(5-x)\,dx$

1 mark

2009 Exam 2

Question 6

The continuous random variable X has a normal distribution with mean 14 and standard deviation 2.

If the random variable Z has the standard normal distribution, then the probability that X is greater than 17 is equal to

A. $\Pr(Z > 3)$

B. $\Pr(Z < 2)$

C. $\Pr(Z < 1.5)$

D. $\Pr(Z < -1.5)$

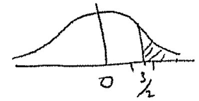
E. $\Pr(Z > 2)$

$\Pr(x > 17)$

$= \Pr\left(z > \frac{17-14}{2}\right)$

$= \Pr\left(z > \frac{3}{2}\right)$

$= \Pr(z < -1.5)$



2009 Exam 2

Question 11

The continuous random variable X has a probability density function given by

$$f(x)=\begin{cases}\pi\sin(2\pi x) & \text{if } 0\le x\le \frac{1}{2}\\ 0 & \text{elsewhere}\end{cases}$$

The value of a such that $\Pr(X>a)=0.2$ is closest to

$\int_a^{\frac{1}{2}} \pi(\sin 2\pi x)\,dx = 0.2$

A. 0.26
B. 0.30
C. 0.32
D. 0.35
E. 0.40

2009 Exam 2

Question 3

The Bouncy Ball Company (BBC) makes tennis balls whose diameters are normally distributed with mean 67 mm and standard deviation 1 mm. The tennis balls are packed and sold in cylindrical tins that each hold four balls. A tennis ball fits into such a tin if the diameter of the ball is less than 68.5 mm.

a. What is the probability, correct to four decimal places, that a randomly selected tennis ball produced by BBC fits into a tin?

$X \sim N(67, 1^2)$

$\Pr(X<68.5) \approx 0.9332$

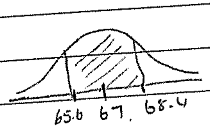
2 marks

BBC management would like each ball produced to have diameter between 65.6 and 68.4 mm.

b. What is the probability, correct to four decimal places, that the diameter of a randomly selected tennis ball made by BBC is in this range?

$\Pr(65.6<X<68.4)$

≈ 0.8385



2 marks

This is worth 1mark.

This question is worth 2 marks. Need to show 1 line of working OR a diagram with correct shaded area.

c. i. What is the probability, correct to four decimal places, that the diameter of a tennis ball which fits into a tin is between 65.6 and 68.4 mm?

$\Pr(\text{want} \mid \text{know}) = \Pr(65.6<X<68.4 \mid \text{fits into tin})$

$= \Pr(65.6<X<68.4 \mid X<68.5)$

$= \dfrac{\Pr(65.6<X<68.4)}{\Pr(X<68.5)}$

$= \dfrac{0.83848\ldots}{0.93319\ldots}$

≈ 0.8985

ii. A tin of four balls is selected at random. What is the probability, correct to four decimal places, that at least one of these balls has diameter outside the desired range of 65.6 to 68.4 mm?

$Y \sim B(4, 0.10148)$ – no. of balls with diameter outside 65.6 and 68.4

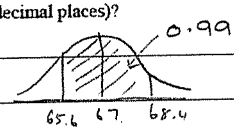
$\Pr(Y\geq 1) \approx 0.3482$

1 + 2 = 3 marks

BBC management wants engineers to change the manufacturing process so that 99% of all balls produced have diameter between 65.6 and 68.4 mm. The mean is to stay at 67 mm but the standard deviation is to be changed.

d. What should the new standard deviation be (correct to two decimal places)?

$X \sim N(67, \sigma^2)$



$\Pr(X<65.6) = \dfrac{0.01}{2} = 0.005$

$\sigma \approx 0.54\text{mm}$

3 marks

2009 Exam 1 had no continuous questions

2008 Exam 2

Question 11

The probability density function for the continuous random variable X is given by

$$f(x)=\begin{cases}|1-x| & \text{if } 0\le x\le 2\\ 0 & \text{elsewhere}\end{cases}$$

The probability that $X<1.5$ is equal to

$\int_0^{1.5} |1-x|\,dx$

A. 0.125
B. 0.375
C. 0.5
D. 0.625
E. 0.75

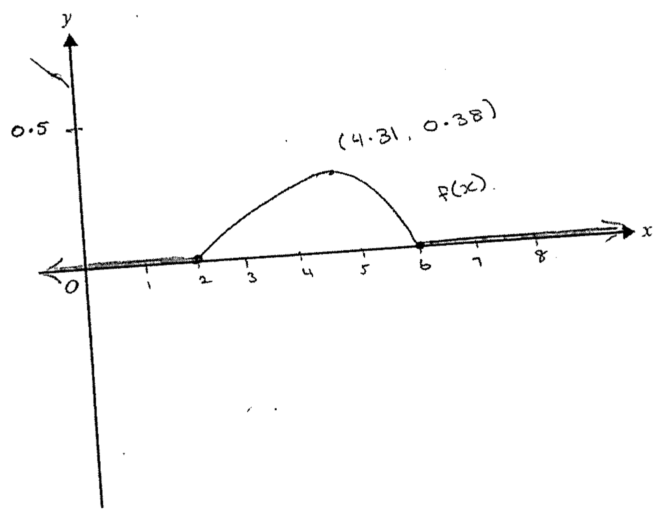
2008 Exam 2

Q1

The time in hours that Sharelle spends training each day is a continuous random variable with probability density function given by

$$f(x)=\begin{cases}\frac{1}{64}(6-x)(x-2)(x+2) & \text{if } 2\le x\le 6\\ 0 & \text{elsewhere}\end{cases}$$

c. i. Sketch the probability density function, and label the local maximum with its coordinates, correct to two decimal places.



ii. What is the probability, correct to four decimal places, that Sharelle spends less than 3 hours training on a particular day?

$Pr(X<3) = \int_2^3 \frac{1}{64}(6-x)(x-2)(x+2)\,dx$

≈ 0.1211

iii. What is the mean time (in hours), correct to four decimal places, that she spends training each day?

$E(X) = \int_2^6 x\,f(x)\,dx$

≈ 4.1333

2 + 2 + 2 = 6 marks

2008 Exam 1

Question 4

The function

$$f(x)=\begin{cases}k\sin(\pi x) & \text{if } x\in[0,1]\\ 0 & \text{otherwise}\end{cases}$$

is a probability density function for the continuous random variable X.

a. Show that $k=\frac{\pi}{2}$.

$\int_0^1 k\sin(\pi x)\,dx = 1$

$k\int_0^1 \sin(\pi x)\,dx = 1$

$k\left[-\frac{1}{\pi}\cos(\pi x)\right]_0^1 = 1$

$k\left(-\frac{1}{\pi}\cos\pi + \frac{1}{\pi}\cos 0\right) = 1$

$k\left(-\frac{1}{\pi}\times -1 + \frac{1}{\pi}\times 1\right) = 1$

$k\left(\frac{1}{\pi}+\frac{1}{\pi}\right) = 1$

$k\times\frac{2}{\pi} = 1$

$k = \frac{\pi}{2}$.

b. Find $\Pr\left(X \le \frac{1}{4} \mid X \le \frac{1}{2}\right)$.

$= \dfrac{\Pr(x \le \frac{1}{4})}{\Pr(x \le \frac{1}{2})}$

$= \dfrac{\int_0^{\frac{1}{4}} \frac{\pi}{2} \sin(\pi x)\, dx}{\int_0^{\frac{1}{2}} \frac{\pi}{2} \sin(\pi x)\, dx}$

$= \dfrac{\frac{\pi}{2}\left[-\frac{1}{\pi}\cos(\pi x)\right]_0^{\frac{1}{4}}}{\frac{\pi}{2}\left[-\frac{1}{\pi}\cos(\pi x)\right]_0^{\frac{1}{2}}}$

$= \dfrac{\left(-\frac{1}{\pi}\cos\frac{\pi}{4} + \frac{1}{\pi}\cos(0)\right)}{\left(-\frac{1}{\pi}\cos\frac{\pi}{2} + \frac{1}{\pi}\cos(0)\right)}$

$= \dfrac{\left(-\cos\frac{\pi}{4} + \cos(0)\right)}{\left(-\cos\frac{\pi}{2} + \cos(0)\right)}$

2 + 3 = 5 marks

$= \dfrac{\left(-\frac{1}{\sqrt{2}} + 1\right)}{(0+1)} = -\frac{1}{\sqrt{2}} + 1 = \frac{\sqrt{2}-1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2-\sqrt{2}}{2}$

2007 Exam 2

Question 7

The random variable X has a normal distribution with mean 11 and standard deviation 0.25.
If the random variable Z has the standard normal distribution, then the probability that X is less than 10.5 is equal to

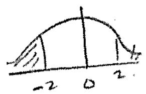
A. $\Pr(Z > 2)$
B. $\Pr(Z < -1.5)$
C. $\Pr(Z < 1)$
D. $\Pr(Z \ge 1.5)$
E. $\Pr(Z < -4)$

$\Pr(x < 10.5)$

$= \Pr\left(z < \frac{10.5 - 11}{0.25}\right)$

$= \Pr(z < -2)$

$= \Pr(z > 2)$



2007 Exam 2

Question 16

If a random variable X has probability density function

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0, 2] \\ 0 & \text{otherwise} \end{cases}$$

then $E(X)$ is equal to

A. $\frac{1}{2}$

B. 1

C. $\frac{4}{3}$

D. $\frac{2}{3}$

E. 2

$E(x) = \int_0^2 x \cdot \frac{x}{2}\, dx$

2007 Exam 2

Question 18

The heights of the children in a queue for an amusement park ride are normally distributed with mean 130 cm and standard deviation 2.7 cm. 35% of the children are not allowed to go on the ride because they are too short.
The minimum acceptable height correct to the nearest centimetre is

A. 126
B. 127
C. 128
D. 129
E. 130

$\Pr(X < a) = 0.35$

Inverse normal

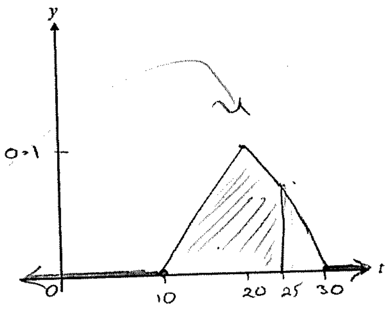
2007 Exam 2

Question 5

In the Great Fun amusement park there is a small train called Puffing Berty which does a circuit of the park. The continuous random variable T, the time in minutes for a circuit to be completed, has a probability density function f with rule

$$f(t) = \begin{cases} \frac{1}{100}(t-10) & \text{if } 10 \le t < 20 \\ \frac{1}{100}(30-t) & \text{if } 20 \le t \le 30 \\ 0 & \text{otherwise} \end{cases}$$

a. Sketch the graph of $y = f(t)$ on the axes provided.



2 marks

b. Find the probability that the time taken by Puffing Berty to complete a full circuit is less than 25 minutes. (Give the exact value.)

$Pr(T < 25) = 1 - \int_{25}^{30} \frac{1}{100}(30-t)\,dt$

$= \frac{7}{8}$

shading the required area gives one method mark!

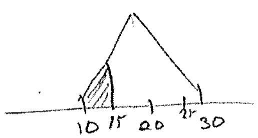
2 marks

c. Find $Pr(T \le 15 \mid T \le 25)$. (Give the exact value.)

$= \frac{Pr(T \le 15)}{Pr(T \le 25)}$

$= \frac{\frac{1}{8}}{\frac{7}{8}} = \frac{1}{7}$

2 marks



The train must complete six circuits between 9.00 am and noon. The management prefers Puffing Berty to complete a circuit in less than 25 minutes.

d. Find the probability, correct to four decimal places, that of the 6 circuits completed, at least 4 of them take less than 25 minutes each.

$Y \sim B(6, \frac{7}{8})$ - no of times a circuit is completed < 25 mins

$Pr(X > 4) \approx 0.9709$

2 marks

For scheduling reasons the management wants to know the time, b minutes, for which the probability of exactly 3 or 4 out of the 6 circuits completed each taking less than b minutes, is maximised.

Let $Pr(T < b) = p$

Let Q be the probability that exactly 3 or 4 circuits completed each take less than b minutes.

e. Show that $Q = 5p^3(1-p)^2(4-p)$.

$Pr(\text{exactly 3 or 4 comp} < b\text{mins}) = \binom{6}{3}(p)^3(1-p)^3 + \binom{6}{4}p^4(1-p)^2$

$= 20p^3(1-p)^3 + 15p^4(1-p)^2$

$= 5p^3(1-p)^2(4(1-p)+3p) = 5p^3(1-p)^2(4-p)$

2 marks

f. i. Find the maximum value of Q and the value of p for which this occurs. (Give the exact value.)

Solve $Q'(p) = 0$

$p = 2 - \sqrt{2}$, $Q = 20(17 - 12\sqrt{2})$

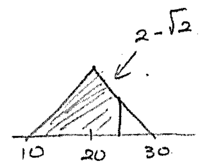
ii. Find, correct to one decimal place, the value of b for which this maximum occurs.

$Pr(T < b) = 2 - \sqrt{2}$

$\int_b^{30} \frac{1}{100}(30-t)\,dt = 1 - (2 - \sqrt{2})$

$b \approx 20.9$

\# You can get methods marks for this question if you draw and shade the required region.



2 + 2 = 4 marks

Total 14 marks

2007 Exam 1 No continuous probability questions

2006 Exam 2

Question 21

The times (in minutes) taken for students to complete a university test are normally distributed with a mean of 200 minutes and standard deviation 10 minutes.

The proportion of students who complete the test in less than 208 minutes is closest to

A. 0.200
B. 0.212
C. 0.758
D. 0.788
E. 0.800

$Pr(X < 208) =$

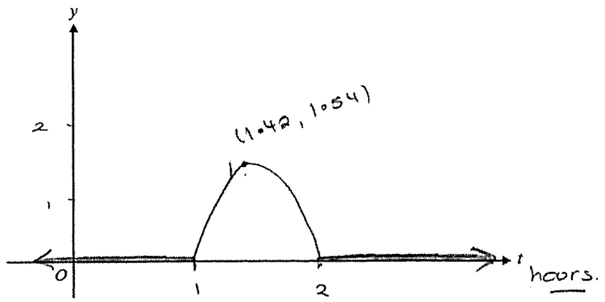
$X \sim N(200, 10^2)$

Q2

When Kim goes to the gym, the time, T hours, that she spends working out is a continuous random variable with probability density function given by

$$f(t) = \begin{cases} 4t^3 - 24t^2 + 44t - 24 & \text{if } 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

c. Sketch the graph of $y = f(t)$ on the axes below. Label any stationary points with their coordinates, correct to two decimal places.



3 marks

d. What is the probability, correct to three decimal places, that she spends less than 75 minutes working out when she goes to the gym?

75 mins = 1.25 hrs

$\int_1^{1.25} (4t^3 - 24t^2 + 44t - 24)\,dt \approx 0.191$

2 marks

e. What is the probability, correct to two decimal places, that she spends more than 75 minutes working out on 4 out of the 5 next times she goes to the gym?

$X \sim B(5, 0.191..)$ – no of times she works out > 75m

$Pr(X=4) \approx 0.41$

2 marks

f. Find the median time, to the nearest minute, that she spends working out in the gym.

Solve $\int_1^m f(t)\,dt = 0.5$

$m \approx 1.4588$ hrs

$= 88$ mins

3 marks

Total 14 marks

Question 5

Let X be a normally distributed random variable with a mean of 72 and a standard deviation of 8. Let Z be the standard normal random variable. Use the result that $\Pr(Z < 1) = 0.84$, correct to two decimal places, to find

a. the probability that X is greater than 80

$Pr(X > 80) = Pr\left(Z > \frac{80-72}{8}\right)$

$= Pr(Z > 1)$

$= 0.16$

1 mark

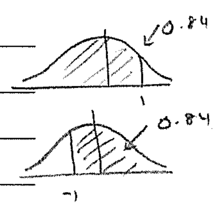
b. the probability that $64 < X < 72$

$Pr(64 < X < 72) = Pr\left(\frac{64-72}{8} < Z < \frac{72-72}{8}\right)$

$= Pr(-1 < Z < 0)$

$= 0.84 - 0.5$

$= 0.34$



1 mark

c. the probability that $X < 64$ given that $X < 72$.

$$\frac{Pr(x<64)}{Pr(x<72)} = \frac{0.16}{0.50} = \frac{.16}{.50} = 0.32$$

1.

2 marks

2006 Exam 1

Question 6

The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{x}{12} & 1 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

a. Find Pr ($X < 3$).

$$Pr(x<3) = \int_1^3 \frac{1}{12} x \, dx$$
$$= \frac{1}{12} \int_1^3 x \, dx$$
$$= \frac{1}{12} \left[\frac{1}{2} x^2 \right]_1^3$$
$$= \frac{1}{12} \left(\frac{9}{2} - \frac{1}{2} \right)$$
$$= \frac{1}{3}$$

2 marks

b. If Pr ($X \ge a$) $= \frac{5}{8}$, find the value of a.

$$\int_a^5 \frac{x}{12} \, dx = \frac{5}{8}$$
$$\frac{1}{12} \left[\frac{1}{2} x^2 \right]_a^5 = \frac{5}{8}$$
$$\frac{1}{12} \left(\frac{25}{2} - \frac{a^2}{2} \right) = \frac{5}{8}$$
$$\frac{25 - a^2}{2} = \frac{60}{8}$$
$$25 - a^2 = \frac{120}{8}$$
$$25 - a^2 = 15$$
$$a = \sqrt{10} \quad \text{since } a > 0$$
$$a \neq -\sqrt{10}$$

2 marks